LIOUVILLE-TYPE RESULTS FOR THE LANE-EMDEN EQUATION
RISULTATI TIPO LIOUVILLE PER L'EQUAZIONE DI LANE-EMDEN

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Abstract. We present some Liouville-type result for the Lane-Emden equation in the subcritical and in the critical regimes. In particular, we focus on the so-called critical $p-$Laplace equation.

Sunto. Presentiamo alcuni risultati di tipo Liouville per l'equazione di Lane-Emded nei casi sottocritico e critico. In particolare, ci concentriamo sull'equazione critica del $p-$Laplaciano.

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1. Introduction

In this survey we focus on the so-called *generalized Lane-Emden equation*, i.e. the following quasilinear equation:

$$\Delta_p u + |u|^{q-1}u = 0 \quad \text{in } \mathbb{R}^n\,, \tag{1}$$

where

$$n \geq 2\,, \quad 1 < p < n\,, \quad q > 1\,,$$

and Δ_p is the usual $p-$*Laplace operator*

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)\,.$$

In particular, we are interested in *Liouville-type results*, i.e. classification and non-existence results for (positive) solutions to (1).

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As we will see an important role in the Liouville-type results that we are going to present is played by the exponent

$$q = p^* - 1\,, \tag{2}$$

where p^* is the usual *Sobolev critical exponent*

$$p^* := \frac{np}{n-p}\,.$$

Basically we have two different behaviours: when $q < p^* - 1$ there are no positive solutions to (1), while when $q = p^* - 1$ the unique positive solutions are the so-called Talentiane or Aubin-Talenti bubbles (see (7) and (10) below). We will refer to the case $q < p^* - 1$ as the *sub-critical regime* and to the case $q = p^* - 1$ as *the critical regime*[1].

We mention that the study of (1) in $\mathbb{R}^n$ (or, more in general in unbounded domains) comes from the fact that this equation naturally arises not only in analysis, but also in physics and in geometry. For instance, for $n = 3$, (1) arises in the study of stellar structure in astrophysics (see e.g. [4] and [7]); while in the critical regime (i.e. when (2) holds), (1) plays and important role in the study of conformal problems, like the Yamabe problem of prescribed scalar curvature, and in the study of extremals of the Sobolev inequality (see e.g. [4, 45] and also [40, Section 3]). Moreover, we mention that classification of solutions to (1) in the critical regime is related, and it is of crucial importance in many applications such as a priori estimates, blow-up analysis and asymptotic analysis (we refer to [19, 23, 45] for the case $p = 2$ and to [41, 51] for the case $p \neq 2$).

Finally, we emphasize that the *critical* $p-$*Laplace equation*

$$\begin{cases} \Delta_p u + u^{p^*-1} = 0 & \text{in } \mathbb{R}^n \\ u > 0 \end{cases} \tag{3}$$

is also interesting from the point of view of the calculus of variations. Since the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ is not compact, the classical tools of the calculus of variations (e.g. the Mountain Pass Lemma or the direct method) do not apply to the functional

$$\mathcal{J}(u) := \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p\, dx - \frac{1}{p^*}\int_{\mathbb{R}^n} u^{p^*}\, dx\,, \tag{4}$$

[1]for this reason equation (1), in this case, is usually called the *critical* $p-$*Laplace equation.*

which is the energy functional associated to problem (3)[2].

The paper is organized as follows: in Section 2 we present the Liouville-type results in the semilinear case (i.e. $p = 2$) and in the quasilinear case (i.e. $p \neq 2$); then we focus on the so-called finite energy assumption and we present a conjecture in this context. In Section 3 we give a sketch of the proof of the conjecture when $n = 2$. Finally, in Section 4 we present some open problems that are related to the Liouville-type results we consider in the paper.

2. Liouville-type results: main results

2.1. **Semilinear case.** We start by considering the semilinear case (i.e. $p = 2$) and the sub-critical regime (i.e. $q < 2^* - 1$), the first Lioville-type result is due to Gidas and Spruck in [26] and reads as follows:

Theorem 2.1. *Let $u \in C^2(\mathbb{R}^n)$ be a solution of*

$$\begin{cases} \Delta u + u^q = 0 & \text{in } \mathbb{R}^n \\ u \geq 0 \end{cases} \tag{5}$$

with

$$1 \leq q < 2^* - 1 = \frac{n+2}{n-2}$$

then

$$u \equiv 0 \,.$$

The proof of this theorem is based on a test functions argument and on integral identities. We mention that the same result holds true in the Riemannian setting, in particular the same result holds in complete non-compact Riemannian manifolds with non-negative Ricci curvature (see [26]).

[2]this means that problem (3) is the Euler-Lagrange equation of the functional (4).

In the critical case (i.e. $q = 2^* - 1$) the scenario is different because of the existence of an explicit family of (classical) solutions to

$$\begin{cases} \Delta u + u^{2^*-1} = 0 & \text{in } \mathbb{R}^n \\ u > 0 \end{cases} \tag{6}$$

given by the so-called *Talentiane* or *Aubin-Talenti bubbles*: these are a two parameters family of functions of the following form

$$\mathcal{U}_{\lambda,x_0}(x) := \left(\frac{\sqrt{n(n-2)}\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{n-2}{2}}, \tag{7}$$

where $\lambda > 0$ and $x_0 \in \mathbb{R}^n$. Observe that with the normalization in (7) it turns out that the functions (7) solves exactly (6). These functions have been constructed indipendently by Aubin in [3] and Talenti in [48] as *minimizers of the Sobolev constant*:

$$\mathcal{S}_2 := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^n} u^{2^*} \, dx \right)^{2/2^*}},$$

where the space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is the homogeneous[3] Sobolev space defined by

$$\mathcal{D}^{1,2}(\mathbb{R}^n) := \left\{ u \in L^{2^*}(\mathbb{R}^n) \, : \, \nabla u \in L^2(\mathbb{R}^n) \right\}.$$

Hence the natural question is whether the functions (7) are the only solutions to (6) or not.

This question has attracted a lot of interest in geometric analysis and PDE's communities. The first results in this directions are the ones contained in the papers [25] and [37] where the authors proved that the only solutions to (6) are the Talentiane (7), provided

$$u(x) = O\left(\frac{1}{|x|^{n-2}} \right), \qquad \text{for } |x| \to \infty.$$

The result without any further assumption is due to Caffarelli, Gidas and Spruck in [4] (see also [9] and [29]) and is contained in the following

Theorem 2.2. *Let $u \in C^2(\mathbb{R}^n)$ be a solution of (6), then $u(x) = \mathcal{U}_{\lambda,x_0}(x)$, for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$.*

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[3] it is homogeneous in the same sense the Sobolev inequality is homogeneous under the rescaling $f_\lambda(\cdot) := f(\cdot/\lambda)$

The proof of this theorem is based on the Kelvin transform and on a variation of the method of moving planes (introduced in [1] in the context of constant mean curvature hypersurfaces and transplanted to the study of qualitative properties of solutions of PDE's in [42] and in [24]). We mention that, also in this case, the same result holds true in the Riemannian setting; in particular, the same result holds in complete non-compact Riemannian manifolds with non-negative Ricci curvature as recently shown in [5].

We emphasize that thanks to the strong maximum principle (see e.g. [27]) we can consider (strictly) positive solutions to (6), indeed every non-negative solution to (6) is strictly positive, unless $u \equiv 0$.

2.2. **Quasilinear case.** The quasilinear case (i.e. $p \neq 2$) is more difficult and complicated and we have to take into account the nonlinear nature of the $p-$Laplace operator, the lack of regularity of the solutions and the fact a Kelvin type transform is not available (as in the case of the Laplace operator). In particular, in this case we cannot consider classical solutions to (1) but we have to deal with weak solutions[4]. For clarity we report here the definition of weak solution to (1).

Definition 2.1. *A weak solution u to* (1) *is a function $u \in W^{1,p}_{loc}(\mathbb{R}^n) \cap L^\infty_{loc}(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx - \int_{\mathbb{R}^n} |u|^{q-1}u\,\varphi \, dx = 0\,, \qquad \text{for all } \varphi \in W^{1,p}_c(\mathbb{R}^n),$$

where $W^{1,p}_c(\mathbb{R}^n)$ denotes the space of compactly supported functions in $W^{1,p}(\mathbb{R}^n)$.

In the sub-critical case (i.e. $q < p^* - 1$) the analogue of Theorem 2.1 has been proved by Serrin and Zou in [43] and is the following

[4]indeed, it is well-known that solutions to quasilinear equations are not smooth; indeed let $p > 1$ and $p \neq 2$ and consider the function

$$u(x_1, \ldots, x_n) = \frac{|x_1|^q}{q}\,, \qquad \text{where } \frac{1}{p} + \frac{1}{q} = 1$$

which clearly solves

$$\Delta_p u = 1\,.$$

Theorem 2.3. *Let u be a weak solution of*

$$\begin{cases} \Delta_p u + u^q = 0 & \textit{in } \mathbb{R}^n \\ u \geq 0 \end{cases} \tag{8}$$

with

$$1 < p < n \quad \textit{and} \quad 1 \leq q < p^* - 1$$

then

$$u \equiv 0\,.$$

In the critical case (i.e. $q = p^* - 1$) the analogue of Theorem 2.2 has been recently proved by Damascelli, Merchán, Montoro and Sciunzi in [14] for $\frac{2n}{n+2} < p < 2$ and by Vétois in [51] and by Sciunzi in [41] for $1 < p < 2$ and for $2 < p < n$, respectively.

These results are summarized and presented in the following

Theorem 2.4. *Let u be a weak solution of* (3)*, with*

$$1 < p < n\,,$$

and such that

$$u \in \mathcal{D}^{1,p}(\mathbb{R}^n) := \left\{ u \in L^{p^*}(\mathbb{R}^n) \,:\, \nabla u \in L^p(\mathbb{R}^n) \right\}\,. \tag{9}$$

Then $u(x) = \mathcal{U}_{\lambda, x_0}(x)$, for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, where

$$\mathcal{U}_{\lambda, x_0}(x) = \left(\frac{n^{\frac{1}{p}} \left(\frac{n-p}{p-1} \right)^{\frac{p-1}{p}} \lambda}{1 + \lambda^{\frac{p}{p-1}} |x - x_0|^{\frac{p}{p-1}}} \right)^{\frac{n-p}{p}}\,. \tag{10}$$

The proof is based on asymptotic bounds on u and $|\nabla u|$ and, again, on the method of moving planes. We refer to the paper [10] for an alternative proof, based on integral identities, which is suitable to be generalized also in the anisotropic setting and in the context of convex cones of $\mathbb{R}^n$ (see also [31] for a previous result)

We mention that in the context of quasilinear equations the strong maximum principle holds true (see [49]) and so we can consider (strictly) positive solutions to (3), indeed every non-negative solution to (3) is strictly positive, unless $u \equiv 0$.

The functions (10) are the generalization of the functions (7) and have been constructed by Aubin in [3] and Talenti in [48] as *minimizers of the $p-$Sobolev constant*:

$$\mathcal{S}_p := \inf_{u\in\mathcal{D}^{1,p}(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\nabla u|^p\,dx}{\left(\int_{\mathbb{R}^n} u^{p^*}\,dx\right)^{p/p^*}}\,.$$

2.3. **The finite energy assumption.** We now comment about the hypothesis of Theorems 2.2 and 2.4. The common hypothesis in both theorems is the following

$$u>0\,,$$

and it is fundamental, indeed it is possible to construct infinitely-many sign-changing solutions to

$$\Delta_p u + |u|^{p^*-2}u = 0 \quad \text{in } \mathbb{R}^n\,,$$

which are not radial (see e.g. [15, 17, 18, 33, 34, 36] for the semilinear equation and [11] for the quasilinear equation, provided $n \geq 4$).

On the contrary, the main difference is the hypothesis (9) in Theorem 2.4. Usually, condition (9) is called *finite energy condition* (recall the energy functional (4)).

Hence, the desired and conjectured result should be the following

Conjecture 2.1. *Let u be a weak solution of* (3)*, with*

$$1<p<n\,,$$

Then $u(x) = \mathcal{U}_{\lambda,x_0}(x)$, for some $\lambda>0$ and $x_0\in\mathbb{R}^n$.

Very recently, positive and partial answers to this conjecture have been provided: in [6] we show that the conjecture is true in the case

$$n=2,3 \quad \text{with} \quad \frac{n}{2}<p<n-1\,,$$

in [38] the author shows that the conjecture is true in the case

$$n\geq 2 \quad \text{with} \quad \frac{n+1}{3}<p<n\,,$$

and in [50] the author shows that the conjecture is true in the case

$$n\geq 4 \quad \text{with} \quad p_n<p<n\,,$$

where $p_n = \frac{8}{5}$ if $n = 4$ and $p_n = \frac{4n+3-\sqrt{4n^2+12n-15}}{6}$. To the best of our knowledge these are the only results in literature facing Conjecture 2.1.

The proof of the these results is based on integral identities and is inspired by the already cited papers [5, 10, 26, 43]. In the next section, we are going to present the proof, contained in [6], of Conjecture 2.1 in the case $n = 2$.

We mention that in [6], we also deal with Conjecture (2.1) for $n \geq 4$ under assumptions on the growth of the energy or assuming a suitable control of the solutions at infinity. These conditions are (much) weaker that the finite energy assumption and we refer to [6] for further details. Finally, in [6] we also consider the Riemannian setting and we prove analogue results, in particular we consider a complete, non-compact Riemannian manifold with non-negative Ricci curvature if $1 < p < 2$ and with non-negative sectional curvatures if $2 < p < n$. We refer to [6, Appendix A] for further details.

3. Proof of Conjecture 2.1 with $n = 2$

The proof of Conjecture 2.1 with $n = 2$ provided in [6] (and also in [38]) is based on a key integral estimate which is an adaptation of the argument in [43]. In order to state it we need to introduce some notations; let $n = 2$ and $1 < p < 2$, we consider u to be the weak solution of (3) and we define the following vector fields:

$$\mathbf{u} := |\nabla u|^{p-2}\nabla u \quad \text{ and } \quad \mathbf{v} := u^{-\frac{2(p-1)}{2-p}}\,\mathbf{u}\,.$$

Then, we set

$$\mathbf{U} := \begin{cases} \nabla \mathbf{u} & \text{in } \Omega_{cr}^c \\ 0 & \text{in } \Omega_{cr} \end{cases} \quad \text{ and } \quad \mathbf{V} := \begin{cases} \nabla \mathbf{v} & \text{in } \Omega_{cr}^c \\ 0 & \text{in } \Omega_{cr}\,, \end{cases}$$

where

$$\Omega_{cr} := \{x \in \mathbb{R}^n \,:\, \nabla u(x) = 0\}\,,$$

is the set of critical points of u.

Finally, we recall the notion of *traceless matrix* associated to $\mathbf{V}$ denoted by $\mathring{\mathbf{V}}$:

$$\mathring{\mathbf{V}} := \mathbf{V} - \frac{\operatorname{tr}\mathbf{V}}{n}\operatorname{Id}_n\,,$$

where Id_n denotes the $n \times n$ identity matrix.

With these notations, we have the following fundamental integral estimate.

Proposition 3.1. *Let $n = 2$, $1 < p < 2$ and let u be a weak solution of* (3). *Then*

$$\int_{\mathbb{R}^2} u^{\frac{p}{2-p}} |\mathring{\mathbf{V}}|^2 \phi \, dx \leq - \int_{\mathbb{R}^2} u^{\frac{p}{2-p}} \langle \mathbf{v} \cdot \mathring{\mathbf{V}}, \nabla \phi \rangle \, dx \,, \tag{11}$$

for all $0 \leq \phi \in C_0^\infty(\mathbb{R}^n)$, where the expression $\mathbf{v} \cdot \mathring{\mathbf{V}}$ is interpreted as the vector with components $(\mathbf{v} \cdot \mathring{\mathbf{V}})_i = \sum_{j=1}^n \mathbf{v}_j \mathring{\mathbf{V}}_{ij}$, for $i = 1, \dots, n$.

We refer to [6, Proposition 2.2] and to [43, Section 6] for its proof, but we mention that all the computations can be performed thanks to the regularity of the solution proved e.g. in the recent paper [2].

An immediate consequence of the previous proposition is when one takes $\phi = \eta^2$, where $0 \leq \eta \in C_0^\infty(\mathbb{R}^n)$ and uses Cauchy-Schwarz or Hölder inequality in (11). The following integral estimates are the keys to deduce the rigidity result.

Corollary 3.1. *Let $n = 2$, $1 < p < 2$ and let u be a weak solution of* (3). *Then,*

$$\int_{\mathbb{R}^2} u^{\frac{p}{2-p}} |\mathring{\mathbf{V}}|^2 \eta^2 \, dx \leq C \int_{\mathbb{R}^2} u^{\frac{4-3p}{2-p}} |\nabla u|^{2(p-1)} |\nabla \eta|^2 \, dx \,, \tag{12}$$

and

$$\int_{\mathbb{R}^2} u^{\frac{p}{2-p}} |\mathring{\mathbf{V}}|^2 \eta^2 \, dx \leq \tag{13}$$
$$C \left(\int_{\mathrm{supp}|\nabla \eta|} u^{\frac{p}{2-p}} |\mathring{\mathbf{V}}|^2 \eta^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} u^{\frac{4-3p}{2-p}} |\nabla u|^{2(p-1)} |\nabla \eta|^2 \, dx \right)^{\frac{1}{2}} ,$$

for all $0 \leq \eta \in C_0^\infty(\mathbb{R}^n)$.

Now we can prove Conjecture 2.1 with $n = 2$

Proof of Conjecture 2.1 with $n = 2$. We aim to prove the following

$$\int_{\mathbb{R}^2} u^{\frac{p}{2-p}} |\mathring{\mathbf{V}}|^2 \eta^2 \, dx = 0 \,. \tag{14}$$

Indeed, once (14) holds we conclude that

$$\mathring{\mathbf{V}} = \nabla \mathbf{v} - \frac{\mathrm{div}\, \mathbf{v}}{n} \, \mathrm{Id}_n \equiv 0 \qquad \text{in } \Omega_{cr}^c \,. \tag{15}$$

Then, we denote with $\Omega_0 \subseteq \Omega_{cr}^c$ a connected component of Ω_{cr}^c. Since $0 < u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$ (see e.g. [2]), then $v := u^{-\frac{p}{2-p}} \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$. From the fact that

$$\mathbf{v} = -\left(\frac{2-p}{p}\right)^{p-1} |\nabla v|^{p-2}\nabla v\,,$$

we get

$$\text{div}\mathbf{v} = -\left(\frac{2-p}{p}\right)^{p-1} \Delta_p v$$

which can be rewritten in terms of u in the following way

$$\text{div}\mathbf{v} = -u^{\frac{p}{n-p}} - \frac{n(p-1)}{n-p} u^{-\frac{p(n-1)}{n-p}} |\nabla u|^p\,,$$

where we used the fact that u solves (3). We observe that the right-hand side of the previous identity is in $C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n)$, hence by standard elliptic regularity, we have $v \in C^{2,\alpha}_{\text{loc}}(\Omega_0)$, $u \in C^{2,\alpha}(\Omega_0)$ and so $\text{div}\mathbf{v} \in C^{1,\alpha}_{\text{loc}}(\Omega_0)$. Differentiating (15), we get

$$\partial_i\left(\text{div}\mathbf{v}\right) = n\,\partial_i\left(\text{div}\mathbf{v}\right).$$

Therefore

$$\text{div}\mathbf{v} = \text{const} \quad \text{on } \Omega_0$$

and thus

$$\mathbf{v}(x) = C(x - x_0)$$

on Ω_0 and for some $C \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Thus

$$v(x) = C_1 + C_2|x - x_0|^{\frac{p}{p-1}}$$

on Ω_0, for some $C_1, C_2 > 0$. Then $u(x) = \mathcal{U}_{\lambda,x_0}(x)$ on Ω_0 for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$. Since the argument above holds whenever $\nabla u \neq 0$, we must have $\Omega_0 = \mathbb{R}^n \setminus \{x_0\}$ and the result follows.

It remains to prove (14). In order to do this, we choose $0 \leq \eta \in C^\infty_0(\mathbb{R}^n)$ in Corollary 3.1 as follows: for any $R > 1$ let $\eta = 1$ in B_R, $\eta = 0$ in B^c_{2R}, $0 \leq \eta \leq 1$ on $\mathbb{R}^n$ and such that

$$|\nabla\eta|^2 \leq \frac{C}{R^2} \quad \text{in } B_{2R} \setminus B_R\,.$$

Then (12) reads as

$$\int_{\mathbb{R}^2} u^{\frac{p}{2-p}} |\mathring{\mathbf{V}}|^2 \eta^2 \, dx \le \frac{C}{R^2} \int_{B_{2R}\setminus B_R} u^{\frac{4-3p}{2-p}} |\nabla u|^{2(p-1)} \, dx$$
$$= \frac{C}{R^2} \int_{B_{2R}\setminus B_R} u \left(u^{-\frac{p}{2-p}} |\nabla u|^p \right)^{\frac{2(p-1)}{p}} \, dx \,.$$

From Hölder inequality we obtain (recall that $1 < p < 2$)

$$\int_{\mathbb{R}^2} u^{\frac{p}{2-p}} |\mathring{\mathbf{V}}|^2 \eta^2 \, dx \le \frac{C}{R^2} \left(\int_{B_{2R}\setminus B_R} u^{-\frac{p}{2-p}} |\nabla u|^p \, dx \right)^{\frac{2(p-1)}{p}} \left(\int_{B_{2R}\setminus B_R} u^{\frac{p}{2-p}} \, dx \right)^{\frac{2-p}{p}} ,$$

moreover, from Young inequality we get

$$\int_{\mathbb{R}^2} u^{\frac{p}{2-p}} |\mathring{\mathbf{V}}|^2 \eta^2 \, dx \le \frac{C}{R^2} \left(\int_{B_{2R}\setminus B_R} u^{-\frac{p}{2-p}} |\nabla u|^p \, dx + \int_{B_{2R}\setminus B_R} u^{\frac{p}{2-p}} \, dx \right) . \tag{16}$$

Now we estimate the right-hand side of (16) in the following way: take

$$\varphi = u^{-\frac{p}{2-p}+1} \eta^2$$

as test function in the weak formulation of (3) to get

$$- \int_{\mathbb{R}^2} u^{\frac{p}{2-p}} \eta^2 \, dx = - \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \left(u^{-\frac{p}{2-p}+1} \eta^2 \right) \, dx$$
$$= \frac{2(p-1)}{2-p} \int_{\mathbb{R}^n} u^{-\frac{p}{2-p}} |\nabla u|^p \eta^2 \, dx - 2 \int_{\mathbb{R}^n} u^{-\frac{p}{2-p}+1} |\nabla u|^{p-2} \eta \langle \nabla u, \nabla \eta \rangle \, dx \,.$$

From Cauchy-Schwarz and Young inequalities we obtain

$$\begin{aligned} - \int_{\mathbb{R}^2} u^{\frac{p}{2-p}} \eta^2 \, dx \ge & \frac{2(p-1)}{2-p} \int_{\mathbb{R}^2} u^{-\frac{p}{2-p}} |\nabla u|^p \eta^2 \, dx \\ & - 2 \int_{\mathbb{R}^2} u^{-\frac{p}{2-p}+1} |\nabla u|^{p-1} \eta |\nabla \eta| \, dx \\ \ge & \frac{2(p-1)}{2-p} \int_{\mathbb{R}^2} u^{-\frac{p}{2-p}} |\nabla u|^p \eta^2 \, dx \\ & - 2\varepsilon \int_{\mathbb{R}^2} u^{-\frac{p}{2-p}} |\nabla u|^p \eta^2 \, dx - C_\varepsilon \int_{\mathbb{R}^2} u^{\frac{p(1-p)}{2-p}} \eta^{2-p} |\nabla \eta|^p \, dx \,, \end{aligned}$$

for all $\varepsilon > 0$, i.e.

$$- \int_{\mathbb{R}^2} u^{\frac{p}{2-p}} \eta^2 \, dx \ge 2 \left(\frac{p-1}{2-p} - \varepsilon \right) \int_{\mathbb{R}^2} u^{-\frac{p}{2-p}} |\nabla u|^p \eta^2 \, dx - C_\varepsilon \int_{\mathbb{R}^2} u^{\frac{p(1-p)}{2-p}} \eta^{2-p} |\nabla \eta|^p \, dx \,, \tag{17}$$

for all $\varepsilon > 0$.

In order to tackle the second integral on the right-hand side of (17) we use the following lower bound for the function u: since u is a $p-$superharmonic function then there exist $C, \rho > 0$ such that

$$u(x) \geq \frac{C}{|x|^{\frac{2-p}{p-1}}} \qquad \text{for every } x \in \mathbb{R}^n \setminus B_\rho\,,$$

we refer to [43, Lemma 2.3] for the proof. Hence, (17) becomes, for $R > \rho$,

$$-\int_{B_R} u^{\frac{p}{2-p}}\,dx \geq 2\left(\frac{p-1}{2-p} - \varepsilon\right)\int_{B_R} u^{-\frac{p}{2-p}}|\nabla u|^p\,dx - C_\varepsilon R^2\,,$$

for all $\varepsilon > 0$ and for some $C_\varepsilon > 0$. Choosing ε small enough and reordering terms , we get

$$\int_{B_R} u^{-\frac{p}{2-p}}|\nabla u|^p\,dx + \int_{B_R} u^{\frac{p}{2-p}}\,dx \leq CR^2\,.$$

So, we have that the right-hand side of (16) is uniformly bounded in R. Hence,

$$\int_{\mathbb{R}^2} u^{\frac{p}{2-p}}|\mathring{\mathbf{V}}|^2\eta^2\,dx < \infty\,.$$

By using (13) and passing to the limit as $R \to \infty$, we obtain (14).

□

4. Open problems

In this section we state some open problems which are related to the Liouville-type results that we presented in the paper.

4.1. **The Lane-Emden conjecture.** A natural generalization of the Lane-Emden equation is the so-called Lane-Emden system:

$$\begin{cases} \Delta u + v^p = 0 & \text{in } \mathbb{R}^n, \\ \Delta v + u^q = 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{18}$$

where $p, q > 0$. The pair (p, q) is called *sub-critical* if

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{n}\,. \tag{19}$$

In [35], the author proves that the system (18) has no positive radial solutions if and only if the pair (p, q) is subcritical. This implies that the following conjecture holds true for non-negative radial solutions (see e.g. [16, 20, 44])

Lane-Emden Conjecture. *If the pair* (p, q) *is subcritical, then the only non-negative solutions to system* (18) *are* $u, v \equiv 0$.

We mention that (19) is optimal for proving nonexistence results: indeed, in the critical case (i.e. (19) with " $=$ ") and supercritical case (i.e. (19) with " $<$ ") the system (18) admits (bounded) positive radial classical solutions (see e.g. [35] and [44]).

Regarding the Lane-Emden Conjecture partial results are know: in dimensions $n = 1$ and $n = 2$ it follows from the results in [35, 44, 47]; moreover, in dimensions $n = 3$ and $n = 4$ the conjecture has been recently solved in [39] and in [46], respectively. In higher dimensions $n \geq 5$ the conjecture is open and is known to be true only in some subregions of the subcritical range given by (19) (we refer to the papers [46] and [30] and to the references therein for further details).

4.2. **The anisotropic setting.** In the anisotropic context, i.e. $\mathbb{R}^n$ endowed with a generic anisotropic norm H, i.e. $H : \mathbb{R}^n \to \mathbb{R}$ is a positive, positively homogeneous of degree one 1 and convex function. In this setting the natural generalization of the Lane-Emden equation is the following

$$\Delta_p^H u + u^q = 0 \quad \text{in } \mathbb{R}^n \,, \tag{20}$$

where, for $1 < p < n$, Δ_p^H denotes the so-called *anisotropic* $p-$*Laplace operator*:

$$\Delta_p^H u := \operatorname{div}(H^{p-1}(\nabla u)\nabla H(\nabla u)) \,;$$

observe that in the Euclidean case, i.e when $H(\cdot) = |\cdot|$ one has

$$H^{p-1}(\nabla u)\nabla H(\nabla u) = |\nabla u|^{p-2}\nabla u \,,$$

and so Δ_p^H reduces to the usual $p-$Laplace operator. In this setting the Liouville-type theorem in the critical case (i.e. $q = p^* - 1$) is given in [10] where we prove that positive solutions to (20) with finite energy are given by

$$\mathcal{U}^H_{\lambda,x_0}(x) := \left(\frac{n^{\frac{1}{p}} \left(\frac{n-p}{p-1}\right)^{\frac{p-1}{p}} \lambda}{1 + \lambda^{\frac{p}{p-1}} H_0(x_0 - x)^{\frac{p}{p-1}}} \right)^{\frac{n-p}{p}} \,, \tag{21}$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ and where H_0 denotes the dual norm of H, i.e.

$$H_0(\zeta) := \sup_{H(\xi)=1} \zeta \cdot \xi \,, \qquad \text{for all } \zeta \in \mathbb{R}^n.$$

For the sake of completeness, we mention that the functions (21) have been constructed as minimizers of the anisotropic Sobolev constant in [12] by using the optimal transport (see also [40] for further details).

It would be of interest to consider the anisotropic subcritial case (i.e. (20) with $1 \leq q < p^* - 1$) and prove the analogue of Theorem 2.3. Moreover, it would be of interest to remove the finite energy assumptions in the critical case proved in [10].

4.3. **The non-local setting.** Given $0 < \alpha < n$ the non-local version of the Lane-Emden equation is the following:

$$-(-\Delta)^{\alpha/2} u + u^q = 0 \quad \text{in } \mathbb{R}^n \,, \tag{22}$$

where $(-\Delta)^{\alpha/2}$ is the usual *non-local Laplace operator*. Observe that when $\alpha = 2$, (22) reduces to

$$\Delta u + u^q = 0 \quad \text{in } \mathbb{R}^n \,.$$

In this setting the Liouville-type theorem in the critical case (i.e. $q = \frac{n+\alpha}{n-\alpha}$) is given in [8], where the authors prove that positive solutions to (22) such that $u \in L^{\frac{2n}{n-\alpha}}_{loc}(\mathbb{R}^n)$ are given by

$$\mathcal{U}^{(\alpha)}_{\lambda,x_0}(x) = c(n,\alpha) \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2} \right)^{\frac{n-\alpha}{2}} \,, \tag{23}$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$. We mention that the functions (23) have been constructed as minimizers of the Hardy-Littlewood-Sobolev constant in [28]. We refer to [8] for further details. In the subcritical case (i.e. $1 \leq q < \frac{n+\alpha}{n-\alpha}$) it has been proved, in [13], that the only non-negative solution to (22) is $u \equiv 0$. We refer to [13] for further details and previous results.

Both results are based on an adaptation of the method of moving planes and it would be of interest to find an alternative proof suitable to be adapted to the non-local $p-$Laplace operator.

4.4. **The Heisenberg setting.** We recall that the Heisenberg group $\mathbb{H}^n$ is the Lie group $(\mathbb{R}^{2n+1}, \circ)$ with the composition defined in the following way: given $\xi, \xi' \in \mathbb{H}^n$ we denote $\xi = (z,t) = (x,y,t)$ and $\xi = (z',t') = (x',y',t')$, where $z = (x,y), z' = (x',y') \in \mathbb{R}^{2n}$ and $t, t' \in \mathbb{R}$ then

$$\xi \circ \xi' := ((z + z', t + t' + 2 [\langle x', y\rangle - \langle x, y'\rangle]) \ ,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$. The linear second order partial differential operator

$$\Delta_{\mathbb{H}^n} := \sum_{i=1}^{n} \left(X_i^2 + Y_i^2 \right) \ ,$$

where

$$X_i = \partial_{x_i} + 2y_i \partial_t \, , \quad Y_i = \partial_{y_i} - 2x_i \partial_t \quad j \in \{1, \ldots, n\} \, ,$$

is called the *Kohn-Laplace operator on* $\mathbb{H}^n$. In this setting the Lane-Emden equation is the following

$$\Delta_{\mathbb{H}^n} u + u^q = 0 \quad \text{ in } \mathbb{H}^n \, . \tag{24}$$

In the critical case $q = 1 + \frac{2}{n}$, in the paper [21], the authors proved that the only positive solutions to (24) such that $u \in L^{\frac{2n+2}{n}}(\mathbb{H}^n)$ are (up to left translations and dilations) of the following form

$$\mathcal{W}(\xi) = \frac{c_0}{(t^2 + (1 + |z|^2)^2)^{\frac{n}{2}}} \, . \tag{25}$$

We mention that the functions (25) have been constructed as minimizers of the so-called Folland-Stein constant in [21]. Moreover, (24) is also related to so-called CR Yamabe problem. For further details on these two topics we refer to the original papers [21] and [22].

In the subcritical case, $1 < q < 1 + \frac{2}{n}$ it has been recently proved that the only non-negative solution to (24) is $u \equiv 0$ (see [32] and the references therein for previous results).

It would be of interest to weaken the (finite energy) assumption $u \in L^{\frac{2n+2}{n}}(\mathbb{H}^n)$ in the critical case and to prove analogue Liouville-type results for the $p-$Kohn-Laplace operator on $\mathbb{H}^n$.

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