Optimal Volume-Sensitive Bounds for Polytope Approximation

Sunil Arya

Department of Computer Science and Engineering, The Hong Kong University of Science and Technology, Hong Kong, China

David M. Mount

Department of Computer Science and Institute for Advanced Computer Studies, University of Maryland, College Park, MD, USA

Abstract

Approximating convex bodies is a fundamental question in geometry and has a wide variety of applications. Consider a convex body K of diameter Δ in $\mathbb{R}^d$ for fixed d. The objective is to minimize the number of vertices (alternatively, the number of facets) of an approximating polytope for a given Hausdorff error ε. It is known from classical results of Dudley (1974) and Bronshteyn and Ivanov (1976) that $\Theta((\Delta/\varepsilon)^{(d-1)/2})$ vertices (alternatively, facets) are both necessary and sufficient. While this bound is tight in the worst case, that of Euclidean balls, it is far from optimal for skinny convex bodies.

A natural way to characterize a convex object's skinniness is in terms of its relationship to the Euclidean ball. Given a convex body K, define its *volume diameter* Δ_d to be the diameter of a Euclidean ball of the same volume as K, and define its *surface diameter* Δ_{d-1} analogously for surface area. It follows from generalizations of the isoperimetric inequality that $\Delta \geq \Delta_{d-1} \geq \Delta_d$.

Arya, da Fonseca, and Mount (SoCG 2012) demonstrated that the diameter-based bound could be made surface-area sensitive, improving the above bound to $O((\Delta_{d-1}/\varepsilon)^{(d-1)/2})$. In this paper, we strengthen this by proving the existence of an approximation with $O((\Delta_d/\varepsilon)^{(d-1)/2})$ facets.

This improvement is a result of the combination of a number of new ideas. As in prior work, we exploit properties of the original body and its polar dual. In order to obtain a volume-sensitive bound, we explore the following more general problem. Given two convex bodies, one nested within the other, find a low-complexity convex polytope that is sandwiched between them. We show that this problem can be reduced to a covering problem involving a natural intermediate body based on the harmonic mean. Our proof relies on a geometric analysis of a relative notion of fatness involving these bodies.

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1 Introduction

Approximating convex bodies by polytopes is a fundamental problem, which has been extensively studied in the literature (see, e.g., Bronstein [11]). We are given a convex body K in Euclidean d-dimensional space and an error parameter $\varepsilon > 0$. The problem is to determine the minimum combinatorial complexity of a polytope that is ε-close to K according to some measure of similarity. In this paper, we define similarity in terms of the Hausdorff distance [11], and we define combinatorial complexity in terms of the number of facets. Throughout, we assume that the dimension d is a constant.

Approximation bounds presented in the literature are of two common types. In both cases, it is shown that there exists $\varepsilon_0 > 0$ such that the bounds hold for all $\varepsilon \leq \varepsilon_0$. The first of these are *nonuniform bounds*, where the value of ε_0 may depend on properties of K, for example, bounds on its maximum curvature [8, 12, 16, 19, 29, 32]. This is in contrast to *uniform bounds*, where the value of ε_0 is independent of K (but may depend on d).

Examples of uniform bounds include the classical work of Dudley [13] and Bronshteyn and Ivanov [10]. Dudley showed that, for $\varepsilon \leq 1$, any convex body K can be ε-approximated by a polytope P with $O((\Delta/\varepsilon)^{(d-1)/2})$ facets, where Δ is K's diameter. Bronshteyn and Ivanov showed the same bound holds for the number of vertices. Constants hidden in the O-notation depend only on d. These results have numerous applications in computational geometry, for example the construction of coresets [1, 3, 5].

The approximation bounds of both Dudley and Bronshteyn-Ivanov are tight in the worst case up to constant factors (specifically when K is a Euclidean ball) [11]. However, these bounds may be significantly suboptimal if K is "skinny". A natural way to characterize a convex object's skinniness is in terms of its relationship to the Euclidean ball. Given a convex body K, define its *volume diameter* Δ_d to be the diameter of a Euclidean ball of the same volume as K, and define its *surface diameter* Δ_{d-1} analogously for surface area. These quantities are closely related (up to constant factors) to the classical concepts of *quermassintegrals* and of *intrinsic volumes* of the convex body [20, 21]. It follows from generalizations of the isoperimetric inequality that $\Delta \geq \Delta_{d-1} \geq \Delta_d$ [21].

Arya, da Fonseca, and Mount [4] proved that the diameter-based bound could be made surface-area sensitive, improving the above bound to $O((\Delta_{d-1}/\varepsilon)^{(d-1)/2})$. In this paper, we strengthen this to the following volume-sensitive bound.

▶ **Theorem 1.1.** *Consider real d-space, $\mathbb{R}^d$. There exists a constant c_d (depending on d) such that for any convex body $K \subseteq \mathbb{R}^d$ and any $\varepsilon > 0$, if the width of K in any direction is at least ε, then there exists an ε-approximating polytope P whose number of facets is at most*

$$\left(\frac{c_d \Delta_d}{\varepsilon}\right)^{\frac{d-1}{2}}.$$

This bound is the strongest to date. For example, in $\mathbb{R}^3$, the area-sensitive bound yields better bounds for pencil-like objects that are thin along two dimensions, while the volume-sensitive bound yields better bounds for pancake-like objects as well, which are thin in just one dimension.

The minimum-width assumption seems to be a technical necessity, since it is not difficult to construct counterexamples where this condition does not hold. But this is not a fundamental impediment. If the body's width is less than ε in some direction, then by projecting the body onto a hyperplane orthogonal to this direction, it is possible to reduce the problem to a convex approximation problem in one lower dimension. This can be repeated until the body's width is sufficiently large in all remaining dimensions, and the stated bound can be applied in this lower dimensional subspace, albeit with volume measured appropriate to this dimension.

While our uniform bound trivially holds in the nonuniform setting, we present a separate (and much shorter) proof that the same bounds hold in the nonuniform setting, assuming that K's boundary is C^2 continuous. This is presented in the full version.

▶ **Theorem 1.2.** *Consider real d-space, $\mathbb{R}^d$. There exists a constant c_d (depending on d) such that for any convex body $K \subseteq \mathbb{R}^d$ of C^2 boundary, as ε approaches zero, there exists an ε-approximating polytope P whose number of facets is at most*

$$\left(\frac{c_d \Delta_d}{\varepsilon}\right)^{\frac{d-1}{2}}.$$

2 Overview of Techniques

Broadly speaking, the problem of approximating a convex body by a polytope involves "sandwiching" a polytope between two nested convex bodies, call them K_0 and K_1. For example, K_0 may be the original body to be approximated and K_1 is an expansion based on the allowed error bound. Most of the prior work in this area has focused on the specific manner in which K_1 is defined relative to K_0, which is typically confined to Euclidean space (for Hausdorff distance) or affine space (for the Banach-Mazur distance).

Recent approaches to convex approximation have been based on covering the body to be approximated with convex objects that respect the local shape of the body being approximated [2, 6]. Macbeath regions have been a key tool in this regard. Given a convex body K and a point x in K's interior, the Macbeath region at x, $M_K(x)$, is the largest centrally symmetric body nested within K and centered at x (see Figure 1(a)). A Macbeath region that has been shrunken by some constant factor λ is denoted by $M_K^\lambda(x)$. Shrunken Macbeath regions have nice packing and covering properties, and they behave much like metric balls.

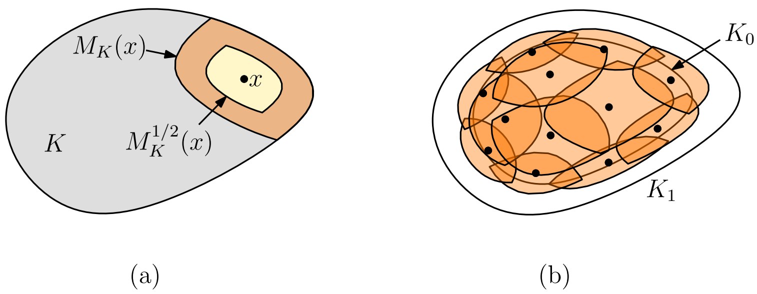

Figure 1 (a) Macbeath regions and (b) covering K_0 by Macbeath regions.

A natural way to construct a sandwiching polytope between two nested bodies K_0 and K_1 is to construct a collection of shrunken Macbeath regions that cover K_0 but lie entirely within K_1 (see Figure 1(b)). If done properly, a sandwiching polytope can be constructed by sampling a constant number of points from each of these Macbeath regions, and taking the convex hull of their union. Thus, the number of Macbeath regions provides an upper bound on the number of vertices in the sandwiched polytope.

The "sandwiching" perspective described above yields additional new challenges. Consider the two bodies K_0 and K_1 shown in Figure 2, where K_0 is a diamond shape nested within the square K_1. Consider 1/2-scaled Macbeath region centered at a point x that lies at the top vertex of K_0. Observe that almost all of its volume lies outside of K_0. This is problematic because our analysis is based on the number of Macbeath regions needed to cover the boundary of a body, in this case ∂K_0. We want a significant amount of the volume of each Macbeath region to lie within K_0. In cases like that shown in Figure 2, only a tiny fraction of the volume can be charged in this manner against K_0.

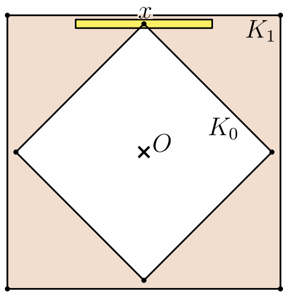

Figure 2 Relative fatness.

Intuitively, while the body K_0 is "fat" in a standard sense[1], it is not fat "relative" to the enclosing body K_1. To deal with this inconvenience, we will replace K_1 with an intermediate body between K_0 and K_1 that satisfies this property. In Section 3.5 we formally define this notion of relative fatness, and we present an intermediate body, called the *harmonic-mean body*, that satisfies this notion of fatness. We will see that this body can be used as a proxy for the sake of approximation.

3 Preliminaries

In this section, we introduce terminology and notation, which will be used throughout the paper. This section can be skipped on first reading (moving directly to Section 4).

Let us first recall some standard notation. Given vectors $u, v \in \mathbb{R}^d$, let $\langle u, v \rangle$ denote their dot product, and let $\|v\| = \sqrt{\langle v, v \rangle}$ denote v's Euclidean length. Throughout, we will use the terms *point* and *vector* interchangeably. Given points $p, q \in \mathbb{R}^d$, let $\|pq\| = \|p - q\|$ denote the Euclidean distance between them. Let $\mathrm{vol}(\cdot)$ and $\mathrm{area}(\cdot)$ denote the d-dimensional and $(d-1)$-dimensional Lebesgue measures, respectively.

3.1 Polarity and Centrality Properties

Given a bounded convex body $K \subseteq \mathbb{R}^d$ that contains the origin O in its interior, define its *polar*, denoted K^*, to be the convex set

$$K^* \;=\; \{u : \langle u, v \rangle \leq 1, \text{ for all } v \in K\}.$$

The polar enjoys many useful properties (see, e.g., Eggleston [14]). For example, it is well known that K^* is bounded and $(K^*)^* = K$. Further, if K_1 and K_2 are two convex bodies both containing the origin such that $K_1 \subseteq K_2$, then $K_2^* \subseteq K_1^*$.

Given a nonzero vector $v \in \mathbb{R}^d$, we define its "polar" v^* to be the hyperplane that is orthogonal to v and at distance $1/\|v\|$ from the origin, on the same side of the origin as v. The polar of a hyperplane is defined as the inverse of this mapping. We may equivalently define K^* as the intersection of the closed halfspaces that contain the origin, bounded by the hyperplanes v^*, for all $v \in K$.

Given a convex body $K \subseteq \mathbb{R}^d$ and $x \in \mathrm{int}(K)$, there are many ways to characterize the property that x is "central" within K [17, 31]. For our purposes, we will make it precise using the concept of Mahler volume. Define K's *Mahler volume*, denoted $\mu(K)$, to be the

[1] That is, the largest ball enclosed in K_0 and the smallest ball containing K_0 differ in size by a constant.

product $\mathrm{vol}(K) \cdot \mathrm{vol}(K^*)$. The Mahler volume is well studied (see, e.g. [28, 22, 30]). It is invariant under linear transformations, and it depends on the location of the origin within K. We say that K is *well-centered* with respect to a point $x \in \mathrm{int}(K)$ if the Mahler volume $\mu(K - x)$ is at most $O(1)$. When x is not specified, it is understood to be the origin. We have the following lemma [6, 23].

▶ **Lemma 3.1.** *Any convex body K is well-centered with respect to its centroid.*

Lower bounds on the Mahler volume have also been extensively studied and it is known that the following bound holds irrespective of the location of the origin [9, 18, 25].

▶ **Lemma 3.2.** *Given a convex body $K \subseteq \mathbb{R}^d$ whose interior contains the origin, $\mu(K) = \Omega(1)$.*

3.2 Caps, Rays, and Relative Measures

Consider a compact convex body K in d-dimensional space $\mathbb{R}^d$ with the origin O in its interior. A *cap* C of K is defined to be the nonempty intersection of K with a halfspace. Letting h_1 denote a hyperplane that does not pass through the origin, let $\mathrm{cap}_K(h_1)$ denote the cap resulting by intersecting K with the halfspace bounded by h_1 that does not contain the origin (see Figure 3(a)). Define the *base* of C, denoted $\mathrm{base}(C)$, to be $h_1 \cap K$. Letting h_0 denote a supporting hyperplane for K and C parallel to h_1, define an *apex* of C to be any point of $h_0 \cap K$.

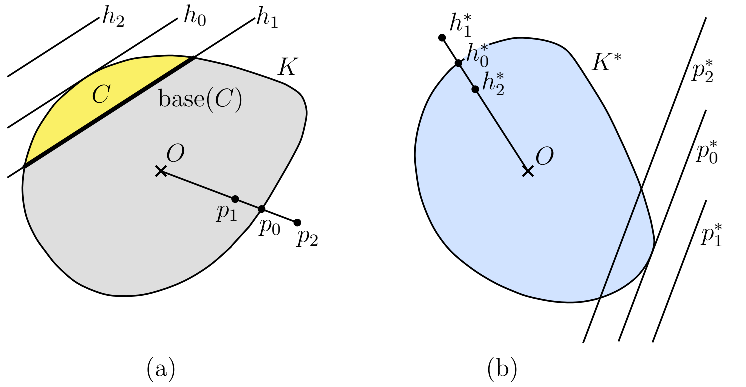

Figure 3 Convex body K and polar K^* with definitions used for width and ray.

We define the *absolute width* of cap C to be $\mathrm{dist}(h_1, h_0)$. When a cap does not contain the origin, it will be convenient to define distances in relative terms. Define the *relative width* of such a cap C, denoted $\mathrm{wid}_K(C)$, to be the ratio $\mathrm{dist}(h_1, h_0)/\mathrm{dist}(O, h_0)$ and, to simplify notation, define $\mathrm{wid}_K(h_1) = \mathrm{wid}_K(\mathrm{cap}_K(h_1))$. Observe that as a hyperplane is translated from a supporting hyperplane to the origin, the relative width of its cap ranges from 0 to a limiting value of 1.

We also characterize the closeness of a point to the boundary in both absolute and relative terms. Given a point $p_1 \in K$, let p_0 denote the point of intersection of the ray Op_1 with the boundary of K. Define the *absolute ray distance* of p_1 to be $\|p_1p_0\|$, and define the *relative ray distance* of p_1, denoted $\mathrm{ray}_K(p_1)$, to be the ratio $\|p_1p_0\|/\|Op_0\|$. Relative widths and relative ray distances are both affine invariants, and unless otherwise specified, references to widths and ray distances will be understood to be in the relative sense.

We can also define volumes in a manner that is affine invariant. Recall that $\mathrm{vol}(\cdot)$ denotes the standard Lebesgue volume measure. For any region $\Lambda \subseteq K$, define the *relative volume* of Λ with respect to K, denoted $\mathrm{vol}_K(\Lambda)$, to be $\mathrm{vol}(\Lambda)/\mathrm{vol}(K)$.

With the aid of the polar transformation we can extend the concepts of width and ray distance to objects lying outside of K. Consider a hyperplane h_2 parallel to h_1 that lies beyond the supporting hyperplane h_0 (see Figure 3(a)). It follows that $h_2^* \in K^*$, and we define $\mathrm{wid}_K(h_2) = \mathrm{ray}_{K^*}(h_2^*)$ (see Figure 3(b)). Similarly, for a point $p_2 \notin K$ that lies along the ray Op_1, it follows that the hyperplane p_2^* intersects K^*, and we define $\mathrm{ray}_K(p_2) = \mathrm{wid}_{K^*}(p_2^*)$. By properties of the polar transformation, it is easy to see that $\mathrm{wid}_K(h_2) = \mathrm{dist}(h_0, h_2)/\,\mathrm{dist}(O, h_2)$. Similarly, $\mathrm{ray}_K(p_2) = \|p_0p_2\|/\|Op_2\|$. Henceforth, we will omit references to K when it is clear from context.

Some of our results apply only when we are sufficiently close to the boundary of K. Given $\alpha \le \frac{1}{2}$, we say that a cap C is *α-shallow* if $\mathrm{wid}(C) \le \alpha$, and we say that a point p is *α-shallow* if $\mathrm{ray}(p) \le \alpha$. We will simply say *shallow* to mean α-shallow, where $\alpha \le \frac{1}{2}$ is a sufficiently small constant.

3.3 Macbeath Regions and MNets

Given a convex body K and a point $x \in K$, and a scaling factor $\lambda > 0$, the *Macbeath region* $M_K^\lambda(x)$ is defined as

$$M_K^\lambda(x) \;=\; x + \lambda((K - x) \cap (x - K)).$$

It is easy to see that $M_K^1(x)$ is the intersection of K with the reflection of K around x, and so $M_K^1(x)$ is centrally symmetric about x. Indeed, it is the largest centrally symmetric body centered at x and contained in K. Furthermore, $M_K^\lambda(x)$ is a copy of $M_K^1(x)$ scaled by the factor λ about the center x (see Figure 1(a)). We will omit the subscript K when the convex body is clear from the context. As a convenience, we define $M(x) = M^1(x)$.

The following lemma states that points in a shrunken Macbeath region all have similar ray distances. The proof appears in [7, Section 2.5].

▶ **Lemma 3.3.** *Let K be a convex body. If x is a $\frac{1}{2}$-shallow point in K and $y \in M^{1/5}(x)$, then* $\mathrm{ray}(x)/2 \le \mathrm{ray}(y) \le 2\,\mathrm{ray}(x)$.

The next lemma shows that translated copies of a Macbeath region act as proxies for Macbeath regions in the vicinity. The proof appears in Section 3.3 of the full version.

▶ **Lemma 3.4.** *Let $\lambda \le 1/2$ and $\gamma \le 1/10$. Let x be a point in a convex body K. Let $R = M(x) - x$. Let y be a point in $x + \lambda R$. Then $y + \gamma R \subseteq M^{2\gamma}(y)$.*

We employ Macbeath region-based coverings in our polytope approximation scheme. In particular, we employ the concept of MNets, as defined in [6]. Let $K \subseteq \mathbb{R}^d$ be a convex body, let Λ be an arbitrary subset of $\mathrm{int}(K)$, and let $c \ge 5$ be any constant. Given $X \subseteq K$, define $\mathcal{M}_K^\lambda(X) = \{M_K^\lambda(x) : x \in X\}$. Define a *$(K, \Lambda, c)$-MNet* to be any maximal set of points $X \subseteq \Lambda$ such that the shrunken Macbeath regions $\mathcal{M}_K^{1/4c}(X)$ are pairwise disjoint. We refer to c as the expansion factor of the MNet. The following lemma, proved in [6], summarizes the key properties of MNets.

▶ **Lemma 3.5** ([6])**.** *Given a convex body $K \subseteq \mathbb{R}^d$, $\Lambda \subset \mathrm{int}(K)$, and $c \ge 5$, a (K, Λ, c)-MNet X satisfies the following properties:*

- *(Packing) The elements of $\mathcal{M}_K^{1/4c}(X)$ are pairwise disjoint.*
- *(Covering) The union of $\mathcal{M}_K^{1/c}(X)$ covers Λ.*
- *(Buffering) The union of $\mathcal{M}_K(X)$ is contained within K.*

For the purposes of this paper, c will be any sufficiently large constant, specifically $c \geq 5$. To simplify notation, we use (K, Λ)-MNet to refer to such an MNet.

As mentioned before, we reduce our polytope approximation problem to that of finding a polytope which is sandwiched between two convex bodies. In turn we tackle this problem using MNets as indicated in the next lemma. The proof appears in Section 3.3 of the full version.

▶ **Lemma 3.6.** *Let $K_0 \subset K_1$ be two convex bodies. Let X be a $(K_1, \partial K_0)$-MNet. Then there exists a polytope P with $O(|X|)$ vertices such that $K_0 \subseteq P \subseteq K_1$.*

The following lemma bounds the sizes of MNets in important special cases involving points at roughly the same ray distance. These bounds will be useful in obtaining our volume-sensitive bounds. The proof appears in Section 4 of the full version.

▶ **Lemma 3.7.** *Let $0 < \varepsilon \leq 1/2$ be sufficiently small and let $K \subseteq \mathbb{R}^d$ be a well-centered convex body. Let Λ be the points of K at ray distances between ε and 2ε, and let X be a (K, Λ)-MNet. Then:*

(i) $|X| = O(1/\varepsilon^{(d-1)/2})$.

(ii) *For any positive real $f \leq 1$, let $X_f \subseteq K$ be such that the total relative volume of the Macbeath regions of $\mathcal{M}^{1/4c}(X_f)$ is $O(f\varepsilon)$. Then $|X_f| = O(\sqrt{f}/\varepsilon^{(d-1)/2})$.*

3.4 Concepts from Projective Geometry

In this section we present some relevant standard concepts from projective geometry. For further details see any standard reference (e.g., [27]). Given four collinear points, a, b, c, d (not necessarily in this order), the *cross ratio* $(a, b; c, d)$ is defined to be $(\|ac\|/\|ad\|)/(\|bc\|/\|bd\|)$, where these are understood to be signed distances determined by the orientations of the segments along the line. We follow the convention of using symbols $a, b, c, d, \ldots$ for points, and the distinction from other uses (such as d for the dimension) should be clear from the context.

It is well known that cross ratios are preserved under projective transformations. If the cross ratio $(a, b; c, d)$ is -1, we say that this quadruple of points forms a *harmonic bundle* (see Figure 4). This is an important special case which occurs frequently in constructions. In this case, the points lie on the line in the order of a, d, b, c and the ratio in which a divides c and d externally (i.e., $\|ac\|/\|ad\|$) is the same as the ratio in which b divides c and d internally (i.e., $\|bc\|/\|bd\|$). The sign is negative since bc and bd have opposite directions. If the point a is at infinity, the cross ratio degenerates to $\|bd\|/\|bc\|$, implying that b is midway between c and d.

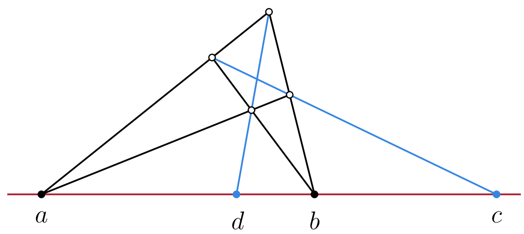

Figure 4 Harmonic bundle (from the quadrilateral construction [27]).

3.5 Intermediate Bodies

In this section we explore the concept of relative fatness, which was introduced in Section 2. Given two convex bodies K_0 and K_1 such that $K_0 \subset K_1$ and $0 < \gamma < 1$, we say that K_0 is *relatively γ-fat* with respect to K_1 if, for any point $p \in \partial K_0$, and any scaling factor $0 < \lambda \leq 1$, at least a constant fraction γ of the volume of the Macbeath region $M = M^{\lambda}_{K_1}(p)$ lies within K_0, that is, $\mathrm{vol}(M \cap K_0)/\mathrm{vol}(M) \geq \gamma$. We say that K_0 is *relatively fat* with respect to K_1 if it is relatively γ-fat for some constant γ. Relative fatness will play an important role in our analyses. Since an arbitrary nested pair $K_0 \subset K_1$ may not necessarily satisfy this property, it will be useful to define an intermediate body sandwiched between K_0 and K_1 that does.

There are a few natural ways to define such an intermediate body. Given two convex bodies K_0 and K_1, where $K_0 \subseteq K_1$, the *arithmetic-mean body*, $K_A(K_0, K_1)$, is defined to be the convex body $\frac{1}{2}(K_0 \oplus K_1)$, where "$\oplus$" denotes Minkowski sum. Equivalently, for any unit vector u consider the two supporting halfspaces of K_0 and K_1 orthogonal to u, and take the halfspace that is midway between the two. The arithmetic-mean body is obtained by intersecting such halfspaces for all unit vectors u.

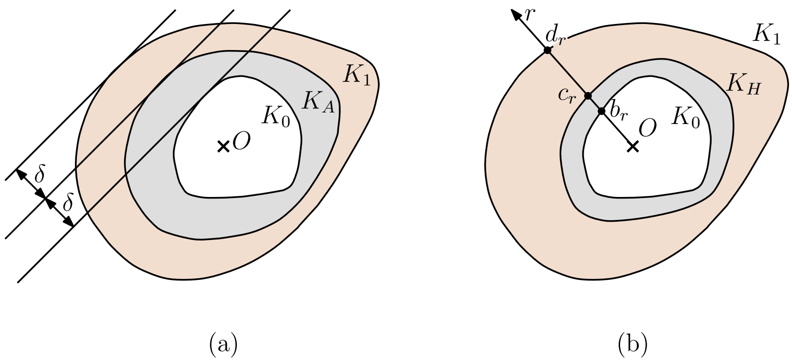

Figure 5 (a) Arithmetic and (b) harmonic-mean bodies.

Another natural choice arises from a polar viewpoint. Assume that $K_0 \subset K_1$ and the origin $O \in \mathrm{int}(K_0)$. The *harmonic-mean body*, $K_H(K_0, K_1)$, was introduced by Firey [15] and is defined as follows. For any ray r from the origin O, let b_r and d_r denote the points of intersection of r with ∂K_0 and ∂K_1, respectively (see Figure 5(b)). Let c_r be the point on the ray such that $1/\|Oc_r\| = (1/\|Ob_r\| + 1/\|Od_r\|)/2$. Equivalently, the cross ratio $(O, c_r; d_r, b_r)$ equals -1, that is, this quadruple forms a harmonic bundle. Clearly, c_r lies between b_r and d_r, and hence the union of these points over all rays r defines the boundary of a body that is sandwiched between K_0 and K_1. This body is the harmonic-mean body. By considering the supporting hyperplanes orthogonal to the ray r, it is easy to see that the arithmetic-mean body of K_0 and K_1 is mapped to the harmonic-mean body of K_0^* and K_1^* under polarity, that is, $(K_A(K_0, K_1))^* = K_H(K_0^*, K_1^*)$. Therefore, $K_H(K_0, K_1)$ is convex. When K_0 and K_1 are clear from context, we will just write K_A and K_H, omitting references to their arguments.

In order to understand why these intermediate bodies are useful to us, recall the diamond and square bodies K_0 and K_1 from Figure 2 (see Figure 6(a)). Recall the issue that a large fraction of the volume of the Macbeath region $M^{1/2}_{K_1}(x)$ lies outside of K_0. If we replace K_1 with $K_H = K_H(K_0, K_1)$ and compute the Macbeath region with respect to K_H instead (see Figure 6(b) and (c)), we see that a constant fraction of the volume of the Macbeath region lies within K_0 and so relative fatness is satisfied.

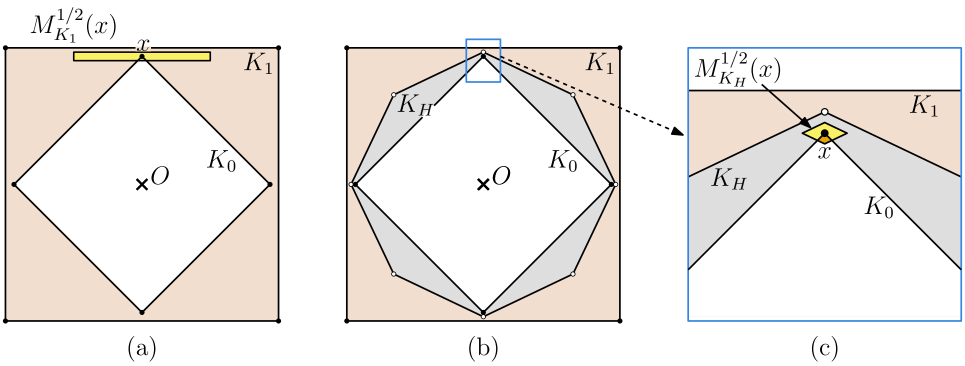

Figure 6 Relative fatness of K_H.

In Section 4, we will present an important result by showing that the inner body K_0 is relatively fat with respect to the harmonic-mean body $K_H(K_0, K_1)$. The proof makes heavy use of concepts from projective geometry, such as the harmonic bundle. This fact will be critical to establishing the volume-sensitive bounds given in this paper.

4 Relative Fatness and the Harmonic-Mean Body

In this section, we establish properties of the harmonic-mean body that are critical to the main results of this paper. In particular, given two bodies $K_0 \subset K_1$, we show that K_0 is relatively fat with respect to K_H. In fact, we present a stronger result in Lemma 4.4, which implies relative fatness as an immediate consequence. We will employ this stronger result in Section 5 to obtain our volume-sensitive bounds for polytope approximation.

The proof of Lemma 4.4 is based on the following technical lemma. For constant λ, it implies that for any point $b \in K_0$ that is not too close to the boundary of K_0, the Macbeath regions centered at b with respect to K_0 and K_H, respectively, are roughly similar up to a constant scaling factor. This is formally stated in the corollary following the lemma.

▶ **Lemma 4.1.** *Let $0 < \lambda < 1$ be a parameter. Let $K_0 \subset K_1$ be two convex bodies, where the origin O lies in the interior of K_0. Let K_H denote the harmonic-mean body of K_0 and K_1. Consider any ray emanating from the origin O. Let c and d denote the points of intersection of this ray with ∂K_0 and ∂K_1, respectively (see figure). Let $b \in K_0$ be a point on this ray such that the cross ratio $(O, c; d, b) \leq -\lambda$. Consider any line passing through b. Let c' and c'' denote the points of intersection of this line with ∂K_H. Then*

$$\min(\|bc' \cap K_0\|, \|bc'' \cap K_0\|) \;\geq\; s(\lambda) \cdot \min(\|bc'\|, \|bc''\|), \qquad \textit{where } s(\lambda) = \lambda/6.$$

Proof. We sketch the key ideas and present a complete proof in the full version. Consider the two dimensional flat that contains the origin and the line ℓ that passes through the points c', b, and c''. Henceforth, let K_0, K_1, K_H refer to the two dimensional convex bodies obtained by intersecting the respective bodies with this flat. Let b' and d' denote the points of intersection of the ray Oc' with ∂K_0 and ∂K_1, respectively, and define b'' and d'' analogously for Oc''. All these points lie on the flat, and it follows from the definition of the harmonic-mean body that $(O, c'; d', b') = (O, c''; d'', b'') = -1$ (see Figure 7(a)).

By rotating space, we may assume that ℓ is horizontal and above the origin. Through an infinitesimal perturbation, we may assume that there is a supporting line for K_1 at d that is not parallel to ℓ. Without loss of generality, we may assume that it intersects ℓ to the left of

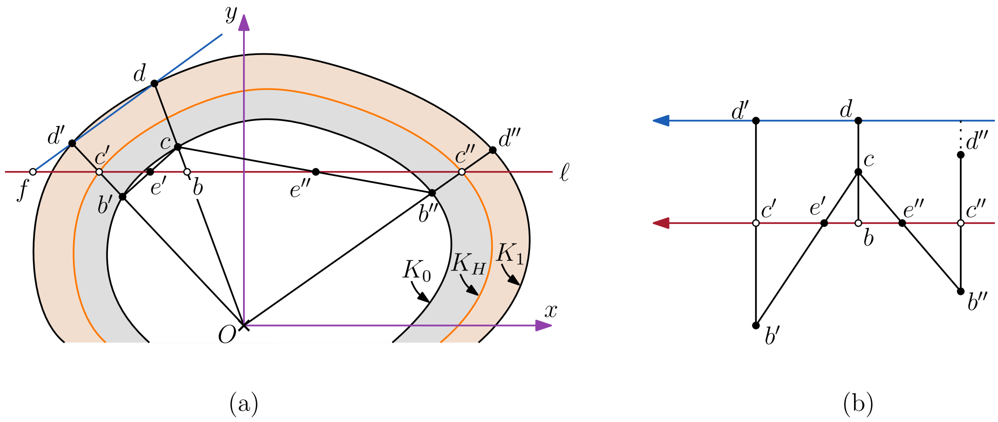

Figure 7 Lemma 4.1 and its proof.

b. Since c' and c'' are symmetrical in the statement of the lemma, we may assume that c' lies to left of b and c'' lies to its right. Let f denote the intersection point of the line dd' with ℓ (see Figure 7(a)). Clearly, the left-to-right order of points along ℓ is $\langle f, c', b, c'' \rangle$. Observe that the points c, d, d', and d'' all lie strictly above ℓ, and the points b' and b'' lie strictly below.

Let e' denote the point of intersection of the segment cb' with segment bc', and define e'' analogously for segment cb''. Since c, b' and b'' all lie on ∂K_0, by convexity, e' and e'' are contained in K_0. Thus, to prove the lemma, it suffices to show that

$$\min(\|be'\|, \|be''\|) \;\geq\; s(\lambda) \cdot \min(\|bc'\|, \|bc''\|). \tag{1}$$

We begin by proving bounds on two cross ratios:

(i) $-(f, e'; c', b) \geq \lambda/2$, and

(ii) $-(f, e''; c'', b) \geq \lambda/2$.

Because projective transformations preserve cross ratios, it will be convenient to prove these bounds after first applying a projective transformation. In particular, this transformation maps O and f to infinity so that lines through O map to vertical lines and lines through f map to horizontal lines (see Figure 7(b)). After this transformation, Oc', Oc, and Oc'' are vertical and directed upwards and $d'd$ and $c'b$ are horizontal and directed to the right. Clearly, $\|c'd'\| = \|bd\|$. Since d'' lies above ℓ and below the line $d'd$ we have $\|c''d''\| \leq \|bd\|$. By definition of b, we have $(O, c; d, b) = -1/(\|cd\|/\|cb\|) \leq -\lambda$. Since $\|cb\| + \|cd\| = \|bd\|$, we have $\|cb\| \geq \|bd\|\lambda/(1+\lambda)$.

Given that f is at infinity, the above cross ratios reduce to simple ratios. Thus, it suffices to show:

(i) $\|e'b\|/\|e'c'\| \geq \lambda/2$, and

(ii) $\|e''b\|/\|e''c''\| \geq \lambda/2$.

To show (i), observe that since $(O, c'; d', b') = -1$ and since O is at infinity and c' lies between b' and d', this is equivalent to $1/(\|c'd'\|/\|c'b'\|) = 1$, that is, $\|c'b'\| = \|c'd'\|$. By similar triangles $\triangle e'bc$ and $\triangle e'c'b'$, the fact that $\|c'b'\| = \|c'd'\| = \|bd\|$, and our bounds on λ, we have

$$\frac{\|e'c'\|}{\|e'b\|} \;=\; \frac{\|c'b'\|}{\|cb\|} \;\leq\; \frac{\|bd\|}{\|bd\|\lambda/(1+\lambda)} \;=\; \frac{1+\lambda}{\lambda} \;\leq\; \frac{2}{\lambda}, \tag{2}$$

which implies (i).

The analysis for (ii) is essentially the same as above. Since $(O, c''; d'', b'') = -1$ we have $\|c''b''\| = \|c''d''\|$. By similar triangles $\triangle e''bc$ and $\triangle e''c''b''$ and the fact that $\|c''b''\| = \|c''d''\| \leq \|bd\|$, the inequalities of Eq. (2) (with double primes for single primes) show that

$$\frac{\|e''c''\|}{\|e''b\|} \;\leq\; \frac{2}{\lambda},$$

which implies (ii).

These inequalities hold only in transformed configuration, but the cross ratios of (i) and (ii) hold unconditionally. Returning to the original configuration and using (i), we can show that $\|be'\|/\|bc'\| \geq \lambda/3$ and from (ii), we can show that either $\|be''\|/\|bf\| \geq \lambda/6$ or $\|be''\|/\|e''c''\| \geq \lambda/5$. We omit the details of this calculation, which can be found in the full version. In both cases, we are able to establish Eq. (1), as desired. ◀

The following corollary is immediate from the definition of Macbeath regions.

▶ **Corollary 4.2.** *Assume all entities to be as defined in the statement of Lemma 4.1. Then* $M^{s(\lambda)}_{K_H}(b) \subseteq M_{K_0}(b)$, *where* $s(\lambda) = \lambda/6$.

We have the following lemma which in conjunction with Corollary 4.2 will be useful in proving Lemma 4.4. The proof is presented in the full version.

▶ **Lemma 4.3.** *Let* λ, K_0, K_1, K_H, *the origin* O, *and points* c *and* d *be as in Lemma 4.1. Let* h *denote the point of intersection of the ray* Oc *with the boundary of* K_H. *Then:*

(i) $\|Oc\| \geq \|hc\|$.

(ii) *Let* b *be a point on segment* Oc, *which is not contained in the interior of* $M^{\lambda}_{K_H}(c)$. *Then* $(O, c; d, b) \leq -\lambda/2$.

We now have all the key ingredients to present the main result of this section. The relative fatness of K_0 with respect to K_H is an immediate consequence of parts (i) and (ii) of this lemma. In order to state part (iii), we need a definition. Given a convex body K with the origin O in its interior and a region $R \subseteq K$, define the *shadow* of R with respect to K, denoted $\text{shadow}_K(R)$, to be the set of points $x \in K$ such that the segment Ox intersects R.

▶ **Lemma 4.4.** *Let* $0 < \beta \leq 1$ *be a real parameter. Let* $K_0 \subset K_1$ *be two convex bodies, let the origin* O *lie in the interior of* K_0, *and let* K_H *denote the harmonic-mean body of* K_0 *and* K_1. *Let* c *be any point on the boundary of* K_0 *and let* $M = M^{\beta}_{K_H}(c)$. *Then there exists a convex body* M' *such that*

(i) $\text{vol}(M') = \Omega(\text{vol}(M))$,

(ii) $M' \subseteq M \cap K_0$, *and*

(iii) $\text{shadow}_{K_0}(M') \subseteq M$.

Proof. We sketch the proof of (i) and (ii) here. A complete proof appears in the full version. For the sake of convenience, assume that the ray Oc is directed vertically upwards. Let h be the point of intersection of the ray Oc with ∂K_H. Let $R = M_{K_H}(c) - c$ be the recentering of $M_{K_H}(c)$ about the origin. By definition, $M = M^{\beta}_{K_H}(c) = c + \beta R$. Let b be the point of intersection of the segment Oc with the boundary of $M^{\lambda}_{K_H}(c) = c + \lambda R$, where $\lambda = \beta/\kappa$ for a suitable large constant $\kappa \geq 2$ (independent of dimension). Recalling from Lemma 4.3(a) that $\|ch\| \leq \|Oc\|$, it follows that b is vertically below c at a distance of $\lambda\|ch\|$. Recalling $s(\lambda)$ from Corollary 4.2, let $M' = b + \gamma R$ for

$$\gamma \;=\; \frac{s(\lambda/2)}{10} \;=\; \frac{s(\beta/2\kappa)}{10} \;=\; \frac{\beta}{120\kappa}$$

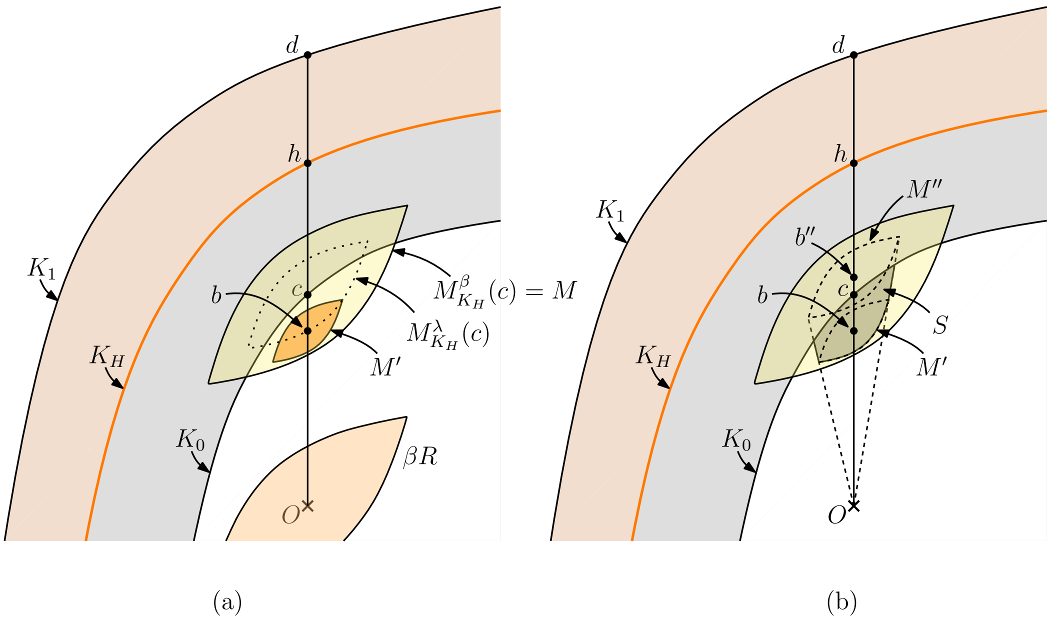

Figure 8 Proof of Lemma 4.4. (Objects are not drawn to scale.)

(see Figure 8(a)). Since M' and M are translated copies of R scaled by a factor of γ and β, respectively, we have $\mathrm{vol}(M') = (\gamma/\beta)^d \,\mathrm{vol}(M) = (1/120\kappa)^d \,\mathrm{vol}(M)$. This proves (i).

To prove (ii), we will show that $M' \subseteq M$ and $M' \subseteq K_0$. Since $b \in c+\lambda R$ and $M' = b+\gamma R$, it follows that $M' \subseteq c+(\lambda+\gamma)R$. For large κ, we have $\lambda+\gamma \leq \beta$, and thus $M' \subseteq c+\beta R = M$.

Next we show that $M' \subseteq K_0$. Let d denote the point of intersection of the ray Oc with ∂K_1. Applying Lemma 4.3(b), it follows that the cross ratio $(O, c; d, b) \leq -\lambda/2$. Applying Corollary 4.2 with $\lambda/2$ in place of λ and recalling that $s(\lambda/2) = 10\gamma$, we have $M^{10\gamma}_{K_H}(b) \subseteq M_{K_0}(b)$. Also, by Lemma 3.4, we have $M' = b + \gamma R \subseteq M^{2\gamma}_{K_H}(b)$. Thus $M' \subseteq M^{1/5}_{K_0}(b)$. By definition of Macbeath regions, $M_{K_0}(b) \subseteq K_0$, and so $M' \subseteq K_0$, as desired. ◀

The following corollary is immediate from parts (i) and (ii) of the above lemma.

▶ **Corollary 4.5.** *Let $K_0 \subset K_1$ be two convex bodies, let the origin O lie in the interior of K_0, and let K_H denote the harmonic-mean body of K_0 and K_1. Then K_0 is relatively fat with respect to K_H.*

5 Uniform Volume-Sensitive Bounds

In this section, we present the proof of Theorem 1.1. Let $\varepsilon > 0$ and let K_0 denote the convex body K described in this theorem. Let $K_1 = K_0 \oplus \varepsilon$ denote the Minkowski sum of K_0 with a ball of radius ε. Also recall that $\Delta_d(K_0)$ denotes the *volume diameter* of K_0. Let $C(K_0, \varepsilon)$ be a shorthand for $(\Delta_d(K_0)/\varepsilon)^{(d-1)/2}$, the desired number of facets.

We will show that there exists a polytope with $O(C(K_0, \varepsilon))$ facets sandwiched between K_0 and K_1. As mentioned above, we will transform the problem by mapping to the polar. Through an appropriate translation, we may assume that the origin O coincides with the centroid of K_0. Note that the arithmetic-mean body K_A of K_0 and K_1 is given by $K_0 \oplus \frac{\varepsilon}{2}$, and recall from Section 3.5 that $K_H = K_A^*$ is the harmonic-mean body of K_1^* and K_0^*.

Our construction is based on Lemma 5.1, which shows that there is a $(K_H, \partial K_1^*)$-MNet X of size $O(C(K_0, \varepsilon))$. Applying Lemma 3.6, it follows that there exists a polytope P sandwiched between K_1^* and K_H with $O(|X|)$ vertices. By polarity, this implies that P^* is a polytope sandwiched between K_A and K_1 having $O(|X|)$ facets. Since $K_0 \subseteq K_A$, this polytope is also sandwiched between K_0 and K_1, which proves Theorem 1.1.

All that remains is showing that $|X| = O(C(K_0, \varepsilon))$. For this purpose, we will utilize the tools for bounding the sizes of MNets in conjunction with the relative fatness of the harmonic-mean body (established in Section 4).

▶ **Lemma 5.1.** *Let $\varepsilon > 0$ and let K_0, K_1, K_A, K_H be convex bodies as defined above. Let X be a $(K_H, \partial K_1^*)$-MNet. Then $|X| = O(C(K_0, \varepsilon))$.*

Proof. We begin by showing that $\mathrm{vol}(K_H) = \Omega(1/\mathrm{vol}(K_0))$, and its Mahler volume $\mu(K_H)$ is at most $O(1)$ (implying that K_H is well-centered). To see this, recall that the width of K_0 in any direction is at least ε and $K_A = K_0 \oplus \frac{\varepsilon}{2}$. It is well-known that the ratio of the distances of the centroid from any pair of supporting hyperplanes is at most d [17, 24, 26]. It follows that a ball of radius $\varepsilon/(d+1)$ centered at the origin lies within K_0. Thus, a constant-factor expansion of K_0 contains K_A, implying that $\mathrm{vol}(K_A) = O(\mathrm{vol}(K_0))$. Also, because $K_H = K_A^*$, by Lemma 3.2, $\mathrm{vol}(K_A) \cdot \mathrm{vol}(K_H) = \Omega(1)$. Thus, $\mathrm{vol}(K_H) = \Omega(1/\mathrm{vol}(K_0))$. To upper bound $\mu(K_H)$, note that by polarity, $K_H \subseteq K_0^*$, and thus

$$\mu(K_H) \;=\; \mathrm{vol}(K_A) \cdot \mathrm{vol}(K_H) \;=\; O(\mathrm{vol}(K_0) \cdot \mathrm{vol}(K_0^*)) \;=\; O(\mu(K_0)) \;=\; O(1),$$

where in the last step, we have used Lemma 3.1 and our assumption that the origin coincides with the centroid of K_0.

To simplify notation, for the remainder of the proof we assume that ray distances, Macbeath regions, and volumes are defined relative to K_H, that is, $\mathrm{ray} \equiv \mathrm{ray}_{K_H}$, $M \equiv M_{K_H}$, and $\mathrm{vol} \equiv \mathrm{vol}_{K_H}$.

For any point $p \in \partial K_1^*$, let p' denote the point of intersection of the ray Op with ∂K_H. We first establish a bound on the relative ray distance $\mathrm{ray}(p)$. Observe that since p and p' lie on ∂K_1^* and ∂K_H, respectively, their polar hyperplanes, p^* and ${p'}^*$, are supporting hyperplanes for K_1 and $K_H^* = K_A$, respectively. Letting r denote the distance between ${p'}^*$ and the origin, it follows from the definition of K_A that the distance between p^* and the origin is $r + \frac{\varepsilon}{2}$. The distance of p' and p from the origin are the reciprocals of these. Therefore, we have

$$\mathrm{ray}(p) \;=\; \frac{\|pp'\|}{\|Op'\|} \;=\; \frac{\|Op'\| - \|Op\|}{\|Op'\|} \;=\; \frac{\frac{1}{r} - \frac{1}{r+(\varepsilon/2)}}{\frac{1}{r}} \;=\; 1 - \frac{r}{r+(\varepsilon/2)} \;=\; \frac{\varepsilon/2}{r+(\varepsilon/2)}.$$

Since $\frac{1}{\|Op'\|} = r = \Omega(\varepsilon)$, we have $\mathrm{ray}(p) = \Theta(\varepsilon/r) = \Theta(\varepsilon\|Op'\|)$. (It is noteworthy and somewhat surprising that this relative ray distance is not a dimensionless quantity, since it depends linearly on $\|Op'\|$.)

To analyze $|X|$, we partition it into groups based on $\|Ox'\|$ for each $x \in X$. Define $R_0 = (\mathrm{vol}(K_H))^{1/d}$. By our earlier remarks, $\mathrm{vol}(K_H) = \Omega(1/\mathrm{vol}(K_0))$, and so $R_0 = \Omega(1/\Delta_d(K_0))$. For any integer i (possibly negative), define $R_i = 2^i R_0$ and $\varepsilon_i = \varepsilon R_i$. We can express X as the disjoint union of sets X_i, where X_i consists of points x such that $R_i \leq \|Ox'\| < 2R_i$. Recall that for any $x \in X_i$, we have $\mathrm{ray}(x) = \Theta(\varepsilon\|Ox'\|) = \Theta(\varepsilon R_i) = \Theta(\varepsilon_i)$.

We will bound the contributions of the $|X_i|$ to $|X|$ based on the sign of i. Let us first consider the nonnegative values of i. We remark that $|X_i| = 0$ for large i (specifically, for $i = \omega(\log(1/\varepsilon R_0)))$ because a ball of radius $\Omega(\varepsilon)$ centered at the origin is contained within K_0, and so by polarity K_0^*, and hence K_1^*, is contained within a ball of radius $O(1/\varepsilon)$. Recalling that K_H is well-centered and applying Lemma 3.7(i), we have (up to constant factors)

$$\begin{aligned}\sum_{i\geq 0}|X_i| &\leq \sum_{i\geq 0}\left(\frac{1}{\varepsilon_i}\right)^{\frac{d-1}{2}} = \sum_{i\geq 0}\left(\frac{1}{\varepsilon 2^i R_0}\right)^{\frac{d-1}{2}} \leq \sum_{i\geq 0}\left(\frac{\Delta_d(K_0)}{\varepsilon 2^i}\right)^{\frac{d-1}{2}} \\ &= \left(\frac{\Delta_d(K_0)}{\varepsilon}\right)^{\frac{d-1}{2}}\sum_{i\geq 0}\left(\frac{1}{2}\right)^{\frac{i(d-1)}{2}} \leq \left(\frac{\Delta_d(K_0)}{\varepsilon}\right)^{\frac{d-1}{2}} = O(C(K_0,\varepsilon)).\end{aligned}$$

In order to bound the contributions to $|X|$ for negative values of i, we need a more sophisticated strategy. Our approach is to first bound the total relative volume of the Macbeath regions of $\mathcal{M}^{1/4c}(X_i)$, which we assert to be $O(\varepsilon_i 2^{id})$. Assuming this assertion for now, we complete the proof as follows. By applying Lemma 3.7(ii) with $f = O(2^{id})$ and recalling that $\varepsilon_i = \varepsilon R_i = 2^i \varepsilon R_0$, we have (up to constant factors)

$$\begin{aligned}\sum_{i<0}|X_i| &\leq \sum_{i<0}\frac{\sqrt{f}}{\varepsilon_i^{(d-1)/2}} = \sum_{i<0}\frac{2^{id/2}}{(2^i\varepsilon R_0)^{(d-1)/2}} = \sum_{i<0}\frac{2^{i(d-(d-1))/2}}{(\varepsilon R_0)^{(d-1)/2}} \\ &= \sum_{i<0}\frac{2^{i/2}}{(\varepsilon R_0)^{(d-1)/2}} = \sum_{i<0}2^{i/2}C(K_0,\varepsilon) = C(K_0,\varepsilon)\sum_{i>0}\left(\frac{1}{2}\right)^{\frac{i}{2}} \\ &= O(C(K_0,\varepsilon)).\end{aligned}$$

It remains only to prove the assertion on the total relative volume of $\mathcal{M}^{1/4c}(X_i)$. Let $x \in X_i$ and let $M_x = M^{1/4c}(x)$. By Lemma 4.4 (with x, K_1^*, and K_H playing the roles of c, K_0, and K_H, respectively), there is an associated convex body M'_x such that

(i) $\mathrm{vol}(M'_x) = \Omega(\mathrm{vol}(M_x))$, (ii) $M'_x \subseteq M_x \cap K_1^*$, and (iii) $\mathrm{shadow}_{K_1^*}(M'_x) \subseteq M_x$.

We will use S_x as a shorthand for $\mathrm{shadow}_{K_1^*}(M'_x)$. Since $\mathrm{vol}(M_x) = O(\mathrm{vol}(M'_x)) = O(\mathrm{vol}(S_x))$, it suffices to show that the total relative volume of the shadows $\{S_x : x \in X_i\}$ is $O(\varepsilon_i 2^{id})$.

For $x \in X_i$, we define cone Ψ_x to be the intersection of K_H with the infinite cone consisting of rays emanating from the origin that contain a point of S_x (see Figure 9). Since the Macbeath regions of $\mathcal{M}^{1/4c}(X_i)$ are disjoint, it follows from (iii) that the associated shadows intersect ∂K_1^* in patches that are also disjoint. Thus the set of cones $\Psi = \{\Psi_x : x \in X_i\}$ are disjoint.

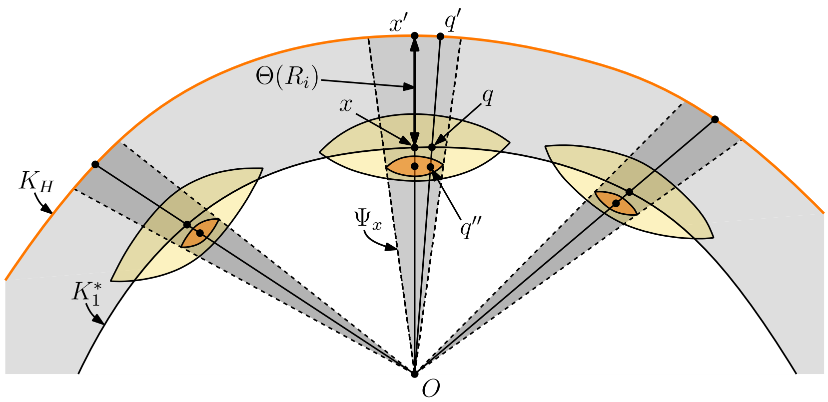

Figure 9 Proof of Lemma 5.1.

Consider a ray emanating from the origin that is contained in any cone Ψ_x. Let q and q' be the points of intersection of this ray with ∂K_1^* and ∂K_H, respectively. Let q'' be any point on this ray that lies inside shadow S_x. Since $q'' \in M_x$, by Lemma 3.3, we have $\mathrm{ray}(q'') = \Theta(\mathrm{ray}(x)) = \Theta(\varepsilon_i)$. By the same reasoning, $\mathrm{ray}(q) = \Theta(\varepsilon_i) = \Theta(\varepsilon R_i)$. Also, recalling our earlier bounds on the relative ray distance of points on ∂K_1^*, we have $\mathrm{ray}(q) = \Theta(\varepsilon \|Oq'\|)$. Equating the two expressions for $\mathrm{ray}(q)$, we obtain $\|Oq'\| = \Theta(R_i)$.

Since the cones of Ψ are disjoint and any ray emanating from the origin and contained in a cone of Ψ has length $\Theta(R_i)$, it follows that the total volume of these cones is $O(R_i^d)$. Further, since only a fraction ε_i of any such ray is contained in the associated shadow, it follows that the total volume of all the shadows $\{S_x : x \in X_i\}$ is $O(\varepsilon_i R_i^d)$. Recalling that $\mathrm{vol}(K_H) = R_0^d$ and $R_i = 2^i R_0$, it follows that the total relative volume of these shadows is $O(\varepsilon_i R_i^d / R_0^d) = O(\varepsilon_i 2^{id})$. This establishes the assertion on the total relative volume of $\mathcal{M}^{1/4c}(X_i)$ and completes the proof. ◀

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