Structural Model Theory, Definable Measures, and Representations

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This thesis is dedicated to Anna, Maman and Papa, for all their love and support. Thank you for always encouraging me to pursue what I am passionate about, and thank you for your patience when my mind wanders.

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Contents

This thesis contains two chapters. They can be read independently and they are concerned with quite distinct aspects of model theory. Nevertheless, these two chapters are connected at a fundamental level via stability theory in Hilbert spaces and the use of model theory at the level of individual formulas to find meaningful structure inside analytical and combinatorial objects. This brief introduction aims to clarify the connections between the two chapters and to highlight the fundamental motivations behind each project. Detailed introductions to each project are given at the start of each chapter.

The first chapter uses stability theory in Hilbert spaces to analyse the fine structure of piecewise interpretable Hilbert spaces. These are Hilbert spaces which are direct limits of imaginary sorts of continuous logic structures. They arise in model theory in a variety of situations where they can code different kinds of information, such as Galois theoretic, representation theoretic, or measure theoretic information. There is some prior work in the literature (most notably [Tsa12], [Hru12] and [Iba21]) which seeks to understand interactions between Hilbert spaces and arbitrary model theoretic structures, but the work contained in this thesis is the first systematic study of these Hilbert spaces in a general setting.

In this thesis, we take the point of view that piecewise interpretable Hilbert spaces are interesting objects of model theory in their own right and that it is meaningful and fruitful to find general structure theorems for piecewise interpretable Hilbert spaces. This philosophy can be compared to the deep study of definable groups which has driven many aspects of research in model theory. We will see that our general approach leads to new results in representation theory and in the study of absolute Galois groups and definable measures, and we will find surprising interactions of piecewise interpretable Hilbert spaces with various aspects of model theory, including the NFCP, one-basedness and strong minimality.

The second chapter in this thesis is concerned with a specific theory, ACFA, and seeks to use the rich model theory of ACFA to derive new results in combinatorics and algebraic geometry. This chapter builds on the work of [Tao12] which established a Szemerédi-style regularity lemma for definable graphs in pseudofinite fields. Tao asked if this result can be extended to hypergraphs, and we answer this question positively.

Tao's original result is best viewed as an application of stability in Hilbert spaces. Indeed, Tao's theorem relies crucially on the definability of the counting measure in pseudofinite fields, a result due to [CvdDM92]. Stability in Hilbert spaces implies that the formulas $\mu_x(\phi(x,a) \wedge \psi(x,b))$ in pseudofinite fields are stable. Hence these formulas carry the usual stationarity properties of stable formulas, and this implies that the generic value of $\mu_x(\phi(x,a) \wedge \psi(x,b))$ is controled by the type of a and the type of b. We call this the 'stationarity theorem' and this is the fundamental technical fact which Tao uses to derive his regularity lemma. This proof of the stationarity theorem, which is different from Tao's, is originally due to Hrushovski and to [PS13].

Since stability in Hilbert spaces is a fundamentally *binary* phenomenon, a different approach is required to derive a regularity lemma for hypergraphs. For this reason, we will carry out a geometric analysis of the definable measure of [CvdDM92] and we will show that the geometric point of view leads to the stochastic independence theorem and the general stationarity theorem in ACFA. The stationarity theorem in ACFA says that the value of the measure $\mu_x(\phi(x,a,b) \wedge \psi(y,a,c) \wedge \chi(x,b,c))$ is determined by $\mathrm{tp}(a,b)$, $\mathrm{tp}(a,c)$ and $\mathrm{tp}(b,c)$, under some reasonable conditions. We will also gain some useful probabilistic information about definable sets in ACFA. The algebraic hypergraph regularity lemma follows in a straightforward manner by using results of Gowers. The stationarity theorem in ACFA in its current form is the first result of its kind in model theory.

From a rough conceptual point of view, the stationarity theorem in ACFA draws on amalgamation in algebraically closed fields and model completeness of ACFA to find probabilistic independence between definable sets. Therefore, in our study of hypergraphs in ACFA, the setting of Hilbert spaces is replaced by the much stronger setting of algebraically closed fields, but stability at the level of individual formulas is still the main driver of our results.

Therefore, both chapters in this thesis stem from the realisation that stability in Hilbert spaces leads to fruitful applications of model theory to a wide variety of topics in mathematics. In the first chapter, the aim was to find a general framework which unifies all instances of Hilbert space stability in model theory. In the second chapter, the aim was to move beyond the black-box application of the stationarity theorem in Tao's regularity lemma to understand the geometric content of Tao's result and to generalise it to hypergraphs.

While the two projects in this thesis started with a common idea and moved in opposite directions, one might hope that the two points of view studied in this thesis can eventually be joined again.

On the one hand, because of its reliance on geometry, it is not yet clear if it is possible to find a general framework in which the proof of the algebraic hypergraph regularity lemma takes place. On the other hand, the work on piecewise interpretable Hilbert spaces raises the very open-ended question of whether other analytical objects can be found to govern interesting model theoretic phenomena in a similar way. This is an exciting direction for future research.

Chapter 1

Piecewise Interpretable Hilbert Spaces

The material in this chapter is joint work with Ehud Hrushovski and will be published as such. Section 1.3 is entirely my own work and all other sections contain contributions from both authors.

In this chapter, we define piecewise interpretable Hilbert spaces and study some of their properties. These are Hilbert spaces which arise as direct limits of imaginary sorts of a first-order continuous logic theory T. While we employ continuous logic, the case where T itself is a discrete first order logic theory is already of interest, and the introduction may be read with such theories in mind. We will use tools of stability theory to study the structure of such Hilbert spaces, obtaining information about the underlying theory. The stability emanates from the Hilbert space inner product formulas themselves, so no stability assumptions on the theory T are required.

Definable measures, which themselves play a central role in recent model-theoretic literature, provide one rich source of examples. If μ is such a measure, the Hilbert space $L^2(\mu)$ is piecewise interpretable in T. The stability-theoretic viewpoint already gives useful information here; notably it was used previously to prove an independence, or 3-amalgamation theorem for definable measures (see [Hru15]; a more basic version was the main engine of [Hru12], Theorem 2.22).

A different class of piecewise interpretable Hilbert spaces arises via the absolute Galois group G of a field K, with $T = Th(K, K^{alg})$. G is essentially a projective limit of definable groups in T. This was discovered by [CvdDM80]. Here no definable measure need be present, but the Hilbert space $L^2(G, \mathrm{Haar})$ is piecewise interpretable in T.

Any piecewise interpretable Hilbert space gives rise to a functor from the category of models of a theory T to the category of Hilbert spaces. In particular, we obtain a homomorphism from the automorphism group G of any model to the unitary group of a Hilbert space, i.e. a unitary reprsentation of

G. Conversely, a basic lemma of [Tsa12] implies in the ω-categorical case that all unitary representations of the automorphism group of the countable model arise in this way, giving a third and very interesting connection to a deep field.

Our initial inspiration was the classification theorem for unitary representations of automorphism groups of ω-categorical structures from [Tsa12]. Changing the viewpoint to that of piecewise interpretable Hilbert spaces, it becomes natural to ask whether they admit a structure theorem under more general hypotheses than ω-categoricity; and if so, what form the statement would take. The answers we found were, respectively, the notions of *scatteredness* and *asymptotic freedom*.

Both scatteredness and asymptotic freedom concern a complete type p within a piecewise interpretable Hilbert space H. *Scatteredness* is defined in 1.3.5; we mention for now a special case (referred to as 'strict interpretability' in 1.3.17): whenever there are only finitely many values achieved by the inner product between elements of p, p is scattered. This subclass already includes a rich class of examples: these include the ω-categorical case of [Tsa12], the Hilbert spaces arising from definable measures over pseudo-finite fields ([CvdDM92]) or over measurable classes ([MS08]), and the Galois-theoretic examples mentioned above.

p is *asymptotically free* if for any $a \in p$, and for any $b \in p$ that is not algebraic over a, the elements a, b are orthogonal as vectors in the Hilbert space. In general, asymptotic freedom means that the Hilbert space structure is defined using only information within the algebraic closure (bounded closure in the continuous logic case) of an element of p. In fact, the interpretation of the subspace generated by p factors through a disintegrated, strongly minimal reduct of T with p as its universe.

We prove a number of structure theorems that analyse scattered representations in terms of asymptotically free ones. Our core result is Theorem 1.3.14:

> *Let $\mathcal{H}$ be a piecewise interpretable Hilbert space in T. Let $\mathcal{H}_p$ be a piecewise $\bigwedge$-interpretable subspace of $\mathcal{H}$ generated by a scattered type-definable set p. Then $\mathcal{H}_p$ is the orthogonal sum of piecewise $\bigwedge$-interpretable subspaces $(\mathcal{H}_\alpha)_{\alpha<\kappa}$ such that for all $\alpha < \kappa$, $\mathcal{H}_\alpha$ is generated by an asymptotically free complete type.*

In particular, Theorem 1.3.14 fully recovers Tsankov's structure theorem in the classical logic ω-categorical case.

Even with the above mentioned notions at hand however, the proof is not a direct generalisation from the ω-categorical case. It proceeds instead via a local stability analysis. The interaction of the stable but highly non-discrete Hilbert space formulas, with the type provided by the underlying theory T turns out to imply not only rank properties but also a certain local modularity of forking that forms the key to the later analysis. A general discussion of these rank properties can be found in Section 1.7.

Using a theorem of Howe and Moore, we show that for any algebraic group G over $\mathbb{Q}$, any irreducible representation of $G(\mathbb{Q}_p)$ or $G(\mathbb{R})$ can essentially be obtained as a piecewise interpretable Hilbert space generated by an asymptotically free type (see Section 1.6.3 for details). As the connection is made directly to the conclusion of our theorem, we do not obtain any new implications for the irreducible representations of these classical groups; but this does show that asymptotic freedom includes both settings for the unitary representation theory of oligomorphic groups as well as algebraic groups.

A significant chapter of abstract model theory concerns interpretable groups, usually referred to as definable groups. First studied for their own sake, especially in the stable context, the study of groups interpreted in a theory T was found to return significant structural information on T. Our treatment of Hilbert spaces as objects of the definable world is partly inspired by this analogy. We hope they may prove to illuminate other aspects of the theory, and we view our results as an indication of the possibility of such a development.

1.1 Structure of the chapter

In Section 1.2, we define and prove some general model theoretic facts about piecewise interpretable Hilbert spaces. We give examples of piecewise interpretable Hilbert spaces in Section 1.2.2.

In Section 1.3 we begin a study of the fine structure of piecewise interpretable Hilbert spaces under the assumption of scatterdness, defined in Definition 1.3.5. Asymptotic freedom is defined in Definition 1.3.13. Our main structure theorem is Theorem 1.3.14. In Section 1.3.2, we study the case where T is a classical logic theory and $\mathcal{H}$ is determined by classical logic formulas, in which case we say $\mathcal{H}$ is strictly interpretable. We will extend Theorem 1.3.14 to this context in Corollary 1.3.21. In Section 1.3.3 we give some concrete examples of the decomposition promised by Theorem 1.3.14.

In Section 1.4, we show that the L^2-spaces associated to definable measures are strictly interpretable. We discuss the asymptotically free decomposition of the L^2-space associated to the random graph, and we prove the strong germ property for pseudofinite fields and ω-categorical Macpherson-Steinhorn measurable structures.

In Section 1.5 we introduce a new construction in classical logic and a new source of piecewise interpretable Hilbert spaces. Let M be a classical logic $\mathcal{L}$-structure and $K \subseteq M$ a definably closed subset. We show that M interprets the projective system of finite quotients of $Gal(K)$ in the language $\mathcal{L}_P$ with an additional predicate P for K. This generalises the classical construction of

[CvdDM80] to a wide variety of settings. This construction yields interpretations of $L^2(Gal(K), Haar)$ and of various subspaces of interest. We will see that there are delicate interpretability issues surrounding these Hilbert spaces, which makes them all the more interesting.

In Section 1.6 we study the connection to unitary group representations. We show that representations generated by asymptotically free complete types are induced representations and we prove a Mackey-style irreducibility criterion for these representations. In Section 1.6.2 we discuss representations of automorphism groups of ω-categorical structures. In Section 1.6.3 we adapt the approach of Theorem 1.3.14 for arbitrary group representations with an orbit whose weak closure is locally compact. We show thanks to the theorem of Howe-Moore that interpretable Hilbert spaces with asymptotically free types capture the unitary representations of p-adic Lie groups.

In this chapter, we use continuous logic for metric spaces but we will not be reliant on any high-level results of continuous logic. In the appendix to this chapter, we have included an exposition of continuous logic in the style of [HI02]. A similar approach was recently used in [GP21]. We list there the basic notions which we will use in this paper. These are mostly based on [BYBHU08] but we lay down explicit definitions consistent with our choice of syntax.

The entire chapter has been written in such a way that all arguments can easily be translated into the formualism of [CK66] or [BYBHU08] if the reader wishes to do so. Outside of the basic notions presented in the appendix, we will develop all the continuous logic theory we need as we go along.

1.2 Introduction to Piecewise Interpretable Hilbert Spaces

In this section, we give a general exposition of piecewise interpretable Hilbert spaces. We use the GNS construction to show that piecewise interpretable Hilbert spaces are defined at the level of the theory T. In Section 1.2.2 we give some key examples of piecewise interpretable Hilbert spaces and in Section 1.2.3 we give an alternative treatment of piecewise interpretable Hilbert spaces which allows us to deduce an important proposition about forking independence in Section 1.2.4.

1.2.1 Basic definitions

In this section, M is a continuous logic structure in an arbitrary language. In this thesis, *definable* always means $\emptyset$*-definable.*

Definition 1.2.1. *A piecewise interpretable Hilbert space $H(M)$ in M is a direct limit of imaginary sorts $(M_j)_{j\in J}$ of M such that $H(M)$ is a Hilbert space and the inner product is definable between all the pieces M_j. More explicitly, for all $i, j \in J$, writing h_i, h_j for the direct limit maps from M_i and M_j to $H(M)$, the map $M_i \times M_j \to \mathbb{R}, (x, y) \mapsto \langle h_i x, h_j y\rangle$ is definable.*

In the above definition, J is a directed partial order. When discussing direct limits of sorts $(M_j)_{j\in J}$ of M, we always assume that for any $i \leq j$ in J, the transition maps $M_i \to M_j$ are definable. We do not require them to be injective. We say that each sort M_i is a *piece* of $H(M)$ and for any $i, j \in J$, we say that the definable map $f_{ij} : M_i \times M_j \to \mathbb{R}$ given by $f_{ij}(x, y) = \langle h_i x, h_j y\rangle$ is an *inner product map.*

Definition 1.2.1 does not explicitly require that the sum and scalar multiplication operations on $H(M)$ be definable between the pieces $(M_j)_J$. However, we show in Lemma 1.2.10 that the Hilbert space operations are always definable in an appropriate sense.

Suppose $H(M)$ is piecewise interpretable in M and suppose that $H(M)$ is the direct limit of the sorts $(M_j)_{j\in J}$. For $i, j \in J$ write $f_{ij} : M_i \times M_j \to \mathbb{R}$ for the inner product maps. Let T be a continuous logic theory, not necessarily complete, such that $M \models T$, the transition maps $M_i \to M_j$ and the inner product maps f_{ij} are definable in T. Suppose also that T proves that the maps f_{ij} factor through the transition maps of the partial order $(M_j)_J$ and that for all $j \in J$, $m \geq 1$ and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$,

$$T \vdash \forall x_1 \dots \forall x_m \sum_{n,p\leq m} \lambda_n \lambda_p f_{jj}(x_n, x_p) \geq 0$$

where the variables x_j range in the sort of the piece M_i. For every $N \models T$, write N_j for the sort in N corresponding to M_j. Since all data is definable, the direct limit $(N_j)_{j\in J}$ with the same inner product maps is a piecewise interpretable Hilbert space $H(N)$ in N.

Therefore, $H(M)$ gives rise to a *piecewise interpretable Hilbert space $\mathcal{H}$ in T.* We will be careful to distinguish a piecewise interpretable $\mathcal{H}$ in T and its interpretation $H(M)$ in a model M of T. Many properties of $H(M)$ transfer to $\mathcal{H}$ and vice versa.

A useful way of defining piecewise interpretable Hilbert spaces is given by the GNS theorem (named after Gelfand, Naimark and Segal, see Appendix C in [BdlHV08]), which we recall below.

Definition 1.2.2. *Let X be a set. A function $f : X \times X \to \mathbb{R}$ is said to be positive-semidefinite if f is symmetric and for all $n \geq 1$, for all $x_1, \dots, x_n \in X$ and for all $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, we have $\sum_{i,j} \lambda_i \lambda_j f(x_i, x_j) \geq 0$.*

Theorem 1.2.3 (GNS Theorem)**.** *Let X be a set and let $f : X \times X \to \mathbb{R}$ be positive-semidefinite. Then there is a Hilbert space H and a map $F : X \to H$ such that $F(X)$ has dense span in H and for all $x, y \in X$, $\langle F(x), F(y)\rangle = f(x, y)$.*

F and H are unique in the sense that if F' and H' satisfy the same statement, then there is a surjective unitary map $G : H \to H'$ such that $F' = G \circ F$.

We will adapt the GNS theorem to our context:

Proposition 1.2.4. *Suppose we are given a Hilbert space H, a collection of imaginary sorts $(M_i)_{i\in I}$ of M and functions $F_i : M_i \to H$ for all $i \in I$ such that for all $i, j \in I$, the map $M_i \times M_j \to \mathbb{R}, (x, y) \mapsto \langle F_i x, F_j y\rangle$ is definable.*

Then there is a piecewise interpretable Hilbert space $H(M)$ in M such that

1. *the sorts $(M_i)_I$ are pieces of $H(M)$*
2. *writing $h_i : M_i \to H(M)$ for the direct limit maps, $\bigcup h_i(M_i)$ has dense span in $H(M)$*
3. *for any $i, j \in I$ and $x \in M_i$, $y \in M_j$, we have $\langle h_i x, h_j y\rangle_{H(M)} = \langle F_i x, F_j y\rangle_H$.*

Proof. To make notation lighter, we write F for all functions F_i, $i \in I$. It will always be clear from the variable which map F_i we are using. By passing to a closed subspace of H, we can assume without loss of generality that $\bigcup F(M_i)$ has dense span in H. Let $x \in H$. There is an increasing function $\eta : \mathbb{N} \to \mathbb{N}$ and a uniformly Cauchy sequence $(\sum_{i=\eta(n)}^{\eta(n+1)-1} \lambda_i^n F x_i^n)$ which converges to x such that $x_i^n \in \bigcup_j M_j$. We can assume that $\eta(n)$ is large enough so that $|\lambda_i^n| \leq \eta(n)$ for all $i \leq n$. We will decompose H according to the growth rate of η.

Fix $\eta : \mathbb{N} \to \mathbb{N}$ strictly increasing and fix an arbitrary countable sequence (i_n) in I. We define an imaginary sort $M^\eta_{(i_n)}$ as follows. $M^\eta_{(i_n)}$ will be the metric completion of the countable Cartesian product

$$\prod_{n\geq 0} \Big(\prod_{k=\eta(n)}^{\eta(n+1)-1} [-\eta(n), \eta(n)] \times M_{i_k} \Big)$$

under the metric δ defined below. We write elements $(\overline{x}_n) \in M^\eta_{(i_n)}$ as tuples $(\lambda_{\eta(n)}, x_{\eta(n)}, \ldots, \lambda_{\eta(n+1)-1}, x_{\eta(n+1)-1})$. For all n, define

$$F(\overline{x}_n) = \sum_{i=\eta(n)}^{\eta(n+1)-1} \lambda_i F x_i.$$

We define inductively maps $g_k : M^\eta_{(i_n)} \to H$. $g_0((\overline{x}_n))$ is just $F(\overline{x}_0)$. Given g_k, define $g_{k+1}((\overline{x}_n)) = F(\overline{x}_{k+1})$ if $\|g_k((\overline{x}_n)) - F(\overline{x}_{k+1})\| \leq 2^{-n}$. Otherwise, define[1]

$$g_{k+1}((\overline{x}_n)) = g_k((\overline{x}_n)) + \frac{2^{-n}}{\|F(\overline{x}_{k+1}) - g_k((\overline{x}_n))\|}(F(\overline{x}_{k+1}) - g_k((\overline{x}_n)))$$

Then $(g_k((\overline{x}_n)))_k$ is a uniformly Cauchy sequence in H. It is straightforward to check that for all k, the map $((\overline{x}_n), (\overline{y}_n)) \mapsto \langle g_k((\overline{x}_n)), g_k((\overline{y}_n))\rangle$ is definable. We obtain a definable map

$$\beta : ((\overline{x}_n), (\overline{y}_n)) \mapsto \lim_k \langle g_k((\overline{x}_n)), g_k((\overline{y}_n))\rangle.$$

β is positive-semidefinite and hence β induces a pseudo-metric δ. We quotient $M^\eta_{(i_n)}$ by this pseudo-metric and take the metric completion, so that $M^\eta_{(i_n)}$ is identified with a subset of H.

We now define the direct limit structure. Fix an arbitrary ordering $\leq_I$ of I. Let J be the set of pairs $(\eta, (i_n))$ such that $\eta : \mathbb{N} \to \mathbb{N}$ is strictly increasing and (i_n) is a sequence in I such that for all n, $(i_{\eta(n)}, \ldots, i_{\eta(n+1)-1})$ is increasing with respect to the order $\leq_I$. We could have chosen such an (i_n) when we constructed $M^\eta_{(i_n)}$ above. We define a partial ordering on J as follows: we say that $(\eta, (i_n)) \leq (\mu, (j_n))$ if and only if for all n, $\eta(n+1)-\eta(n) \leq \mu(n+1) - \mu(n)$ and $(i_{\eta(n)}, \ldots, i_{\eta(n+1)-1})$ is a subtuple of $(j_{\mu(n)}, \ldots, j_{\mu(n+1)-1})$. We have definable maps $M^\eta_{(i_n)} \to M^\mu_{(j_n)}$ for $(\eta, (i_n)) \leq (\mu, (j_n))$ by taking the obvious inclusions and by using the scalar 0 to pad the image of $M^\eta_{(i_n)}$ in $M^\mu_{(j_n)}$. $H(M)$ is defined as the direct limit of the sorts $M^\eta_{(i_n)}$. □

We now clarify some technical issues around the construction of Proposition 1.2.4. We define isomorphisms of piecewise interpretable Hilbert spaces and show uniqueness of the piecewise Hilbert space of Proposition 1.2.4 with respect to this notion.

Definition 1.2.5. *Let $H(M), H'(M)$ be piecewise interpretable Hilbert spaces in M. Suppose $H(M)$ is the direct limit of $(M_j)_{j\in J}$ and $H'(M)$ is the direct limit of $(M_{j'})_{J'}$. Write $h_j : M_j \to H(M)$ and $h'_j : M_{j'} \to H'(M)$ for the direct limit maps.*

We say that a map $F : H(M) \to H'(M)$ is an embedding of piecewise interpretable Hilbert spaces if F is a unitary map and for all $j \in J$ and $j' \in J'$ and all $\epsilon \geq 0$, the set $\{(x, y) \in M_j \times M'_{j'} \mid \|F(h_j x) - h'_{j'} y\| \leq \epsilon\}$ is type-definable.

If F is also surjective, we say that F is an isomorphism of piecewise interpretable Hilbert spaces.

Lemma 1.2.6. *Let $H(M), H'(M)$ be piecewise interpretable Hilbert spaces in M. Let $(M_i)_{i\in I}$ and $(M_{i'})_{i'\in I'}$ be pieces of $H(M)$ and $H'(M)$ respectively which*

[1] This is analogous to Definition 3.6 in [BYU10]

generate $H(M)$ and $H'(M)$. Write $h_i : M_i \to H(M)$ and $h'_{i'} : M_{i'} \to H'(M)$ for the direct limit maps.

A unitary map $F : H(M) \to H'(M)$ is an embedding of interpretable Hilbert spaces if and only if for all $i \in I$ and $i' \in I'$ the map $M_i \times M_{i'} \to \mathbb{R}, (x, y) \mapsto \langle F(h_i x), h'_{i'} y\rangle$ is definable.

Proof. Suppose that F is an embedding of piecewise interpretable Hilbert spaces. Fix $i \in I$, $i' \in I'$ and D a closed bounded subset of $\mathbb{R}$. For every $\epsilon > 0$, let D_ϵ be the closed set $\{x \in \mathbb{R} \mid \exists y \in D, |y - x| \leq \epsilon\}$. Let B be an upper bound on $\{\|h'_{i'} y\| \mid y \in M_{i'}\}$.

By compactness, for every $\epsilon > 0$ we can find $i'_\epsilon \in I'$ such that for every $x \in M_i$, there is $x' \in M_{i'_\epsilon}$ such that $\|h'_{i'_\epsilon} x' - F(h_i x)\| < \epsilon/B$. The set $\{(x, y) \in M_i \times M_{i'} \mid \langle F(h_i x), h'_{i'} y\rangle \in D\}$ is equal to the intersection over $\epsilon > 0$ of the sets

$$\{(x, y) \in M_i \times M_{i'} \mid \exists x' \in M_{i_\epsilon}, \|F(h_i x) - h'_{i'_\epsilon} x'\| \leq \epsilon/B \text{ and } \langle h'_{i'_\epsilon} x', h_{i'} y\rangle \in D_\epsilon\}.$$

Hence this set is type-definable.

Conversely, suppose that $H(M)$ and $H'(M)$ respectively are the direct limits of the sorts $(M_j)_J$ and $(M_{j'})_{J'}$. We write (h_j) and $(h'_{j'})$ for the direct limit maps. Fix $j \in J$, $j' \in J'$. For any $x \in M_j$, the element $F(h_j x)$ is uniquely determined by the collection of maps $M_{i'} \to \mathbb{R}, y \mapsto \langle F(h_j x), h'_{i'} y\rangle$, for $i' \in I'$.

Moreover, by a standard compactness argument, for arbitrary $\epsilon > 0$ there is $n_\epsilon \geq 1$ such that for any $x \in M_j$, $h_j x$ is within distance ϵ of a vector of the form $\sum_{k=1}^{n_\epsilon} \lambda_k h_{i_k} x_{i_k}$ where $i_k \in I$ for all $k \leq n_\epsilon$, $x_{i_k} \in M_{i_k}$ and $|\lambda_k| \leq n_\epsilon$. By considering small enough δ and quantifying over elements $\sum_{k=1}^{n_\delta} \lambda_k h_{i_k} x_{i_k}$, we find that $\{(x, y) \in M_j \times M_{j'} \mid \|F(h_j x) - h'_{j'} y\| \leq \epsilon\}$ is type-definable if the maps $\langle F(h_i x), h'_{i'} y\rangle$ are definable.

□

The following sharpening of Proposition 1.2.4 follows easily:

Lemma 1.2.7. *Let M be a continuous logic structure. As in Proposition 1.2.4, suppose we are given a Hilbert space H, a collection of distinct imaginary sorts $(M_i)_{i\in I}$ of M and maps $F_i : M_i \to H$ such that for all $i, j \in I$, the map $M_i \times M_j \to \mathbb{R}, (x, y) \mapsto \langle F_i x, F_j y\rangle$ is definable.*

Then there is a piecewise interpretable Hilbert space $H(M)$ which is unique up to isomorphism such that the sorts $(M_i)_I$ are pieces of $H(M)$, $\bigcup h_i(M_i)$ has dense span in $H(M)$, and for $x \in M_i$, $y \in M_j$, $\langle h_i x, h_j y\rangle_{H(M)} = \langle F_i x, F_j y\rangle_H$.

Proof. Let $H(M)$, $H'(M)$ be two piecewise interpretable Hilbert spaces satisfying the existence claim. Since the inner product maps on the pieces M_i are identical, the uniqueness statement of the GNS theorem applies and we find an isomorphism of Hilbert spaces $F : H(M) \to H'(M)$. Lemma 1.2.6 applies directly so F is an isomorphism of interpretable Hilbert spaces. □

We are only interested in discussing piecewise interpretable Hilbert spaces in M up to isomorphism. Therefore, in order to fix a piecewise interpretable Hilbert space $H(M)$, it will be enough to specify a pair $(H, (h_i)_{i\in I})$ where H is a Hilbert space and the maps $h_i : M_i \to H$ are as in Proposition 1.2.4 and the span of $\bigcup h_i(M_i)$ is dense in H.

Equivalently, a piecewise interpretable Hilbert space $H(M)$ can be described up to isomorphism by fixing sorts $(M_i)_I$ of M and taking for every pair $i, j \in I$ a definable map $f_{ij} : M_i \times M_j \to \mathbb{R}$ such that the concatenation of all maps $(f_{ij})_I$ is positive-semidefinite on $(\bigcup_I M_i)^2$. These various presentations correspond to the following equivalence of categories:

Lemma 1.2.8. *Let $M \models T$. The categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ defined below are all equivalent:*

Let $\mathcal{A}$ be the category of piecewise interpretable Hilbert spaces in M with embeddings of interpretable Hilbert spaces.

Let $\mathcal{B}$ be the category of pairs $(H, (h_i)_I)$ where H is a Hilbert space and for every $i \in I$, $h_i : M_i \to H$ is a map on an imaginary sort of M such that

1. *the set $\bigcup_I h_i(M_i)$ has dense span in H*
2. *for all $i_1, i_2 \in I$, the map $M_{i_1} \times M_{i_2} \to \mathbb{R}, (x, y) \mapsto \langle h_{i_1}x, h_{i_2}y\rangle$ is definable.*

The morphisms between objects $(H, (h_i)_I)$ and $(H', (h_{i'})_{I'})$ of $\mathcal{B}$ are unitary maps $F : H \to H'$ such that for all $i \in I$, $i' \in I'$ the map $M_i \times M_{i'} \to \mathbb{R}, (x, y) \mapsto \langle F(h_i x), h_{i'}y\rangle$ is definable.

Let $\mathcal{C}$ be the category of pairs $((M_i)_I, (f_{ij})_{i,j\in I})$ where $(M_i)_I$ is a set of imaginary sorts of M and the $f_{ij} : M_i \times M_j \to \mathbb{R}$ are definable functions such that their concatenation $f : \bigcup_I M_i \times \bigcup_I M_i \to \mathbb{R}$ is positive semidefinite. The morphisms between objects $((M_i)_I, (f_{ij}))$ and $((M_{i'})_{I'}, f_{i'j'})$ of $\mathcal{C}$ are piecewise definable functions $G : \bigcup_{k\in I\cup I'} M_k \times \bigcup_{k\in I\cup I'} M_k \to \mathbb{R}$ such that

1. *G is positive semidefinite and G extends each function f_{ij} where $i, j \in I$ or $i, j \in I'$*
2. *writing H_I, $H_{I\cup I'}$ for the Hilbert spaces induced by $\bigcup_I M_i$, $\bigcup_{k\in I\cup I'} M_k$ respectively as in the GNS theorem, the resulting Hilbert space embedding $F : H_{I'} \to H_{I\cup I'}$ induced from G is surjective [2]*

Composition of morphisms in $\mathcal{C}$ is induced by the GNS theorem[3].

[2] The GNS theorem gives Hilbert spaces H_I, $H_{I'}$ and $H_{I\cup I'}$ and maps $a : \bigcup_I M_i \to H_I$, $b : \bigcup_{I'} M_{i'} \to H_{I'}$, $c : \bigcup_{k\in I\cup I'} M_k \to H_{I\cup I'}$ and Hilbert space embeddings $\phi : H_I \to H_{I\cup I'}$ and $\phi' : H_{I'} \to H_{I\cup I'}$ such that the corresponding diagram commutes. So we have $G(x, y) = \langle \phi \circ a(x), \phi' \circ b(y)\rangle$ when $x \in \bigcup_I M_i$ and $y \in \bigcup_{I'} M_{i'}$

[3] It is possible to give a definition of the morphisms of $\mathcal{C}$ and their composition which does not rely explicitly on the GNS theorem by using the Gram-Schmidt orthogonalisation process and Bessel's inequality. The details are left to the reader.

Recall that we did not require direct limits of sorts of M to have injective direct limit maps. We now show that this does not present any significant disadvantage.

Lemma 1.2.9. *Let $H(M)$ be a piecewise interpretable Hilbert space in M. Then $H(M)$ is isomorphic to a piecewise interpretable Hilbert space $H'(M)$ which is a direct limit of imaginary sorts $(M_{j'})_{J'}$ with isometric direct limit maps.*

Proof. Suppose that $H(M)$ is the direct limit of the sorts (M_j) with inner product maps (f_{ij}). Write $h_{ij} : M_i \to M_j$ for the transition maps between the sorts M_i, M_j when $i \leq j$ and $h_j : M_j \to H(M)$ for the direct limit maps. For every $j \in J$, let M'_j be the imaginary sort of canonical parameters of the inner product map f_{jj}, where we view $f_{jj}(x, y)$ as a function in y with a parameter x. For $i \leq j \in J$, define the transition map $h'_{ij} : M'_i \to M'_j$ as the map which takes the canonical parameter for the map $f_{ii}(a, .)$ to the canonical parameter for the map $f_{jj}(h_{ij}a, .)$. Observe h'_{ij} is well-defined because $f_{ii}(a, .) = f_{ii}(b, .)$ if and only if $h_i a = h_i b$. It is clear that the direct limit $H'(M)$ of $(M'_j)_J$ satisfies the lemma. □

As a direct application of the construction in Proposition 1.2.4 and Lemma 1.2.9, we have:

Lemma 1.2.10. *Let $H(M)$ be a piecewise interpretable Hilbert space in M. Then $H(M)$ is isomorphic to a piecewise interpretable Hilbert space $H'(M)$ in M which is a direct limit of $(M_{j'})_{j' \in J'}$ with isometric direct limit maps $h'_{j'}$ such that the Hilbert space operations on $H'(M)$ are piecewise bounded*[4]. *This means that*

1. *for every $i, j \in J'$, there is $k \in J'$ such that $i, j \leq k$ and $h'_i(M_i) + h'_j(M_j) \subseteq h'_k(M_k)$ and the map $M_i \times M_j \to M_k, (x, y) \mapsto (h'_k)^{-1}(x + y)$ is definable*

2. *for every $i \in J'$ and $n \geq 0$, there is k such that for $x \in M_i$ and $\lambda \in [-n, n]$, $\lambda h'_i x \in h'_i M_k$ and the map $[-n, n] \times M_i \to M_k$, $(\lambda, x) \mapsto (h'_i)^{-1}(\lambda h'_i x)$ is definable.*

Proof. Apply the construction of Proposition 1.2.4 to all pieces $(M_j)_J$ of $H(M)$ with the direct limit maps $M_j \to H(M)$ to obtain $H_1(M)$ isomorphic to $H(M)$. Observe that the construction of Proposition 1.2.4 implies that the operations on $H_1(M)$ are piecewise bounded. Now apply Lemma 1.2.9 to obtain $H'(M)$ with isometric direct limit maps. □

Finally we show that an isomorphism of interpretable Hilbert spaces in an ω-saturated structure induces an isomorphism at the level of the theory T.

[4] Thanks to Arturo Rodríguez Fanlo for this terminology

Lemma 1.2.11. *Let T be a complete theory and let $\mathcal{H}$, $\mathcal{H}'$ be two piecewise interpretable Hilbert spaces in T. Suppose that for some ω-saturated $M \models T$, $H(M)$ and $H'(M)$ are isomorphic as interpretable Hilbert spaces. Then for every $N \models T$, $H(N)$ and $H'(N)$ are isomorphic as piecewise interpretable Hilbert spaces.*

Proof. Suppose that $H(M)$ and $H'(M)$ are the direct limits of $(M_j)_J$ and $(M_{j'})_{J'}$ respectively. We write h_j and $h'_{j'}$ for the direct limit maps $M_j \to H(M)$, $M_{j'} \to H'(M)$ respectively. Write $F_M : H(M) \to H'(M)$ for the isomorphism of interpretable Hilbert spaces. Fix $N \models T$, $j \in J$, and take $a \in N_j$. Find $x \in M_j$ with $\mathrm{tp}(x) = \mathrm{tp}(a)$ and fix $j' \in J'$ such that $F_M h_j x \in h'_{j'} M_{j'}$.

$F_M h_j x$ is uniquely determined by the value $\lambda = \langle F_M h_j x, F_M h_j x\rangle$ and the x-definable function $f_x : M_{j'} \to \mathbb{R}, y \mapsto \langle F_M h_j x, h'_{j'} y\rangle$. By elementarity, for every n, there is $b_n \in N_{j'}$ such $|\langle h'_{j'} b_n, h'_{j'} b_n\rangle - \lambda| < 2^{-n}$ and for all $y \in N_{j'}$ $|f_a(y) - \langle h'_{j'} b_n, h'_{j'} y\rangle| < 2^{-n}$. Then $(h'_{j'} b_n)$ is Cauchy and there is $b \in N_{j'}$ such that $h'_{j'} b$ is the limit of $(h'_{j'} b_n)$. Define $F_N h_j a = h'_{j'} b$. It is straightforward to check that F_N is well-defined, definable, and is an isomorphism of piecewise interpretable Hilbert spaces. $\square$

When the direct limit maps are isometries, we identify the pieces M_j with subsets of $H(M)$. By a *type-definable subset p of $H(M)$* we mean a type-definable subset of some piece M_j of $H(M)$ with an isometric direct limit map $M_j \to H(M)$. We stress that a type-definable subset p of $H(M)$ is contained in a single piece of $H(M)$. This will allow us to quantify over the piece containing p and to use compactness arguments.

A type-definable subset of $H(M)$ is not to be confused with a piecewise $\bigwedge$-interpretable subspace of $H(M)$, which we define as follows:

Definition 1.2.12. *A piecewise $\bigwedge$-interpretable subspace $V(M)$ of $H(M)$ is a subspace $V(M)$ such that if $H(M)$ is the direct limit of the sorts $(M_j)_{j\in J}$ with direct limit maps $h_j : M_j \to H(M)$, then for all $j \in J$ the set $h_j^{-1}(V(M))$ is type-definable in M_j.*

If p is a type-definable subset of some piece M_j, we write $H_p(M)$ for the piecewise $\bigwedge$-interpretable subspace of $H(M)$ consisting of the closed span of the set $h_j(p)$ in $H(M)$.

If $\mathcal{H}$ is the piecewise interpretable Hilbert space in T corresponding to $H(M)$, we write $\mathcal{H}_p$ for the piecewise $\bigwedge$-interpretable Hilbert space in T corresponding to $H_p(M)$.

We have chosen to define piecewise interpretable Hilbert spaces in such a way that pieces are always imaginary sorts of T. As remarked above, this allows us to quantify over the pieces of $H(M)$. However, as we will see in Section 1.6.2, it is often natural to consider piecewise interpretable Hilbert spaces with pieces which are distance-definable sets. We recall some notions from [BYBHU08] and we introduce a general construction which shows that there is no loss of generality in only considering piecewise interpretable Hilbert spaces whose pieces are sorts of M.

Definition 1.2.13 ([BYBHU08] 9.16)**.** *Let T be a complete continuous logic theory. Let M be an ω_1-saturated model of T.*

Let r be a type-definable set. We say that r is distance-definable if the function $d(x, r) = \inf\{d(x, y) \mid y \models r\}$ is definable in M, where x is in the sort of r.

Distance-definability is the continuous logic equivalent of 'definability' in classical logic (as distinguished from 'type-definability'). 9.18 in [BYBHU08] shows that distance-definability of p is not model-dependent and if $r \neq \emptyset$ in M, then $r \neq \emptyset$ in any model of T.

Definition 1.2.14. *Let T be an arbitrary complete continuous logic theory. Let r be a non-empty distance-definable set contained in a sort S of T.*

Define an expansion T^r of T as follows: we add a sort X_r and a map $f_r : X_r \to S$. T^r extends T and says that f_r is an isometry with dense image in r, i.e. for every $0 < \epsilon$, if $d(x, r) < \epsilon$ then there is $z \in X_r$ with $d(x, f_r(z)) \leq \epsilon$.

For every $M \models T$, we write M^r for the extension of M to a model of T^r. Since non-empty distance-definable sets are realised in every model, this extension exists. This extension is clearly unique. It is straightforward to check that condition (4) of Lemma 1.8.14 is satisfied by T inside T^r, so T is stably embedded in T^r. If one wishes to work with a piecewise interpretable Hilbert space $H(M)$ such that r is a piece of $H(M)$, we can move to the theory T^r and work with the sort X_r instead. This shows that there is no loss of generality in assuming that piecewise interpretable Hilbert spaces always have entire sorts as pieces.

Convention: *In this thesis, we will only consider 'piecewise interpretable Hilbert spaces' and 'piecewise $\bigwedge$-interpretable subspaces', so we will now refer to them simply as 'interpretable' or '$\bigwedge$-interpretable'.*

1.2.2 Examples

We give some elementary examples which illustrate a variety of interpretable Hilbert spaces. See Sections 1.4, 1.5 and 1.6 for rich sources of examples which are of wider relevance to model theory and representation theory.

1. In classical logic, let T^∞ be the theory of an infinite set. Writing S for the main sort of T, we define the inner product map $f : S^2 \to \mathbb{R}$ by $f(x, x) = 1$ and $f(x, y) = 0$ if $x \neq y$. This gives an interpretable Hilbert space $\mathcal{H}$ such that for any $M \models T^\infty$, M is an orthonormal set in $H(M)$ with dense span.

Define also the inner product maps $g(x,x) = 2$ and $g(x,y) = 1$ if $x \neq y$. Define also $h(x,x) = 4$ and $h(x,y) = 3$ if $x \neq y$. These also give interpretable Hilbert spaces $\mathcal{H}'$ and $\mathcal{H}''$ respectively. Observe that $\mathcal{H}'$ and $\mathcal{H}''$ are isomorphic, but they are not isomorphic to $\mathcal{H}$. One way of proving this is to note that for any $M \models T$, $H'(M)$ and $H''(M)$ have an invariant vector under the action of $\mathrm{Aut}(M)$, but this is not true of $H(M)$. This will be discussed further in Section 1.6.

2. Let $T = Th(\mathbb{Z}, \leq)$. Define $f(x,x) = 2$, $f(x,y) = 1$ if x, y are consecutive, and $f(x,y) = 0$ otherwise. f is positive definite and defines an interpretable Hilbert space.

For a more complicated example with the same flavour, let $V = \ell^2(\mathbb{Z})$ and for $n \in \mathbb{Z}$ let $v_n = (2^{-|k+n|})_{k \in \mathbb{Z}}$. The sequence $(v_n)_{\mathbb{Z}}$ generates $\ell^2(\mathbb{Z})$. Define the map $h : \mathbb{Z} \to V, n \mapsto v_n$. Then for $x, y \in \mathbb{Z}$, $\langle hx, hy \rangle$ depends only on the distance between x and y, so the inner product is definable on $\mathbb{Z}$ in $Th(\mathbb{Z}, \leq)$. Then we are in the situation of Proposition 1.2.4 so h induces an interpretation of $\ell^2(\mathbb{Z})$ in $(\mathbb{Z}, \leq)$.

For yet another example, let S be an arc of the circle S^1. $\mathbb{Z}$ acts on $L^2_{\mathbb{C}}(S)$ via $f \mapsto z^n f$. Let V be the subspace of $L^2_{\mathbb{C}}(S)$ generated by the orbit of 1 under $\mathbb{Z}$. Then Proposition 1.2.4 shows that V is interpretable in $\mathbb{Z}$.

3. Given two interpretable Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, we can form their sum $\mathcal{H} + \mathcal{H}'$ as follows. Say $\mathcal{H}$ and $\mathcal{H}'$ are the direct limits of $(S_i)_{i \in I}$, $(S_j)_{j \in J}$ with direct limit maps $h_i : S_i \to \mathcal{H}$ and $h_j : S_j \to \mathcal{H}'$ respectively. For any $M \models T$, let H be the orthogonal sum of $H(M)$ and $H'(M)$. Then the system of maps $h_i : M_i \to H$ and $h_j : M_j \to H$ is as in Proposition 1.2.4 and hence they define an interpretable Hilbert space $\mathcal{H} + \mathcal{H}'$ in T. It is clear that for all $N \models T$, $(H + H')(N)$ is the orthogonal sum of $H(N)$ and $H'(N)$.

We can also define the orthogonal sum of infinitely many interpretable Hilbert spaces in the same way. In Section 1.3, we will see that it is sometimes possible to recognise an interpretable Hilbert space in T as the orthogonal sum of a family of interpretable Hilbert spaces with interesting properties.

4. Suppose that $\mathcal{H}$ and $\mathcal{H}'$ are interpretable in T. Then their tensor $\mathcal{H} \otimes \mathcal{H}'$ is interpretable in T. For any $M \models T$, write $H(M)$ and $H'(M)$ as the direct limits of $(M_j)_{j \in J}$ and $(M_{j'})_{j' \in J'}$ respectively. For any $j \in J$ and $j' \in J'$, define the map $F_{j,j'} : M_j \times M'_j \to H(M) \otimes H'(M)$ by $F_{j,j'}(x,y) = h_j x \otimes h'_{j'} y$. The image of the maps $(F_{j,j'})$ have dense span in $H(M) \otimes H'(M)$ and for any (j_1, j'_1) and (j_2, j'_2),

$$\langle F_{j_1,j'_1}(x_1,y_1), F_{j_2,j'_2}(x_2,y_2) \rangle = \langle h_{j_1} x_1, h_{j_2} x_2 \rangle_{H(M)} \langle h_{j'_1} x'_1, h_{j'_2} x'_2 \rangle_{H'(M)}.$$

This is a definable map $M_{j_1} \times M_{j'_1} \times M_{j_2} \times M_{j'_2} \to \mathbb{R}$ so Proposition 1.2.4 applies.

1.2.3 Prolonging piecewise interpretable Hilbert spaces

Let $\mathcal{H}$ be an interpretable Hilbert space in T. In this section, we fix a family of pieces $(S_i)_{i\in I}$ of $\mathcal{H}$ with direct limit maps h_i and inner product maps f_{ij} $(i, j \in I)$ such that $\bigcup h_i(S_i)$ has dense span in $\mathcal{H}$. I is not necessarily a directed partial order.

We have seen that in practice $\mathcal{H}$ is often characterised by such a collection of pieces. In contrast, the full direct limit structure of $\mathcal{H}$ can be complicated to describe. Therefore, we present here a construction which allows us to work with interpretable Hilbert spaces in a setting which is closer to the classical presentation of Hilbert spaces in continuous logic.

Under this construction, the balls $B(0, n)$ in $H(M)$ become subsets of Hilbert space balls which we add to the theory T as new sorts. These subsets are not definable, but this construction helps simplify the discussion of interpretable Hilbert spaces and easily yields the important results about model-theoretic independence in interpretable Hilbert spaces in Section 1.2.4.

Suppose that T is a theory in the language $\mathcal{L}$. We define an extension $T^{\mathcal{H}}$ of the theory T in a language $\mathcal{L}^{\mathcal{H}}$ as follows. We add to $\mathcal{L}$ all the sorts and functions which are used in the presentation of Hilbert spaces in continuous logic, as in Appendix 1.8.3, and $T^{\mathcal{H}}$ says that these new sorts form an infinite dimensional Hilbert space. For each $i \in I$, we also add to our language a function symbol h_i from S_i to one of the Hilbert space balls with radius greater than $\sup_x \sqrt{f_{ii}(x, x)}$. For $i, j \in I$, $T^{\mathcal{H}}$ contains the additional axioms

$$\forall x \in S_i, \forall y \in S_j, f_{ij}(x, y) = \langle h_i(x), h_j(y)\rangle$$

where $\langle ., .\rangle$ is the inner product on the ball which h_i maps into. We also add axioms saying that the orthogonal complement of $\bigcup h_i(M_i)$ is infinite dimensional. We will refer to the maps (h_i) as the interpretation maps in $T^{\mathcal{H}}$.

Any model of $T^{\mathcal{H}}$ has the form (M, H) where $M \models T$ and where H is an infinite dimensional Hilbert space containing $H(M)$ as a subspace in the obvious way. It follows that any model (M, H) of $T^{\mathcal{H}}$ is uniquely determined by its T-part and the dimension of the orthogonal complement of $H(M)$ in H. One deduces that $T^{\mathcal{H}}$ is a complete theory by applying standard saturation arguments. We now show that moving from $M \models T$ to the $\mathcal{L}^{\mathcal{H}}$-structure (M, H) does not add any extra structure to M.

Proposition 1.2.15. *T is stably embedded in $T^{\mathcal{H}}$.*

If $\mathcal{H}$ is the direct limit of the sorts $(S_j)_{j\in J}$, if the sorts S_j are real sorts of T and if the direct limit map on each S_j is injective, then for any $(M, H) \models T^{\mathcal{H}}$ and $C \subseteq (M, H)$, if f is a C-definable function between sorts of T in $\mathcal{L}^{\mathcal{H}}$, then f is $\mathrm{dcl}_{\mathcal{L}^{\mathcal{H}}}(C) \cap M$-definable in $\mathcal{L}$.

Hence, without assuming that $\mathcal{H}$ is the direct limit of real sorts of T, T is fully embedded in $T^{\mathcal{H}}$ in the sense that for any $(M, H) \models T^{\mathcal{H}}$ and any function f between sorts of T definable in $\mathcal{L}^{\mathcal{H}}$ over (M, H) is definable in $\mathcal{L}$ over M.

Proof. First let $(M,H) \models T^{\mathcal{H}}$ be κ-saturated with $|(M,H)| = \kappa$ where $\kappa \geq \omega_1$. We show that (4) from Lemma 1.8.14 holds. Let α be an automorphism of M. By the GNS-theorem, α induces an automorphism of $H(M)$. Define $\beta : (M,H) \to (M,H)$ extending α as follows: on $H(M)$, β is the Hilbert space isomorphism induced by α, and on $H(M)^{\perp}$, β is the identity. Then β respects all the basic relations in $\mathcal{L}^{\mathcal{H}}$ so β is an isomorphism and T is stably embedded in $T^{\mathcal{H}}$.

Now assume that the pieces of $\mathcal{H}$ belong to T and that the direct limit maps are injective. Let (M,H) be any model of $T^{\mathcal{H}}$. Let $C \subseteq (M,H)$ and let f be a C-definable function into $\mathbb{R}$ on a sort of T. We can assume that $f(x) = g(x,C)$ where g is 0-definable and C is a finite tuple of (M,H). We can also write the tuple C as a pair ab where $a \subseteq M$ and $b \subseteq H$. Finally, we can assume that $b = b_0 b_1$ where $b_0 \subseteq H(M)$ and $b_1 \subseteq H(M)^{\perp}$, by expressing elements of b are sums of elements of b_0 and b_1 and by noting that $b_0 b_1$ and b are inter-definable.

Let (N,H') be any elementary extension of (M,H). As above, we can construct an automorphism of (N,H') by taking any automorphism of N and extending it by any unitary automorphism of $H(N)^{\perp}$. It follows that $\mathrm{tp}(b_1/M)$ is completely determined by the values of the inner product between elements of b_1 and by the partial type which says $b_1 \subseteq H(M)^{\perp}$. In particular $\mathrm{tp}(b_1/M)$ is the unique extension to M of $\mathrm{tp}(b_1)$. It follows that f is definable over ab_0.

Since we have assumed that the sorts (S_j) are part of T and that the direct limit maps are injective, each element of b_0 is inter-definable with an element of M. Hence we have proved that f is definable over $\mathrm{dcl}_{\mathcal{L}^{\mathcal{H}}}(C) \cap M$ in $\mathcal{L}^{\mathcal{H}}$.

In order to show that f is definable over $\mathrm{dcl}_{\mathcal{L}^{\mathcal{H}}}(C) \cap M$ in $\mathcal{L}$, it is enough to note that $S^{\mathcal{L}}(C') \cong S^{\mathcal{L}^{\mathcal{H}}}(C')$ via the restriction map for any $C' \subseteq M$, where $S^{\mathcal{L}}(C')$ and $S^{\mathcal{L}^{\mathcal{H}}}(C')$ are type-spaces in the sort of f over C' in $\mathcal{L}$ and $\mathcal{L}^{\mathcal{H}}$ respectively. This is seen by noting that in a sufficiently saturated extension N of M, if $a, b \in M$ are conjugate over C', then they are conjugate in any $(N,H) \models T^{\mathcal{H}}$.

Now let f be a C-definable function between any two sorts X and X' of T. We need to show that its graph is $\mathrm{dcl}_{\mathcal{L}^{\mathcal{H}}}(C) \cap M$-definable in $\mathcal{L}$. Our result for functions $X \times X' \to \mathbb{R}$ shows that $S^{\mathcal{L}}_{X \times X'}(\mathrm{dcl}_{\mathcal{L}^{\mathcal{H}}}(C) \cap M) \cong S^{\mathcal{L}^{\mathcal{H}}}_{X \times X'}(C)$ so f is indeed $\mathrm{dcl}_{\mathcal{L}^{\mathcal{H}}}(C) \cap M$-definable in $\mathcal{L}$.

Finally, if we do not assume that $\mathcal{H}$ is the direct limit of real sorts of T, then the final part of the proposition follows by choosing parameters in M containing all necessary imaginaries in their definable closure. $\square$

1.2.4 Forking independence in interpretable Hilbert spaces

In this section, T is a complete continuous logic theory.

This section aims to prove Proposition 1.2.17 which will be essential in the remainder of this chapter. Proposition 1.2.17 says that we can use stability of the inner product maps and stable embeddedness of T in T' to obtain full forking independence in the sense of Lemma 1.8.17.

We begin with a general lemma. Throughout this chapter, 'forking' is always meant with respect to stable definable functions (see the appendix for basic definitions).

Lemma 1.2.16. *Let $\mathcal{L}_0$ be a language and $\mathcal{L}$ an extension of $\mathcal{L}_0$, possibly with new sorts. Let T be a complete $\mathcal{L}$-theory and T_0 the reduct of T to $\mathcal{L}_0$. Suppose that T_0 has weak elimination of imaginaries.*

Let $(M, N) \models T$ where $M \models T_0$ and N denotes the new sorts in $\mathcal{L}$ added to $\mathcal{L}_0$. Take $A \subseteq B \subseteq (M, N)$ such that $A = \mathrm{bdd}_{\mathcal{L}}(A)$. Let p be a type over B in a fragment of $\mathcal{L}_0$-formulas which does not fork over A in the sense of T. Then p does not fork over $A \cap M$ in the theory T_0.

Proof. Let q be a non-forking extension of p to M in $\mathcal{L}_0$. We know that q is $\mathcal{L}$-definable over A. The usual proof of the theorem which says that stable partial types over models are definable tells us that q is $\mathcal{L}_0$-definable over M. Let $f(x, y)$ be a stable $\mathcal{L}_0$-definable function.

Let α be a canonical parameter of $d_q f(y)$, viewed as an $\mathcal{L}_0$-M-definable function, so that α is an imaginary element of M. Working in $(M, N)^{eq}$, since q is $\mathcal{L}$-definable over A, α is in the $\mathcal{L}$-definable closure of A. By weak elimination of imaginaries in T_0, α is in the $\mathcal{L}_0$-definable closure of some $C \subseteq \mathrm{bdd}_{\mathcal{L}_0}(\alpha)$, so $d_q f(y)$ is C-definable in $\mathcal{L}_0$. Since adding imaginaries does not affect the definable or bounded closure, in (M, N) we have $C \subseteq \mathrm{bdd}_{\mathcal{L}}(A) \cap M = A \cap M$ and q is $\mathcal{L}_0$-definable over $A \cap M$. □

If $H(M)$ is intepretable in M and is the direct limit of the sorts $(M_j)_{j \in J}$ where the direct limit maps are injective and M_j is a real sort of M, and if $A \subseteq M$ is definably closed, we write P_A for the orthogonal projection onto the subspace of $H(M)$ given by $A \cap H(M)$. The following proposition follows from full embeddedness of T in $T^{\mathcal{H}}$ and the charactersiation of forking in Hilbert spaces:

Proposition 1.2.17. *Take $M \models T$. Suppose that $H(M)$ is the direct limit of the sorts $(M_j)_{j \in J}$ and that the direct limit maps are injective. Suppose that the sorts M_j belong to M, so that elements of $H(M)$ are identified with elements of M.*

Let A be a subset of $H(M)$ and let $B \subseteq C \subseteq M$ be bdd*-closed. If $A \mathop{\smile\!\!\!\!\!\!|}_B C$ in the sense of T with respect to the inner product maps between the pieces (M_j), then for all $a \in A$, $P_{C \cap H(M)} a = P_{B \cap H(M)} a$.*

Proof. Suppose that $A \mathop{\smile\!\!\!\!\!\!|}_B C$ with respect to the inner product maps. Take $(M, H) \models T^{\mathcal{H}}$ where the $T^{\mathcal{H}}$ construction is applied to all pieces (M_j). Let

$B' = \mathrm{bdd}_{\mathcal{L}^{\mathcal{H}}}(B)$ and $C' = \mathrm{bdd}_{\mathcal{L}^{\mathcal{H}}}(C)$. By full embeddedness of M in (M, H), $B' \cap M = B$. Suppose that $v \in H \cap B'$. Then $v \in H(M)$ and we can find $x \in M$ such that $v = h_j x$. Since the interpretation maps are assumed to be injective, $x \in \mathrm{dcl}_{\mathcal{L}^{\mathcal{H}}}(B)$. Hence $B' = \mathrm{dcl}_{\mathcal{L}^{\mathcal{H}}}(B)$, and similarly for C'.

Write $h(A)$ for the result of mapping A in H by the appropriate interpretation maps h_j. By considering definable bijections, we find that $h(A) \downarrow_{B'} C'$ with respect to the maps $\langle x, y\rangle$ where x and y range in the balls of H. Hilbert spaces have weak elimination of imaginaries so Lemma 1.2.16 applies and we have $h(A) \downarrow_{B' \cap H} C' \cap H$ in the theory of Hilbert spaces.

Note that $B' \cap H = B' \cap H(M)$ and similarly for C'. Now the lemma follows by the characterisation of forking independence in Hilbert spaces (see Lemma 1.8.17). □

1.3 Structure Theorems for Scattered Interpretable Hilbert Spaces

In this section, we give our structure theorems for interpretable Hilbert spaces. These reduce the notion of scatteredness to the stronger notion of asymptotic freedom; see the definitions below. The main structure theorem is 1.3.14. In section 1.3.2 we show how the decomposition of Theorem 1.3.14 can be almost recovered when working in classical logic with the weak NFCP. This is Corollary 1.3.21. In Section 1.3.3 we give concrete examples of the decomposition promised by 1.3.14 and 1.3.21.

Fix a continuous logic theory T and take $\mathcal{H}$ an interpretable Hilbert space in T. In this section, we will freely move to imaginary sorts of T, so we assume that the direct limit maps on pieces of $\mathcal{H}$ are isometries, and for any $M \models T$, we identify pieces of $H(M)$ with subsets of $H(M)$.

When $M \models T$ and $A \subseteq H(M)$, we write P_A for the orthogonal projection onto the closed subspace of $H(M)$ generated by A.

Definition 1.3.1. *Let $M \models T$ and let p be a type-definable set in $H(M)$. We define $\mathcal{P}(p) \subseteq H(M)$ to be the closure of the set of realisations of p in the weak topology.*

The notation '$\mathcal{P}(p)$' stands for the partial order which we will define below. In the next lemma, we characterise the set $\mathcal{P}(p)$ from a Hilbert space point of view.

Lemma 1.3.2. *Let $M \models T$ and let p be a type-definable set in $H(M)$.*

When M is ω-saturated, $\mathcal{P}(p)$ is equal to the set of weak limit points of p in $H(M)$.

When M is ω_1-saturated, $\mathcal{P}(p)$ is equal to the set $\{P_{\mathrm{bdd}(A)}(b) \mid b \models p, A \subseteq M\}$ and is closed under the maps $P_{\mathrm{bdd}(A)}$ for arbitrary $A \subseteq M$.

For any $M \models T$, $\mathcal{P}(p)$ is closed under the maps $P_{\mathrm{bdd}(A)}$ for $A \subseteq M$.

Proof. Recall that the weak topology on $H(M)$ has a basis of open sets of the form $\bigcap_{i=1}^{n}\{v \in H(M) \mid \langle v, v_i\rangle \in U_i\}$ where $v_i \in H(M)$ and U_i is an open interval in $\mathbb{R}$.

Suppose M is ω-saturated and take w in $\mathcal{P}(p)$. Find $a_0 \models p$ such that $\langle a_0, w\rangle = \langle w, w\rangle$. Given $a_0, \ldots, a_n$, find $a_{n+1} \models p$ such that $\langle a_{n+1}, w\rangle = \langle a_j, w\rangle$ and for all $j \leq n$ $\langle a_{n+1}, a_j\rangle = \langle w, a_j\rangle$. By Lemma 1.8.18, $(a_n) \rightharpoonup w$ so $\mathcal{P}(p)$ is the set of weak limit points of p.

Now suppose M is ω_1-saturated. With $w \in \mathcal{P}(p)$ and (a_i) as constructed above, write $A = \mathrm{bdd}(w)$. Since the inner product maps are stable, we can use classical stability theory to find $b \models p$ realising the eventual type of (a_i) over A with respect to the inner product maps. Then $w = P_A b$ and $\mathcal{P}(p)$ is contained in $\{P_{\mathrm{bdd}(A)}b \mid b \models p, A \subseteq M\}$. For the converse inclusion, if $w = P_{\mathrm{bdd}(A)}b$, we can assume that A is separable and we can take a Morley sequence (b_n) in $\mathrm{tp}(b/\mathrm{bdd}(A))$. Proposition 1.2.17 shows that (b_n) is a Morley sequence in the sense of Hilbert spaces. By Lemma 1.8.19, $b_n \rightharpoonup w$ and hence $w \in \mathcal{P}(p)$.

Finally, take any $M \models T$, $A \subseteq M$ with $\mathrm{bdd}(A) = A$ and $w \in \mathcal{P}(p)$. We can assume that A is separable. Since $\mathcal{P}(p)$ is type-definable, we can move to an elementary extension and we can assume that M is ω_1-saturated. Let (a_n) be a sequence in p such that (a_n) converges weakly to w. Consider the following partial type in x over A:

$$\{|\langle x, v\rangle - \langle w, v\rangle| \leq \epsilon \mid \epsilon > 0, v \in \mathrm{bdd}(A) \cap H\}$$

This is finitely satisfiable in (a_n), so by saturation we can find a realisation b in p. It follows that $P_A b = P_A w$. By our previous result, we deduce $P_A w \in \mathcal{P}(p)$. $\square$

The next lemma shows that $\mathcal{P}(p)$ is a type-definable set of $\mathcal{H}$. In Section 1.7.1, it is shown that $\mathcal{P}(p)$ can also be constructed as a type-definable set in an imaginary sort of T coding canonical bases for $\langle x, y\rangle$-types consistent with p.

Lemma 1.3.3. *Let p be a type-definable set in $\mathcal{H}$ and let $M \models T$ be ω-saturated. Then $\mathcal{P}(p)$ is a type-definable set in a piece of $\mathcal{H}$.*

Proof. Lemma 1.3.2 shows that $\mathcal{P}(p)$ is the set of weak limit points of Hilbert space indiscernible sequences in p. If (a_n) is a sequence in p which is Hilbert space indiscernible and (a_n) converges weakly to $b \in H(M)$, then it is easy to show that $(\sum_{k=1}^{n} a_k/n)$ converges to b in the norm topology. Note that the inner product is bounded on $p \times p$, so we can control the rate of convergence of $(\sum_{k=1}^{n} a_k/n)$ uniformly for all indiscernible sequences (a_n) in p by considering subsequences of the form $(\sum_{k=1}^{\eta(n)} a_k/\eta(n))$, where η is an increasing function

$\mathbb{N} \to \mathbb{N}$. It is then possible to use the construction of Proposition 1.2.4 to construct a piece of $H(M)$ containing $\mathcal{P}(p)$.[5] $\square$

Definition 1.3.4. *Let $M \models T$ be ω_1-saturated and let p be a type-definable set in $H(M)$. We define the partial order $\leq$ on $\mathcal{P}(p)$ as follows: we say that $v \leq w$ in $\mathcal{P}(p)$ is there is a finite sequence of* bdd-*closed subsets $A_1, \ldots, A_n$ of M such that $v = P_{A_n} \ldots P_{A_1} w$.*

The partial order on $\mathcal{P}(p)$ is especially interesting under the assumption of scatteredness, which is one of the fundamental notions of this chapter.

Definition 1.3.5. *Let $M \models T$ be ω_1-saturated and let p be a type-definable set in $H(M)$. We say that p is scattered if $\mathcal{P}(p)$ is locally compact in the norm topology.*

While scatteredness is the most general assumption we will work with, the following stronger condition is of special interest in many model-theoretic situations:

Definition 1.3.6. *Let p, q be type-definable sets in $\mathcal{H}$. We say that the inner product map on $p \times q$ is strictly definable if it takes only finitely many values on $p \times q$.*

Strict definability of the inner product map is often easier to verify than scatteredness. For example, if the type-definable sets p, q are pieces of $\mathcal{H}$, then strict definability of the inner product on $p \times q$ is not model dependent.

Lemma 1.3.7. *Let p be a partial type in $\mathcal{H}$. If the inner product map f is strictly definable on $p \times p$ then in any $M \models T$, $\mathcal{P}(p)$ is a discrete metric space in $H(M)$ and hence p is scattered.*

Proof. Let $v, w \in \mathcal{P}(p)$. We showed in Lemma 1.3.2 that $\mathcal{P}(p)$ is the set of weak limit points of p so there are sequences (a_n) and (b_n) in p such that $a_n \rightharpoonup v$ and $b_n \rightharpoonup w$. Then $\langle v, w \rangle = \lim_n \lim_m \langle a_n, b_m \rangle$. By stability of the inner product and strict definability, $\langle a_n, a_m \rangle$ must be eventually constant. Therefore $\langle v, w \rangle$ is one of the finitely many values already achieved by the inner product on $p \times p$. It follows that $\mathcal{P}(p)$ is a discrete set and hence it is locally compact. $\square$

Finally, we note that saying that p is scattered is strictly weaker than saying that the set of realisations of p is locally compact in $H(M)$, for $M \models T$ ω_1-saturated. Consider the following example. Let T be a two sorted structure (S_1, S_2) where the sort S_1 is an infinite set with the discrete metric and S_2 is the surface of the unit ball in an infinite dimensional Hilbert space. We add

[5] this piece may not have been present in the original direct limit of $H(M)$, but in this section we freely add imaginary sorts to our structure.

to T the inner product map on S_2 and a function $f : S_1 \to S_2$. T says that f has dense image in S_2 and that every fibre of f is infinite. Define the positive-definite map $b(x, y)$ on $S_1 \times S_1$ by $b(x, x) = 2$ and $b(x, y) = \langle f(x), f(y)\rangle$. Let $\mathcal{H}$ be the interpretable Hilbert space generated by S_1 with inner product map b..

Then for any ω_1-saturated $M \models T$, S_1 is a discrete set in $H(M)$ but $\mathcal{P}(S_1)$ contains S_2 and hence S_1 is not scattered.

1.3.1 Decomposition into $\bigwedge$-interpretable subspaces

Until the end of Section 1.3.1, we make the following assumptions and notational conventions. We fix an ω_1-saturated $M \models T$ and we fix a type-definable set p in $\mathcal{H}$. We assume that p is scattered. Recall that we write $H_p(M)$ for the $\bigwedge$-interpretable subspace of $H(M)$ generated by the set p (see Definition 1.2.12).

The next theorem is the basic fact which shows that it is interesting to look at $\mathcal{P}(p)$ as a partial order. It also shows that types over bdd-closed subsets of M contained in p are one-based in a restricted sense. See Section 1.7.2 for a detailed discussion.

Theorem 1.3.8. *Let A, B be small subsets of M such that $A = \operatorname{bdd}(A)$ and $B = \operatorname{bdd}(B)$. Then $A \cap H_p(M)$ and $B \cap H_p(M)$ are orthogonal over $A \cap B \cap H_p(M)$. Equivalently, for any $v \in A \cap H_p(M)$, we have $P_B v = P_{A\cap B} v$. Equivalently, for any $v \in H_p(M)$,*

$$P_B P_A v = P_A P_B v = P_{A\cap B} v$$

Proof. It is enough to check the statement for arbitrary $v \models p$. Define $x_0 = P_A v$, $y_n = P_B x_n$ and $x_{n+1} = P_A y_n$. It is well-known that the sequences (x_n) and (y_n) converge to $w = P_{A\cap B} v \in \mathcal{P}(p)$ in the norm topology. See Theorem 13.7 in [VN50] for more details.

Suppose for a contradiction that x_n and y_n are distinct from w for all n. Then $y_n \notin A$ and $x_n \notin B$. By Lemma 1.3.2, for every n we can find infinite sequences $(x_n^k)_k$ and $(y_n^k)_k$ such that $(x_n^k)_k$ is a sequence in $\operatorname{tp}(x_n/B)$ converging weakly to y_n and similarly for $(y_n^k)_k$. Then for any $\epsilon > 0$ there is $n \geq 0$ such that the sequence $(x_n^k)_k$ is within distance ϵ of w. Since we are assuming that $\mathcal{P}(p)$ is locally compact, this is a contradiction and $(x_n), (y_n)$ are eventually constant equal to w.

Take $n \geq 1$ such that $y_n \in A \cap B$. We now show that $x_n \in A \cap B$. Write

$x_n = y_n + \alpha$ where $\alpha \perp B$ and $y_{n-1} = x_n + \beta$ where $\beta \perp A$. We have

$$\begin{aligned}\langle \alpha, y_{n-1} \rangle &= 0 \\ &= \langle \alpha, y_n + \alpha + \beta \rangle \\ &= \langle \alpha, \alpha \rangle + \langle \alpha, y_n \rangle + \langle \alpha, \beta \rangle \\ &= \langle \alpha, \alpha \rangle + \langle \alpha, \beta \rangle \\ &= \langle \alpha, \alpha \rangle + \langle x_n - y_n, \beta \rangle \\ &= \langle \alpha, \alpha \rangle \text{ since } x_n - y_n \in A.\end{aligned}$$

So $\alpha = 0$, $x_n = y_n$ and $x_n \in A \cap B$. We use a similar calculation to show that $y_{n-1} \in A \cap B$ when $x_n \in A \cap B$, with $n \geq 1$. This proves by induction that $y_0 \in A$. Hence $P_B P_A v = P_{A \cap B} v$ and the theorem is proved. □

Lemma 1.3.9. *For every $v \in \mathcal{P}(p)$, $\{w \in \mathcal{P}(p) \mid w \leq v\}$ is uniformly type-definable over v. Therefore, this set is compact in the norm topology.*

Proof. Recall from Lemma 1.3.3 that $\mathcal{P}(p)$ is a type-definable set in $H(M)$. Take $v \in \mathcal{P}(p)$. By Theorem 1.3.8, $\{w \in \mathcal{P}(p) \mid w \leq v\} = \{P_{\mathrm{bdd}(A)} v \mid A \subseteq M\}$. As in Lemma 1.3.2, $\{P_{\mathrm{bdd}(A)} v \mid A \subseteq M\}$ is the set of weak limit points of Hilbert space indiscernible sequences in $\mathrm{tp}(v)$ which begin at v. It is straightforward to check that this is a uniformly type-definable set over v.

By Theorem 1.3.8, $\{w \in \mathcal{P}(p) \mid w \leq v\}$ is contained in $\mathrm{bdd}(v)$ in the language $\mathcal{L}^{\mathcal{H}}$. By type-definability, this set is compact in the norm topology. □

Lemma 1.3.10. *$\mathcal{P}(p)$ is well-founded with respect to the partial order from Definition 1.3.1.*

Proof. Suppose (v_n) is an infinite decreasing sequence in $\mathcal{P}(p)$. By Theorem 1.3.8, we can write $v_n = P_{V_n} v_0$ where V_n is a bdd-closed subspace of $H(M)$ and $V_{n+1} \subseteq V_n$. Since $\{w \in \mathcal{P}(p) \mid w < v_0\}$ is metrically compact, the sequence (v_n) is convergent and it follows that it converges to $z := P_V v_0$ with $V = \bigcap_n V_n$. Then $z < v_n$ for all n and $v_n \notin \mathrm{bdd}(z)$. For every $\epsilon > 0$ we can find n such that $\|v_n - z\| < \epsilon$ and we can take an infinite indiscernible sequence in $\mathrm{tp}(v_n/z)$ which must lie in $\mathcal{P}(p)$, by Lemma 1.3.3. Hence $\mathcal{P}(p)$ is not locally compact around z and this contradicts scatteredness of p. Therefore any decreasing sequence in $\mathcal{P}(p)$ is eventually constant. □

We use Lemma 1.3.10 to decompose $H_p(M)$. Fix an enumeration $(q_\alpha)_{\alpha<\kappa}$ of the complete types in $\mathcal{P}(p)$ with the property that for any $a, b \in \mathcal{P}(p)$, if $b < a$ in $\mathcal{P}(p)$ then $\mathrm{tp}(b)$ comes before $\mathrm{tp}(a)$ in the sequence (q_α). Such an enumeration exists by Lemmas 1.3.3 and 1.3.10. For any $\alpha < \kappa$, let V_α be the subspace of $H(M)$ generated by the realisations of $\bigcup_{\beta<\alpha} q_\beta$ (set $V_0 = \{0\}$).

For every α, find a complete type $\tilde{q}_\alpha$ in $\mathcal{H}$ such that for some (any) $x \models q_\alpha$, there is $y \models \tilde{q}_\alpha$ such that $y = P_{V_\alpha^\perp} x$.

Lemma 1.3.11. *For every α, the relation $P_{V_\alpha^\perp}x = y$ is type-definable on $q_\alpha \times \tilde{q}_\alpha$.*

Proof. Fix $x \models q_\alpha$ and write $d \geq 0$ for the distance between x and V_α. Since q_α is a complete type and V_α is generated by a union of $\bigwedge$-interpretable Hilbert spaces, d does not depend on x. Moreover, $P_{V_\alpha}x$ is the unique element $v \in V_\alpha$ such that $\|x - v\| = d$. Note also that $P_{V_\alpha^\perp}x = x - P_{V_\alpha}x$

For every $\epsilon > 0$, there is $n_\epsilon \geq 0$ such that for every $i \leq n_\epsilon$ we can find a type q_{α_i}, $x_i \in q_{\alpha_i}$ and $\lambda_i \in [-n_\epsilon, n_\epsilon]$ satisfying $\|\sum_{i \leq n_\epsilon} \lambda_i x_i - x\| \leq d + \epsilon$. n_ϵ, the types q_{α_i} and the scalars λ_i do not depend on x. Write $\phi_\epsilon(x, y)$ for the type-definable set

$$\exists x_1, \ldots x_{n_\epsilon} \Big(\bigwedge_{i \leq n_\epsilon} q_{\alpha_i}(x_i) \wedge \|\sum \lambda_i x_i - x\| \leq d + \epsilon \wedge \|x - \sum \lambda_i x_i - y\| \leq \epsilon \Big)$$

The relation $P_{V_\alpha^\perp}x = y$ is defined on $q_\alpha \times \tilde{q}_\alpha$ by the intersection of all $\phi_\epsilon(x, y)$. $\square$

It is easy to prove by induction that for every $\alpha \leq \kappa$ the Hilbert space generated by $\bigcup_{\beta < \alpha} \tilde{q}_\beta$ is equal to V_α. Therefore, $H_p(M)$ is the orthogonal sum of the spaces generated by each $\tilde{q}_\alpha$. Finally, the types $\tilde{q}_\alpha$ have the following important property:

Lemma 1.3.12. *For every $\alpha < \kappa$, for every $x, y \in \tilde{q}_\alpha$, we have $\langle x, y \rangle = 0$ or $y \in \mathrm{bdd}(x)$.*

Proof. Suppose $x, y \in \tilde{q}_\alpha$ and $y \notin \mathrm{bdd}(x)$. Let (y_n) be an infinite indiscernible sequence in $\mathrm{tp}(y/\mathrm{bdd}(x))$. For all n, find z_n in q_α such that $y_n = P_{V_\alpha^\perp} z_n$. For all n, we have

$$\langle y_n, x \rangle = \langle P_{V_\alpha^\perp} z_n, x \rangle = \langle z_n, x \rangle = \langle w, x \rangle.$$

where w is the weak limit of the sequence (z_n). By the choice of the enumeration (q_α), we know that $w \in V_\alpha$ so $\langle w, x \rangle = 0$ and the lemma follows. $\square$

Definition 1.3.13. *For q a type-definable set in $\mathcal{H}$, we say that q is asymptotically free if for any $M \models T$, $x \models q$ in M and $\epsilon \geq 0$, the set $\{y \models q \mid |\langle x, y \rangle| \geq \epsilon\}$ is compact.*

Equivalently, when $M \models T$ is ω-saturated, for any $x, y \models q$, either $\langle x, y \rangle = 0$ or $x \in \mathrm{bdd}(y)$.

Note that an asymptotically free type-definable set q is always scattered, since $\mathcal{P}(q) = q \cup \{0\}$. We collect our results so far and repeat the assumptions we are working with:

Theorem 1.3.14. *Let $\mathcal{H}$ be an interpretable Hilbert space in T. Let $\mathcal{H}_p$ be a $\bigwedge$-interpretable subspace of $\mathcal{H}$ generated by a scattered type-definable set p. Then $\mathcal{H}_p$ is the orthogonal sum of $\bigwedge$-interpretable Hilbert spaces $(\mathcal{H}_\alpha)_{\alpha < \kappa}$ such that for all $\alpha < \kappa$, $\mathcal{H}_\alpha$ is generated by an asymptotically free complete type.*

Proof. For $M \models T$ ω_1-saturated, we constructed the asymptotically free types $(\tilde{q}_\alpha)_{\alpha<\kappa}$ such that $H_p(M)$ is the orthogonal sum of the spaces $H_\alpha(M)$ generated by the asymptotically free complete type $\tilde{q}_\alpha$. We only need to show that this decomposition is not model-dependent.

Let $N \models T$ be an arbitrary model. We can assume $N \prec M$. If p is not realised in N, the theorem holds vacuously. Suppose $a \models p$ in N.

Working in M, let b be an element contained in a type $\tilde{q}_\alpha$ such that $\langle a, b\rangle \neq 0$. If $b \notin \mathrm{bdd}(a)$, then we find an infinite indiscernible sequence (b_n) in $\mathrm{tp}(b/\mathrm{bdd}(a))$. Since $\tilde{q}_\alpha$ is asymptotically free, (b_n) is an infinite orthogonal sequence in $H(M)$ with $\langle a, b_n\rangle \neq 0$, and this is a contradiction. Therefore $b \in \mathrm{bdd}(a)$ and hence $b \in N$.

Working again in M, we can find a bdd-independent family (b_n) in types $\tilde{q}_{\alpha_n}$ such that $\langle a, b_n\rangle \neq 0$ and a is in the closed span of $\bigcup(\mathrm{bdd}(b_n) \cap \tilde{q}_{\alpha_n})$. Since each b_n is in N, each set $\mathrm{bdd}(b_n) \cap \tilde{q}_{\alpha_n}$ is contained in N and the spaces $(H_\alpha(N))_{\alpha<\kappa}$ generate $H_p(N)$. □

The following corollary shows that Theorem 1.3.14 is not restricted to the $\bigwedge$-interpretable subspace $\mathcal{H}_p$ generated by a single type-definable set.

Corollary 1.3.15. *Suppose that $\mathcal{H}$ contains $\bigwedge$-interpretable subspaces $(\mathcal{H}_i)_I$ such that each $\mathcal{H}_i$ is generated by a scattered type-definable set. Then the subspace of $\mathcal{H}$ generated by all $\mathcal{H}_i$ can be expressed as the orthogonal sum of $\bigwedge$-interpretable subspaces $(\mathcal{H}_j)_{j\in J}$ such that for all $j \in J$, $\mathcal{H}_j$ is generated by an asymptotically free complete type q_j.*

Proof. Take $M \models T$ ω_1-saturated as before. Let q be a complete type of T with an asymptotically free interpretation map h into $\mathcal{H}$. Let V be a subspace of $H(M)$ generated by an arbitrary collection of $\bigwedge$-definable sets. As in Lemma 1.3.11, the projection $P_{V^\perp}$ is definable on q and we can find a type $\tilde{q}$ in $\mathcal{H}$ which is the image of q under $P_{V^\perp}$. It is enough to check that $\tilde{q}$ is asymptotically free.

Take $a, b \models \tilde{q}$ with $a \notin \mathrm{bdd}(b)$ and suppose $a = P_{V^\perp}x$ where $x \models q$. Then $\langle a, b\rangle = \langle x, b\rangle$ and $x \notin \mathrm{bdd}(b)$. Take (x_n) an infinite indiscernible sequence in $\mathrm{tp}(x/b)$. Then (x_n) is an infinite orthogonal sequence in $H(M)$ and hence $\langle x, b\rangle = 0$.

Therefore the subspace of $H(M)$ generated by all $H_i(M)$ is the orthogonal sum of $\bigwedge$-interpretable subspaces $H_j(M)$ generated by asymptotically free complete types. The same proof as in Theorem 1.3.14 shows that this decomposition is not model dependent. □

1.3.2 Strictly interpretable Hilbert spaces

In Section 1.3.2, we fix an interpretable Hilbert space $\mathcal{H}$ in T. In the following results, there are no unstated assumptions on T or $\mathcal{H}$.

In this section, we investigate interpretable Hilbert spaces generated by strictly definable inner product maps. The next proposition shows that we can find an asymptotically free generating set for such an interpretable Hilbert space on which the inner product is still strictly definable.

Proposition 1.3.16. *Let p be a type-definable set in $\mathcal{H}$. If the inner product map is strictly definable on $p \times p$, then there is an asymptotically free type-definable set q generating $\mathcal{H}_p$ such that the inner product map is strictly definable on $q \times q$.*

Moreover, if p is a finite union of complete types, then q is a finite union of complete types.

Proof. Take $M \models T$ ω_1-saturated. We have seen in Lemma 1.3.7 that $\mathcal{P}(p)$ is a discrete type-definable set. It follows from Lemma 1.3.9 that for any $v \in \mathcal{P}(p)$, the set $\{w \in \mathcal{P}(p) \mid w \leq v\}$ is finite and uniformly definable over v. Additionally if p is a finite union of complete types, then $\mathcal{P}(p)$ is a finite union of complete types.

For every $v \in \mathcal{P}(p)$ write $\pi(v)$ for the finite set $\{w \in \mathcal{P}(p) \mid w < v\}$. Write $V(v)$ for the subspace of $H(M)$ spanned by $\pi(v)$. Let q be the type-definable set in $\mathcal{H}$ of elements of the form $P_{V(v)^\perp}v$ for $v \in \mathcal{P}(p)$. Since $P_{V(v)^\perp}v$ is a linear combination of $\{v\} \cup \pi(v)$ and the coefficients in this linear combination only depend on the values of the inner product between elements in this set, the inner product on $q \times q$ is strictly definable. We only have to show that q is asymptotically free.

Fix $x, y \in q$ and find $v \in \mathcal{P}(p)$ such that $x = P_{V(v)^\perp}v$. To simplify notation, write $V = V(v)$ and $Y = \mathrm{bdd}(y)$. We will prove that if $x \notin Y$ then $P_Y x = 0$. Since $P_Y x = P_Y v - P_Y P_V v$, it is enough to prove that $P_Y P_V v = P_Y v$. We show that $\|P_Y P_V v - P_Y v\|^2 = 0$. Expanding the left hand side gives $\langle P_Y P_V v, P_Y P_V v\rangle + \langle P_Y v, P_Y v\rangle - 2\langle P_Y v, P_Y P_V v\rangle$. Now we have:

$$\langle P_Y v, P_Y P_V v\rangle = \langle P_V P_Y v, v\rangle = \langle P_Y v, v\rangle = \langle P_Y v, P_Y v\rangle$$

and similarly

$$\begin{aligned}\langle P_Y P_V v, P_Y P_V v\rangle = \langle P_V P_Y P_V v, v\rangle = \langle P_Y P_V v, v\rangle = \langle v, P_V P_Y v\rangle = \langle v, P_Y v\rangle \\ = \langle P_Y v, P_Y v\rangle\end{aligned}$$

The proposition follows for M. To see that q generates $\mathcal{H}_p$ independently of a choice of model, we observe that the realisations of q are contained in $\mathrm{bdd}(p)$. □

We now focus on the case where T is a classical discrete logic theory and $\mathcal{H}$ is generated by classical sorts of T with strictly definable inner product maps. We investigate to what extent the decomposition of Theorem 1.3.14 can be recovered inside the classical logic sorts of T.

Definition 1.3.17. *Let T be a classical logic theory. If $\mathcal{H}$ is an interpretable Hilbert space in T generated by classical imaginary sorts of T with strictly definable inner product maps, we say that $\mathcal{H}$ is a strictly interpretable Hilbert space in T.*

In Proposition 1.3.16, even if T is a classical logic theory and p is a definable set in a classical sort of T, the asymptotically free set q with its strictly definable inner product map is not always a definable set. We show that this can be obtained with additional assumptions on T.

Definition 1.3.18. *In a classical logic theory T, a formula $\phi(x,y)$ (possibly with parameters) has the finite cover property (the FCP) if for all $n \geq 1$ there are $a_1, \ldots, a_n$ such that $\bigwedge_{i \leq n} \phi(x, a_i)$ is inconsistent but for every $l \leq n$, $\bigwedge_{i \neq l} \phi(x, a_i)$ is consistent. If ϕ does not have the FCP, we say ϕ has the NFCP.*

We say that T has the weak NFCP if all stable formulas of T have the NFCP.

We will use the following easy lemma about NFCP formulas, which is a weak version of Theorem II.4.6 in [She78]:

Lemma 1.3.19. *Let T be a classical logic theory and let $M \models T$ be ω-saturated. Let $\phi(x,y)$ be a formula with the NFCP over $A \subseteq M$. There is $n \in \mathbb{N}$ such that, for all $n \leq \alpha < \omega$, any sequence $(a_i)_{i<\alpha}$ such that $\models \phi(a_i, a_j)$ for all $i \neq j$ can be extended to a sequence $(a_i)_{i<\omega}$ with the same property.*

Proof. Take n as given by the definition of NFCP for $\phi(x,y)$. Given $(a_i)_{i<\alpha}$, the partial type $\{\phi(x, a_i) \mid i < \alpha\}$ is n-consistent, so it is consistent. Take a_α a realisation of this partial type. □

The next lemma can be seen as a strengthening of Lemma 1.3.3 to the present context.

Lemma 1.3.20. *Suppose that T is a classical logic theory with the weak NFCP and $\mathcal{H}$ is strictly interpretable in T. Let S be a piece of $\mathcal{H}$ which is a classical sort of T with a strictly definable inner product map.*

For any $M \models T$, $\mathcal{P}(S)$ is a definable set in a piece of $\mathcal{H}$ which is a classical sort of T with strictly definable inner product map.

Proof. Throughout this proof, we write $\overline{x}$ for tuples of variables and x for single variables. We write R for the finite set of values achieved by the inner product on $S \times S$. Let M be an arbitrary model of T.

Recall from elementary stability theory that there is a number N such that for any sequence (x_n) in S and $y \in S$, there is a unique λ in R such that $|\{i \in \mathbb{N} \mid \langle x_i, y \rangle \neq \lambda\}| < N/2$. Let S' be the imaginary sort S^N/E where E is the equivalence relation defined by

$$\forall z \in S,\ \mathrm{Med}_{i \leq N} \langle x_i, z \rangle = \mathrm{Med}_{i \leq N} \langle y_i, z \rangle$$

and where $\mathrm{Med}_{i\le N}\langle x_i, z\rangle$ is the median of the set of values $\{\langle x_i, z\rangle \mid i \le N\}$. Let S^+ be the type-definable subset of S' consisting of elements z such that there is a sequence (x_n) in S, possibly constant, such that (x_n) is Hilbert space indiscernible and for all $k_0 < \ldots < k_N$, the E-class of $(x_{k_0}, \ldots, x_{k_N})$ equals z.

Claim 1.3.20.1. *S^+ is a definable subset of S'*

Proof of claim. For any $\lambda \in R$, write $F_\lambda(\overline{x}, \overline{y})$ for the formula on $S^N \times S^N$ which says

1. $\overline{x}$, $\overline{y}$ and E-equivalent
2. For all $i, j \le N$, $\langle x_i, y_j\rangle = \lambda$
3. For all $i \ne j \le N$, $\langle x_i, x_j\rangle = \langle y_i, y_j\rangle = \lambda$
4. For all $i, j \le N$, $\langle x_i, x_i\rangle = \langle y_j, y_j\rangle$.

F_λ is an equivalence relation so F_λ is stable . By the weak NFCP there is a number n_λ such that for all $k \ge n_\lambda$ and $\overline{y}_1, \ldots, \overline{y}_k$, if $\{F_\lambda(\overline{x}, \overline{y}_i) \mid i \le k\}$ is n_λ-consistent then it is consistent. Take $m > n_\lambda$ for all $\lambda \in R$.

Let $S_0(\overline{x})$ be the definable set in S^N

$$\exists \overline{y}_1, \ldots, \overline{y}_m \bigvee_{\lambda \in R} \Big(\bigwedge_{i \le m} F_\lambda(\overline{x}, \overline{y}_i) \wedge \bigwedge_{i \ne j} F_\lambda(\overline{y}_i, \overline{y}_j) \Big)$$

and let S_1 be S_0/E. We check that S_1 is in fact equal to S^+. S_1 contains S^+ because any indiscernible sequence (x_n) witnessing the definition of S^+ can be broken down into m N-tuples which witness the definition of S_1. Conversely, suppose $\overline{a} \models S_0$, take $\overline{b}_1, \ldots, \overline{b}_m$ as given by the definition of S_0 and fix λ such that these satisfy

$$\bigwedge_{i \le n_0} F_\lambda(\overline{a}, \overline{b}_i) \wedge \bigwedge_{i \ne j} F_\lambda(\overline{b}_i, \overline{b}_j).$$

By Lemma 1.3.19, we can construct an sequence of tuples $(\overline{b}_n)$ which satisfy $F_\lambda(\overline{b}_i, \overline{b}_j) \wedge F_\lambda(\overline{a}, \overline{b}_i)$ for $i \ne j$. Concatenate the tuples $\overline{b}_i$ to form a sequence (c_k) in S. By construction, (c_k) is Hilbert space indiscernible. It follows easily that (c_k) witnesses the definition of S^+. $\square$

Take $z \in S^+$ and let (x_n) be a sequence in S as in the definition of S^+ for z. Let v be the weak limit of (x_n). Note that $v \in \mathcal{P}(S)$. Then for all $y \in S$, $\langle v, y\rangle = \lim\langle x_n, y\rangle = \mathrm{Med}_{i\le N}\langle x_i, y\rangle$ and this last value only depends on z. Therefore v only depends on z and we can define the definable map $h : S^+ \to \mathcal{P}(S)$ which maps z to v. Note that h is injective.

An application of the weak NFCP similar to the one in the claim shows that in any $M \models T$, $\mathcal{P}(S)$ is the set of weak limit points of S. It follows that $h : S^+ \to \mathcal{P}(S)$ is bijective. Since S^+ is definable, we can extend h to S' by mapping the complement of S^+ to 0. The lemma follows. $\square$

Combining Lemma 1.3.20 and Proposition 1.3.16, we obtain the following corollary:

Corollary 1.3.21. *Let T be a classical logic theory with the weak NFCP. If $\mathcal{H}$ is a strictly interpretable Hilbert space in T,then $\mathcal{H}$ is generated by classical imaginary asymptotically free sorts of T with strictly definable interpretation maps.*

Corollary 1.3.21 does not give the same kind of information as Theorem 1.3.14 because it does not decompose the strictly interpretable $\mathcal{H}$ into orthogonal subspaces generated by strictly interpretable asymptotically free types. It is an open question if Corollary 1.3.21 can be improved in this direction.

It is certainly possible to improve Corollary 1.3.21 when T is an ω-categorical classical logic theory. In this case, inner product maps on classical sorts of T are always strictly definable, since type spaces are finite. Lemma 1.3.20 applies and we find that the sets $\mathcal{P}(p)$ are definable sets in classical sorts of T. Moreover, the decomposition procedure of Theorem 1.3.14 produces definable sets in classical imaginary sorts of T, on which the inner product maps are necessarily strictly definable. Therefore we have the corollary:

Corollary 1.3.22. *Let T be an ω-categorical classical logic theory and let $\mathcal{H}$ be a strictly interpretable Hilbert space in T. Then $\mathcal{H}$ is isomorphic to an orthogonal sum of interpretable Hilbert spaces each generated by asymptotically free complete types in classical imaginary sorts of T.*

We will see in Section 1.6.2 that Corollary 1.3.22 is equivalent to the classification theorem of Tsankov for unitary representations of oligomorphic groups.

1.3.3 Some elementary examples and counterexamples

The asymptotically free type-definable sets of Theorem 1.3.14 or Corollary 1.3.21 live in imaginary sorts of T which may be difficult to identify in practice. Nevertheless, in many special cases it is possible to give a presentation of $\mathcal{H}$ which satisfies our various decomposition theorems without having to go through the proofs of these results.

1. Let T be the theory of an infinite set X and let $\mathcal{H}_1$ be the interpretable Hilbert space generated by X with inner product map $f(x, y) = 0$ for $x \neq y$ and $f(x, x) = 1$. Then $\mathcal{H}_1$ already satisfies the conclusion of Corollary 1.3.22.

Let $\mathcal{H}_n$ be the interpretable Hilbert space generated by X^n with inner product map $f(\overline{x}, \overline{y}) = k$ if $\overline{x}, \overline{y}$ share k entries, ignoring order. Then we can take the integer N from Lemma 1.3.20 to be equal to $2n + 1$ so the factors from Corollary 1.3.22 are given by a quotients of X^N. Going through the decomposition procedure of Theorem 1.3.14 leads to the easy observation that $\mathcal{H}_n$ is the orthogonal sum of n copies of $\mathcal{H}_1$.

Let $\mathcal{H}_1'$ be the interpretable Hilbert space generated by X with inner product map $f(x,y) = 1$ if $x \neq y$ and $f(x,x) = 2$. Going through the decomposition procedure of Theorem 1.3.14 again leads to the easy observation that $\mathcal{H}_1'$ is the orthogonal sum of a copy of $\mathcal{H}_1$ and a one-dimensional interpretable Hilbert space.

2. For $n \geq 2$, let T_n be the theory of the set of unordered n-tuples over an infinite set X. For $k \leq n$, T_n has predicates $P_k(x,y)$ to say that x and y have exactly k elements in common. Let $\mathcal{H}$ be the interpretable Hilbert space generated by the main sort S with inner product map f defined by $f(x,y) = k$ if and only if $P_k(x,y)$. Then the structure of $\mathcal{H}$ is similar to the structure of $\mathcal{H}_n$ from the previous example, but imaginary sorts of T are needed to give the decomposition into orthogonal subspaces generated by asymptotically free definable sets.

3. Let $T = Th(\mathbb{Z}, \leq)$. All three interpretable Hilbert spaces considered in Section 1.2.2, Example 2 already satisfy Theorem 1.3.14, as they are generated by an asymptotically free complete type.

4. We show the failure of Corollary 1.3.21 when some stable formula of T has the FCP. Let T be the classical logic theory of an equivalence relation E on a sort S such that, for each n, E has exactly one equivalence class of cardinality n. Define the positive definite function f by $f(x,x) = 2$ for all x, $f(x,y) = 1$ if $x \neq y$ and xEy and $f(x,y) = 0$ if $\neg xEy$. Write $\mathcal{H}$ for the interpretable Hilbert space in T generated by S with inner product map f. Let $M \models T$ be ω_1-saturated.

The set $\mathcal{P}(S)$ consists of the set S together with 0 and the weak limit point of each infinite E-class. In the notation of Lemma 1.3.20, $N = 2$ and $S' = S^2$ quotiented out by the equivalence relation E' on pairs (x,y) with classes $x = y$, $x \neq y \wedge xEy$, and $\neg xEy$. S^+ is the type-definable subset of S' containing all E'-classes except those represented by pairs (x,y) such that $x \neq y \wedge xEy$ and such that the E-class of x is finite. S^+ is not definable.

Observe that there is no way of extending h to a definable set D containing S^+ in such a way that the inner product map on D is strictly definable. Since there are no other natural candidates for generating sets, this suggests that the conclusion of Corollary 1.3.21 is false in this case. Furthermore, we have the following classical result:

Lemma 1.3.23 ([She78], II 4.4)**.** *Let T be an arbitrary classical logic theory. If T does not have the weak NFCP, then there is a definable equivalence relation $E((x,z),(y,z'))$ such that for all $n \geq 1$ there is a tuple c_n such that the formula $E((x,c_n),(y,c_n))$ is an equivalence relation with more than n but only finitely many equivalence classes.*

Therefore, any theory without the weak NFCP has an interpretable Hilbert space with the same properties as $\mathcal{H}$ constructed above.

1.4 Definable Measures and L^2-spaces

In this section, we discuss interpretable Hilbert spaces in classical logic theories T generated by a definable measure μ. Examples include pseudofinite fields and MS-measurable theories. We show that $L^2(\mu)$ is interpretable in T and we prove the strong germ property for $L^2(\mu)$ for pseudofinite fields and ω-categorical measurable structures.

Suppose T is a classical logic theory with elimination of imaginaries and with a Keisler measure μ on a sort X of T. This means that for all $M \models T$, μ is a finitely additive probability measure on the Boolean algebra $Def_x(M)$ of M-definable subsets in the variable x, where x ranges in X. We view $Def_x(M)$ as an algebra of subsets of the type space $S_x(M)$. Suppose in addition that μ is definable, in the sense that for any formula $\phi(x, y)$ and any $\lambda \geq 0$, the set of a in M such that $\mu(\phi(x, a)) = \lambda$ is a definable set. One important example is the theory T of pseudofinite fields with the counting measure μ. μ was first shown to be definable in [CvdDM92]. MS-measurable classes introduced in [MS08] generalise the case of pseudofinite fields and offer a rich source of examples.

Given $M \models T$, the measure μ on the algebra $Def_x(M)$ extends to a countably additive probability measure on the σ-algebra $\mathcal{D}_x(M)$ generated by $Def_x(M)$. We view $\mathcal{D}_x(M)$ as a σ-algebra over the type space $S_x(M)$ but when M is ω_1-saturated we can also view $\mathcal{D}_x(M)$ as a σ-algebra of definable subsets of X itself. Write $L^2(X(M), \mu)$ for the space of square-integrable functions on $S_x(M)$ with respect to $\mathcal{D}$ and μ. $L^2(X(M), \mu)$ is densely generated by functions of the form $\mathbb{1}_{\phi(x,a)}$.

For any formula $\phi(x, y)$, let S_ϕ be an imaginary sort of T which is a Cartesian product of sorts of T corresponding to the tuple y. If $\phi(x, y)$ and $\psi(x, y)$ are different formulas, we take the sorts S_ϕ and S_ψ to be distinct, although they are copies of each other.

For $M \models T$ and for any formula $\phi(x, y)$, define the map $h_\phi : S_\phi \to L^2(X(M), \mu)$ by $h_\phi(a) = \mathbb{1}_{\phi(x,a)}$. By definability of the measure, we are in the setting of Proposition 1.2.4 and the system of maps (h_ϕ) gives an interpretable Hilbert space $\mathcal{H}$. $\mathcal{H}$ satisfies the following easy proposition:

Proposition 1.4.1. *Let $\mathcal{H}$ be as defined above. For any $N \models T$, there is a Hilbert space isomorphism $F : H(N) \to L^2(X(N), \mu)$ such that for every formula $\phi(x, y)$, the map $F \circ h_\phi : S_\phi \to L^2(X(N), \mu)$ takes the element $a \in S_\phi$ to the vector $\mathbb{1}_{\phi(x,a)}$.*

By Proposition 1.4.1, it is natural to say that the functor $L_2(X, \mu)$ *is interpretable in T*. By definability of the measure μ, $L^2(X, \mu)$ is strictly interpretable, in the sense of Definition 1.3.17. Therefore the decomposition Theorem 1.3.14 applies and we know that $L^2(X, \mu)$ is an orthogonal sum of $\bigwedge$-interpretable subspaces generated by asymptotically free complete types.

While it is not yet clear if it is possible to give a systematic presentation of $L^2(X,\mu)$ which satisfies Theorem 1.3.14, we can give an explicit example when T is the theory of the random graph. Write X for the main sort of T and let μ be the unique definable measure on X such that for any distinct parameters $a_1,\dots,a_n$ and $b_1,\dots,b_k$,

$$\mu\Big(\bigwedge R(x,a_i)\wedge\bigwedge\neg R(x,b_j)\Big)=1/2^{n+k}$$

(see [Alb94] for a classification of the definable measures on the random graph). For every $n\geq 1$, let X_n be the imaginary sort of n-element subsets of X. For $M\models T$, define the maps $h_n: X_n\to L^2(X(M),\mu)$ by

$$h_n(\alpha)=(-1)^{|\{y\in\alpha|R(x,y)\}|}$$

and write $\mathcal{H}_n$ for the interpretable subspace of $L^2(X,\mu)$ generated by each sort X_n with the map h_n. A direct computation shows that the Hilbert spaces $\mathcal{H}_n$ are pairwise orthogonal and that each X_n is an orthogonal set in $\mathcal{H}_n$. Let $\mathcal{H}_0$ be the one-dimensional subspace of $L^2(X,\mu)$ spanned by the constant function 1. Let $a_1,\dots,a_k$ and $b_1,\dots,b_n$ be in X and pairwise distinct. It is straightfoward to check by induction on $k+n$ that the indicator function of the set $\bigwedge_{i\leq k}R(x,a_i)\wedge\bigwedge_{j\leq n}\neg R(x,b_j)$ is in the orthogonal sum of the spaces $\mathcal{H}_m$ for $0\leq m\leq n+k$ $L^2(X,\mu)$. Therefore $L^2(X,\mu)$ is the orthogonal sum of the spaces $\mathcal{H}_n$, $n\geq 0$.

A goal which is more modest than finding explicit asymptotically free decompositions of the interpretable Hilbert spaces $L^2(X,\mu)$ is to identify the bdd-closed subspaces of $L^2(X,\mu)$. We saw that these play an important role in the proof of Theorem 1.3.14 and we saw that Theorem 1.3.8 proves modularity for the lattice of these subspaces. We show that in some interesting cases, the bdd-closed subspaces of $L^2(X,\mu)$ are the subspaces we would expect.

Recall that T is a classical logic theory with a strictly definable measure μ on the sort X. We write acl(0) for the algebraic closure of the emptyset, in the sense of classical logic. This is not to be confused with bdd(0), which requires us to view T as a continuous logic theory and can be identified with certain hyperimaginaries of T.

Definition 1.4.2. *Let T and $L^2(X,\mu)$ be as above. For $M\models T$, write V_0 for the subspace of $L^2(X(M),\mu)$ generated by vectors $\mathbb{1}_\phi$ where ϕ is a definable subset of X over* acl(0)*.*

We say that T has the strong germ property if for some (any) $M\models T$, $V_0=\mathrm{bdd}(0)$ in $H(M)$.

Let $M\models T$. Note that V_0 and bdd(0) are subspaces of $L^2(X(M),\mu)$ of measurable functions with respect to certain σ-algebras and that we always have $V_0\leq\mathrm{bdd}(0)$. Therefore, the strong germ property is a certain form of elimination of hyperimaginaries, since it asserts that the vectors of $\mathrm{bdd}(0)$ are in the dense span of $\mathrm{acl}(0)$.

If $\phi(x,a)$ is a definable subset of X, $P_{\mathrm{bdd}(0)}\mathbb{1}_{\phi(x,a)}$ and $P_{V_0}\mathbb{1}_{\phi(x,a)}$ are the Radon-Nikodym derivatives of $\mathbb{1}_{\phi(x,a)}$ with respect to the appropriate σ-algebras. While we do not pursue this further, the next lemma asserts a form of probabilistic independence with respect to a certain disintegration of μ along $\mathrm{bdd}(0)$. See [Hru15] for a discussion.

Lemma 1.4.3. *Let T and $L^2(X,\mu)$ be as above. Let $\phi(x,y)$ and $\psi(x,z)$ be formulas of T where x ranges in X. Let $M \models T$ and suppose $a,b \in M$ are independent over* $\mathrm{bdd}(0)$ *with respect to the inner product formulas of $L^2(X,\mu)$. Then $P_{\mathrm{bdd}(0)}\mathbb{1}_{\phi(x,a)\wedge\psi(x,b)} = (P_{\mathrm{bdd}(0)}\mathbb{1}_{\phi(x,a)})(P_{\mathrm{bdd}(0)}\mathbb{1}_{\psi(x,b)})$ almost everywhere, viewed as functions on the probability space $(S(M),\mu)$.*

Proof. Let $g \in L^2(X(M),\mu)$ be a function $S(M) \to \mathbb{R}$ which is in $\mathrm{bdd}(0)$. To make notation lighter, we write $H_0 = \mathrm{bdd}(0) \cap H(M)$. It is enough to show

$$\langle P_{H_0}\mathbb{1}_{\phi(x,a)\wedge\psi(x,b)}, g\rangle = \langle (P_{H_0}\mathbb{1}_{\phi(x,a)})(P_{H_0}\mathbb{1}_{\psi(x,b)}), g\rangle.$$

Note that $\langle P_{H_0}\mathbb{1}_{\phi(x,a)\wedge\psi(x,b)}, g\rangle = \langle \mathbb{1}_{\phi(x,a)\wedge\psi(x,b)}, g\rangle = \langle \mathbb{1}_{\phi(x,a)}, g\mathbb{1}_{\psi(x,b)}\rangle$. Since $g\mathbb{1}_{\psi(x,b)} \in \mathrm{bdd}(b)$ and $\mathrm{bdd}(a), \mathrm{bdd}(b)$ are independent over $\mathrm{bdd}(0)$, we have

$$\begin{aligned}\langle \mathbb{1}_{\phi(x,a)}, g\mathbb{1}_{\psi(x,b)}\rangle &= \langle P_{H_0}\mathbb{1}_{\phi(x,a)}, gP_{H_0}\mathbb{1}_{\psi(x,b)}\rangle \\ &= \langle (P_{H_0}\mathbb{1}_{\phi(x,a)})(P_{H_0}\mathbb{1}_{\psi(x,b)}), g\rangle.\end{aligned}$$

□

The argument for the next proposition is adapted from the independence theorem for probability logic of [Hru15]. We assume that T carries a definable measure on all definable sets and we assume that the measure satisfies Fubini (see Definition 3.1 in [EM08]). The main case where these assumptions hold is when T is an MS-measurable structure.

Proposition 1.4.4. *Let T and $L^2(X,\mu)$ be as above, with μ satisfying Fubini. Let $M \models T$ and suppose that for any formula $\phi(x,a) \subseteq X$ with $\phi(x,y) \subseteq X \times Y$, the following holds: there is a positive-measure* $\mathrm{acl}(0)$*-definable set $Y' \subseteq Y$ containing a such that*

1. *for every* $\mathrm{acl}(0)$*-definable set $Z \subseteq X$, for all $a' \in Y'$, $\mu(\phi(x,a') \wedge Z(x)) = \mu(\phi(x,a) \wedge Z(x))$*

2. *for every* $\mathrm{acl}(0)$*-definable $Z \subseteq X$ and any two pairs $(a_1,a_2), (a_1',a_2')$ in $(Y')^2$ with a_1 independent from a_2 over* $\mathrm{acl}(0)$ *with respect to the stable definable function $\mu_x(\phi(x,y_1) \wedge \phi(x,y_2) \wedge Z(x))$ and similarly for a_1', a_2', we have*

$$\mu(\phi(x,a_1) \wedge \phi(x,a_2) \wedge Z(x)) = \mu(\phi(x,a_1') \wedge \phi(x,a_2') \wedge Z(x)).$$

Then T has the strong germ property.

Proof. We can work inside an ω_1-saturated model M. Write V_0 for the subspace of $L^2(X(M), \mu)$ generated by acl(0) definable sets and $H_0 = \mathrm{bdd}(0)$, so that $V_0 \leq H_0$. In this proof, we also write $\phi(x, a)$ instead of the indicator function $\mathbb{1}_{\phi(x,a)}$.

We show that $P_{V_0}(P_{H_0}\phi(x, a))^2 = (P_{V_0}\phi(x, a))^2$. Given this identity, we will have

$$\|P_{H_0}\phi(x,a)\|^2 = \int_X (P_{H_0}\phi(x,a))^2 d\mu = \int_X P_{V_0}(P_{H_0}\phi(x,a))^2 d\mu = \|P_{V_0}\phi(x,a)\|^2$$

and the result will follow. It is enough to show that for any acl(0)-definable set $Z(x)$, we have

$$\int_X P_{V_0}(P_{H_0}\phi(x,a))^2 Z(x) d\mu(x) = \int_X (P_{V_0}\phi(x,a))^2 Z(x) d\mu(x). \tag{1.1}$$

Take a' in M independent from a over bdd(0) realising tp(a/bdd(0)). Note that $P_{H_0}\phi(x, a) = P_{H_0}\phi(x, a')$. By Lemma 1.4.3, we have

$$\begin{aligned}
\int_X P_{V_0}(P_{H_0}\phi(x,a))^2 Z(x) d\mu(x) &= \int_X (P_{H_0}\phi(x,a))^2 Z(x) d\mu(x) \\
&= \int_X P_{H_0}(\phi(x,a)\phi(x,a')) Z(x) d\mu(x) \\
&= \int_X \phi(x,a)\phi(x,a') Z(x) d\mu(x) \\
&= \frac{1}{\mu(Y')^2} \int_{y\in Y'} \int_{y'\in Y'} \int_X \phi(x,y)\phi(x,y') Z(x) d\mu(y') d\mu(y) d\mu(x) \\
&= \frac{1}{\mu(Y')^2} \int_X \Big(\int_{Y'} \phi(x,y) d\mu(y) \Big)^2 Z(x) d\mu(x)
\end{aligned}$$

To deduce (1.1), it is enough to show that $\int_{Y'} \phi(x, y)d\mu(y) = \mu(Y')P_{V_0}(\phi(x, a))$ on X. Since $\int_{Y'} \phi(x, y)d\mu(y) = \mu_y(Y'(y) \wedge \phi(x, y))$, which is a function in V_0, this follows by the computation:

$$\begin{aligned}
\mu(Y') \int_X P_{V_0}\phi(x,a) Z(x) d\mu(x) &= \mu(Y') \int_X \phi(x,a) Z(x) d\mu(x) \\
&= \int_X (\int_{Y'} \phi(x,y) d\mu(y)) Z(x) d\mu(x)
\end{aligned}$$

□

Remark: The conditions of Proposition 1.4.4 always hold when T is an ω-categorical MS-measurable structure. The discussion of the definable measure in ACFA in Chapter 2 of this thesis shows that the conditions of Proposition 1.4.4 also hold in that setting. This follows from Theorem 2.2.21 and the Stationarity Theorem 2.3.3.

1.5 Absolute Galois Groups and Associated Hilbert Spaces

In this section, we study a particular source of interpretable Hilbert spaces: the L^2-spaces associated to the absolute Galois group of a definably closed subset K of a classical logic structure M.

We start by generalising the classical result of [CvdDM80] about the interpretation of the absolute Galois group of a perfect field inside its algebraic closure to a quite general first order setting. Assuming elimination of finite imaginaries, we show that the definable projective system associated to $Gal(K)$ is canonically interpretable in M in the language of M with an extra predicate P for K. See Proposition 1.5.6.

We then show that $L^2(Gal(K))$ is interpretable canonically in M in the language with an extra predicate P for K and we find a canonical asymptotically free decomposition. Using this asymptotically free decomposition, we also show that there is a sense in which K can be said to interpret $L^2(Gal(K))$ in the language induced from M, although this interpretation is not canonical. See Proposition 1.5.11. Finally, we show that for arbitrary $k \geq 1$, the space of $Gal(K)$-invariant functions on $Gal(K)^k$ is canonically interpretable in K in the language induced from M.

In this section, we work with classical logic. We fix a language $\mathcal{L}$ and a complete $\mathcal{L}$-theory T. We will assume that T admits quantifier elimination. We assume that T has *elimination of finite imaginaries*, which is a weakening of elimination of imaginaries:

Definition 1.5.1. *T admits elimination of finite imaginaries if for every finite product S of sorts of T and any $l \geq 1$, there is a definable set $C_{\leq l}(S)$ and a definable relation $R \subseteq C_{\leq l}(S) \times S$ such that the elements a of $C_{\leq l}(S)$ code the $\leq l$-element subsets of S in the following sense: for any $n \leq l$ and any $x_1, \ldots, x_n \in S$, there is a unique $a \in C_{\leq l}(S)$ such that $R(a, y)$ is exactly the set $\{x_1, \ldots, x_n\}$, and every $a \in C_{\leq l}(S)$ codes such a set.*

We consider an enriched language $\mathcal{L}_P$ where P stands for a collection of unary predicates (P_i) in distinct sorts of T. Let T_P be the $\mathcal{L}_P$-theory containing T which says that $P = \mathrm{dcl}(P)$ (i.e. if $S_1, \ldots, S_n$ and S' are sorts of T and $f : S_1 \times \ldots \times S_n \to S'$ is a definable function, then the image of $P_{S_1} \times \ldots \times P_{S_n}$ under f is contained in $P_{S'}$).

Assume that T admits elimination of finite imaginaries. Then for every finite product S of sorts of $\mathcal{L}$ and $l \geq 1$, there is an $\mathcal{L}_P$-definable set $D_{\leq l}(S)$ such that for any $M \models T_P$, $D_{\leq l}(S)$ is a set of codes for the $P(M)$-definable sets contained in S which contain at most l elements. Namely, take $D_{\leq l}(S) = C_{\leq l}(S) \wedge P$.

Let $M \models T$ and let $K \subseteq M$. We will say that K is a *substructure of* M if K is *definably closed in* M. With a choice of substructure K, M is naturally a model of T_P with $K = P(M)$. If K is a substructure of M, we will view K as an $\mathcal{L}$-structure and we will write $Th(K)$ for the complete theory of K in the language $\mathcal{L}$.

When we make no special reference to T or T_P , we will be working in $\mathcal{L}$ and viewing K simply as a subset of M. In particular, when we write dcl and acl, we mean the definable and algebraic closure in M in the language $\mathcal{L}$.

1.5.1 Interpretation of the inverse system of $Gal(K)$

In this section, we assume that T has quantifier elimination and elimination of finite imaginaries. Working with $M \models T$ and K a definably closed substructure, we show that $Gal(K)$ is the inverse limit of an $\mathcal{L}_P$-definable system of finite definable groups. Although this definable inverse system is not coded in K, we show that K codes the profinite space of $Gal(K)$-conjugacy classes of $Gal(K)^k$ for arbitrary $k \geq 1$.

Let $M \models T$ and let $K \subseteq M$ be a definably closed substructure. Define $Gal(K)$ to be the group of elementary automorphisms of $\mathrm{acl}(K)$ which fix K pointwise. Note that elements of $Gal(K)$ might not extend to automorphisms of M. $Gal(K)$ is a profinite group with a basis of open normal subgroups given by the family $\mathrm{Aut}(\mathrm{acl}(K)/K, \phi(x, K))$ where $\phi(x, K)$ is the set of realisations of a complete algebraic type over K. The family of groups $\mathrm{Aut}(\phi(x, K)/K)$ where $\phi(x, K)$ is a complete algebraic type forms a projective system with quotient maps $\mathrm{Aut}(\phi(x, K)/K) \to \mathrm{Aut}(\psi(y, K)/K)$ when $\mathrm{Aut}(\mathrm{acl}(K)/K, \phi(x, K))$ is a normal subgroup of $\mathrm{Aut}(\mathrm{acl}(K)/K, \psi(y, K))$. We say that the family of groups $\mathrm{Aut}(\phi(x, K)/K)$ is the *inverse system of finite quotients of* $Gal(K)$.

Remark: This definition of $Gal(K)$ is sensitive to the language $\mathcal{L}$ in several ways; in particular it concerns a quotient of the Shelah-Galois group corresponding to those sorts represented in $\mathcal{L}$. It yields the full Shelah-Galois group, the automorphism group of algebraic imaginary elements, when T admits full elimination of imaginaries. As there is no additional reason to assume full elimination of imaginaries, we will work with the weaker notion of elimination of finite imaginaries. For example, with T the theory of algebraically closed valued fields formulated in a single-sorted language $\mathcal{L}$ referring to the field sort, $Gal(K)$ will give the field-theoretic absolute Galois group of a perfect Henselian subfield K; while in a language $\mathcal{L}'$ with an additional sort for the residue field, a substructure K can be a Henselian subfield along with a perfect field extension of its residue field, and $Gal(K)$ would give their combined Galois groups.

We want to show that under the assumptions on T mentioned above, M interprets in T_P the inverse system of finite quotients of $Gal(K)$. This will generalise the classical result of [CvdDM80] in the case where K is a perfect field and $M = K^{alg}$. A detailed exposition of the construction for perfect fields can be found in the appendix of [Cha02].

Definition 1.5.2. *Let $M \models T$ and let K be a definably closed substructure of M. A K-definable finite structure c in M is a pair $c = (u(c), R(c))$ where $u(c)$ is a finite K-definable set in M of size $n \geq 1$ and $R(c)$ is a non-empty K-definable n-ary relation on $u(c)$ such that every element of $R(c)$ is an enumeration of $u(c)$. Here $u(c)$ is a subset of a finite Cartesian product of sorts of M. We say that $u(c)$ is the universe of c.*

We say that c is a complete K-definable finite structure if $R(c)$ is minimal, in the sense that for any finite K-definable structure c' with the same universe as c, if $R(c') \subseteq R(c)$ then $R(c') = R(c)$

For c a complete finite K-definable structure, let $Gal(c/K)$ be the finite group of elementary automorphisms of $u(c) \cup K$ which fix K.

We say that c is Galois if $Gal(c/K)$ acts sharply transitively on $u(c)$.

If c is a complete finite K-definable structure in M, then there is some enumeration $u_1, \ldots, u_n$ of $u(c)$ such that $R(c)$ is the set of realisations of the type $\mathrm{tp}(u_1, \ldots, u_n/K)$. Note that $Gal(c/K)$ acts sharply transitively on $R(c)$. Therefore, taking a finite K-definable structure d with universe $R(c)$ produces a Galois K-definable finite structure.

Finally, observe that $Gal(K)$ is the projective limit of the system $Gal(d/K)$ where d is a Galois K-definable finite structure. Therefore we will focus on Galois K-definable finite structures in what follows. This is not necessary from a technical point of view but it makes the analogy with the case of fields clearer.

We will now code uniformly the Galois structures definable over K as elements of M. Their Galois groups are also coded uniformly as $\mathcal{L}_P$-definable sets that we will call $G(c)$.

Lemma 1.5.3. *For every finite Cartesian product S of sorts of T and $n \geq 1$, there is an $\mathcal{L}_P$-definable set $C_n^{comp}(S)$ such that for any $M \models T_P$, $C_n^{comp}(S)$ codes the complete n-element K-definable structures contained in S. $C_n^{comp}(S)$ is contained in K.*

Moreover, there are $\mathcal{L}_P$-definable sets $G_n(S)$ such that for every $M \models T_P$ and $c \in C_n^{comp}(S)$, there is a K-definable group $G(c) \subseteq G_n(S)$ with a K-definable action on $u(c)$ such that $G(c) = Gal(c/K)$ as a group of permutations of $u(c)$.

Therefore there are $\mathcal{L}_P$-definable sets $C_n^{gal}(S)$ such that for any $M \models T_P$, $C_n^{gal}(S) \subseteq C_n^{comp}(S)$ codes the Galois K-definable structures of size n in S.

Proof. Suppose $u(c)$ has cardinality n and is in S. Then $R(c)$ is a subset of S^n containing at most $n!$ elements such that every element in $R(c)$ enumerates the same n-element set. There is also an $\mathcal{L}_P$-formula asserting that an element $x \in D_{\leq n!}(S^n)$ codes a K-definable relation which is minimal in the sense of Definition 1.5.2. Hence all n-element complete K-definable structures in S are coded in some definable subset of $D_{\leq n!}(S^n)$.

Consider the set of pairs $(x, y) \in S^{2n}$ such that there is $c \in C_n^{comp}(S)$ such that x, y both belong to $R(c)$. The set of such pairs is 0-definable. Observe that any such pair (x, y) determines an element of $Gal(c/K)$ and that a definable equivalence relation decides whether two such pairs determine the same element of $Gal(c/K)$. By elimination of finite imaginaries, we can find a definable set $G_n(S)$ of codes for automorphisms of n-element complete K-definable structures in S.

By construction of $G_n(S)$, we see that for every $c \in C_n^{comp}(S)$ there is a c-definable group $G(c)$ contained in $G_n(S)$ with a c-definable action on $u(c)$, and $G(c)$ is canonically identified with $Gal(c/K)$ in the obvious way. The definability of $C_n^{gal}(S)$ follows. □

Note that in Lemma 1.5.3, for $M \models T_P$ and $K = P(M)$, the definable sets $G_n(S)$ are not usually contained in K. See the discussion following Proposition 1.5.6.

We now show that the projective limit structure on the system of groups $(Gal(c/K))$ is definable:

Definition 1.5.4. *Let $M \models T$ and let $K \subseteq M$ be a definably closed substructure. For c, c' finite K-definable structures, we write $c \leq c'$ if $u(c) \subseteq \mathrm{dcl}(K, u(c'))$.*

It is clear that for $c \in C_n^{gal}(S)$ and $c' \in C_m^{gal}(S')$, $c \leq c'$ if and only if $\mathrm{Aut}(\mathrm{acl}(K)/K, c)$ is a normal subgroup of $\mathrm{Aut}(\mathrm{acl}(K)/K, c')$.

Lemma 1.5.5. *The relation $c \leq c'$ is definable in T_P between sets $C_n^{gal}(S)$ and $C_m^{gal}(S')$.*

Proof. Let $M \models T_P$ and take $c \in C_n^{gal}(S)$ and $c' \in C_m^{gal}(S')$. Let a and b be arbitrary enumerations of $u(c)$ and $u(c')$. Then $\mathrm{tp}(ab/K)$ is algebraic so we can find $d \in C_k^{gal}(S^n \times S'^m)$ such that $u(d)$ consists of the realisations of $\mathrm{tp}(ab/K)$. Then $c \leq c'$ if and only if the action of $G(d)$ on $u(d)$ is determined by its restriction to the coordinates in S'^m. This is a definable property of d. Finally, we quantify-out d to obtain a definition of $c \leq c'$. □

The proof of Lemma 1.5.5 shows that for any sets $C_n^{gal}(S)$ and $C_m^{gal}(S')$, there is a sort S'' and $k \leq nm$ such that for any $c \in C_n^{gal}(S)$, $c' \in C_m^{gal}(S')$, there is $c'' \in C_k^{gal}(S'')$ with $c \leq c''$ and $c' \leq c''$. It follows that $\bigcup_{n,S} C_n^{gal}(S)$ forms a directed preorder under $\leq$. It is a preorder because we may have $c \neq c'$, $c \leq c'$ and $c' \leq c$ for c, c' in the same set $C_n^{gal}(S)$ or in distinct sets.

Moreover, Lemma 1.5.5 shows that if $c \in C_n^{gal}(S)$ and $c' \in C_m^{gal}(S')$ then the relation $c \leq c'$ determines a canonical K-definable surjective homomorphism $G(c') \to G(c)$. It is easy to check that if we have $c \leq c' \leq c$, then the resulting homomorphism $G(c) \to G(c)$ is the identity. We say that the projection maps are compatible with the preorder $\leq$ on $\bigcup_{n,S} C_n^{gal}(S)$. Hence we have proved the following proposition:

Proposition 1.5.6. *Let $M \models T$ and let K be a definably closed substructure. The family of groups $G(c)$ indexed by the set*

$$\bigcup \{C_n^{gal}(S) \mid n \geq 2, S \text{ finite product of sorts of } T\}$$

forms a strict $\mathcal{L}_P$-piecewise-definable projective system of finite $\mathcal{L}_P$-definable groups with the directed preorder $x \leq y$ and the induced definable homomorphisms $G(y) \to G(x)$. The inverse limit of this projective system is canonically isomorphic to $Gal(K)$.

In Proposition 1.5.6, we say that the family $G(c)$ is a *piecewise definable* projective system of finite definable groups because the underlying preorder is given by a family of definable sets. We stress that this projective system is indexed by a preorder. We say that the definable projective system is *strict* because the group homomorphisms $G(y) \to G(x)$ for $x \leq y$ are surjective.

Observe that the $\mathcal{L}_P$-piecewise-definable projective system is only sensitive to the part of M which contains $\mathrm{acl}(K)$. Therefore, we can define a language $\mathcal{L}_P^*$ which contains only the quantifier-free definable relations of $\mathcal{L}$ which occur between elements of $\mathrm{acl}(K)$ and define T_P^* to be the theory of the $\mathcal{L}_P^*$-structure $\mathrm{acl}(K)$ in the obvious way. The definable projective system of groups of Proposition 1.5.6 is definable in T_P^*. We will not make any further references to $\mathcal{L}_P^*$ but every interpretability result about M as an $\mathcal{L}_P$-structure in this section and in section 1.5.2 also holds for $\mathrm{acl}(K)$ as an $\mathcal{L}_P^*$-structure. This stronger fact can be seen by applying those results to T_P^* in place of T and to the $\mathcal{L}_P^*$-structure $\mathrm{acl}(K)$ in place of M while keeping the same substructure K.

Finally, we remark that Proposition 1.5.6 entails trivially that $Gal(K)^n$ is also the inverse limit of an $\mathcal{L}_P$-definable projective system of finite groups in M, for any $n \geq 1$.

Let $M \models T$ and let K be a substructure. Using quantifier-elimination in T, an inspection of the proofs of Lemmas 1.5.3 and 1.5.5 will show that the sets $C_n^{gal}(S)$ and the relation $c \leq c'$ are $\mathcal{L}$-definable subsets of K, which we view as an $\mathcal{L}$-structure. As a result, $Th(K)$ has partial access to the definable projective system of grouos $(G(c))$. For example, there are quantifier-free $\mathcal{L}$-formulas which determine in $Th(K)$ the isomorphism type of the group $G(c)$ for $c \in C_n^{gal}(S)$.

However, [Dit18] shows in the context of fields that the $\mathcal{L}_P$-definable projective system of groups $(G(c))$ is usually not interpretable in $Th(K)$. Indeed, when $c \leq c'$, $Th(K)$ does not code a projection $G(c') \to G(c)$. This is because $Th(K)$ does not determine an embedding of the sets $u(c)$ and $u(c')$ in M. See [Dit18] for proofs and a detailed discussion.

Nonetheless, for any $c \in C_n^{gal}(S)$ and $k \geq 1$, K contains codes for each orbit on $G(c)^k$ under diagonal conjugation by $G(c)$. We say that these are the $G(c)$-conjugacy classes of $G(c)^k$. To see this, take $\phi \in Gal(c/K) = G(c)$ and $g = (g_1, \dots, g_k) \in G(c)^k$. Then $\phi(g)$ is the conjugate $(g_1^\phi, \dots, g_k^\phi)$. Therefore, ϕ fixes the conjugacy class $(g_1, \dots, g_k)^{G(c)}$. By elimination of finite imaginaries, this conjugacy class is coded by an element of K. As in Lemma 1.5.3, we can construct a definable set $Conj_n^k(S) \subseteq K$ such that for every $c \in C_n^{gal}(S)$, there is a finite K-definable set $Conj^k(c) \subseteq Conj_n^k(S)$ coding the $G(c)$-conjugacy classes of $G(c)^k$.

If $c \in C_n^{gal}(S)$, $c' \in C_m^{gal}(S')$ and $c \leq c'$, then the canonical projection $\pi : G(c') \to G(c)$ induces a definable relation $\sqsubseteq$ between $Conj^k(c)$ and $Conj^k(c')$ as follows: we write $a \sqsubseteq b$ where $a \in Conj^k(c')$ and $b \in Conj^k(c)$ if a is contained in $\pi^{-1}(b)$. This relation is $\mathcal{L}$-definable in $Th(K)$. Hence we have the proposition:

Proposition 1.5.7. *Let $M \models T$, let K be a definably closed substructure and let $k \geq 1$. Working in $Th(K)$, the family of $\mathcal{L}$-definable finite sets $Conj^k(c)$ indexed by the set $\bigcup_{n,S} C_n^{gal}(S)$ forms a strict piecewise-definable projective system with the directed preorder $x \leq y$ on $\bigcup_{n,S} C_n^{gal}(S)$ and the induced relation $\sqsubseteq$.*

The inverse limit of this projective system is canonically isomorphic to the profinite space of $Gal(K)$-conjugacy classes of $Gal(K)^k$.

We end this section with a comment about the opposite group to $G(c)$. Fix $M \models T$ and K a substructure of M. For $c \in C_n^{gal}(S)$, define $G^{op}(c)$ the c-definable group of permutations of $u(c)$ which commute with $G(c)$. Since c is Galois, it is easy to show that $G^{op}(c)$ is isomorphic to $G(c)$, but not canonically: for any $a \in u(c)$, we have an isomorphism $G(c) \to G^{op}(c)$, $g \mapsto h^{-1}$ where $ga = ha$. Only the conjugacy classes of $G(c)$ and $G^{op}(c)$ are in a canonical bijection.

Since $G(c)$ is equal to $Gal(c/K)$, $G^{op}(c)$ is pointwise K-invariant and hence $G^{op}(c)$ is contained in K. In fact, we can use the opposite groups to give a more canonical characterisation of K-definable finite Galois structures. We can define $C_n^{gal}(S)$ as the set of elements c in $D_{\leq n}(S)$ such that the set coded by c does not contain any smaller K-definable set and such that the set of permutations of c which belong to K acts sharply transitively on c. This approach is more canonical than the approach through Definition 1.5.2 but it does not allow any strengthening of our results.

This is because the identification of $G^{op}(c)$ with $G(c)$ is not canonical and there is no canonical system of projections $G^{op}(c') \to G^{op}(c)$ when $c \leq c'$. If there is a K-definable surjection $u(c') \to u(c)$, then it is true that we have a

K-definable surjective homomorphism $G^{op}(c') \to G^{op}(c)$, but different choices of surjections $u(c') \to u(c)$ induce incompatible homomorphisms between the opposite groups. Therefore, although it is possible to work in $Th(K)$ with an arbitrary finite collections of opposite groups, the family $(G^{op}(c))$ does not form an $\mathcal{L}$-definable projective system in $Th(K)$.

1.5.2 Hilbert spaces associated to $Gal(K)$

As before, we assume that T eliminates finite imaginaries and has quantifier elimination and T_P is the expansion of T by a predicate which names a definably closed substructure. In this section we construct the interpretable Hilbert spaces $\mathcal{H}_{gal}$, $\mathcal{H}'_{gal}$ and $\mathcal{H}_{class,k}$ for $k \geq 1$. We summarise our results about these interpretable Hilbert spaces in the following theorem. The rest of this section is devoted to proving Theorem 1.5.8.

Theorem 1.5.8. *In the following, we write K for the definably closed substructure of T_P named by the predicates P. In (3) and (4), K is considered as the universe of an $\mathcal{L}$-structure.*

1. $L^2(Gal(K), \mathrm{Haar})$ *is strictly interpretable in T_P.*
2. $L^2(Gal(K), \mathrm{Haar})$ *is the orthogonal sum of a family of finite-dimensional Hilbert spaces determined by a piecewise-interpretable family of asymptotically free $\mathcal{L}_P$-definable sets.*
3. *For every $k \geq 1$, $L^2(Gal(K)^k, \mathrm{Haar})^{Gal(K)}$, the space of class functions of $Gal(K)^k$, is strictly interpretable in the substructure K in the language $\mathcal{L}$.*
4. $L^2(Gal(K)^k, \mathrm{Haar})^{Gal(K)}$ *is the orthogonal sum of a family of finite-dimensional Hilbert spaces determined by a piecewise-interpretable family of asymptotically free $\mathcal{L}$-definable sets in K.*

We will write $\mathcal{H}_{gal} = L^2(Gal(K), \mathrm{Haar})$ and $\mathcal{H}_{class,k} = L^2(Gal(K)^k, \mathrm{Haar})^{Gal(K)}$, as this will make notation less ambiguous.

There is also a stricly interpretable Hilbert space $\mathcal{H}'_{gal}$ in $\mathcal{L}$ such that for any $M \models T$ and any definably closed substructure $K \subseteq M$, $H'_{gal}(K)$ is a Hilbert space interpretable in K which is abstractly isomorphic to $L^2(Gal(K), \mathrm{Haar})$ in a way that respects the orthogonal decomposition given in (2) above.

Proof. (1) is proved in Proposition 1.5.9. (2) is proved in Lemma 1.5.10. (3) is proved in Proposition 1.5.13 and (4) is proved in the discussion following 1.5.13. $\mathcal{H}'_{gal}$ is discussed in Proposition 1.5.11. $\square$

We always consider $Gal(K)$ as a measure space with the Haar measure, so we suppress the reference to the Haar measure in what follows.

We use the notation that we set up in Section 1.5.1. Take $M \models T$ with a definably closed substructure K and take $g \in G(c) \subseteq G_n(S)$ in M. Then we have seen that $G(c) = Gal(K)/Fix(u(c))$ and g is canonically identified with a coset of $Fix(u(c))$ in $Gal(K)$. In order to simplify notation, we will also write g for this coset.

Proposition 1.5.9. *There is a strictly interpretable Hilbert space $\mathcal{H}_{gal}$ in T_P generated by the definable sets $(G_n(S))$ where $n \geq 2$ and S ranges over finite Cartesian products of sorts of T such that for any $M \models T_P$, $H_{gal}(M)$ can be canonically identified with $L^2(Gal(K))$ in the following sense:*

Writing $h_{S,n} : G_n(S) \to H_{gal}(M)$ for the direct limit maps, there is a Hilbert space isomorphism $F : H_{gal}(M) \to L^2(Gal(K))$ such that for any $g \in G(c) \subseteq G_n(S)$, $F \circ h_{S,n}(g) = \mathbb{1}_g$. F is necessarily unique.

Proof. We only need to show that for any choice of $G_n(S)$ and $G_m(S')$, the inner product map $G_n(S) \times G_m(S') \to \mathbb{R}, (g, g') \mapsto \langle \mathbb{1}_g, \mathbb{1}_{g'} \rangle$ is definable, where g, g' are identified with cosets in $Gal(K)$ as explained above.

For any $g \in G(c)$ and $g' \in G(c')$, choose any c'' such that $c, c' \leq c''$ and write $\pi : G(c'') \to G(c)$, $\pi' : G(c'') \to G(c')$ for the canonical homomorphisms. Then quotienting by $Fix(c'')$ we have

$$\langle \mathbb{1}_g, \mathbb{1}_{g'} \rangle = \frac{|\pi^{-1}(g) \cap \pi'^{-1}(g')|}{|G(c'')|}.$$

Since π, π' are (c, c', c'')-definable, the above relation is also (c, c', c'')-definable. Quantifying-out c'', we find that the inner product maps are $\mathcal{L}_P$-definable and this defines $\mathcal{H}_{gal}$. □

We say that $\mathcal{H}_{gal}$ is the *interpretation of* $L^2(Gal(K))$ *in* T_P. Compare with Proposition 1.4.1 in the case of definable measures.

We show how to find a natural asymptotically free decomposition of $\mathcal{H}_{gal}$. We have seen that the equivalence relation $(c \leq c') \wedge (c' \leq c)$ on each $C_n^{gal}(S)$ induces an equivalence relation on $G_n(S)$ and we can quotient out the sets $C_n^{gal}(S)$ and $G_n(S)$ by these equivalence relations to obtain injective direct limit maps for $\mathcal{H}_{gal}$. We continue to write $G_n(S)$ for the resulting imaginary sorts and we identify $G_n(S)$ with a subset of $\mathcal{H}_{gal}$.

Fix $M \models T$ and K a definably closed substructure. For every c-definable group $G(c) \subseteq G_n(S)$, let $H(c)$ be the finite-dimensional subspace of $L^2(Gal(K))$ generated by functions which are constant on cosets of $Fix(u(c))$. $H(c)$ can be canonically identified with $L^2(G(c)) = L^2(Gal(c/K))$ with the normalised counting measure. Let $W(c)$ be the subspace of $H(c)$ generated by the sum of the spaces $H(c')$ for c' in arbitrary sets $C_m^{gal}(S')$ such that $c' < c$.

Let $\tilde{G}_n(S) = \{P_{W(c)^\perp} g \mid g \in G(c), c \in C_n^{gal}(S)\}$. By an argument similar to the one given in Proposition 1.3.16, $\tilde{G}_n(S)$ can be identified with a definable set in a classical imaginary sort of T_P with a strictly definable inner product map.

Lemma 1.5.10. *Each definable set $\tilde{G}_n(S)$ above is asymptotically free in $\mathcal{H}_{gal}$. Therefore $\mathcal{H}_{gal}$ is the completed orthogonal sum of the finite dimensional spaces $H(c) \cap W(c)^{\perp}$.*

Proof. It is enough to show that if g, g' are automorphisms of $c, c' \in C_n^{gal}(S)$ such that c, c' are not inter-definable, then $P_{W(c)^{\perp}} g \perp P_{W(c')^{\perp}} g'$. Fix $c, c' \in C_n^{gal}(S)$ such that $c' \not\geq c$. We can find a K-definable finite Galois structure d such that $c, c' \leq d$ and we identify $H(c)$ and $H(c')$ with subspaces of $H(d)$.

Let $G_0 = Gal(K)/Fix(u(c))Fix(u(c'))$. Then we identify $G(d)$ with the fibre product $G(c) \times_{G_0} G(c')$, where we have quotient maps $G(d) \to G(c) \to G_0$ and $G(d) \to G(c') \to G_0$. Write $N, N', N_0 \trianglelefteq G(d)$ for the kernels of the maps $G(d) \to G(c), G(c'), G_0$ respectively.

Let $v \in H(c) \cap W(c)^{\perp}$ and $v' \in H(c')$, viewed as functions $G(d) \to \mathbb{R}$. We will show that $v \perp v'$. v is a function which is constant on N-cosets in $G(d)$ and such that the sum of the values of v over any N_0-coset is 0. v' is a function which is constant on N'-cosets in $G(d)$. For any N_0-coset $C \subseteq G(d)$, it is enough to show $\langle v\mathbb{1}_C, v'\mathbb{1}_C \rangle = 0$. There are subsets $A \subseteq G(c)$ and $B \subseteq G(c')$ such that C is identified with $A \times B$ via the isomorphism $G(d) \to G(c) \times_{G_0} G(c')$. Therefore, we have

$$\langle v\mathbb{1}_C, v'\mathbb{1}_C \rangle = \sum_{g \in A} \sum_{g' \in B} v(g)v'(g') = (\sum_{g \in A} v(g))(\sum_{g' \in B} v'(g')) = 0$$

□

Remarks: 1. We have expressed $\mathcal{H}_{gal}$ as a strictly interpretable Hilbert space generated by asymptotically definable sets. This matches the description of Corollary 1.3.21, although it is not clear whether the weak NFCP occurs in the present context.

2. Although each set $\tilde{G}_n(S)$ is asymptotically free, we cannot guarantee that these sets are pairwise orthogonal because of possible identifications between $C_n^{gal}(S)$ and $C_n^{gal}(S')$. Observe also that different complete types in $G_n(S)$ are not pairwise orthogonal.

3. The asymptotically free decomposition constructed above only depends on $Gal(K)$ being the inverse limit of a piecewise-definable projective system of finite groups. Therefore, the same approach gives an asymptotically free decomposition of $L^2(Gal(K)^k)$ for any $k \geq 1$.

For $M \models T$ and K a definably closed substructure, it is unlikely that K as an $\mathcal{L}$-structure interprets $L^2(Gal(K))$ in the same sense as in Proposition 1.5.9. However we can use Lemma 1.5.10 to show that K interprets a Hilbert space which is abstractly isomorphic to $L^2(Gal(K))$ in a way that respects the orthogonal decomposition of Lemma 1.5.10.

Indeed, with the same notation as before, $L^2(Gal(K))$ is the orthogonal sum of the finite-dimensional Hilbert spaces $H(c) \cap W(c)^\perp$. Moreover, there are quantifier-free $\mathcal{L}$-formulas contained in each $C_n^{gal}(S)$ which determine the dimension of $H(c) \cap W(c)^\perp$ for $c \in C_n^{gal}(S)$. Write $dim(x) = k$ for these formulas (where $k \leq n$).

For every set $C_n^{gal}(S) \subseteq K$ and $k \leq n$, let $D_k^i(x_i)$ $(1 \leq i \leq n)$ be disjoint copies of the set $dim(x) = k$. For every c satisfying $dim(c) = k$, choose an orthonormal basis $v_1(c), \dots, v_k(c)$ of $H(c) \cap W(c)^\perp$ and define maps $h_{S,k,i} : D_k^i \to L^2(Gal(K))$ by $h_{S,k,i}(c_i) = v_i(c)$ for $i \leq k$, where c_i is the copy of c in $D_k^i(x_i)$. For $k < i \leq n$, define $h_{S,k,i}(c_i) = 0$. The maps $(h_{S,k,i})$ define an interpretable Hilbert space $H_{S,n}$ isomorphic to the orthogonal sum of the spaces $H(c) \cap W(c)^\perp$ as c ranges over $C_n^{gal}(S)$.

Working now across different sets $C_n^{gal}(S)$ and $C_n^{gal}(S')$, if c and c' are inter-definable so that $H(c) = H(c')$, we can choose the same orthonormal bases $v_1(c) = v_1(c'), \dots, v_k(c) = v_k(c')$ for $H(c) \cap W(c)^\perp = H(c') \cap W(c')^\perp$. Writing $(D_k^i(x_i))$ and $(\tilde{D}_k^j(y_j))$ for the copies of $dim(x) = k$ in $C_n^{gal}(S)$ and $C_n^{gal}(S')$ respectively, we set $h_{S,k,i}(c_i) = h_{S',k,i}(c_i')$. The resulting inner product maps $D_k^i \times \tilde{D}_k^j \to \mathbb{R}$ are definable. Note that c, c' are inter-definable if and only if $c \leq c'$ and $c' \leq c$ and this relation is definable in $Th(K)$. Therefore, the inner product maps are defined independently of the particular identifications which take place in K between $C_n^{gal}(S)$ and $C_n^{gal}(S')$.

Carrying out this construction over all sets $C_n^{gal}(S)$ and making coherent choices of bases, we obtain a strictly interpretable Hilbert space $\mathcal{H}'_{gal}$ satisfying the following proposition:

Proposition 1.5.11. *For any $M \models T$ and K a definably closed substructure, there is a Hilbert space isomorphism $F : H'_{gal}(K) \to L^2(Gal(K))$ satisfying the following property: for every $c \in C_n^{gal}(S)$, there is a finite dimensional subspace $U(c)$ of $H'_{gal}(K)$ such that $U(c) \subseteq \operatorname{dcl}(c)$ and F maps $U(c)$ isomorphically to $L^2(Gal(c/K))$.*

$U(c)$ is defined at the level of $\mathcal{H}'_{gal}$, in the sense $U(c)$ is determined by formulas over c which do not depend on any particular choice of substructure K of M.

Remarks: 1. The strictly interpretable Hilbert space $\mathcal{H}'_{gal}$ defined above satisfies the conclusion of Corollary 1.3.21 by construction.

2. We can also view $\mathcal{H}'_{gal}$ as an interpretable Hilbert space on T_P since K is interpretable in T_P. In that case, we point out that $\mathcal{H}_{gal}$ and $\mathcal{H}'_{gal}$ are not isomorphic interpretable Hilbert spaces on T_P.

When G is a profinite group with Haar measure μ and $k \geq 1$, write $L^2(G^k)^G$ for the Hilbert space of class functions on G^k. These are the functions $G^k \to \mathbb{R}$ or $\mathbb{C}$ which are invariant under conjugation by G. In the following, we work with class functions on G^k into $\mathbb{R}$ but all the results go through without modification to class functions into $\mathbb{C}$.

Lemma 1.5.12. *Let G be a profinite group with Haar measure μ. Then $L^2(G^k)^G$ is generated as a Hilbert space by functions $f : G^k \to \mathbb{R}$ for which there is an open normal $N \trianglelefteq G$ and a G-conjugacy class A of G^k/N^k such that $f = \mathbb{1}_{\pi^{-1}(A)}$ where π is the quotient map $G^k \to G^k/N^k$.*

Proof. Let f be an arbitrary class function on G^k. Then we can find a simple function h which is arbitrarily close to f in the L^2-norm. Let $h'(x) = \int_G h(x^g)dg$. Then

$$\begin{aligned}\|h' - f\|^2 &\leq \int_{x\in G^k}\int_{g\in G}(f(x^g) - h(x^g))^2 dgdx \\ &= \int_G\int_{G^k}(f(x^g) - h(x^g))^2 dxdg \\ &= \|f - h\|^2\end{aligned}$$

It is easy to check that h' is a simple function. Since h' is a class function and a simple function, there is $N \trianglelefteq G$ such that h factors through N^k. The lemma follows. □

Now let T be a theory as before, with elimination of finite imaginaries and quantifier elimination. Recall from Proposition 1.5.7 that there is a $\mathcal{L}$-piecewise-definable projective system such that for any $M \models T$ with K a definably closed substructure, the inverse limit of this projective system in K is canonically identified with the profinite space of $Gal(K)$-conjugacy classes of $Gal(K)^k$. This piecewise-definable projective system is given by the collection of definable sets $Conj^k_n(S)$.

Proposition 1.5.13. *For every $k \geq 1$, there is a strictly interpretable Hilbert space $\mathcal{H}_{class,k}$ generated by the definable sets $(Conj^k_n(S))$ such that for any $M \models T$ with K a definably closed substructure, writing $G = Gal(K)$, $H_{class,k}(K)$ can be canonically identified with $L^2(G^k)^G$ in the following sense:*

Writing $h_{S,n} : Con^k_n(S) \to H_{class,k}(K)$ for the direct limit maps, there is a Hilbert space isomorphism $F : H_{class,k}(K) \to L^2(G^k)^G$ such that for any $a \in Conj^k(c) \subseteq Conj^k_n(S)$, $F \circ h_{S,n}(a) = \mathbb{1}_{\pi^{-1}(a)}$ where $\pi : G^k \to Gal(c/K)^k$ is the quotient map.

Proof. By Proposition 1.5.9, the inner product maps $Conj^k_n(S)\times Conj^k_m(S') \to \mathbb{R}$, $(a, b) \mapsto \langle \mathbb{1}_{\pi^{-1}(a)}, \mathbb{1}_{\pi'^{-1}(b)}\rangle$ are definable. Since quantification only takes place over P, these maps are $\mathcal{L}$-definable in K. This defines $\mathcal{H}_{class,k}$. □

We can also view $\mathcal{H}_{class,k}$ as an interpretable Hilbert space in T_P. In that case, $\mathcal{H}_{class,k}$ embeds definably into $\mathcal{H}_{gal}$ but not into $\mathcal{H}'_{gal}$.

We show that the asymptotically free decomposition of $L^2(Gal(K)^k)$ in M given by Lemma 1.5.10 also provides an asymptotically free decomposition of $\mathcal{H}_{class,k}$. Fix $M \models T$ and K a definably closed substructure. For every

$c \in C_n^{gal}(S)$, define $H_{class,k}(c)$ the Hilbert space of class functions on $G(c)^k$, viewed as a subspace of $L^2(G(c)^k) := H(c)$.

Recall that $W(c)$ is defined as the subspace of $H(c)$ generated by the sum of the $H(c')$ where $c' < c$. Let $U(c) = W(c) \cap H_{class,k}(c)$. Since $W(c)$ is invariant under conjugation, for any $v \in H_{class,k}(c)$ we have $P_{U(c)^\perp}(v) = P_{W(c)^\perp}(v)$. Therefore we can express $L^2(Gal(K)^k)^{Gal(K)}$ as the orthogonal sum of the finite dimensional spaces $P_{U(c)^\perp}(H_{class,k}(c))$. We obtain a decomposition of $\mathcal{H}_{class,k}$ similar to Lemma 1.5.10.

1.6 Unitary Representations

Throughout this chapter so far, we were interested in interpretable Hilbert spaces. These are defined at the level of the theory rather than individual models. In this section, we will look at the connection with representation theory, which requires fixing a group. In our context, this amounts to fixing a sufficiently homogeneous model.

In Section 1.6.1, we show that the notion of irreducibility for representations on interpretable Hilbert spaces does not depend on the choice of model, and in fact is quite local in nature, in the sense that it is witnessed by the representation of Aut(bdd(a)) *for a a finite tuple. See Propositions 1.6.7 and 1.6.9. In Section 1.6.2 we review the special case of ω-categorical structures and we recover the classification theorem of [Tsa12]. In Section 1.6.3 we adapt our decomposition theorem 1.3.14 to general unitary group representations with an orbit whose weak closure is locally compact and we use a theorem of Howe and Moore to uncover a new source of interpretable Hilbert spaces generated by asymptotically free types.*

1.6.1 Unitary representations of automorphism groups

In this section, we show that the theorems of Section 1.3 shed light on some of the representation theory of automorphism groups of theories with scattered interpretable Hilbert spaces. In this section, T is an arbitrary complete continuous logic theory with an interpretable Hilbert space $\mathcal{H}$.

Definition 1.6.1. *Let G be a group and H a Hilbert space, real or complex. A unitary representation σ of G on H is a group action $G \times H \to H$ such that for every $g \in G$ $\sigma(g)$ is a unitary map if H is a complex Hilbert space and $\sigma(g)$ is an orthogonal map if H is a real Hilbert space.*

When G is a topological group, we say that σ is continuous if $\sigma : G \times H \to H$ is continuous. This is equivalent to $\sigma(\cdot, v)$ being continuous for every $v \in V$.

Convention: In this thesis, we only consider continuous unitary representations of topological groups, so we just say 'representation' instead of 'continuous unitary representation'.

Definition 1.6.2. *Let G be a group and let σ, σ' be two representations of G on the Hilbert spaces H and H' respectively. σ and σ' are equivalent if there is a surjective isometry $U : H \to H'$ such that for all $g \in G$ and $v \in H$, $U(\sigma(g)v) = \sigma'(g)U(v)$. We say that U intertwines σ and σ'.*

Let $M \models T$ and write $G = \mathrm{Aut}(M)$. Then G is a topological group with the topology of pointwise convergence. G has a basis at the identity consisting of subsets of the form $\{g \in G \mid d(gA, A) < \epsilon\}$ where A ranges over finite subsets of M and $\epsilon > 0$. When T is a classical logic theory, this is a basis of subgroups. Note we can add imaginary sorts to M without changing the topology on G.

Suppose $\mathcal{H}$ is an interpretable Hilbert space in T. For any $M \models T$, $\mathrm{Aut}(M)$ has a *canonical unitary representation* π on $H(M)$ given by $\pi(g)hx = h(gx)$ where h is the direct limit map on a piece of $\mathcal{H}$. By our previous discussion, π is continuous. The following lemma is an easy definition chase.

Lemma 1.6.3. *If $\mathcal{H}$, $\mathcal{H}'$ are isomorphic interpretable Hilbert spaces in T, then for any $M \models T$, the representations of $\mathrm{Aut}(M)$ on $H(M)$ and $H'(M)$ are equivalent.*

We turn to a discussion of irreducibility for canonical representations.

Definition 1.6.4. *Let $\mathcal{H}_0$ be an $\bigwedge$-interpretable subspace of $\mathcal{H}$. We say that $\mathcal{H}_0$ is irreducible if there do not exist complete types q, q' in $\mathcal{H}_0$ such that $q(x) \cup q'(y)$ implies $\langle x, y \rangle = 0$.*

We will study irreducibility for $\bigwedge$-interpretable Hilbert spaces in relation with the notion of ω-near homogeneity, which occurs naturally in continuous logic:

Definition 1.6.5 ([BYBHU08] 8.7, 12.11)**.** *Let N be a model of T realising all types. If p, q are complete types of N, we define $d(p, q) = \inf\{d(a, b) \mid a \models p, b \models q, a, b \in N\}$.*

We say that $M \models T$ is ω-near-homogeneous if for any two finite tuples a and b in M, for every $\epsilon > 0$ there exists $g \in \mathrm{Aut}(M)$ such that $d(g(a), b) < d(\mathrm{tp}(a), \mathrm{tp}(b)) + \epsilon$.

Remark: Recall that we work in continuous logic for metric structures so that every sort of T comes equiped with a metric. ω-near-homogeneity is defined with respect to these metrics. As a result, ω-near-homogeneity for an arbitrary model M is *not* preserved under adding imaginary sorts to T. Therefore, we make the following assumption on $\mathcal{H}$:

> *We assume that the pieces of $\mathcal{H}$ are real sorts of T and that the direct limit maps on each piece of $\mathcal{H}$ are isometries.*

With this assumption, if $M \models T$ is ω-near-homogeneous, then the characterisation of ω-near-homogeneity applies to types in $\mathcal{H}$. We also observe that if M realises all types of T and is ω-near-homogeneous, then expansions of M by imaginary sorts remain ω-near-homogeneous. In particular, ω-near-homogeneity is a robust notion for models of ω-categorical theories.

Lemma 1.6.6. *Let $\mathcal{H}_0$ be a $\bigwedge$-interpretable subspace of $\mathcal{H}$.*

1. *if $\mathcal{H}_0$ is irreducible then for any ω-near-homogeneous $M \models T$ the canonical representation π of* $\mathrm{Aut}(M)$ *on $H_0(N)$ is irreducible.*
2. *if there is some $M \models T$ realising all types of T such that the canonical representation π of* $\mathrm{Aut}(M)$ *on $H_0(N)$ is irreducible, then $\mathcal{H}_0$ is irreducible*

Proof. (1) Let $M \models T$ be ω-near-homogeneous. Let $v, w \in H(M)$ be two nonzero vectors. Write q_1, q_2 for their respective types. By irreducibility of $\mathcal{H}_0$ and the assumption following Definition 1.6.5, $d(q_1, q_2)^2 < \|v\|^2 + \|w\|^2$. By ω-near-homogeneity, we can find $g \in \mathrm{Aut}(M)$ such that $\|gv - w\|$ is arbitrarily close to $d(q_1, q_2)$. Then gv and w are not orthogonal and the representation of $\mathrm{Aut}(M)$ on $H_0(M)$ is irreducible.

(2) Take $M \models T$ as in the statement. Let p, q be types of $\mathcal{H}_0$ and let v, w be realisations in $H_0(M)$. There is $g \in \mathrm{Aut}(M)$ such that $\langle gv, w\rangle \neq 0$ so $\mathcal{H}_0$ is irreducible. □

Proposition 1.6.7. *Let p be a complete type in $\mathcal{H}$. Take $M \models T$ ω-near-homogeneous and realising all types.*

Let $a \models p$ in $H(M)$, let K be the subgroup of $\mathrm{Aut}(M)$ *fixing* $\mathrm{bdd}(a)$ *setwise and let $A = H_p(M) \cap \mathrm{bdd}(a)$. If the canonical representation of K on A is irreducible, then $\mathcal{H}_p$ is irreducible.*

Proof. Let q, q' be two types in $\mathcal{H}_p$. Let v, w be realisations of q, q' in $H(M)$. Choosing $\mathrm{Aut}(M)$-conjugates of v and w if necessary, we can assume that $P_A v$ and $P_A w$ are nonzero. Conjugating v by an element of K, we can assume that $\langle P_A v, P_A w\rangle \neq 0$.

Consider the nonforking extension r of $\mathrm{tp}(v/A)$ to $\mathrm{bdd}(Aw)$ with respect to the inner product maps of $\mathcal{H}$. If z realises r in an ω_1-saturated elementary extension N of M, we have $\langle z, w\rangle = \langle P_A v, P_A w\rangle \neq 0$. Let $r' = \mathrm{tp}(z, w)$.

By our assumption on M, r' is realised in M by some pair (z', w'). By ω-near-homogeneity, we can assume that w' is arbitrarily close to w. Now $\mathrm{tp}(z') = \mathrm{tp}(v)$ so we can find $g \in \mathrm{Aut}(M)$ taking v arbitrarily close to z'. Now we have $\langle gv, w\rangle \approx \langle z', w'\rangle = \langle z, w\rangle \neq 0$. □

We will now show how asymptotically free types give rise to induced representations. We begin by recalling the notion of induced representation in the special case where we induce from an open subgroup. See [BdlHV08] for more details. Let G be a topological group and take K an open subgroup of G. Let σ be a representation of K on the Hilbert space V. We suppose that V is a real Hilbert space (the case of complex Hilbert spaces is similar). Write $\mathbb{R}G$ for the free vector space on G. We define $G \otimes_\sigma V$, the σ-tensor of G and V, to be the vector space $\mathbb{R}G \otimes V$ quotiented by a suitable subspace so that $gk \otimes_\sigma v = g \otimes \sigma(k)v$ for all $g \in G$ and $k \in K$.

We define an inner product on $G \otimes_\sigma V$ as follows. For any $g, g' \in G$ and $v, v' \in V$, if $gK \neq g'K$, then $\langle g \otimes_\sigma v, g' \otimes_\sigma v' \rangle = 0$. If $gK = g'K$, then find $k \in K$ such that $g' = gk$ and define $\langle g \otimes_\sigma v, g' \otimes_\sigma v' \rangle = \langle v, \sigma(k)v' \rangle$. Observe that if we choose a set of coset representatives for G/K, we can identify $G \otimes_\sigma V$ with the orthogonal sum of copies of V indexed by G/K.

We will always work with the Hilbert space completion of $G \otimes_\sigma V$. We also write $G \otimes_\sigma V$ for this completion. We define the induced representation of G from σ, denoted $\mathrm{Ind}_K^G(\sigma)$, as the unitary representation of G on $G \otimes_\sigma V$ given by $g \cdot (g' \otimes_\sigma v) = gg' \otimes_\sigma v$. Since K is open in G, the induced representation is continuous.

Proposition 1.6.8. *Suppose p is an asymptotically free complete type in $\mathcal{H}$. For $x, y \models p$, write $x \sim y$ if $\mathrm{bdd}(x) = \mathrm{bdd}(y)$ and write $[x]$ for the equivalence class of x under $\sim$.*

Let $M \models T$, write $G = \mathrm{Aut}(M)$ and suppose that for some (any) $a \models p$, the orbit of a under G is metrically dense in p. Fix $a \models p$ in $H(M)$ and write K for the open subgroup of elements of G which fix $[a]$ setwise. Then the canonical representation π of G on $H_p(M)$ is equivalent to $\mathrm{Ind}_K^G(\sigma)$ where σ is the restriction of π to K on the Hilbert space V spanned by $[a]$.

Proof. Since p is asymptotically free, p is metrically locally compact and hence there is $\epsilon > 0$ such that for any $x, y \models p$, if $d(x, y) < \epsilon$ then $[x] = [y]$. Therefore K is open in G. Let $A = [a]$ and write $\{A_i \mid i \in I\}$ for the orbit of A under G setwise (we ignore permutations of A). For every $i \in I$ pick $g_i \in \mathrm{Aut}(M)$ which maps A to A_i. Then $\{g_i \mid i \in I\}$ is a list of representatives for the left cosets of K in $\mathrm{Aut}(M)$. Since p is asymptotically free, the sets A_i are pairwise orthogonal in $H_p(M)$.

Write π for the canonical representation of $\mathrm{Aut}(M)$ on $H_p(M)$ and let σ be the restriction of π to K on V, the vector space spanned by A. Let $\iota = \mathrm{Ind}_K^G(\sigma)$ and write $W = G \otimes_\sigma V$. We show that ι and π are equivalent. Write also $g_i \otimes_K V$ for the subspace of W given by $\{g_i \otimes_\sigma v \mid v \in V\}$. Take $w \in W$ and write $w = \sum w_i$ where $w_i \in g_i \otimes_\sigma V$. Let $P_i : g_i \otimes_\sigma V \to V$ be the Hilbert space isomorphism taking $g_i \otimes_\sigma v$ to v. We define

$$U(w) = \sum \pi(g_i) P_i(w_i).$$

Since $A_i \perp A_j$ for $i \neq j$, we have $\pi(g_i)P_i(w_i) \perp \pi(g_j)P_j(w_j)$, so U is well-defined, and it is easy to check that U is in fact a surjective isometry. U

intertwines ι and π: take $g \in G, i \in I, v \in V$. Then, writing $gg_i = g_j k$, we have

$$U(\iota(g)(g_i \otimes_K v)) = U(g_j \otimes_K \sigma(k)v) = \pi(g_j)(\sigma(k)v) = \pi(g_j k)v = \pi(g)U(g_i \otimes_K v).$$

$\square$

Remark: Taking a and K as in Proposition 1.6.8, we note that $\mathrm{Aut}(M/[a])$ is a normal subgroup of K contained in the kernel of σ. Write G_1 for the group of automorphisms of the set $[a]$. Then σ factors through $Aut(M/[a])$ to a representation of a subgroup of G_1. Since $[a]$ is a separable locally compact metric space, the closure of $K/\mathrm{Aut}(M/[a])$ in G_1 is locally compact with respect to the topology of pointwise convergence. We say that the canonical representation of G is obtained from the representation of $K/\mathrm{Aut}(M/[a])$ by inflation.

We prove a version of Mackey's irreducibility criterion for the representations arising in interpretable Hilbert spaces generated by asymptotically free types. This strengthens Proposition 4.1 in [Tsa12].

Proposition 1.6.9. *Suppose p is an asymptotically free complete type in $\mathcal{H}$. For $x, y \models p$, write $x \sim y$ if $\mathrm{bdd}(x) = \mathrm{bdd}(y)$ and write $[x]$ for the equivalence class of x under $\sim$.*

Let $M \models T$ be ω-near-homogeneous and realise all types. Write $G = \mathrm{Aut}(M)$. Fix $a \models p$ in M and write K for the subgroup of elements of G which fix $[a]$ setwise. Let σ be the restriction of the canonical representation π of G to K on the Hilbert space V spanned by $[a]$. Then σ is irreducible if and only if $\mathcal{H}_p$ is irreducible.

Proof. By Proposition 1.6.8 and easy facts about induced representations, if $\sigma = \sigma_1 \oplus \sigma_2$, then $\pi = \mathrm{Ind}_K^G(\sigma_1) \oplus \mathrm{Ind}_K^G(\sigma_2)$, so $\mathcal{H}_p$ is reducible by Lemma 1.6.6.

Conversely, suppose that σ is irreducible. If we move to an ω_1-strongly homogeneous and ω_1-saturated elementary extension M' of M, then the representation of the subgroup of $\mathrm{Aut}(M')$ which fixes $[a]$ setwise is also irreducible on V. By Lemma 1.6.6, we can assume that M is ω_1-strongly homogeneous.

Recall that in Proposition 1.6.8 we expressed $H_p(M)$ as the orthogonal sum of subspaces $g_i \otimes_\sigma V$ where $(g_i)_{i \in I}$ is a set of coset representatives of G/K. Suppose 0 is an indexing element in I with $g_0 = e$ so that $V = g_0 \otimes_\sigma V$.

Suppose that we have a G-invariant subspace Z of $H_p(M)$ Then the orthogonal projection P_Z commutes with G. Fix a nonzero $v \in V$. We can write $P_Z v = \sum_{j \in J} u_j$ where $J \subseteq I$ is countable and $u_j \in g_j \otimes_\sigma V$ is nonzero. Write u_0 for the element of $\{u_j \mid j \in J\}$ which lies in V. Since the u_j are pairwise orthogonal, $u_0 = 0$ would imply that $P_Z v = 0$. Switching if necessary to $Z^\perp$, we can assume that $P_Z v \neq 0$.

Write $J_0 = \{j \in J \mid g_j a \notin \mathrm{bdd}(a)\}$ and $J_1 = \{j \in J \mid a \notin \mathrm{bdd}(g_j a)\}$. Then $J = \{0\} \cup J_0 \cup J_1$. We show that $J_0 = J_1 = \emptyset$. By ω_1-strong homogeneity and saturation of M, we can find a sequence (α_n) in $\mathrm{Aut}(M/\mathrm{bdd}(a))$ such for any $n \neq m$

$$\alpha_n\big\{[g_j a] \mid j \in J_0\big\} \cap \alpha_m\big\{[g_j a] \mid j \in J_0\big\} = \emptyset.$$

Hence for all $n \neq m$, the sets $\alpha_n\{u_j \mid j \in J_0\}$ and $\alpha_m\{u_j \mid j \in J_0\}$ are orthogonal. For all n and $j \in J \setminus J_0$, we also have $\alpha_n u_j = u_j$. Therefore $(\alpha_n P_Z(v))$ converges weakly to $\sum_{j \in J\setminus J_0} u_j$. Since Z is G-invariant, we have $\sum_{j\in J\setminus J_0} u_j \in Z$. Since $\sum_{j\in J\setminus J_0} u_j$ is at least as close to v as $\sum_{j\in J} u_j$, we conclude that $J_0 = \emptyset$.

Suppose for a contradiction that we have $j_1 \in J_1$. Let $J_2 = \{j \in J \mid g_j a \notin \mathrm{bdd}(g_{j_1} a)\}$. Note that $0 \in J_2$. By the same argument as above, we have $\sum_{j\in J\setminus J_2} u_j \in Z$. Therefore $\sum_{j\in J_2} u_j = \sum_{j\in J} u_j - \sum_{j \in J\setminus J_2} u_j \in Z$. Since $j_1 \notin J_2$, $\sum_{j\in J_2} u_j$ is an element of Z closer to v than $P_Z v$ and this is a contradiction. Hence $J_1 = \emptyset$ and $P_Z v \in V$.

Therefore, $Z \cap V$ is a nonempty G-invariant subspace of V. Since σ is irreducible, we have $Z \cap V = V$. By invariance of Z, we have $g_i \otimes_\sigma V \subseteq Z$ for all g_i and hence $Z = H_p(M)$. This proves that π is irreducible.

□

1.6.2 Unitary representations of automorphism groups of ω-categorical structures

In this section, we recall some results of [Tsa12] and [Iba21] about unitary representations of automorphisms groups of ω-categorical structures and we show that Corollary 1.3.22 combined with Proposition 1.6.8 recovers the classification theorem in [Tsa12].

Recall from [BYBHU08] 12.2 that a complete type p in a continuous logic theory T is *principal* if it is distance definable, as in Definition 1.2.13. Recall also the Ryll-Nardzewski theorem in continuous logic which says that T is ω-categorical if and only if every complete type is principal (see [BYBHU08] 12.10). Finally, [BYBHU08] 12.11 shows that if M is the separable model of an ω-categorical theory, then M is ω-near-homogeneous.

Finally, for a general continuous logic theory T, we define the expansion T^{princ} as in Definition 1.2.14 by adding p as a new sort to T, where p is any principal type of T.

The following result is Lemma 1.1 in [Iba21] rephrased in the language of interpretable Hilbert spaces.

Lemma 1.6.10. *Let T be an ω-categorical continuous logic theory and let M be an ω-near-homogeneous model of T. Let σ be a representation of* $\mathrm{Aut}(M)$ *on a Hilbert space H. Then there is an interpretable Hilbert space $\mathcal{H}$ in T^{princ} such that σ is equivalent to the canonical representation of* $\mathrm{Aut}(M)$ *on $H(M)$.*

If T is a classical logic theory, then the T^{princ} construction is not needed, since a principal type is a definable set D and we can define the interpretation map outside of D to be the trivial 0 map. Now the following lemma is a combination of Lemma 3.1 in [Tsa12] and Lemma 1.1 in [Iba21]:

Lemma 1.6.11. *Let T be a classical logic ω-categorical theory and let M be an ω-homogeneous model of T. Let σ be a unitary representation of $\mathrm{Aut}(M)$ on a Hilbert space H. Then there is a strictly interpretable Hilbert space $\mathcal{H}$ in T such that σ is equivalent to the canonical representation of $\mathrm{Aut}(M)$ on $H(M)$.*

Let T be a classical logic ω-categorical theory and let $M \models T$ be ω-homogeneous. Note that if p is a type in a classical imaginary sort of M, then the relation $x \in \mathrm{acl}(y)$ is symmetric and transitive on p. This is because $\mathrm{acl}(x) \cap p$ is a finite set with fixed cardinality. Applying Corollary 1.6.11, Corollary 1.3.22 and Proposition 1.6.8, we deduce directly that every unitary representation of $\mathrm{Aut}(M)$ is an orthogonal sum of representations obtained by inflation from representations of groups of partial automorphisms of finite sets of the form $\mathrm{acl}(a) \cap \mathrm{tp}(a)$, where a is a classical imaginary element of M. This is precisely the classification theorem 5.2 in [Tsa12].

It remains to be seen if it is possible to build on the techniques developed in this chapter in order to find a classification of the unitary representations of continuous logic ω-categorical structures. This is an open question for future research.

1.6.3 Unitary representations with asymptotically free orbits

In this section we show that our analysis in Section 1.3 gives information about all unitary representations containing a cyclic vector such that the weak closure of its orbit is locally compact. We also show that asymptotic freedom is a common property of unitary group representations.

First, we introduce a general technique for constructing continuous logic structures with prescribed automorphism groups.

Definition 1.6.12. *Let M be a continuous logic structure in a language $\mathcal{L}$ and let G be a subgroup of $\mathrm{Aut}(M)$. We define an expansion M_G of M in a language $\mathcal{L}_G$ as follows.*

For every $n \geq 1$, for every finite Cartesian product X of sorts of M and for every orbit O of G on X, we add function symbols $r_O : X \to [0, \infty)$. M_G is the structure obtained from M by interpreting each function r_O as the distance in X from the metric closure of O.

Note that M_G is the maximal G-invariant expansion of M. The following lemma is straightforward:

Lemma 1.6.13. *Let M be a continuous logic structure, G a subgroup of* $\mathrm{Aut}(M)$. *Then* $\mathrm{Aut}(M_G)$ *is the closure of G in* $\mathrm{Aut}(M)$ *with the topology of pointwise convergence.*

Now let G be an arbitrary group and let H be a Hilbert space with a cyclic faithful representation σ of G. Let v be a cyclic vector and let X be the closure of the orbit of v in H. Let M be the continuous logic structure consisting of X with the inner product map on X induced from H. Observe that the topology of pointwise convergence is the coarsest topology on G under which the representation σ is continuous. We will always work with this topology on G. The following lemma is clear:

Lemma 1.6.14. *Take G, σ, X, and M as above. Then M_G is an atomic model (i.e. all types that are realised are principal) and the action of G on M_G is ω-near-homogeneous.*

X is a complete type in M_G and there is an interpretable Hilbert space $\mathcal{H}$ in $Th(M_G)$ generated by X such that the restriction to G of the canonical representation of $\mathrm{Aut}(M_G)$ *on $H(M_G)$ is equivalent to σ.*

It is also clear that the structure M_G above remains ω-near-homogeneous under expansion by pieces of $\mathcal{H}$. Therefore, we add the piees of $\mathcal{H}$ as imaginary sorts to M_G so that the assumption following Definition 1.6.5 holds in this context.

We are interested in applying Theorem 1.3.14 to the structure M_G. In Section 1.3 we worked with an ω_1-saturated structure. This is an obstacle for a direct application of Theorem 1.3.14 to M_G, since it is not clear how to guarantee scatteredness in an ω_1-saturated elementary extension. Nevertheless, if the weak closure of X is locally compact, it is still possible to replicate much of the proof of Theorem 1.3.14, albeit deriving a weaker conclusion.

In the next lemma, we show that there is enough model theory present in the structure M_G to recover the key characterisation of $\mathcal{P}(X)$ from Lemma 1.3.2 without saturation assumptions. Lemma 1.6.15 only uses the fact that M_G is an atomic model of its theory.

Lemma 1.6.15. *Let G be a group and let σ be a faithful cyclic representation of G on H. Let v be a cyclic vector and let X be the metric closure of the orbit of v in H. We work in the associated structure M_G as described above.*

Take $u, w \in \mathcal{P}(X)$ and $A \subseteq M_G$ such that $w = P_{\mathrm{bdd}(A)}u$. Then there is a sequence (u_n) in $\mathrm{tp}(u)$ *in $H(M_G)$ such that (u_n) converges weakly to w.*

Proof. Take u, w, A such that $w = P_{\mathrm{bdd}(A)}(u)$ and write $p = \mathrm{tp}(u)$. Write S for the piece of $\mathcal{H}$ containing p. Since p is realised in M_G, p is principal. Let $d(x)$ be the definable function on S which gives the distance to p.

Suppose that we have found $u_1, \ldots, u_n$ in M_G satisfying p such that for all $m < n$, $|\langle u_n, u_m\rangle - \langle w, u_m\rangle| \leq 1/n$. Fix $\epsilon > 0$ small enough, to be determined below. By Lemma 1.3.2, w satisfies the formula

$$\forall x_1, \ldots, x_n \in S, \exists y \in S, \left(d(y) \leq \epsilon \wedge \bigwedge_{m \leq n} |\langle y, x_m \rangle - \langle w, x_m \rangle| \leq 1/2n \right)$$

Find a realisation y of this formula over the tuple $u_1, \ldots, u_n$ previously constructed. Then there is $u_{n+1} \models p$ in M_G with $\|u_{n+1} - y\| \leq \epsilon$. Choose $\epsilon > 0$ small enough so that $|\langle u_{n+1}, u_m \rangle - \langle w, u_m \rangle| \leq 1/(n+1)$ for all $m \leq n$. In this way, we construct a sequence (u_n) in p such that (u_n) converges weakly to w. $\square$

We deduce the following general proposition:

Proposition 1.6.16. *Let G be a group and let σ be a faithful cyclic unitary representation of G on H. Let v be a cyclic vector and let X be the metric closure of the orbit of v in H.*

Suppose that the weak closure of X in H is locally compact. Then σ is equivalent to an orthogonal sum of representations (τ_i) such that each τ_i has a cyclic vector v_i satisfying the following: if w is a conjugate of v_i such that $\langle w, v_i \rangle \neq 0$, then the orbit of w under the stabilizer of v_i is precompact (i.e. it has compact closure in the metric topology).

Proof. We work in the associated structure M_G, as above. Let $\mathcal{Q}(X) \subseteq \mathcal{P}(X)$ be the smallest metrically closed subset of $H(M_G)$ containing X and which is closed under the projections $P_{\mathrm{bdd}(A)}$, where $A \subseteq M_G$. Then $\mathcal{Q}(X)$ is locally compact. Note that $\mathcal{Q}(X)$ contains X but may be much smaller than $\mathcal{P}(X)$. We view $\mathcal{Q}(X)$ as a partial order with the order inherited from $\mathcal{P}(X)$.

By Lemma 1.6.15, it is straightforward to adapt the proof of Theorem 1.3.8 to deduce that for any bdd-closed subsets A_1, A_2 of M_G, we have $P_{A_1} P_{A_2} = P_{A_2} P_{A_1} = P_{A_1 \cap A_2}$ in $H(M_G)$. Similarly, we can adapt the proof of Lemma 1.3.10 to show that $\mathcal{Q}(X)$ is a well-founded partial order.

By ω-near-homogeneity, $\mathcal{Q}(X)$ is a union of complete types. Let (q_α) be an enumeration of the complete types in $\mathcal{Q}(X)$. By well-foundedness, we can assume that the enumeration (q_α) respects the partial order on $\mathcal{Q}(X)$.

Let V_α be the closed subspace generated by $\bigcup_{\beta < \alpha} q_\alpha$. As in Lemma 1.3.11, we find complete types $\tilde{q}_\alpha$ such that $\tilde{q}_\alpha = P_{V_\alpha^\perp} q_\alpha$ and the relation $P_{V_\alpha^\perp} x = y$ is definable on $q_\alpha \times \tilde{q}_\alpha$ (since all types are principal, the distances between the subspaces V_α do not change if we move to a saturated elementary extension of M_G).

$H(M_G)$ is the orthogonal sum of the subspaces generated by each $\tilde{q}_\alpha$. We check that the orbits corresponding to the types $\tilde{q}_\alpha$ satisfy the proposition. Fix α and $x, y \models \tilde{q}_\alpha$ such that the orbit of y under the stabilizer of x is not compact and suppose that $|\langle x, y \rangle| \geq \epsilon > 0$. Then $y \notin \mathrm{bdd}(x)$ and we can find $z \models q_\alpha$ such that $y = P_{V_\alpha^\perp} z$ and $z \notin \mathrm{bdd}(x)$. Let $u = P_{\mathrm{bdd}(x)} z \in V_\alpha$. By Lemma 1.6.15, we can find an infinite sequence (z_n) in q_α which converges weakly to u. The proof of Lemma 1.6.15 shows that we can also have $|\langle P_{V_\alpha^\perp} z_n, x\rangle| > \epsilon/2$ for all n. Since $(P_{V_\alpha^\perp} z_n)$ converges weakly to $P_{V_\alpha^\perp} u = 0$, this is a contradiction. $\square$

In the proof of Proposition 1.6.16, we found a type q in M_G such that for $x, y \models q$ in M_G, $x \in \mathrm{bdd}(y)$ or $\langle hx, hy\rangle = 0$. This does not imply that q is asymptotically free, since M_G is not saturated.

We can define asymptotic freedom in purely group theoretic terms: let σ be a representation of a group G on H and take $v \in H$. We say that the orbit of v is asymptotically free if for every $\alpha > 0$, the set of conjugates w of v such that $|\langle w, v\rangle| \geq \alpha$ is precompact.

We have already seen that representations with asymptotically free orbits capture all representations of oligomorphic groups, representations of automorphism groups of measurable structures on the associated L^2-spaces, and representations of absolute Galois groups on various associated L^2-spaces. In a different setting, if G is a locally compact group and σ is a representation of G with vanishing matrix coefficients, then every orbit in σ is asymptotically free. We use the classical result of [HM79] for the following proposition:

Proposition 1.6.17. *All continuous unitary irreducible representations of algebraic groups over a local field of characteristic* 0 *have asymptotically free orbits.*

Proof. Let G be a connected algebraic group over a local field of characteristic 0 with a representation σ on H. Let $P \leq G$ be the preimage under σ of the circle group in $U(H)$. Then [HM79] Theorem 6.1 shows that for any $v \in H$, the map $g \mapsto |\langle \sigma(g)v, v\rangle|$ tends to 0 on G/P. Since the action by P does not affect compactness, we deduce that the orbit of v is asymptotically free.

If G is not connected, we find a connected normal algebraic subgroup G_0 such that G/G_0 is finite. Then σ splits as a finite orthogonal sum of irreducible representations of G_0. For any $v \in H$, the G-orbit of v is a finite union of G_0 orbits in the irreducible subrepresentations and hence the G-orbit of v is asymptotically free. $\square$

See [BM00] for an overview of the Howe-Moore result and its extension to various additional cases.

1.7 Model Theory of Scattered Interpretable Hilbert Spaces

In this section, we discuss Theorems 1.3.8 and 1.3.14 from a model theoretic point of view. We establish a rough dictionary between Hilbert space and representation theoretic notions, and purely model theoretic notions. We connect *weak closure* in Hilbert spaces and *canonical bases* for Hilbert space types;

scattered type-definable sets and a local continuous logic version of *U-rank*; *commuting orthogonal projections* and *one-basedness*; and *asymptotically free type-definable sets* and *strongly minimal sets*. These connections are strong but they are not exact correspondences. We explore the relations between these different notions and highlight some open questions.

1.7.1 Weak closure and canonical bases

We fix a continuous logic T and $\mathcal{H}$ interpretable in T. In this section we freely move to imaginary sorts of T, so for any $M \models T$ we identify vectors in $H(M)$ with elements of M.

Let $p(x)$ a type-definable set in $\mathcal{H}$. For $M \models T$, we will study $\langle x, y\rangle$*-types over M consistent with p*. These are partial types q of the form $\{\langle x, b\rangle = \lambda(b) \mid b \in H(M), \lambda(b) \in \mathbb{R}\}$. Note that q is uniquely determined by the formulas $\langle x, b\rangle = \lambda(b)$ where b ranges in the piece of $H(M)$ containing p, so $q(x)$ is determined by the single function $\langle x, y\rangle$ where y ranges in the same sort as x.

Let $b \in \mathcal{P}(p)$ and let (a_n) be a sequence in p converging weakly to b. Define $p^b(x)$ to be the $\langle x, y\rangle$-type over $H(M)$ consistent with p defined by $\langle x, c\rangle = \lim_n \langle a_n, c\rangle = \langle b, c\rangle$.

Lemma 1.7.1. *For any $M \models T$ ω_1-saturated, $\mathcal{P}(p)$ is a set of canonical bases for $\langle x, y\rangle$-types over M consistent with p. In fact, b is a canonical base for p^b.*

Proof. $p^b(x)$ is a $\langle x, y\rangle$-type over $H(M)$ consistent with p. Conversely, let q be any $\langle x, y\rangle$-type over $H(M)$ consistent with p. Let $M \prec N$ and let a be a realisation of $q(x) \cup p(x)$ in N. Let $c = P_{H(M)}(a)$. Then for all $v \in H(M)$, $\langle a, v\rangle = \langle c, v\rangle$, so q is definable over c. We only need to check that $c \in \mathcal{P}(p)$, so that $q = p^c$.

Let $a' \in M$ be a realisation of $p \cup (q \upharpoonright \mathrm{bdd}(c))$. Then $P_{\mathrm{bdd}(c)}a' = c$ but we already know that $P_{\mathrm{bdd}(c)}a' \in \mathcal{P}(p)$, so we are done. □

Remark: We are making a slightly unconventional use of the term 'canonical base'. Canonical bases are usually dcl-closed sets, and nothing in Lemma 1.7.1 ensures that distinct $b, b' \in \mathcal{P}(p)$ are not inter-definable. Nevertheless, we will say that b is 'the' canonical base of p^b. With this choice of terminology, for any $b \neq b' \in \mathcal{P}(p)$, b and b' are canonical bases for different types over M even though they may be inter-definable.

The next lemma follows directly by applying Lemma 1.7.1 with the type-definable set $\mathcal{P}(p)$ instead of p. It gives a model theoretic justification for working with the set $\mathcal{P}(p)$.

Lemma 1.7.2. *$\mathcal{P}(p)$ has built-in canonical bases for $\langle x, y\rangle$-types: for any $M \models T$, for any $\langle x, y\rangle$-type $q(x)$ over M consistent with $\mathcal{P}(p)$, there is a unique*

$b \in \mathcal{P}(p)$ in M such that $q(x)$ is defined over b by $\langle x, v\rangle = \langle b, v\rangle$ for every $v \in H(M)$. Therefore $q = p^b$.

We define a partial order which gives a model-theoretic characterisation of the partial order $\leq_{\mathcal{P}}$ on $\mathcal{P}(p)$.

Definition 1.7.3. *Define the relation $L_1(x, y)$ on $\mathcal{P}(p) \times \mathcal{P}(p)$ in M by saying that $L_1(b, c)$ if and only if p^b extends $p^c \restriction \mathrm{bdd}(c)$. Define the relation $\leq_1$ on $\mathcal{P}(p) \times \mathcal{P}(p)$ as the transitive closure of L_1.*

For any $c \in \mathcal{P}(p)$, since $\langle x, y\rangle$ is a stable relation, p^c is the unique nonforking extension of $p^c \restriction \mathrm{bdd}(c)$ to M. Therefore, $L_1(b, c)$ and $b \neq c$ if and only if p^b is a forking extension of $p^c \restriction \mathrm{bdd}(c)$.

Lemma 1.7.4. *For any $b, c \in \mathcal{P}(p)$, $L_1(b, c)$ if and only if there exists a small* bdd*-closed $A \subseteq M$ such that $c = P_A b$. Hence the partial order $\leq_1$ is anti-isomorphic to $\leq_{\mathcal{P}}$.*

Proof. For any small bdd-closed A, $c = P_A b$ if and only if $c = P_{\mathrm{bdd}(c)} b$ if and only if p^b extends $p^c \restriction \mathrm{bdd}(c)$. $\square$

Next we show that the partial order $\leq_{\mathcal{P}}$ can be seen as a natural partial order between $\langle x, y\rangle$-types themselves.

Definition 1.7.5. *Define the relation L_2 on $\langle x, y\rangle$-types over* bdd*-closed subsets of M consistent with $\mathcal{P}(p)$ as follows: if r_1 is over B and r_2 is over C, then $L_2(r_1, r_2)$ if and only if $(r_1 \restriction M) \restriction C = r_2$, where $r_1 \restriction M$ is the unique nonforking extension of r_1 to M. Define the relation $\leq_2$ between types as the transitive closure of L_2.*

For the next lemma, recall that types r_1, r_2 over B, C respectively are said to be *parallel* if the nonforking extensions $r_1 \restriction M$ and $r_2 \restriction M$ are equal.

Lemma 1.7.6. *Let B, C be* bdd*-closed subsets of M. Let r_1, r_2 be $\langle x, y\rangle$-types over B, C respectively consistent with p with canonical parameters b, c in $\mathcal{P}(p)$. Then $L_2(r_1, r_2)$ if and only if $L_1(b, c)$.*

Hence $\leq_2$ induces a partial order on parallelism classes of $\langle x, y\rangle$-types over small bdd*-closed subsets of M consistent with p. This partial order is anti-isomorphic to $\leq_{\mathcal{P}}$.*

Proof. If $A \subseteq M$ is a bdd-closed subset and r is a $\langle x, y\rangle$-type over A consistent with p, then r has a canonical base $b \in A$ and Lemma 1.7.2 shows that $r = p^b \restriction A$.

Therefore $L_2(r_1, r_2)$ if and only if $p^b \restriction C = r_2$ if and only if for all $v \in C$, $\langle b, v\rangle = \langle c, v\rangle$ if and only if $P_C b = c$. By Lemma 1.7.4, this is equivalent to $L_1(b, c)$.

r_1, r_2 are parallel if and only if $r_1 \restriction M = r_2 \restriction M$ if and only if $p^b = p^c$ if and only if $b = c$, by Lemma 1.7.2. Hence $\leq_2$ induces a partial order on parallelism classes and these parallelism classes are in an obvious bijection with $\mathcal{P}(p)$. $\square$

1.7.2 Ranks in scattered Hilbert spaces

We show that the conclusion of Theorem 1.3.8 is equivalent to a local form of one-basedness.

Definition 1.7.7. *For small* bdd*-closed* $A \subseteq H(M)$*, a* $\langle x, y\rangle$*-type* $q(x)$ *over* A *is one-based if for any realisation* a *of* q *in* M*,* q *is definable over* $A \cap \mathrm{bdd}(a)$*.*

Lemma 1.7.8. $\langle x, y\rangle$*-types over* bdd*-closed subsets of* M *consistent with* p *are one-based if and only for all* bdd*-closed* $A, B \subseteq M$*,* $P_A P_B = P_{A\cap B}$*.*

Proof. By Lemma 1.7.2, we can replace p by $\mathcal{P}(p)$ in the statement of the lemma.

Suppose first that $\langle x, y\rangle$-types consistent with p are one-based and take $B, C \subseteq M$ bdd-closed. Let $a \models p$ and let $b \in \mathcal{P}(p)$ be the canonical base of the $\langle x, y\rangle$-type of a over B. Then $P_C P_B a = P_C b$, which is the canonical parameter of the $\langle x, y\rangle$-type of b over C. By one-basedness $P_C b \in C \cap \mathrm{bdd}(b) \subseteq B \cap C$. Therefore $P_C P_B a \in B \cap C$. By the same computation as in the proof of Theorem 1.3.8, we have $P_B a \in B \cap C$ and hence $P_C P_B a = P_{B\cap C}$ on $H_p(M)$.

Conversely, let q be a $\langle x, y\rangle$-type over a bdd-closed $A \subseteq M$ consistent with p. Let a be a realisation of $q \cup p$ in M. Let $b = P_A a$ be the canonical parameter of q. Then $b = P_A P_{\mathrm{bdd}(a)} a \in A \cap \mathrm{bdd}(a)$. □

In Section 1.3, one-basedness is the key technical observation which makes it possible to prove that if p is scattered, then $(\mathcal{P}(p), \leq_{\mathcal{P}})$ is well-founded. We now turn to a general discussion of $\mathcal{P}(p)$ as a partial order.

Definition 1.7.9. *If* $(\mathcal{Q}, \leq_{\mathcal{Q}})$ *is an arbitrary partial order, we define the foundation rank* $F_{\mathcal{Q}}(x)$ *of* $x \in \mathcal{Q}$ *as follows:*

1. $F_{\mathcal{Q}}(x) \geq 0$ *for all* x
2. $F_{\mathcal{Q}}(x) \geq \lambda$ *for limit ordinal* λ *if* $F(x) \geq \alpha$ *for all* $\alpha < \lambda$
3. $F_{\mathcal{Q}}(x) \geq \alpha + 1$ *if there is* $y <_{\mathcal{Q}} x$ *such that* $F_{\mathcal{Q}}(y) \geq \alpha$

We say that $F_{\mathcal{Q}}(x) = \infty$ *if* $F_{\mathcal{Q}}(x) \geq \alpha$ *for every ordinal* α *and* $F_{\mathcal{Q}}(x) = \alpha$ *if* $F_{\mathcal{Q}}(x) \geq \alpha$ *and* $F_{\mathcal{Q}}(x) \not\geq \alpha + 1$*.*

We write $F_{\mathcal{Q}}(\mathcal{Q}) = \sup\{F_{\mathcal{Q}}(x) \mid x \in \mathcal{Q}\}$*.*

Proposition 1.7.10. *Let* $F_{\mathcal{P}}$ *be the foundation rank of* $(\mathcal{P}(p), \leq_{\mathcal{P}})$*. If for every* $x \models p$ *we have* $F_{\mathcal{P}}(hx) < \infty$*, then* $\langle x, y\rangle$*-types consistent with* p *are one-based.*

Proof. This is similar to the argument behind Theorem 1.3.8. For any bdd-closed $A, B \subseteq M$, for any $a \models p$, the sequence of alternating projections $P_A P_B \dots P_A P_B a$ must eventually be constant, since $F_{\mathcal{P}}$ is ordinal-valued. The argument of Theorem 1.3.8 shows that we must then have $P_A P_B = P_{A\cap B}$ so $\langle x, y\rangle$-types are one-based. □

In light of the anti-isomorphism between $\leq_{\mathcal{P}}$ and the partial orders $\leq_1$ and $\leq_2$, it is natural to define a local continuous logic version of U-rank which captures forking inside the Hilbert space. We call this the V-rank:

Definition 1.7.11. *For any* bdd*-closed subset A of M and q a $\langle x, y\rangle$-type over A in $\mathcal{H}$, define the relation $V(q) \geq \alpha$ for α an ordinal as follows:*

1. *$V(q) \geq 0$ for all q*
2. *$V(q) \geq \lambda$ for limit ordinal λ if for every $\alpha < \lambda$, $V(q) \geq \alpha$*
3. *$V(q) \geq \alpha + 1$ if there is some $B \supseteq A$ and q' over B extending q such that $V(q') \geq \alpha$ and q' forks over A with respect to the function $\langle x, y\rangle$.*

If $V(q) \geq \alpha$ for all α, we say $V(q) = \infty$. We say $V(q) = \alpha$ if $V(q) \geq \alpha$ and $V(q) \not\geq \alpha + 1$.

For $a \in H(M)$, write $V(a/A)$ for the V-rank of the $\langle x, y\rangle$-type of a over A.

Lemma 1.7.12. *Let A be a small* bdd*-closed subspace of $H(M)$ and let q be a $\langle x, y\rangle$-type over A consistent with p. Let b be the canonical parameter of q in $\mathcal{P}(p)$. Then $V(q)$ is equal to the foundation rank of b in $(\mathcal{P}(p), \leq_1)$.*

Proof. Note that $b \in A$. We have already pointed out that for any $c \in \mathcal{P}(p)$, $L_1(b, c)$ and $b \neq c$ if and only if p^b is a forking extension of $p^c \restriction \mathrm{bdd}(c)$. The lemma follows easily. $\square$

When p is a piece of $\mathcal{H}$ and the inner product is strictly definable, the V-rank coincides with the Shelah ω-local rank defined in [She78]. We recall the definition here:

Definition 1.7.13 ([She78], II.1.1)**.** *In an arbitrary theory T, let $\Delta(x, y)$ be a set of formulas and $p(x)$ a type-definable set, possibly with parameters. For α an ordinal, we define $R_\Delta(p) \geq \alpha$ as follows:*

1. *$R_\Delta(p) \geq 0$ if p is consistent*
2. *For λ a limit ordinal $R_\Delta(p) \geq \lambda$ if $R_\Delta(p) \geq \alpha$ for all $\beta < \alpha$*
3. *$R_\Delta(p) \geq \alpha + 1$ if for every finite $p' \subseteq p$ and every $n < \omega$ there are Δ-types $(q_m(x))_{m<n}$ such that:*
   (a) *for $m \neq m' < n$, $q_m(x) \cup q_{m'}(x)$ is inconsistent*
   (b) *$R_\Delta(p' \cup q_m) \geq \alpha$ for all m.*

We write $R_\Delta(p) = \alpha$ if $R_\Delta(p) \geq \alpha$ and $R_\Delta(p) \not\geq \alpha + 1$.

Proposition 1.7.14. *Suppose S is a piece of $\mathcal{H}$ such that the inner product takes only finitely many values on S. Let Δ be the finite set of formulas $\langle x, y\rangle = \lambda$ where x, y range in S. Then for every $a \in \mathcal{P}(S)$ and $A \subseteq M$* bdd-*closed,* $V(a/A) = R_\Delta(a/A) < \omega$.

Proof. This is proved entirely by using results of [She78]. We sketch the proof here for convenience. Let q be a $\langle x, y\rangle$-type over A consistent with p. Define the rank $R^*_\Delta(p)$ in the same way as in Definition 1.7.13 except that we say $R^*_\Delta(q) \geq \alpha + 1$ if there are Δ-types $(q_i)_{i<\omega}$ which are pairwise inconsistent and $R^*_\Delta(q \cup q_i) \geq \alpha$ for all i.

Now all references are from [She78]. In general for Δ a finite set of stable formulas, we can assume that Δ is a single stable formula by II.2.1. By II.2.7, $R_\Delta(q) < \omega$. We see from definitions that $R^*_\Delta(q) \leq R_\Delta(q)$. By II.2.9, the property $R_\Delta(q) \geq n$ is type-definable. It then follows by a compactness argument that $R^*_\Delta(q) \geq R_\Delta(q)$. Therefore $R_\Delta(q) = R^*_\Delta(q)$.

Now it follows directly from the definition of R^*_Δ that if $R_\Delta(q) = n+1$, there is a Δ-type q' over some $B \supseteq A$ such that $R_\Delta(p \cup q) = n$. Furthermore, II.1.2 and III.4.1 show that q' does not fork over A if and only if $R_\Delta(q') = R_\Delta(q)$. The proposition follows. □

We are interested in understanding the relations between scatteredness, one-basedness, the $F_\mathcal{P}$-rank, and the V-rank. We say in Lemma 1.3.10 that if p is scattered, then $(\mathcal{P}(p), \leq_\mathcal{P})$ is well-founded so the $F_\mathcal{P}$-rank is ordinal valued. It is an open question whether the $F_\mathcal{P}$-rank can achieve infinite values when p is scattered. We record some partial results which explore the connections between these different notions.

Proposition 1.7.15. *If p is scattered then for every $x \in \mathcal{P}(p)$, $V(x) < \infty$. Equivalently, if p is scattered, there is no infinite $\leq_\mathcal{P}$-increasing sequence in $\mathcal{P}(p)$.*

Proof. Suppose that p is scattered and that (a_n) is an infinite increasing sequence in $\mathcal{P}(p)$. Then $a_n = P_{\mathrm{bdd}(a_n)} a_{n+1}$ for all n and by one-basedness, $\mathrm{bdd}(a_n) \subseteq \mathrm{bdd}(a_{n+1})$ for all n. Write V_n for the subspace $\mathrm{bdd}(a_n)$ and W_n for the orthogonal complement of V_n in V_{n+1}. Then for all n, $a_{n+1} = a_n + w_n$ where $w_n \in W_n$ is orthogonal to a_n. Hence $a_n = a_0 + \sum_{i=0}^{n-1} w_i$.

Since every a_n is in the type-definable set $\mathcal{P}(p)$, $\|a_n\|$ is bounded above and the sequence $(\|a_n\|)$ is convergent. Now $\|a_n\|^2 = \|a_0\|^2 + \sum_{k=0}^{n-1} \|w_k\|^2$ since the vectors (w_k) and a_0 are pairwise orthogonal. For $n \geq m$,

$$\|a_n - a_m\|^2 = \| \sum_{k=m}^{n-1} w_n \|^2 = \sum_{k=m}^{n-1} \|w_n\|^2 = \|a_n\|^2 - \|a_m\|^2$$

so (a_n) is Cauchy and hence convergent to some $a \in \mathcal{P}(p)$ such that $a_n \leq_\mathcal{P} a$ for all n. Let $\epsilon > 0$ and take n such that $\|a - a_n\| < \epsilon$. There is an indiscernible sequence (b_k) in $\mathrm{tp}(a/\mathrm{bdd}(a_n))$ starting at a and converging weakly to a_n. For all k, $\|b_k - a\|^2 = 2\|a\|^2 - 2\langle b_k, a\rangle = 2\|a\|^2 - 2\langle a_n, a\rangle$. Choosing ϵ small enough, we find that (b_k) is arbitrarily close to a and hence $\mathcal{P}(p)$ is not scattered, a contradiction. $\square$

Proposition 1.7.16. *If p is a complete type and is not locally compact, then $F_{\mathcal{P}}(a) \geq \omega$ for any $a \models p$.*

Proof. Take $a \models p$ in M. Note that $a \neq 0$ and $\{b \in \mathcal{P}(p) \mid b < a\}$ is nonempty. Suppose that we have $b_0 <_{\mathcal{P}} \dots <_{\mathcal{P}} b_n <_{\mathcal{P}} a$ in $\mathcal{P}(p)$. Suppose that $\|a - b_n\| = \epsilon$. Since p is not locally compact around a, we can find an infinite indiscernible sequence (a_k) with $a_0 = a$ such that $\|a_k - a_j\| = \delta < \epsilon$ for all $k \neq j$, for arbitrarily small δ. Write c for the weak limit of (a_k). Then $\|a - c\| < \|a - b_n\|$ so $c \neq b_n$. Since $b_n = P_{\mathrm{bdd}(b_n)}a$ and $c = P_{\mathrm{bdd}(c)}a$, we see that $c \notin \mathrm{bdd}(b_n)$. Moreover, $\|P_{\mathrm{bdd}(b_n)}c - b_n\| \leq \|c - a\| = \delta$, so we can assume that $P_{\mathrm{bdd}(b_n)}c \notin \mathrm{bdd}(b_{n-1})$ by taking δ small enough.

Taking into account all distances $\|b_{n+1} - b_n\|$, choosing δ small enough and writing $c_{n+1} = c$, $c_i = P_{\mathrm{bdd}(b_i)}c_{i+1}$, we have $c_0 <_{\mathcal{P}} \dots <_{\mathcal{P}} c_{n+1} <_{\mathcal{P}} a$. This proves that $F_{\mathcal{P}}(a) \geq \omega$. $\square$

Proposition 1.7.17. *There are examples of theories T with interpretable Hilbert spaces $\mathcal{H}$ and type-definable sets p satisfying any of the following:*

1. *$(\mathcal{P}(p), \leq_{\mathcal{P}})$ has foundation rank ω but $(\mathcal{P}(p), \leq_1)$ is not well-founded*
2. *$(\mathcal{P}(p), \leq_1)$ has foundation rank ω, $(\mathcal{P}(p), \leq_{\mathcal{P}})$ is not well-founded, every type in $\mathcal{P}(p)$ is locally compact and $\langle x, y\rangle$-types consistent with p are one-based*
3. *$\langle x, y\rangle$-types consistent with p are one-based but $(\mathcal{P}(p), \leq_{\mathcal{P}})$ and $(\mathcal{P}(p), \leq_1)$ are not well-founded.*

Proof. (1) Let T be the classical logic theory of a collection of equivalence relations E_n on a set X such that $E_0(x, y)$ is the trivial equivalence relation $x = x$ and E_{n+1} refines each E_n-class into infinitely many infinite E_{n+1}-classes. Let $M \models T$ and for every $n \geq 0$ let $(\alpha_k^n)_{k<\kappa}$ be an enumeration of the E_n-classes of M. Let H be the free Hilbert space on the set $\bigcup_{k,n\geq 0} \alpha_k^n$, so that $\{\alpha_k^n \mid k, n \geq 0\}$ is an orthonormal basis of H.

Define $h : X \to H$ by $h(x) = \sum_{n\geq 0} \alpha_{k(x)}^n / 2^n$ where $\alpha_{k(x)}^n$ is the E_n-class of x. Then h gives rise to an interpretation of H in M. Let p be the quotient of X by the equivalence relation $h(x) = h(y)$. We view p as a subset of h. Then $\mathcal{P}(p) = p \cup \{\sum_{n\leq m} \alpha_{k(x)}^n / 2^n \mid x \in X, m \geq 0\}$.

(2) Let T be the classical logic theory of a collection of equivalence relations E_n on a set X such that $E_0(x, y)$ is the trivial equivalence relation $x = y$ and

each E_{n+1}-class is an infinite union of E_n-classes. Let $M \models T$ and for every $n \geq 0$ let $(\alpha_k^n)_{k<\kappa}$ be an enumeration of the E_n-classes of M. Let H be the free Hilbert space on the set $\bigcup_{k,n\geq 0} \alpha_k^n$.

Define $h : X \to H$ by $h(x) = \sum_{n\geq 0} \alpha_{k(x)}^n / 2^n$ where $\alpha_{k(x)}^n$ is the E_n-class of x. Note that h is injective and that the set $h(X)$ is discrete in H, since $x \neq y$ implies $\|x - y\| \geq 1$. We also have $\mathcal{P}(X) = \{0\} \cup \{\sum_{n\geq m} \alpha_{k(x)}^n / 2^n \mid x \in X, m \geq 0\}$. $\mathcal{P}(X)$ is locally compact everywhere except around 0.

(3) Let T be the classical logic theory of an infinite set X. Let $\mathcal{H}$ be the interpretable Hilbert space such that X is an orthonormal basis of $\mathcal{H}$. Let $M \models T$ be ω_1-saturated as a continuous logic theory. Let (e_n) be an infinite sequence in X and let p be the complete type of the vector $\sum_{n\geq 0} e_n/2^n$. Then $\mathcal{P}(p) = \{\sum_{n\in J} e_n/2^n \mid J \subseteq \mathbb{N}\}$ but $\mathcal{H}$ is one-based because X is scattered. □

1.7.3 Asymptotically free types are strongly minimal

We now show how asymptotically free types give strongly minimal local reducts of T.

Definition 1.7.18. *We say that a continuous logic theory T with a sort X is strongly minimal if for every $M \models T$, every M-definable function $f(x)$ in one variable into $\mathbb{R}$ has a unique generic value α on X, meaning that for any $\beta \neq \alpha$, the set $\{f(x) = \beta\}$ is compact.*

The above definition agrees with [Han20].

Lemma 1.7.19. *Suppose T is strongly minimal. Then for any $M \models T$,* bdd *is a pregeometry.*

Proof. We check the exchange property, i.e. for $A \subseteq M$ and $a, b \in M$, if $a \in \mathrm{bdd}(Ab)$ and $a \notin \mathrm{bdd}(A)$, then $b \in \mathrm{bdd}(Aa)$.

Take a, b as above. Recall that a complete type q over Ab is uniquely determined by the values $f(q)$ where f is any Ab-definable function into $\mathbb{R}$. By strong minimality, there is an Ab-definable function f with generic value α such that the value $f(a)$ is not generic. By a standard approximation argument, we can find an A-definable function $g(x, y)$ such that $g(a, b)$ is not the generic value of $g(x, b)$.

Observe that there is some α_y such that for any $c \notin \mathrm{bdd}(A)$, α_y is the generic value of $g(c, y)$. There is a corresponding value α_x for $g(x, c)$. Taking c, d bdd-independent over A, we have $g(c, d) = \alpha_x = \alpha_y$. Therefore $g(a, b)$ is not the generic value of $g(a, y)$ and $b \in \mathrm{bdd}(Aa)$.

□

Definition 1.7.20. *Let T be strongly minimal. We say that T is disintegrated if the pregeometry is trivial on T, in the sense that for any $M \models T$ and $A, B \subseteq M$,* $\mathrm{bdd}(A \cup B) = \mathrm{bdd}(A) \cup \mathrm{bdd}(B)$

Now let T be a continuous logic theory with an interpretable Hilbert space $\mathcal{H}$. Let p be a complete asymptotically free type of $\mathcal{H}$. We add the inner product map as a function symbol to the language. Let $M \models T$ be ω_1-saturated and let T_p be the theory of the set of realisations of p in M in the language with the inner product as its only function symbol.

Proposition 1.7.21. *T_p is a strongly minimal disintegrated continuous logic theory.*

Proof. Write X for the set of realisations of p in M viewed as a model of T_p. We consider the equivalence relation $\sim$ on X defined to be the transitive closure of the relation $\langle x, y\rangle \neq 0$. For $x \in X$, write $[x]$ for the $\sim$-class of x. Since X is a complete type in M, any two $\sim$-classes are isomorphic and the isomorphism type of these classes does not depend on the choice of model M.

Let $(x_i)_{i\in I}$ be a set of representatives of $\sim$-classes in X. Then any bijection $I \to I$ can be extended to an X-automorphism which respects the inner product. Hence in the theory T_p, for $A \subseteq X$, $\mathrm{bdd}(A) = \bigcup_{a\in A}[a]$ and any two $x, y \notin \mathrm{bdd}(A)$ are conjugate by an X-automorphism. It follows directly that T_p is strongly minimal and disintegrated. $\square$

1.8 Appendix to Chapter 1: Continuous logic

In *any* version of continuous logic, one has the notion of a *type-definable set*. The collection of type-definable sets is closed under positive Boolean combinations and under projections. As in the discrete logic case, the projection $(\exists x)p(x, y)$ of a partial type $p(x, y)$ is the partial type $q(y)$ such that, *in a sufficiently saturated model*, $q(b)$ holds iff there exists a with $p(a, b)$. We can thus freely write formulas or sentences involving positive first order operations. These describe partial types in any formulation of continuous logic, regardless of a specific calculus.

With this in mind, we give a presentation of continuous logic based on the approach of [HI02]. A similar approach was recently used in [GP21]. This approach uses the syntax of classical logic and uses classical results of model theory to deduce corresponding results in continuous logic. This formalism is completely equivalent to the approach of [BYBHU08] and this chapter was written in a way to make it easy to translate any argument into the formalism of [BYBHU08] if one wishes to do so.

In 1.8.1, we define syntax and type-spaces and we discuss briefly the relation to [HI02]. In 1.8.2, we recall some model theoretic notions which are used in this thesis. Section 1.8.2 is mainly based on [BYBHU08]. In 1.8.3, we recall some classical facts about the model theory of Hilbert spaces.

1.8.1 Continuous logic and type-spaces

In this section we introduce the basic concepts of continuous logic. Since we take classical discrete logic as our starting point, we will use the qualifiers 'classical logic...' and 'continuous logic...' to highlight the differences between the two logics. We do not use this terminology anywhere else.

In continuous logic, we will work with positive formulas (and their negations):

Definition 1.8.1. *Let $\mathcal{L}$ be an arbitrary language for classical first-order logic. We say that an $\mathcal{L}$-formula $\phi(x)$ is positive if $\phi(x)$ is logically equivalent to a formula which uses only the logical connectives $\wedge$ and $\vee$ and the usual quantifiers $\forall$ and $\exists$.*

Note that if ϕ and ψ are positive $\mathcal{L}$-formulas, then $(\neg\phi \to \psi)$ is positive. In general, positive formulas can have very weak expressive power but we will always work in certain languages and theories where they have strong expressive power. The restrictions we impose on $\mathcal{L}$ are the following.

We always fix a multi-sorted language $\mathcal{L}$ with sorts $(S_i)_{i\in I}$ and $(I_n)_{n\geq 1}$. We refer to the sorts (S_i) as the *metric sorts* and to the sorts I_n as the *value sorts*. $\mathcal{L}$ has an equality relation on every value sort but not on any metric sort. Each sort I_n will be identified with the interval $[-n, n]$, so we add functions $i_{nm} : I_n \to I_m$ for $n \leq m$ which will play the role of inclusion functions. The value sorts are also equipped with the following structure of the real numbers: functions $+, -, \times, \max$ between the appropriate value sorts and predicates $=$ and $\leq$ (note that we choose $\leq$ and not $<$, for reasons which become clear below). In each value sort I_n we also add a constant symbol for each rational number in $[-n, n]$. Each metric sort is equipped with a function $d_i : S_i \times S_i \to I_n$ for some n. $\mathcal{L}$ may contain more function symbols but no other relation symbols.

Whenever we fix $\mathcal{L}$, we also fix a *minimal $\mathcal{L}$-theory $T_{\mathcal{L}}$*. This is a collection of $\mathcal{L}$-sentences $T_{\mathcal{L}} = T_0 \cup T_{\mathbb{R}}$ satisfying the following conditions:

1. $T_{\mathbb{R}}$ says that the value sorts satisfy the full first-order theory of the real numbers (where I_n is identified with $[-n, n]$)

2. T_0 says that every d_i is a pseudometric on S_i and that d_i is bounded by some rational c_i.

3. T_0 says that every function symbol f in $\mathcal{L}$ is a uniformly continuous function from a finite product of metric sorts to a metric sort in the following way: for every rational $\epsilon > 0$ there is a rational $\delta > 0$ such that $T_{\mathcal{L}}$ contains the positive sentence

$$\forall x, y(d(x, y) < \delta \to d_i(f(x), f(y)) \leq \epsilon)$$

where x, y are finite tuples of variables appropriate for f and d is the max-metric on the sorts corresponding to the tuple x and d_i is the metric on the sort of $f(x)$.

When working with $\mathcal{L}$, we only ever consider models of $T_{\mathcal{L}}$.

Definition 1.8.2. *A continuous logic $\mathcal{L}$-structure M is an $\mathcal{L}$-structure in the usual sense of classical first order logic such that $M \models T_{\mathcal{L}}$, the value sorts of M are the standard real numbers, and each metric sort of M is a complete metric space.*

A useful way of constructing continuous logic $\mathcal{L}$-structures is to quotient sufficiently saturated classical $\mathcal{L}$-structures: given any model M of $T_{\mathcal{L}}$, we define $\tilde{M}$ to be the structure obtained by quotienting every sort of M by the $\bigwedge$-definable equivalence relation $E(x, y)$ which says that x and y are within distance $\leq 1/n$ for every $n \geq 0$. It is easy to check that for every function symbol $f \in \mathcal{L}$, the interpretation f^M of f in M induces a uniformly continuous function $f^{\tilde{M}}$ on $\tilde{M}$. Therefore $\tilde{M}$ is a classical $\mathcal{L}$-structure and $\tilde{M} \models T_{\mathcal{L}}$ which respects the uniform continuity conditions of $T_{\mathcal{L}}$.

Therefore, if $M \models T_{\mathcal{L}}$ then $\tilde{M} \models T_{\mathcal{L}}$. If M is ω_1-saturated, then the value sorts of $\tilde{M}$ are the standard real numbers and each sort of $\tilde{M}$ is a complete metric space. Therefore $\tilde{M}$ is a continuous logic $\mathcal{L}$-structure. We say that $\tilde{M}$ is the *standardisation* of M.

The key property of positive formulas is the following: if $\phi(x)$ is any positive formula and $M \models \phi(a)$ where $a \in M$ and M is a classical $\mathcal{L}$-structure, then $\tilde{M} \models \phi(\tilde{a})$ where $\tilde{a}$ is the equivalence class of a in $\tilde{M}$.

For a continuous logic language $\mathcal{L}$, a positive formula $\phi(x)$ in $\mathcal{L}$, and a rational $\epsilon > 0$, we define the *ϵ-approximation of $\phi(x)$*, written $\phi^\epsilon(x)$, as the formula obtained from ϕ by weakening all the bounds mentioned in ϕ by ϵ. This construction is adapted from Section 5 in [HI02]. Explicitly, ϕ^ϵ is defined up to logical equivalence inductively as follows:

1. if $\phi(x)$ is of the form $f(x) = r$ where r is a rational and $f(x)$ is an $\mathcal{L}$-term, then $\phi^\epsilon(x)$ is $|f(x) - r| \leq \epsilon$
2. if $\phi(x)$ is of the form $f(x) \leq r$, then $\phi^\epsilon(x)$ is $f(x) \leq r + \epsilon$
3. if $\phi(x) = \psi(x) \wedge \chi(x)$, then $\phi^\epsilon(x) = \psi^\epsilon(x) \wedge \chi^\epsilon(x)$, and similarly if $\phi(x) = \psi(x) \vee \chi(x)$
4. if $\phi(x) = (\exists y)\psi(x, y)$, then $\phi^\epsilon(x) = (\exists y)\psi^\epsilon(x, y)$, and similarly if $\phi(x) = (\forall y)\psi(x, y)$.

Note that in any $M \models T_{\mathcal{L}}$, we always have $\phi(M) \subseteq \phi^\epsilon(M)$ for any $\epsilon > 0$.

In continuous logic, we are usually interested in knowing if a formula $\phi(x)$ is *approximately satisfied* by a point a in M. This means that for all $\epsilon > 0$,

$M \models \phi^\epsilon(a)$, where $\models$ is meant in the usual sense of classical logic. The difference between satisfaction and approximate satisfaction is the same as the difference between saying $(\exists y)f(a, y) = 0$ and $\inf_y |f(a, y)| = 0$. [HI02] take the route of defining a new relation of approximate satisfaction inside a continuous logic structure (see section 5 in [HI02]). In this thesis, we prefer to keep the usual notion of satisfaction but we work with approximations of formulas.

In Section 5 of [HI02], the authors define the *weak negation* $neg(\phi)(x)$ of a positive $\mathcal{L}$-formula $\phi(x)$ as follows:

1. if $\phi(x)$ is of the form $f(x) = r$, then $neg(\phi)(x)$ is $|f(x) - r| \geq 0$
2. if $\phi(x)$ is of the form $f(x) \leq r$, then $neg(\phi)$ is $f(x) \geq r$
3. if $\phi(x) = \psi(x) \wedge \chi(x)$ then $neg(\phi)(x) = neg(\psi)(x) \vee neg(\chi)(x)$, and similarly if $\phi(x) = \psi(x) \vee \chi(x)$.
4. if $\phi(x) = (\exists y)\psi(x, y)$, then $neg(\phi) = (\forall y)neg(\psi)(x, y)$, and similarly if $\phi(x) = (\forall y)\psi(x, y)$.

Note that $neg(\phi)$ is a positive formula and in any model of $T_{\mathcal{L}}$, $\neg\phi \subseteq neg(\phi)$. It is easy to check that for any continuous logic structure M and $a \in M$, for any positive formula, for all $\epsilon > 0$, either $M \models \phi^\epsilon(a)$ or $M \models neg(\phi^\epsilon(a))$ and that for $\epsilon < \delta$, ϕ^ϵ and $neg(\phi^\delta)$ are inconsistent. This shows that continuous logic languages have strong expressive power.

Definition 1.8.3. *Let T be a consistent collection of $\mathcal{L}$-sentences. We say that T is a continuous logic $\mathcal{L}$-theory if $T \supseteq T_{\mathcal{L}}$ and for every $\phi \in T$, either $\phi \in T_{\mathcal{L}}$ or there is some positive sentence ψ and $\epsilon > 0$ such that $\phi = \psi^\epsilon$*

We say that T is a complete continuous logic $\mathcal{L}$-theory if for every positive $\mathcal{L}$-sentence ϕ, for every $\epsilon > 0$, T contains either ϕ^ϵ or $neg(\phi^\epsilon)$.

When M is a continuous logic $\mathcal{L}$-structure, we write $Th(M) = \{\phi^\epsilon \mid M \models \phi, \epsilon > 0\}$

Therefore, a complete continuous logic theory T is not maximal consistent with respect to positive sentences. Nevertheless, if $M \models T$ is a continuous logic structure and N is a nonprincipal ultrapower of M, then the set of positive sentences true in $\tilde{N}$ is maximal consistent and does not depend on M.

Let M, N be continuous logic structures. We say that N is an *elementary extension* of M (and we write $M \prec N$) if M is a substructure of N in the usual sense and for any tuple a in M and any positive formula $\phi(x)$, if $M \models \phi(a)$ then $N \models \phi(a)$. Note that $Th(M) = Th(N)$ although more positive formulas and sentences may be true in N than in M.

Definition 1.8.4. *Let T be a continuous logic theory. A continuous logic type-definable set $p(x)$ is a collection of positive formulas consistent with T such that for every $\phi(x) \in p(x)$ there is some positive $\psi(x)$ and $\epsilon > 0$ and $\phi = \psi^\epsilon$.*

A continuous logic complete type $p(x)$ is a continuous logic type-definable set such that for any positive formula $\phi(x)$ and $\epsilon > 0$, p contains either ϕ^ϵ or $neg(\phi^\epsilon)$.

When $M \models T$ is a continuous logic structure, $a \in M$ and $A \subseteq M$, we write $\mathrm{tp}(a/A)$ for the continuous logic type of a over A. This is the set of formulas $\phi^\epsilon(x)$ such that $\phi(x)$ is a positive formula over A satisfied by a.

We write $S_x(T)$ for the space of continuous logic complete types in x.

Equivalently, we could define a continuous logic type $p(x)$ as a continuous logic type-definable set such that there is a maximal consistent set $q(x)$ of positive $\mathcal{L}$-formulas with $p = \{\phi^\epsilon \mid \phi \in q\}$.

Saturation is defined as expected for continuous logic structures, with respect to continuous logic types. Note that the existence of saturated continuous logic structures follows from the existence of saturated models in classical first order logic: if $M \models T$ is κ-saturated in the usual sense of classical logic, it is easy to show that the standardisation $\tilde{M}$ is κ-saturated as a continuous logic structure. Homogeneous continuous logic structures are constructed in the same way.

We know from classical logic that $S_x(T)$ is a compact topological space. A basis of closed sets of $S_x(T)$ is given by the formulas contained in the continuous logic types of T. Moreover, our discussion of the weak negation $neg(\phi)$ shows that $S_x(T)$ is Hausdorff.

For the purpose of this appendix, write $S_{pos}(T)$ for the set of maximal consistent types of positive formulas. For every $p(x) \in S_{pos}(T)$, define $\tilde{p} = \{\phi^\epsilon \mid \phi \in p\} \in S_x(T)$. When $M \models T$ is an ω_1-saturated continuous logic structure, every tuple $a \in M$ realises some $p \in S_{pos}(T)$. Therefore, $S_{pos}(T)$ and $S_x(T)$ are homeomorphic topological spaces via the map $p \mapsto \tilde{p}$.

In this thesis, we often work in ω_1-saturated continuous logic structures. In this context, continuous logic is equivalent to working with the fragment of positive formulas in classical logic, as is clear from the definitions we have laid down. Arguments about the type space $S_x(T)$ can be streamlined by discussing the space $S_{pos}(T)$ and working with arbitrary positive formulas. Whenever deducing results about non-saturated continuous logic structures, we are careful to introduce approximations of formulas to transfer the results appropriately.

In the remainder of this appendix, we work in continuous logic, so we say 'structure' instead of 'continuous logic structure', 'theory' instead of 'continuous logic theory', 'type' instead of 'continuous logic type', etc.

1.8.2 Standard facts and definitions in continuous logic

Many basic results in classical logic go through to continuous logic unchanged. We record some definitions and results which are used in this thesis. In this section, T always denotes a complete continuous logic theory.

Bounded and definable closure

Definition 1.8.5. *Let $M \models T$ and take $A \subseteq M$. We say that a tuple $b \in M$ is in the bounded closure of A if for every elementary extension N of M, there is no infinite indiscernible sequence realising* $\mathrm{tp}(b/A)$ *in N. We write* $\mathrm{bdd}(A)$ *for the bounded closure of A in M.*

We say that a tuple $c \in M$ is in the definable closure of A if for every elementary extension N of M, c is the only realisation of $\mathrm{tp}(c/A)$ *in N. We write* $\mathrm{dcl}(A)$ *for the definable closure of A in M.*

See 10.7 and 10.8 in [BYBHU08] for standard results about definable and bounded closure. Note in particular that if $M \prec N$ and $A \subseteq M$ then $\mathrm{bdd}(A)$ is the same set in M and in N, and similarly for $\mathrm{dcl}(A)$.

Definable functions

Definition 1.8.6. *Let M be a κ-saturated model of T and take $A \subseteq M$ of size $< \kappa$. Let X and Y be finite Cartesian products of sorts of T and let p be a type-definable subset of X over A. We say that a function $f : p \to Y$ is definable over A if the set $\{(x, f(x)) \mid x \in p\}$ is type-definable over A.*

When M contains A but isn't sufficiently saturated, we say that f is definable over A if there is a sufficiently saturated elementary extension N of M and a definable function on N which restricts to f.

When $f : p \to Y$ is definable over A, we often identify f with its graph in the type space $S_{xy}(A)$. f is definable on p over A if and only if the function $p \times Y \to \mathbb{R}, (x, y) \mapsto d(f(x), y)$ is definable on p over A. See 9.24 in [BYBHU08].

In the special case where Y is the value sort of T corresponding to the interval $[-n, n]$, the definable functions $p \to Y$ over A are exactly the continuous functions $S_x(A) \cap p \to [-n, n]$. This is because the type-space of a Y can always be identified with the interval of real numbers $[-n, n]$.

Since type-spaces are compact Hausdorff topological spaces, any complete type q in $S_x(A)$ is uniquely determined by the values $f(q)$ where f ranges over the A-definable functions $X \to \mathbb{R}$. Urysohn's lemma also entails that any definable $f : p \to \mathbb{R}$ extends to a continuous function $S_x(A) \to \mathbb{R}$ so the local definition of f on p is not usually relevant. This is a significant difference with general definable functions $p \to Y$.

A useful technical fact is that any A-definable $f : X \to \mathbb{R}$ is the uniform limit of a sequence of A-definable functions (f_n) on X such that for all n, there is a finite tuple a_n in A and a function $g_n(x, y)$ definable by a term in the language $\mathcal{L}$ such that $f_n(x) = g_n(x, a_n)$. To see this, note that such $g_n(x, a_n)$ form a lattice of functions on $S_x(A)$ which separate points. Therefore the Stone-Weierstrass theorem applies. One consequence of this is that any A-definable $f : X \to \mathbb{R}$ is definable over a countable subset of A.

When working with definable functions, we will often write down formulas which contain symbols for these functions, e.g $M \models f(a) = 0$. This is a slight abuse of notation, especially when the functions are only defined on a type-definable set p. *These expressions are meant as shorthand for type-definable sets in $\mathcal{L}$.*

As a final comment on definable functions, let $M \models T$, $A \subseteq N$, let p be a type-definable set over A, and let $f : p \to Y$ be A-definable, where Y is any finite product of sorts of T. We have seen that f is defined at the level of the type-space or equivalently at the level of an elementary extension of M. Nevertheless, if $p(M) \neq \emptyset$ then $f : p(M) \to Y(M)$ is a total function. This is because $f(a) \in \mathrm{dcl}(Aa)$ for all $a \in M$ and we have seen that $\mathrm{dcl}(Aa)$ does not depend on M.

Canonical parameters and imaginaries

In this thesis, we make a slightly non-standard use of the notion of canonical parameter.

Definition 1.8.7. *Let $M \models T$, let $A \subseteq M$ and let $f : X \to Y$ be A-definable. We say that a single element $c \in M$ is a canonical parameter for f if for any elementary extension N of M, any automorphism of N preserves f if and only if it fixes c.*

This definition is slightly non-standard for the reason that we do not allow tuples of elements as canonical parameters. This is because canonical parameters consisting of exactly one element play an important role in our study of interpretable Hilbert spaces. We obtain canonical parameters by adding imaginary sorts whenever we need them. Imaginary sorts in continuous logic are a special case of hyperimaginary sorts in classical logic. See [BYU10] Section 5 for a clear comparison.

Definition 1.8.8. *An imaginary sort S of T is a Cartesian product of at most countably many sorts (S_n) of T endowed with a pseudo-metric d such that there is an increasing sequence (n_k) in $\mathbb{N}$ and definable pseudo-metrics d_k on $\prod_{i=0}^{n_k} S_i$ such that the pseudo-metrics (d_k) converge uniformly to d on S.*

This means that for any $\epsilon > 0$, there is $K \geq 0$ such that for all $k \geq K$, in any $M \models T$, for any $(a_n), (b_n) \in \prod_{n \geq 0} S_n$,

$$|d((a_n), (b_n)) - d_k((a_n)_{n \leq n_k}, (b_n)_{n \leq n_k})| < \epsilon.$$

If S is an imaginary sort of M with metric d, expressed as the product of the sorts (M_n), we can add the sort S to the language with the metric d and projection maps $\pi_n : S \to M_n$. This construction is carried out in detail in [BYU10].

We use imaginary sorts in two ways. Firstly, we can use an imaginary sort to add a countable Cartesian product of metric spaces to the language: if d_n is a metric on M_n with diameter 1 and S is the product of the sorts (M_n), then we can define $d(x,y) = \sum d_n(x_n, y_n)/2^n$.

Secondly, we can use imaginary sorts to add canonical parameters for arbitrary definable functions. If f is a definable function over a countable set $A \subseteq M$, we have seen that we can express f as the uniform limit of functions $g_n(x, a_n)$ where $a_n \subseteq A$ is finite and g is a term in the language. For every n, let S_n be the product of the sorts corresponding to the tuple a_n and let d_n be the definable pseudometric $d_n(y,z) = \sup_x d'(g_n(x,y), g_n(x,z))$ where d' is the metric on the sort of x. The notion of forced limit in [BYU10] shows how to ensure that the metrics d_n are uniformly convergent. Let S be the Cartesian product of the sorts S_n and let d be the limit of the pseudo-metrics d_n. Quotienting out S_n and S to obtain metric spaces, S is an imaginary sort of canonical parameters for f. See [BYU10] for details.

Stability and definable types

Definition 1.8.9. *Let $M \models T$. Take $A \subseteq M$ and let $f : X \times Y \to \mathbb{R}$ be an A-definable function where X and Y are finite Cartesian products of sorts of M and f takes values in $\mathbb{R}$. We say f is unstable if there is some elementary extension N of M, indiscernible sequences $(a_n), (b_n)$ in N and $\epsilon > 0$ such that $|f(a_n, b_m) - f(a_m, b_n)| \geq \epsilon$ for all $n \neq m$. We say f is stable if f is not unstable.*

Definition 1.8.10. *Let $M \models T$ and let $A, B \subseteq M$. Let $p \in S_x(A)$ and let $f(x,y)$ be an A-definable function into $\mathbb{R}$. We say that p is definable over B with respect to f if there is a B-definable function $g(y)$ such that for any tuple a in A in the sort of y, $p(x)$ entails that $f(x,a) = g(a)$. In that case we write $g(y) = d_p f(y)$.*

In this thesis we work with local stable independence. First developed in [She78] and [Pil86] for classical logic, local stable independence for continuous logic has roots in [She75] and was studied in [BYU10]. We only recall the main definition:

Definition 1.8.11. *Let $M \models T$, let $C \subseteq B \subseteq M$ and $p(x) \in S(B)$. Let Δ be a set of stable functions definable over A. We say that $p(x)$ does not fork over C with respect to Δ if we can add imaginary sorts to M and extend p to $\mathrm{bdd}(B)$ so that p is definable over $\mathrm{bdd}(C)$ with respect to Δ.*

Let $A \subseteq M$. We write $A \mathop{\smile\!\!\!\!\!\!\mid}^{\Delta}_{C} B$ if $\mathrm{tp}(A/BC)$ does not Δ-fork over C.

When Δ is the set of all stable functions, we simply say that p does not fork over C and we write $A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B$.

We refer the reader to [Pil86] and [BYU10] for an exposition of the theory of stable independence. We will also make use of Morley sequences in the context of local stability.

Definition 1.8.12. *Let $M \models T$ and let Δ be a set of stable formulas. We say that a sequence (a_n) in M is a Morley sequence over A with respect to Δ if for all n we have $a_{n+1} \mathop{\smile\!\!\!\!\!\!\mid}^{\Delta}_{A} a_0 \dots a_n$ and $\mathrm{tp}(a_n/\mathrm{bdd}(A)) = \mathrm{tp}(a_0/\mathrm{bdd}(A))$.*

Continuous logic and classical logic

Continuous logic is a direct generalisation of classical logic, and there is a canonical way of taking a classical logic theory T and viewing it as a continuous logic theory T^{cont}. The construction of T^{cont} is as follows.

Every sort of T is now viewed as a metric space with the discrete metric with diameter 1. We remove the equality symbol from the sorts of T. Observe that there is no loss of information in doing this, since $x = y$ is equivalent to $d(x, y) \leq 1/2$ and $x \neq y$ is equivalent to $d(x, y) \geq 1/2$, both of which are approximations of positive formulas. We add the usual value sorts to T. Function symbols in the language of T are unchanged. For each relation symbol R in the language of T, we substitute a function symbol f_R which we view as the indicator function of R in the corresponding continuous logic sort. It is then clear how to axiomatise a continuous logic theory T^{cont} so that there is an exact correspondence between models of T and models of T^{cont}.

When we construct a theory T^{cont} from a classical logic theory T, we can distinguish two kinds of imaginary sorts in T^{cont}. Firstly we have the *classical imaginary sorts of T^{cont}* which come from the imaginary sorts of T defined in the usual way (see [TZ12]). Secondly we have the *continuous logic imaginary sorts of T^{cont}* which are imaginary sorts obtained by the constructions sketched in 1.8.8. These correspond to certain hyperimaginary sorts of T. Unless specified otherwise, 'imaginary sort' and 'imaginary element' always refer to imaginaries in continuous logic, as defined in 1.8.8.

Stable embeddedness

We show that the classical notion of stable embeddedness adapts easily from classical logic to continuous logic. An account of stable embeddedness in the context of classical logic can be found in the appendix of [CH99a]. Here we take T to be a complete continuous logic theory.

Definition 1.8.13. *Let $\mathcal{D}$ be a collection of type-definable sets of T. We say that $\bigcup \mathcal{D}$ is stably embedded in T if for any κ-saturated $M \models T$ with $|M| = \kappa$,* *any M-definable function from a finite Cartesian products of sets in $\mathcal{D}$ to $\mathbb{R}$ is definable over $\bigcup \mathcal{D}(M)$.*

In the above definition, saturation is not essential, but it is convenient to include it. Saturation can be eliminated by considering imaginaries as in [CH99a]. We will replicate that argument in Proposition 1.2.15 in a more restricted setting so we only consider saturated structures for now.

Lemma 1.8.14. *Let $\mathcal{D}$ be a collection of distance-definable sets in T (see Definition 1.2.13). The following are equivalent:*

1. *$\bigcup \mathcal{D}$ is stably embedded in T.*
2. *For any κ-saturated $M \models T$ with $|M| = \kappa > |\mathcal{L}|$, for every finite tuple $a \in M$, $\mathrm{tp}(a/\bigcup \mathcal{D}(M))$ is definable over a subset of $\bigcup \mathcal{D}(M)$ of size at most $|\mathcal{L}|$.*
3. *For any κ-saturated $M \models T$ with $|M| = \kappa > |\mathcal{L}|$, for every finite tuple $a \in M$, there is a subset C of $\bigcup \mathcal{D}(M)$ of size at most $|\mathcal{L}|$ such that $\mathrm{tp}(a/C)$ extends uniquely to $\bigcup \mathcal{D}(M)$.*
4. *For any κ-saturated $M \models T$ with $|M| = \kappa > |\mathcal{L}|$, every automorphism of $\bigcup \mathcal{D}(M)$ extends to an automorphism of M.*

Proof. To make notation lighter we can assume that $\mathcal{D}$ is closed under finite Cartesian products. We fix $M \models T$ κ-saturated with $|M| = \kappa$.

(1) $\Rightarrow$ (2) Let $a \in M$. Let $f(x, y)$ be a definable function into $\mathbb{R}$ with y in the sort of $D \in \mathcal{D}$ and x in the sort of a. By (1) there is a $\bigcup \mathcal{D}(M)$-definable function $g(y)$ such that $g(y) = f(a, y)$ on D. g defines $\mathrm{tp}(a/D(M))$ for f. Moreover, g is definable over a countable $A \subseteq D(M)$, so $\mathrm{tp}(a/\bigcup \mathcal{D}(M))$ is definable over a subset B of $\bigcup \mathcal{D}(M)$ with $|B| \leq |\mathcal{L}|$.

(2) $\Rightarrow$ (1): Let $f(x)$ be an M-definable function into $\mathbb{R}$ where x is a finite tuple in the sort of $D \in \mathcal{D}$. We can assume that $f(x) = g(x, b)$ where b is a finite tuple in M and g is 0-definable. $q = \mathrm{tp}(b/\bigcup \mathcal{D}(M))$ is definable over some small $C \subseteq \mathcal{D}(M)$ so we have a C-definable function $d_q g(x) = f(x)$.

(2) $\Rightarrow$ (3): Let $p(x) = \mathrm{tp}(a/\bigcup \mathcal{D}(M))$, $a \in M$. Suppose that p is definable over $C \subseteq \bigcup \mathcal{D}(M)$. Let $f(x)$ be a $\bigcup \mathcal{D}(M)$-definable function into $\mathbb{R}$. We show that the restriction of $p(x)$ to C determines the value of $f(x)$. We can assume that $f(x) = g(x, b)$ where b is a finite tuple in $\bigcup \mathcal{D}(M)$ and $g(x, y)$ is 0-definable.

By (2), p is definable over C with respect to g and we write $d_p g(y)$ for its definition. Write $d(y, D)$ for the definable function which gives the distance to D. An easy compactness argument shows that for every $\epsilon > 0$ there is $\delta > 0$ such that $p(x)$ contains the positive formula over C:

$$\forall y (d(y, D) < \delta \to |g(x, y) - d_p(g(y))| \leq \epsilon)$$

Therefore $p \restriction C$ has a unique extension to $\bigcup \mathcal{D}(M)$.

(3) $\Rightarrow$ (2): Let $a \in M$, write $p(x) = \mathrm{tp}(a/\bigcup \mathcal{D}(M))$ and let $f(x, y)$ be a definable function with y in the sort of $D \in \mathcal{D}$ and x a tuple in the sort of a. By (3), there is $C \subseteq \bigcup \mathcal{D}(M)$ such that $p(x)$ is the unique extension of $p \restriction C$ to $\bigcup \mathcal{D}(M)$. Let N be an elementary extension of M. Suppose there are $c, c' \in N$ both realising $p \restriction C$ and $b \in D(N)$ such that $|f(c, b) - f(c', b)| \geq \delta$ for some $\delta > 0$. Then

$$N \models \exists x, x', y\big((p \restriction C)(x) \wedge (p \restriction C)(x') \wedge D(y) \wedge |f(x, y) - f(x', y)| \geq \delta\big)$$

The above is a type-definable set. Since $M \prec N$ and M is saturated over C, M satisfies the same type-definable set. This contradicts (3) and this proves that p has a unique extension to $\bigcup \mathcal{D}(N)$. Hence for any $M \prec N$, $p(x)$ is C-invariant in N. It follows that $p(x)$ is C-definable.

(3) $\Rightarrow$ (4) : Let σ be an automorphism of $\bigcup \mathcal{D}(M)$ and suppose that we have extended it to an automorphism $\sigma : \bigcup \mathcal{D}(M) \cup A \to \bigcup \mathcal{D}(M) \cup B$ where $|A| < \kappa$. Let $a \in M$. There is $C \subseteq \bigcup \mathcal{D}(M)$ with $|C| < \kappa$ such that $\mathrm{tp}(a/A, C)$ extends uniquely to $A \cup \bigcup \mathcal{D}(M)$. By saturation, we can find $b \in M$ such that σ extends to an automorphism $\bigcup \mathcal{D}(M) \cup Aa \to \bigcup \mathcal{D}(M) \cup Bb$. The result follows by a back-and-forth argument.

(4) $\Rightarrow$ (3): Suppose (3) fails for $M \models T$, where M is κ-saturated with cardinality κ. Fix $a \in M$ which witnesses the failure of (3). Let $(a_i)_{i<\kappa}$ be an enumeration of the realisations of $\mathrm{tp}(a)$ in M. Suppose we have constructed an isomorphism $\sigma : C \to \sigma(C)$ where C is a subset of $\bigcup \mathcal{D}(M)$ such that for some $\alpha < \kappa$ and all $i < \alpha$, the maps $\sigma : aC \to a_i\sigma(C)$ are not isomorphisms.

Suppose that $\sigma_1 : aC \to a_\alpha\sigma(C)$ is an isomorphism. By the failure of (3) there is a' in M and $b \in \bigcup \mathcal{D}(M)$ such that $\mathrm{tp}(a/C) = \mathrm{tp}(a'/C)$ and $\mathrm{tp}(a/bC) \neq \mathrm{tp}(a'/bC)$. Then we can find b' such that $\sigma_1 : a'bC \to a_\alpha b'\sigma(C)$ is an isomorphism. Then we extend σ by putting $\sigma(b) = b'$. Note that now we cannot extend σ to a by sending a to a_α. By enumerating $\bigcup \mathcal{D}(M)$, we can also make sure that after κ iterations of this procedure σ is defined on all of $\bigcup \mathcal{D}(M)$. This contradicts (4). $\square$

1.8.3 Hilbert spaces in continuous logic

We recall here basic facts about the model theory of Hilbert spaces which we use in this chapter. We refer the reader to [BYBHU08] for a more complete summary. We work with Hilbert spaces over $\mathbb{R}$, but all results can be transposed to complex Hilbert spaces without modification.

On one presentation of the model theory of Hilbert spaces, the language of Hilbert spaces in continuous logic consists of countably many metric sorts, which stand for balls with radius n around 0. We add appropriate inclusion maps between the metric sorts. The language consists of the usual vector space structure over $\mathbb{R}$ and a function $\langle \cdot, \cdot \rangle$ on each metric sort into an appropriate value sort which stands for the inner product. The axiomatisation of the theory of infinite dimensional Hilbert spaces T^{Hilb} is as expected. We usually do not disinguish between a Hilbert space and a model of T^{Hilb}.

T^{Hilb} is complete, has quantifier-elimination, is stable, and is totally categorical. The theory of Hilbert spaces does not have elimination of imaginaries, but it has weak elimination of imaginaries:

Lemma 1.8.15 ([BYB04] 1.2)**.** *Let $M \models T^{Hilb}$. Let α be a canonical parameter for an M-definable function f in an arbitrary imaginary sort of T^{Hilb}. Then there is a closed subspace H of M such that each point of H is in* $\mathrm{bdd}(\alpha)$ *and α is definable over H.*

Definition 1.8.16. *If H is a Hilbert space and $V \leq H$ is a closed subspace, write P_V for the orthogonal projection onto V.*

We will make much use of the characterisation of forking independence in Hilbert spaces. See [BYBHU08] for a proof:

Lemma 1.8.17. *Let $M \models T^{Hilb}$. Let $A, B, C \subseteq M$ with $B \subseteq C$. Then* $\mathrm{bdd}(B)$ *is the closed subspace of M generated by B and $A \mathop{\smile\!\!\!\!\!\vert}_B C$ if and only if for all $a \in A$, $P_{\mathrm{bdd}(C)}(a) = P_{\mathrm{bdd}(B)}(a)$.*

Finally, we recall some elementary facts about the weak topology in Hilbert spaces which we use in our proofs. If H is a Hilbert space, recall that the weak topology has a sub-basis consisting of the sets

$$\{v \in H \mid \langle v, w \rangle \in U, w \in H, U \subseteq \mathbb{R} \text{ open}\}.$$

Recall also that the unit ball is compact in the weak topology and that every bounded sequence in H has a weakly convergent subsequence.

Lemma 1.8.18. *Let $M \models T^{Hilb}$ and $A \subseteq M$ and $v \in M$. Let (v_n) be an indiscernible sequence in* $\mathrm{tp}(v/A)$ *in M. Then there is an orthogonal sequence (w_n) in M and $w \in M$ such that for all n, $v_n = w_n + w$, $w_n \perp A$ and $w_n \perp w$. It follows that (v_n) converges weakly to w (we write $v_n \rightharpoonup w$).*

Moreover, w is the unique element of M such that $\langle w, w \rangle = \langle v_1, v_0 \rangle = \langle w, v_n \rangle$ for all n.

Lemma 1.8.19. *Let $M \models T^{Hilb}$. Let (v_n) be a Morley sequence over $A \subseteq M$. Then (v_n) converges weakly to $P_{\mathrm{bdd}A}(v_0)$.*

Proof. It is enough to check that $\lim_n \langle v_n, v_m \rangle = \langle P_{\mathrm{bdd}(A)}(v_0), v_m \rangle$ for all m. For $n > m$ we have $\langle v_n, v_m \rangle = \langle P_{\mathrm{bdd}(Av_0 \ldots v_m)}(v_n), v_m \rangle = \langle P_{\mathrm{bdd}(A)}(v_n), v_m \rangle = \langle P_{\mathrm{bdd}(A)}(v_0), v_m \rangle$. ☐

Chapter 2

An Algebraic Hypergraph Regularity Lemma

The material in this chapter builds on prior work by Elad Levi and will be published as joint work. Levi proved a first algebraic hypergraph regularity lemma in the context of Galois-rigid pseudofinite fields. This result is discussed in Section 2.4.3. The implementation of the étale point of view in this chapter is based on ideas of Hrushovski.

In this chapter, we prove a strong hypergraph regularity lemma for definable sets in finite fields. We offer three main contributions which build on the algebraic regularity lemma of [Tao12]. Firstly, we extend the algebraic regularity lemma to arbitrary definable hypergraphs, of any arity, thus answering a question of Tao. Secondly, we extend the algebraic regularity lemma beyond the context of finite fields by proving our results in the difference fields $K_q = (F_q^{alg}, x^q)$ for definable sets of finite total dimension in the language of rings with a difference operator σ. Thirdly, we offer a new point of view on algebraic regularity, relating combinatorial Szemerédi-style regularity for definable sets to algebraic and geometric properties of associated varieties and function fields. Therefore our hypergraph partitions have a natural geometric interpretation.

The algebraic regularity lemma of [Tao12] says that a strong version of Szemerédi regularity holds for definable sets $\phi(x, y)$ (with parameters) when interpreted in finite fields F_q. Suppose that $\phi(x, y)$ is contained in the definable set $X \times Y$. The algebraic regularity lemma says that there is some $N \geq 1$ and definable sets $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ partitioning X and Y respectively with $n, m \leq N$ such that for all $i \leq n$ and $j \leq m$, the graph $\phi(x, y) \wedge X_i(x) \wedge Y_j(y)$ is $q^{-1/4}$-regular in the field F_q. This means that for sufficiently large prime powers q, taking parameters in F_q, if $\phi \cap X_i \times Y_j \neq \emptyset$ in F_q, then for any $A \subseteq X_i$ and $B \subseteq Y_j$,

$$\left|\frac{|\phi(x, y) \cap A \times B|}{|\phi(x, y) \cap X_i \times Y_j|} - \frac{|A \times B|}{|X_i \times Y_j|}\right| = O(q^{-1/4}).$$

$O(\cdot)$ and N depend only on the formula ϕ and not on the parameters.

The algebraic regularity lemma for graphs strengthens the classical Szemerédi regularity lemma in two ways. Firstly, there is a fixed N such that the error bounds on regularity vanish against a partition of the graph into at most N^2 subgraphs, as $q \to \infty$. As a result, the sizes of the sets X_i and Y_j are of the same order as $|X|$ and $|Y|$ as $q \to \infty$. Secondly, regularity is obtained for all pairs (X_i, Y_j), whereas the classical graph regularity lemma only guarantees regularity for *most* pairs, in an appropriate sense. The Szemerédi regularity lemma first appeared in [Sze78] and the reader is referred to [Gow06] for a general discussion.

The algebraic regularity lemma of [Tao12] has attracted considerable attention from model theorists and it is now well-understood that the main engine behind this result is the stability of the formulas $\mu(\phi(x, a) \wedge \psi(x, b)) = \lambda$, where μ is the counting measure in the variable x, which was first shown to be definable in [CvdDM92]. See [PS13] for a model theoretic discussion of the algebraic regularity lemma for graphs and its extension to MS-measurable structures.

However, generalising Tao's algebraic regularity lemma to definable hypergraphs requires new ideas. In this thesis, we take a fundamentally geometric approach to this problem and we use the model theory of ACFA to derive an algebraic hypergraph regularity lemma which strengthens the classical hypergraph regularity lemma of [Gow07] and [RS04] in the same ways as Tao's algebraic regularity lemma strengthens Szemerédi's.

There are some interesting technical twists to our results, but given a definable hypergraph $\phi(x_1, \ldots, x_n) \subseteq X_1 \times \ldots \times X_n$ in finite fields (or in the difference fields K_q, as discussed below), we find some $N \geq 0$ depending only on ϕ and a partition $\mathcal{W}_u$ of each set $\prod_{i \in u} X_i$ where $u \subseteq \{1, \ldots, n\}$ and $|u| = n-1$ such that $|\mathcal{W}_u| \leq N$ for every u and such that restricting ϕ to any family $(W_u)_{|u|=n-1}$ with $W_u \in \mathcal{W}_u$ gives a $q^{-2^{-n-1}}$-regular hypergraph in the sense of [Gow07]. See Theorem 2.4.13 for a precise statement.

The choice to work with difference equations and ACFA rather than pseudofinite fields is a natural and useful one. Recall that ACFA is the model completion of the theory of algebraically closed inversive difference fields in the language of rings with a difference operator σ. ACFA extends naturally the theory of pseudofinite fields in the sense that for any $K \models ACFA$, the fixed field $\sigma(x) = x$ of K is a model of the theory of pseudofinite fields and is stably embedded inside K. The fundamental results of [Hru22] also show that ACFA is the asymptotic theory of the structures $K_q = (F_q^{alg}, x^q)$ in the same way that the theory of pseudofinite fields is the asymptotic theory of the finite fields F_q.

In ACFA, there is a form of quantifier-elimination which is more natural than the quantifier-elimination of pseudofinite fields: every definable set contained in a variety of finite total dimension is equivalent to the image of a projection $\pi : X \to Y$ where X and Y are difference varieties of finite total dimension and X is a finite cover of Y. We refer to such definable sets as *Galois formulas.* Galois formulas specialise to finite sets in the structures K_q and it is easy to deduce a natural characterisation of the counting measure on Galois formulas using the results of [Hru22]. This unlocks an algebraic characterisation of combinatorial regularity.

We note that it is possible to recover the same kind of quantifier-elimination as in ACFA in the more modest context of pseudofinite fields by enriching the language of rings slightly. If F is a pseudofinite field viewed as a structure in the language of rings, we can add a sort F_n for every Galois extension of F of degree n and on each F_n we add a difference operator σ_n which we interpret as a generator of $Gal(F_n/F)$. See [Joh19] for an example of this approach. Under this approach and in this restricted context, all results in this chapter could be recast using the classical Lang-Weil estimates and the discussion of the counting measure found in the Appendix of [Hru02].

However, instead of working with the expansion of pseudofinite fields described above, it is more natural to work with full models of ACFA. Hrushovski's twisted Lang-Weil estimates directly extend the classical Lang-Weil estimates to this setting, so the technical transition is seamless. Moreover, shifting the focus to difference equations gives access to many new definable sets which do not come from pseudofinite fields. For example, algebraic dynamics, which is concerned with the study of equations of the form $\sigma(x) = f(x)$ where f is rational, produces many new definable sets of great interest. Finite simple groups of Lie type also fall under this umbrella.

In fact, much of the proof of our algebraic hypergraph regularity lemma happens in a setting which is close to the category of algebraic dynamics defined in [CH08]. If ϕ is a Galois formula corresponding to the finite projection of irreducible difference varieties $X \to Y$ over a difference field A, then we associate to ϕ a finite Galois extension $L/A(a)_\sigma$ where $A(a)_\sigma$ is the difference function field of Y and L is the pure field associated to the finite cover Y over $A(a)_\sigma$. Our fundamental object of study is the pair $(A(a)_\sigma, L)$, where L has no specified difference structure. We think of ϕ as asking a question about extensions of σ from $A(a)_\sigma$ to L. In this setting, we find that combinatorial regularity for definable hypergraphs is equivalent to certain algebraic properties of the pair $(A(a)_\sigma, L)$.

We find it useful to discuss *systems of varieties* rather than definable hypergraphs. We say that Ω is a system of varieties on the finite set V over the difference field A if Ω is a functor from the powerset $P(V)$ to difference varieties of finite total dimension such that for every $u \subseteq V$, $\Omega(u)$ is a finite cover of the fibre product $\prod(\Omega(v), v \in P(u)^-)$. See section 2.2.2 for the precise definitions and notation. When Ω is a system of varieties, we are interested in the definable sets $\rho_u\Omega(u)$ obtained by projecting the variety $\Omega(u)$ down onto the fibre product $\Omega(u)^- = \prod(\Omega(v), v \in P(u)^-)$. The definable sets $\rho_u\Omega(u)$ are Galois formulas, by definition.

Systems of varieties relate to hypergraphs as follows. An n-partite n-uniform hypergraph G can be viewed as a functor G on the powerset of $\{1, \ldots, n\}$ together with sets $X_1, \ldots, X_n$ such that for every subset u, $G(u) \subseteq \prod_{i\in u} X_i$ and $G(u)$ is contained in the fibre product of the sets $\{G(v) \mid v \subsetneq u\}$. Here the fibre product is defined with respect to the system of projections $\prod_{i\in u} X_i \to \prod_{i\in v} X_i$ for $v \subseteq u$. To say that $G(u)$ is contained in the fibre product of the sets $\{G(v) \mid v \subsetneq u\}$ is equivalent to saying that $G(u)$ is contained in the set $\{(x_i)_{i\in u} \in \prod_{i\in u} X_i \mid (x_i)_{i\in v} \in G(v), v \subsetneq u\}$. Equivalently, $G(u)$ is contained in the set of cliques of the n-partite $(n-1)$-uniform hypergraph $(G(v))_{v\subsetneq u}$. Under this point of view, it is clear how to translate any statement about a definable hypergraph into a statement about a system of varieties.

However, systems of varieties offer finer control over definable sets, as they allow us to move between the various finite Galois covers inside the system, whereas a definable hypergraph G only gives access to a single Galois cover. As is common in the model theory of pseudofinite fields or ACFA, the correct point of view on definable sets is an *étale point of view*. We will not make any technical use of the term 'étale', but we will use it to underline the philosophy and the context of our results. Therefore, a system of varieties is really the étale analog of a definable hypergraph.

There is a natural correspondence between systems of varieties and *systems of difference fields*, obtained by taking the difference function fields associated to the varieties in a system of varieties Ω. See section 2.2.1 for precise definitions. If $\mathcal{S}$ is a system of difference fields associated to Ω, we will say that Ω is *regular* if for all $u \subseteq V$, $\mathcal{S}(u)$ is linearly disjoint from the composite of fields $(\mathcal{S}(v)^{alg})_{v\in P(u)^-}$ over the the composite $(\mathcal{S}(v))_{v\in P(u)^-}$. This algebraic notion of regularity will be seen to be essentially equivalent to the combinatorial one for hypergraphs, see Propositions 2.2.16, 2.2.20 and 2.3.6. Our choice of terminology results from a happy coincidence, since our algebraic notion of regularity can be seen as a generalisation of the usual notion of regularity for field extensions.

We formulate two hypergraph regularity lemmas, one in the étale setting and one in the "classical" setting. See Theorems 2.4.10 and 2.4.13 respectively. The étale setting provides definable partitions and good error bounds, whereas the classical setting provides the "expected" partitions but these are not definable, and the error bounds become weaker. It is an open question whether it is possible to find a "classical" algebraic hypergraph regularity lemma where the partitions are definable sets. See the end of Section 2.4.3 for a discussion.

Our proof of the algebraic regularity lemma relies on the fundamental result of Gowers which establishes the near-equivalence between *edge-uniformity* and *quasirandomness*. Our terminology is in line with [Gow06] but either edge-uniformity or quasirandomness are often referred to simply as *combinatorial regularity* in the literature. These notions are defined in Definitions 2.3.5, 2.4.2 and 2.4.4. Because of our shift to the étale point of view, we also introduce the notion of *étale-edge-uniformity*, see Definitions 2.2.19 and 2.4.4. Lemma 2.4.7 shows that edge-uniformity and étale-edge-uniformity are equivalent, but the distinction is useful.

The second ingredient of our proof of the algebraic regularity lemma is the Stochastic Independence Theorem 2.3.2. We will see that quasirandomness follows easily from the Stochastic Independence Theorem. This theorem says that certain definable sets contained in regular systems behave like independent random variables. From quasirandomness, it is straightforward to recover étale-edge-uniformity by the Gowers equivalence, and one can then deduce edge-uniformity by taking sections. This is carried out in Section 2.4.3.

From the Stochastic Independence Theorem, we also deduce the Stationarity Theorem 2.3.3. A stationarity theorem was the main tool in Tao's proof of the algebraic regularity lemma but the argument for our algebraic regularity lemma does not rely on the stationarity theorem. Nevertheless, it is a theorem of wider interest.

Recall that Tao's algebraic regularity lemma relied on the fact that the measure of a formula $\phi(x,a) \wedge \psi(x,b)$ is controlled by the type of a and the type of b. As a result, we can construct definable partitions $Y_1, \dots, Y_n$, $Z_1, \dots, Z_k$ such that for any $i \leq n$ and $j \leq k$, $\mu(\phi(x,a) \wedge \psi(x,b))$ is generically constant for $a \in Y_i$ and $b \in Z_j$. It was shown in [PS13] that this result follows from the fact that the formula $\mu_x(\phi(x,y) \wedge \psi(x,z))$ is stable, so that the partitions $(Y_i), (Z_j)$ can be seen to exist by a general stability-theoretic argument. For this reason, we call this a "stationarity theorem".

Our stationarity theorem is somewhat stronger than the one in [Tao12], even restricted to the intersection of two definable sets. Given $\phi(x,y)$ and $\psi(x,z)$, we find partitions $X_1, \dots, X_m$, $Y_1, \dots, Y_n$ and $Z_1, \dots, Z_p$ such that for any (i,j,k), the definable sets $\phi(x,a)$ and $\psi(x,b)$ behave like independent random variables on the probability space X_i, with measure governed by the sets Y_j and Z_k containing a and b respectively. Moreover, the random variables $\phi(x,a)$ and $\psi(x,b)$ are found to be independent in the probabilistic sense from definable sets $\chi(x)$ contained in $X_i(x)$. As a corollary, if $\mu(\phi(x,a) \wedge X_i(x)) > 0$ and $\mu(\psi(x,b) \wedge X_i(x)) > 0$, then $\mu(\phi(x,a) \wedge \psi(x,b) \wedge X_i(x)) > 0$. Therefore our stationarity theorem is an amalgamation theorem.

In the case of three sets $\phi(x,y,z)$, $\psi(x,y,t)$ and $\chi(x,z,t)$, our stationarity theorem says that after applying some suitable base changes to the domain, we find compatible partitions of the triples $X \times Y \times Z$, $X \times Z \times T$ and $X \times Y \times T$ into irreducible varieties such that for a generic triple a,b,c, the sets $\phi(x,a,b)$, $\psi(x,a,c)$ and $\chi(x,b,c)$ behave like independent random variables on any domain in the partition. In particular, the measure $\mu(\phi(x,a,b) \wedge \psi(x,a,c) \wedge$

$\chi(x,b,c))$ is uniquely determined by $\mathrm{tp}(a,b)$, $\mathrm{tp}(a,c)$ and $\mathrm{tp}(b,c)$, and this is a direct generalisation of Tao's original result.

As far as we know, this is the first higher-order stationarity theorem of any form for a definable measure. We hope this theorem will be of interest for future applications.

We conclude this introduction by highlighting some related work concerning Szemerédi regularity in model theory. Model theory has been largely successful in finding settings where the classical Szemerédi graph and hypergraph regularity lemmas can be strengthened. [CS16] shows that the classical hypergraph regularity lemmas can be strengthened for definable hypergraphs in NIP structures. See also [CT20] for an extension of those results to functions and for a proof of a strong regularity lemma when the edge-relation is NIP_2. Our results lie at the opposite end of the model theoretic spectrum, since we work here in a simple unstable theory.

The recent results of [TW21] study hypergraph regularity in finite fields without any definability assumptions on the edge-relations but under some new combinatorial restrictions. These conditions aim to generalise stability in the case of graphs to the setting of hypergraphs. It is not yet clear if our approach for treating definable hypergraphs can be connected to this combinatorial project.

Definable graphs in finite fields coming from difference varieties of finite total dimension have already been considered in [DT17]. In that work, the authors adapt Tao's approach to Szemerédi regularity to the definable measure in ACFA over definable sets of finite total dimension, and they also consider graphons. This measure was first shown to be definable in [RT06], drawing on the twisted Lang-Weil estimates of [Hru22][1]. [DT17] also adapt the results of Tao on polynomial expansion to setting of varieties of finite total dimension. In our work, we use a new approach to derive the algebraic graph and hypergraph regularity lemmas.

Finally, the results of [Tom06] establish probabilistic independence results for definable sets in pseudofinite fields. This can be seen as a weak version of our stationarity theorem. The approach for that theorem in [Tom06] is quite different from the approach we take here and it is not clear if it yields easily the general Stochastic Independence Theorem which we need for our algebraic hypergraph regularity lemma.

The chapter is structured as follows. In the first section, we revisit classical results about quantifier-elimination in ACFA and we give a new characterisation of the definable measure for definable sets of finite total dimension.

In the second section, we introduce systems of difference fields and varieties.

[1] The manuscript we reference dates from 2022 but the twisted Lang-Weil estimates were first proved by Hrushovski in the early 2000s.

We define the notion of *definable étale-edge-uniformity* and prove a natural hypergraph regularity lemma at the level of the theory ACFA. See Theorem 2.2.21.

In the third section, we prove the stochastic independence theorem and the stationarity theorem. We define quasirandomness at the level of ACFA, and we show that regular systems of varieties are quasirandom.

In the fourth section, we apply the twisted Lang-Weil estimates to deduce the algebraic hypergraph regularity lemmas. We prove the classical equivalence of Gowers in the étale setting and we show how to decompose étale systems into disjoint sections to recover classical regular decompositions of hypergraphs. Our main results are Theorems 2.4.10 and 2.4.13.

2.1 The Definable Measure in ACFA

We study the definable measure on definable sets in ACFA of finite total dimension. The existence of this measure is implicit in [Hru22] and a detailed presentation can be found in [RT06]. In that paper, the authors show that it is possible to use the same kind of quantifier-elimination as is used in [CvdDM92] to construct the definable measure. We will need more fine-grained information about this definable measure so we find it easier to repeat the presentation from the beginning.

2.1.1 Some background and notation

We use the language of rings with a difference operator σ. A difference field is a field K where σ is an endomorphism. We say K is inversive if σ is an automorphism of K. In this thesis, we are careful to distinguish the notions of field, difference field, and inversive difference fields. If A is a subset of K, then A_σ will denote the smallest difference field containing A and A^{inv} will denote the smallest inversive difference field containing A.

Let $A \leq K$ be a difference field. A difference variety in K over A is the solution set of a finite set of difference equations in σ over A. Note that difference varieties can use positive powers of σ but not negative powers.

Convention: We will usually discuss difference varieties, so "variety" will refer to difference varieties. We will be careful to say "algebraic variety" if we want to refer to the algebraic setting.

As in the algebraic setting, if X is a variety over A, we say that X is irreducible if X cannot be expressed as a finite union of proper subvarieties. Note that we do not assume varieties to be irreducible. When A is an algebraically closed inversive difference field and X is irreducible over A, then

X is absolutely irreducible, meaning that for every extension B of A, X remains irreducible over B. See [Coh65] or [Lev08] for detailed presentations of difference algebra.

If X is an irreducible variety over a difference field $A \leq K$, then we can define the difference function field over A associated to X in the usual way. We denote it $A(X)_\sigma$. Define the *transformal dimension* of X over A to be the maximal cardinality of a set $S \subseteq A(X)_\sigma$ such that the elements $\sigma^i(x)$ where $x \in S$ and $i \geq 0$ are algebraically independent over A. When X has transformal dimension 0, we define the *total dimension* of X to be the transcendence degree of $A(X)_\sigma$ over A. We write $dim(X)$ for the total dimension of X over A. See section 4.1 of [Hru22] for a more complete account of various notions of dimension in difference algebra.

Recall that ACFA is the model completion of the theory of inversive difference fields. See [CH99b] for an explicit axiomatisation. ACFA is not complete but all completions of ACFA are obtained by specifying the field characteristic and the action of σ on the algebraic closure of the prime field. Recall also that for any $K \models ACFA$ the fixed field $\sigma(x) = x$ is a model of the theory of pseudofinite fields. The reader is referred to the first few pages of [CH99b] for the basic model theory of ACFA. In this chapter we will revisit quantifier elimination down to existential formulas.

The fundamental theorem of [Hru22] says that ACFA is the asymptotic theory of the difference fields K_q, where q is a prime power, K_q is the algebraic closure of the field with q elements and σ is interpreted as $x \mapsto x^q$. We will rely heavily on the twisted Lang-Weil estimates of [Hru22]. In Section 2.1.3 we will use some algebraic corollaries of the main theorem of [Hru22], and we will use the full strength of this theorem in Section 2.4 when we discuss asymptotic counting estimates.

2.1.2 Galois formulas

We review the quantifier-elimination in ACFA. Let A be a difference field. A finite Galois extension L of A will be said to be invariant if σ extends to some endomorphism of L. The join of all invariant finite Galois extensions of A will be called the *effective algebraic closure of* A and we write A^e for this field. Combining Corollary 1.5 and (2.8) in [CH99b] yields the following description of types in ACFA:

Lemma 2.1.1 ([CH99b]). *Let $K \models ACFA$, let $A \leq K$ be an inversive difference field and let a, b be tuples in K. Let $A(a)^{inv}$ and $A(b)^{inv}$ be the inversive closure of the fields $A(a)$ and $A(b)$ respectively. Then* $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$ *if and only if there is a difference field isomorphism between $(A(a)^{inv})^e \to (A(b)^{inv})^e$ fixing A and sending a to b.*

We will deduce a restricted form of quantifier-elimination for A-definable sets contained in A-definable varieties of finite total dimension. We will rely heavily on the next definition for the rest of this chapter.

Definition 2.1.2. *Let A be a difference field. We say that a formula $\phi(x)$ over A is a Galois formula if there are A-definable varieties of finite total dimension $V_2(x)$ and $V_1(x, y)$ where y is a tuple of variables, and a finite family of polynomials $\mathcal{P}$ over $A(x)_\sigma$ such that, in any algebraically closed field K containing A*

1. *the set of realisations of ϕ is equal to the projection $V_1(x, y) \to V_2(x)$*
2. *for every generic $(a, b) \in V_1$, the difference field $A(a, b)_\sigma$ is the splitting field of $\mathcal{P}$ over $A(a)_\sigma$ and is a finite invariant extension of $A(a)_\sigma$.*

We do not set an explicit syntax for Galois formulas, but it is clear how to do this if one so wishes. We will assume that we have decided on some explicit syntax for Galois formulas, and we will assume that the family of polynomials $\mathcal{P}$ in Definition 2.1.2 is always explicitly given.

When ϕ is a Galois formula over A and $\mathcal{P}$, V_1, V_2 are as in the definition and a is a generic point of V_2, we will say that the splitting field L of $\mathcal{P}$ over $A(a)_\sigma$ is *the field extension associated to ϕ*. This terminology applies even if we are in a field K where a does not belong to the set ϕ. Note that in such a situation, L is always a finite invariant extension of $A(a)_\sigma$.

We will often work with the perfect hull $A(a)_\sigma^{insep}$, in which case the *Galois extension associated to ϕ* is the splitting field of $\mathcal{P}$ over $A(a)_\sigma^{insep}$.

Lemma 2.1.3. *Let $K \models ACFA$ and $A \leq K$ an inversive subfield. Let $\phi(x)$ be a definable set over A contained in a variety of finite total dimension. In K, ϕ is equivalent to a Galois formula over A.*

Proof. We use the following claim:

Claim 2.1.3.1. *Suppose $a \in K$ lies in a variety over A of finite total dimension. Then $(A(a)_\sigma^{insep})^e$ is inversive.*

Proof of Claim. It is enough to prove that $\sigma^{-1}(a)$ is in $(A(a)_\sigma^{insep})^e$. Since the extension of $A(a)_\sigma^{insep}$ by $\sigma^{-1}(a)$ is clearly invariant, it is enough to show that $\sigma^{-1}(a)$ is algebraic over $A(a)_\sigma$. Since a is algebraic over $\sigma(a), \ldots, \sigma^n(a)$ for n sufficiently large and A is inversive, the claim follows. □

Observe that, up to logical equivalence, Galois formulas are preserved under disjunctions. By the claim and Lemma 2.1.1, it is enough to show that for $a, b \in K$ contained in A-definable varieties of finite total dimension, $(A(a)_\sigma^{insep})^e \cong_A A(b)_\sigma^{insep})^e$ if and only if a and b satisfy the same Galois formulas over A. This is clear. See Proposition 3.7 in [Joh19] for a similar proof in a similar context. □

When ϕ is a Galois formula over A, we define the total dimension of ϕ over A, or $dim(\phi)$, as the total dimension of the variety V_1 such that ϕ is the projection $V_1 \to V_2$. We also write $cl_A(\phi)$ for the smallest difference variety over A containing ϕ.

2.1.3 The definable measure in ACFA on definable sets of finite total dimension

Let $K \models ACFA$ and let $A \leq K$ be a difference field such that K is $|A|^+$-saturated. The notions of transformal and total dimension clearly extend to quantifier-free definable sets over A. For $n \geq 0$, write $qfDef_n(A)$ for the quantifier-free A-definable sets with total dimension $\leq n$. Write DF_A^n for the family of finitely generated difference field extensions of A of transcendence degree $\leq n$. Write IDF_A^n for the family of finitely generated inversive difference field extensions of A of transcendence degree $\leq n$. There are obvious 1-1-correspondences between DF_A^n, IDF_A^n, and irreducible difference varieties of dimension n over A.

We will see that the transformal function fields provide the correct setting to describe the measure on sets of transformal dimension 0. We begin by using the main theorem of [Hru22] to define the measure on quantifier-free sets of transformal dimension 0. We reserve the asymptotic component of this theorem for Section 2.4, in Theorem 2.4.1.

Theorem 2.1.4 ([Hru22])**.** *let $K \models ACFA$. For every n, there is a definable measure $\mu_n : qfDef_n(K) \to \mathbb{Q}^{\geq 0}$ such that*

1. *$\mu_n(X) = 0$ if X has total dimension $< n$*
2. *For every n, $\mu_n(\bigwedge_{i=1}^n \sigma(x_i) = x_i) = 1$*
3. *Fubini holds for the system of measures (μ_n): if $f : Y \to X$ is a definable surjective map between varieties such that for sufficiently generic $a \in X$, $\mu_n(f^{-1}(a)) = \gamma > 0$ and such that $\mu_k(X) = \lambda > 0$, then $\mu_{n+k}(Y) = \gamma\lambda$.*

Fixing $K \models ACFA$ and $A \leq K$, we see that μ_n on $qfDef_n(A)$ corresponds uniquely to a function $\mu_n : DF_A^n \to \mathbb{Q}^{\geq 0}$ and to a function $IDF_A^n \to \mathbb{Q}^{\geq 0}$. Indeed, even though an element L in IDF_A^n is usually not finitely generated as a difference field over A, it is equal to the inversive hull of some $L \in DF_A^n$ which is uniquely determined up to transformally inseparable extensions.

For L an arbitrary inversive difference field, write L^{tr} for the field L with the automorphism $\sigma^{tr} = \sigma^{-1}$. Then tr is a map $IDF_A^n \to IDF_{A^{tr}}^n$. If $L \in DF_A^n$ and $L' \in DF_{A^{tr}}^n$, we say that L and L' are inversion-dual if $L^{inv} \cong_A ((L')^{inv})^{tr}$, where L^{inv} is the inversive hull of L.

We have additional information about the measure μ_n:

Proposition 2.1.5 ([Hru22]). *Let $K \models ACFA$ and take (μ_n) the definable measures from Theorem 2.1.4.*

1. *For all n, μ_n is invariant under base change in the following sense: let $A \leq A'$ be algebraically closed inversive difference fields, let $L \in DF_A^n$ and let L' be the field of fractions of $L \otimes_A A'$. Then $\mu_n(L/A) = \mu_n(L'/A')$.*

2. *For all n, and for all difference fields $A, A' \leq K$, if $A \leq A' \leq A^{inv}$, then $\mu_n(L/A) = \mu_n(L'/A')$ when $L \in DF_A^n$, $L' \in DF_{A'}^n$, and $L' \leq (L^{insep})^{inv}$, where L^{insep} is the perfect hull of L.*

3. *Let $n \geq 0$, let $A \leq K$ be algebraically closed and inversive and let $L \in DF_A^n$ and $L' \in DF_{A^{tr}}^n$ with L, L' inversion dual. Then*
$$\mu_n(L/A) = \frac{[L' : \sigma(L')]}{[L : \sigma(L)]_{insep}}.$$

We deduce the following important lemma:

Lemma 2.1.6. *Let $K \models ACFA$ and let $A \leq K$ be algebraically closed and inversive. Let $n \geq 1$ and let L, M be difference fields with M/L a finite field extension. Then*

1. *if $L \in DF_A^n$, then $M \in DF_A^n$ and $[L : \sigma(L)] = [M : \sigma(M)]$ and $[L : \sigma(L)]_{insep} = [M : \sigma(M)]_{insep}$*

2. *if $L \in IDF_A^n$, then $M \in IDF_A^n$ and there exist $L_0, M_0 \in DF_A^n$ with M_0/L_0 a finite field extension such that $L_0^{inv} = L$ and $M_0^{inv} = M_0L = M$*

3. *if $M, L \in DF_A^n$, there exists $M', L' \in DF_{A^{tr}}^n$ with M'/L' a finite field extension such that M' is inversion-dual to M and L' is inversion-dual to L.*

4. *if $M, L \in DF_A^n$, $\mu_n(M/A) = \mu_n(L/A)$.*

Proof. (1) If $L \in DF_A^n$, it is clear that $M \in DF_A^n$. Moreover, $L/\sigma(L)$ and $M/\sigma(M)$ are finite field extensions. Then we have $[M : \sigma(M)][\sigma(M) : \sigma(L)] = [M : \sigma(L)] = [M : L][L : \sigma(L)]$ and $[\sigma(M) : \sigma(L)] = [M : L]$ so $[M : \sigma(M)] = [L : \sigma(L)]$. Since the same holds for the separable degree, the statement follows.

(2) If $L \in IDF_A^n$ then $[M : L] = [M : \sigma(L)] = [M : \sigma(M)][\sigma(M) : \sigma(L)]$ and we deduce that $[M : \sigma(M)] = 1$, so $M \in IDF_A^n$.

Let S be a finite set of generators of M over L. For every $a \in S$ there is polynomial h_a over L such that $\sigma(a) = h(a)$. Let F_0 be a finite set of generators of L over A as an inversive difference field which includes the coefficients of each h_a and of the minimal polynomial of each $a \in S$ over L. Define $L_0 = A(F_0)_\sigma$

and $M_0 = L_0(S)_\sigma$. Then M_0/L_0 is a finite extension of fields in DF^n_A. Moreover M_0L is inversive so $M = M_0L$ and hence $M = M_0^{inv}$.

(3) By (2), $ML^{inv} = M^{inv}$ and we can apply (2) to $(M^{inv})^{tr}/(L^{inv})^{tr}$ in $IDF^n_{A^{tr}}$.

(4) By (1), $[L : \sigma(L)]_{insep} = [M : \sigma(M)]_{insep}$. By (3) we can find $M'/L' \in DF^n_{A^{tr}}$ with M, M' and L, L' inversion-dual. By (1), $[L' : \sigma(L')] = [M' : \sigma(M')]$. By Proposition 2.1.5(3), we have $\mu_n(M/A) = \mu_n(L/A)$. □

By [RT06], we know that the measures μ_n from Theorem 2.1.4 extend to definable measures on all definable sets of total dimension n and that these measures have the same properties as in Theorem 2.1.4. Note however that this is straightforward to prove directly from Lemma 2.1.3, expressing such definable sets as projections, breaking them up into irreducible components, and counting the degrees of the projections. Henceforth, μ_n will refer to the definable measures on definable sets of dimension n.

2.1.4 A new characterisation of the definable measure in ACFA

In this section, we give a new characterisation of the definable measures μ_n. In fact we could follow this approach from scratch to define the measures μ_n. As we already have the results of [RT06] or the approach sketched in the previous section, we will only aim to give an alternative characterisation of the μ_n.

If ϕ is a definable set in $K \models ACFA$ and $A \le K$, recall that $cl_A(\phi)$ is the smallest variety over A containing ϕ.

Definition 2.1.7. *Let A be an inversive difference field. Let ϕ be a Galois formula over A and suppose that $cl_A(\phi)$ is absolutely irreducible. Let a be a generic point of $cl_A(\phi)$ and let L be the field extension of $A(a)^{insep}_\sigma$ associated to ϕ. We view L as a field with no difference structure.*

Define $N(\phi)$ to be the number of extensions of σ from $A(a)^{insep}_\sigma$ to an endomorphism τ of L such that $\phi(a)$ is satisfied in the difference field (L, τ). Define $\nu(\phi) = \frac{N(\phi)}{[L:A(a)^{insep}_\sigma]}$.

Fix ϕ and a generic point a of $cl_A(\phi)$ as in the above definition. Choose a partition (c, b) of the tuple a such that L is a regular extension of $A(b)^{insep}_\sigma$. Write $B = (A(b)_\sigma)^{alg}$. Define $M(\phi(x, b))$ to be the number of extensions of σ from $B(a)_\sigma$ to an endomorphism τ of the composite BL such that $\phi(a)$ is satisfied in the structure (BL, τ).

Lemma 2.1.8. *Take ϕ, $a = (c, b)$ and L as above and suppose that L is a regular extension of $A(b)^{insep}_\sigma$. Then $N(\phi) = M(\phi(x, b))$.*

Proof. As before, let $B = (A(b)_\sigma)^{alg}$. Let Σ_1 and Σ_2 respectively be the sets of extensions of σ from $A(a)^{insep}_\sigma$ to L and from $B(a)_\sigma$ to BL such that $\phi(a)$

is satisfied in the corresponding structures. We check that restriction to L induces a bijection $\Sigma_2 \to \Sigma_1$.

Note that σ extends uniquely from $B(a)_\sigma$ to $B(a)_\sigma^{insep}$. Since any $\tau \in \Sigma_2$ must be an endomorphism of L, it is clear that restriction induces an injection $\Sigma_2 \to \Sigma_1$. Conversely, for $\tau \in \Sigma_1$, use the fact that L is a regular extension of $A(b)_\sigma$ to extend σ to an endomorphism τ of BL compatible with σ on B. Then $\phi(a)$ is true in (BL, τ) because ϕ is an existential formula. $\square$

Let $K \models ACFA$ and let A be a small inversive subfield. Let ϕ be a Galois formula over A and let $a = (c, b)$, B and L be as above. Observe that even if $\phi(a)$ is false in K, $\phi(x, b)$ is satisfiable. Indeed, using model completeness and the description of types in Lemma 2.1.1, we can construct a difference field containing B and realising $\phi(x, b)$ and we can embed this field over B in K.

Informally, the next proposition says that with a, B and L as above, the extensions of σ from $B(a)_\sigma$ to L are all equally likely to arise inside a model K of ACFA and this determines the probability that $\phi(a)$ is true, when the measure is normalised by $cl_A(\phi)$. We write $cl_A(\phi)$ for the smallest difference variety over A containing ϕ.

Proposition 2.1.9. *Let $K \models ACFA$ and let $A \leq K$ be an inversive subfield. Let ϕ be a Galois formula over A such that $cl_A(\phi)$ is absolutely irreducible over A. Let $a \in cl_A(\phi)$ be generic over A and let L be the splitting field associated to ϕ over $A(a)_\sigma^{insep}$. Let $a = (c, b)$ be a partition of a and suppose that L is a regular extension of $A(b)_\sigma^{insep}$. Write $B = (A(b)_\sigma)^{alg}$ and suppose that $cl_B(\phi(x, b))$ has total dimension d. Then*

$$\mu_d(\phi(x, b)) = \nu(\phi)\mu_d(cl_B(\phi(x, b))).$$

Proof. This proof is similar to the proof of Proposition 11.1 in [Hru02]. We only consider the definable measure in dimension d so we write $\mu = \mu_d$. Write $B = (A(b)_\sigma)^{alg}$. By Lemma 2.1.8, it is enough to show that

$$\mu(\phi(x, b)) = \nu'(\phi(x, b))\mu(cl_B(\phi(x, b)))$$

where

$$\nu'(\phi(x, b)) = \frac{M(\phi(x, b))}{[BL : B(a)_\sigma^{insep}]} = \frac{M(\phi(x, b))}{[L : A(a)_\sigma^{insep}]}.$$

Suppose that $\phi(x, y)$ is the projection of varieties $\pi : Y \to X$. Write $X(b), Y(b)$ respectively for the subvarieties of X, Y consisting of points which project onto b by restricting to the appropriate coordinates. Therefore, $\phi(x, b)$ is the image of the projection $\pi : Y(b) \to X(b)$. We can assume that $X(b) = cl_B(\phi(x, b))$. In order to calculate $M(\phi(x, b))$, we can consider each irreducible component of $Y(b)$ over B separately, since these correspond to different extensions of σ from $B(a)_\sigma$ to BL. Therefore we assume that $Y(b)$ is irreducible over B. Then the projection $\pi : Y(b) \to X(b)$ has constant degree η.

Let Σ be the set of extensions τ of σ from $B(a)_\sigma$ to BL such that $\phi(a)$ is true in (BL, τ). Write $G = Gal(BL/B(a)_\sigma^{insep})$. Note that G acts on Σ by both left and right translation and hence by conjugation. Fix an element τ_0 of Σ and let $H = C_G(\tau_0) = \{g \in G \mid \tau_0 g = g\tau_0\}$.

Claim 2.1.9.1. $\eta = |H|$.

Proof of claim. We work in (BL, τ_0) and we show that H acts sharply transitively on the pullback of a to $Y(b)$. Let $y_1, y_2 \in \pi^{-1}(a)$. Since $Y(b)$ is irreducible over B, there is $g \in G$ such that $g(y_1) = y_2$. Since BL is Galois over $B(a)_\sigma^{insep}$ and y_1, y_2 both generate BL over $B(a)_\sigma^{insep}$, g is unique.

We check that $g \in H$. Since L is τ_0-invariant, we have $\tau_0(y_1) = P(y_1)$ for some polynomial map P over $A(a)_\sigma^{insep}$. Since $Y(b)$ is irreducible, we also have $\tau_0(y_2) = P(y_2)$. Now we conclude $\tau_0(g(y_1)) = P(y_2) = g(P(y_1)) = g(\tau_0(y_1))$ so $g \in H$. □

Claim 2.1.9.2. $|\Sigma| = [G : H]$.

Proof of claim. If $\tau_1, \tau_2 \in \Sigma$, by irreducibility of $Y(b)$ we have $(BL, \tau_1) \cong_{B(a)_\sigma} (BL, \tau_2)$. Hence there is $g \in G$ with $g\tau_1 = \tau_2 g$. Therefore τ_1 and τ_2 are G-conjugate and the number of such conjugates is $[G : H]$. □

By the first claim and by Fubini, we have

$$\mu(\phi(x, b)) = \frac{\mu(Y(b))}{|H|}.$$

By irreducibility of $Y(b)$ and $X(b)$ and by Lemma 2.1.6(4) we also have $\mu(Y(b)) = \mu(X(b))$. By the second claim, we deduce

$$\mu(\phi(x, b)) = \frac{1}{|H|}\mu(X(b)) = \frac{|\Sigma|}{|G|}\mu(X(b)) = \nu'(\phi(x, b))\mu(X(b))$$

□

In Proposition 2.1.9, the assumption that L is a regular extension of $A(b)_\sigma^{insep}$ is essential. We will usually need to change the domain of definition of ϕ in order to put ourselves in the situation where L is regular over $A(b)_\sigma^{insep}$. Since L is regular over a finite extension of $A(b)_\sigma^{insep}$, this can be achieved by changing the domain of ϕ and lifting the whole definable set to a cover, so that ϕ ranges over tuples of the form (x, y, y') and y' codes Galois information.

Moreover, we will be interested in applying Proposition 2.1.9 to many different partitions of the tuple a at the same time. This motivates the notion of systems of difference fields and difference varieties in the next section.

2.2 Regular Systems and Definable Edge-uniformity

In this section, we often work inside an arbitrary algebraically closed inversive difference field K. We introduce systems of difference fields and of varieties, and we isolate the fundamental notion of a regular system. This is a purely algebraic or geometric notion, but we will see that it corresponds to a certain form of combinatorial regularity in the sense of Szemeredi when K is a model of ACFA.

2.2.1 Regular systems of perfect difference fields

In the following definition, we record some notation which we will use often:

Definition 2.2.1. *V usually denotes a finite set. $P(V)$ is the powerset of V and $P(V)^- = P(V) \setminus \{V\}$. For $k \leq |V|$, $P_k(V) = \{u \in P(V) \mid |u| \leq k\}$. For $u \subseteq V$ and $0 \leq k \leq |u|$, $P(V, u, k) = \{v \subseteq V \mid |v \cap u| \leq k\}$.*

If I is a collection of subsets of V closed under taking subsets, we say that I is downward closed. If in addition $\bigcup I = V$, we say that I is a simplicial complex on V.

If I is downward closed and $u \subseteq V$, define $I_u = \{v \cap u \mid v \in I\}$. If I is a collection of subsets of V, write $\partial I = \{v \mid \exists u \in I, v \subseteq u\}$.

We fix an algebraically closed inversive difference field K. K is not usually assumed to be a model of ACFA and we will be careful to indicate when we need this assumption.

In this thesis, we write composites of fields as products: if $(K_i : i \in I)$ is a collection of fields contained in a field K, $\prod_{i \in I} K_i$ denotes the composite of this family. We never write down Cartesian products of fields so this notation is not ambiguous.

If $A, B, C \subseteq K$, we say that A is independent from B over C if $((AC)^{inv})^{alg}$ is linearly disjoint from $((BC)^{inv})^{alg}$ over $(C^{inv})^{alg}$ and we write $A \downarrow_C B$. When $(B_i)_I$ is a family of subsets of K and $A \subseteq K$, we say that the family $(B_i)_I$ is independent over A if for all disjoint $I_1, I_2 \subseteq I$, $\prod_{i \in I_1} AB_i$ is independent from $\prod_{i \in I_2} AB_i$ over A.

When $K \models ACFA$, this notion of independence coincides with model theoretic independence. It is useful to recall from [CH99b] that ACFA has existence of amalgamation for all orders. This is the Generalised Independence Theorem. We will not use this theorem directly because we will work at the more detailed level of systems of difference fields.

Definition 2.2.2. *Let V be a finite set and let $A \leq K$ be a perfect inversive difference field. Let $\mathcal{S}$ be a functor from $P(V)$ to finitely generated perfect difference field extensions of A of finite total dimension contained in K, where the arrows between elements of $P(V)$ and between the difference field extensions of A are inclusions.*

We say that $\mathcal{S}$ is a system of difference fields on V over A if $\mathcal{S}$ satisfies the following conditions:

1. *$\mathcal{S}(\emptyset) = A$*

2. *The family $\{\mathcal{S}\{i\} \mid i \in V\}$ is independent over A.*

3. *For $u \subseteq V$ with $|u| \geq 2$, $\mathcal{S}(u)$ is a finite invariant Galois extension of $\prod_{v \in P(u)^-} \mathcal{S}(v)$.*

Let I be a simplicial complex on V. We say that $\mathcal{S}$ is an I-system if for all $u \notin I$, $\mathcal{S}(u) = \prod_{v \in P(u)^-} \mathcal{S}(v)$.

If $\mathcal{S}$ is an I-system on V over A, we will often want to restrict $\mathcal{S}$ to smaller simplicial complexes. Let $J \subseteq I$ be downward closed, but not necessarily a simplicial complex. Then we write $\mathcal{S} \restriction J$ for the J-system on $\bigcup J$ over A defined by $(\mathcal{S} \restriction J)(u) = \prod_{J_u} \mathcal{S}(v)$ for any $u \subseteq \bigcup J$.

Definition 2.2.3. *Let $A \leq K$ be a perfect inversive difference field and let $\mathcal{S}$ be a system of difference fields on $P(V)$ over A in K. Let K' be an algebraically closed inversive difference field containing K, let A' be a perfect inversive difference field containing A and let $\mathcal{S}'$ be a system of difference fields on $P(V)$ over A' in K'.*

Let I be a downward-closed collection of subsets of V. We say that $\mathcal{S}'$ is an I-refinement of $\mathcal{S}$ over A if

1. *A' is a finitely generated extension of A*

2. *for every $u \in I$, $\mathcal{S}'(u)$ is a finite invariant Galois extension of $\mathcal{S}(u)$*

3. *for every $u \notin I$, $\mathcal{S}'(u) = \mathcal{S}(u) \prod_{v \in I_u} \mathcal{S}'(v)$.*

Remarks: (1) We usually move freely between K and K' when taking refinements, even if we do not explicitly reference the fields. In fact, we often omit the ambient field K, taking it to be fixed in our background assumptions.

(2) Let I be a simplicial complex on V and $J \subseteq I$ a downward closed subset. Let $\mathcal{S}$ be an I-system on V over A' and let $\mathcal{S}'$ be a J-refinement of $\mathcal{S}$ over A. Then $\mathcal{S}'$ is an I-system on $P(V)$ over A'.

The next definition is fundamental in our work.

Definition 2.2.4. *Let $\mathcal{S}$ be a system of perfect difference fields on V over A. We say that $\mathcal{S}$ is regular if for every nonempty $u \subseteq V$, $\mathcal{S}(u)$ is linearly disjoint from $\prod_{v \in P(u)^-} \mathcal{S}(v)^{alg}$ over $\prod_{v \in P(u)^-} \mathcal{S}(v)$.*

Remark: The notion of a $P(V)^-$-system $\mathcal{S}$ of difference fields over A is closely related to the notion of an amalgamation functor over A. See [Hru06] for a general discussion of amalgamation functors. $\mathcal{S}$ is not exactly the solution of an amalgamation functor on $P(V)^-$ because the fields $\mathcal{S}(v)$ are not algebraically closed, but it is the result of truncating the solution $(\mathcal{S}(v)^{alg})_{v\in P(V)^-}$ down to various subfields.

We will show that systems of difference fields can always be refined to regular systems. We use the following technical lemma:

Lemma 2.2.5. *Let $\mathcal{S}$ be a system of difference fields on V over an algebraically closed difference field A contained in K. Suppose A is existentially closed in K. Let I be a family of subsets of V and let $u \subseteq V$. Then $\mathcal{S}(V)^e \prod_{v\in I} \mathcal{S}(v)^{alg}$ is linearly disjoint from $\mathcal{S}(u)^{alg}$ over $\mathcal{S}(u)^e \prod_{v\in I_u} \mathcal{S}(v)^{alg}$.*

Proof. We first prove the lemma under the assumption that the base A is ω_1-saturated. We will remove this assumption at the end of the proof.

We can assume that $\mathcal{S}$ is an $\{\{i\} \mid i \in V\}$-system. For every $i \in V$, let a_i be a finite tuple such that $\mathcal{S}(i) = A(a_i)_\sigma^{insep}$ and write $a_v = (a_i)_{i\in v}$ for any $v \subseteq V$. Take $d \in \mathcal{S}(V)^e \prod_I \mathcal{S}(v)^{alg} \cap \mathcal{S}(u)^{alg}$. There is a finitely generated algebraically closed difference field B contained in A such that $B(a_u)_\sigma$ is linearly disjoint from A over B and such that $d \in B(a_V)_\sigma^e \prod_I B(a_v)_\sigma^{alg} \cap B(a_u)_\sigma^{alg}$.

By ω_1-saturation and existential closure of A, we can find $a'_{V\setminus u} \in A$ such that the difference fields $B(a_{V\setminus u})_\sigma$ and $B(a'_{V\setminus u})_\sigma$ are isomorphic over B. Since $B(a'_{V\setminus u)}) \subseteq A$, $B(a'_{V\setminus u})_\sigma$ is linearly disjoint from $B(a_u)_\sigma$ over B. Hence the difference fields $B(a_V)_\sigma$ and $B(a_u, a'_{V\setminus u})_\sigma$ are isomorphic over $B(a_u)_\sigma$.

As a pure field, $B(a_V)_\sigma^e$ depends only on the difference field isomorphism type of $B(a_V)_\sigma$. Hence the isomorphism $B(a_V)_\sigma \to B(a_u, a'_{V\setminus u})_\sigma$ extends to an isomorphism of pure fields $B(a_V)_\sigma^{alg} \to B(a_u, a'_{V\setminus u})_\sigma^{alg}$ over $B(a_u)_\sigma^{alg}$ which restricts to an isomorphism $B(a_V)_\sigma^e \to B(a_u, a'_{V\setminus u})_\sigma^e$ over $B(a_u)_\sigma^e$. Since $d \in B(a_u)_\sigma^{alg}$, the isomorphism constructed above fixes d and we deduce that $d \in B(a_u, a'_{V\setminus u})_\sigma^e \prod_I B(a_{v\cap u}, a'_{v\setminus u})_\sigma^{alg}$. Since $a'_{V\setminus u} \subseteq A$, we have $d \in \mathcal{S}(u)^e \prod_{I_u} \mathcal{S}(v)^{alg}$ as desired.

Now suppose that A is not ω_1-saturated. Consider the structure K with an additional predicate for A. Let K^* be an ω_1-saturated elementary extension and let A^* be the ω_1-saturated algebraically closed difference field extending A. Then the independence properties of the tuples a_V are preserved by moving to K^* and A^* is also existentially closed in K^*, so we can apply the lemma to the system $\mathcal{S}^*$ defined by lifting the base to A^*.

Take $d \in \mathcal{S}(V)^e \prod_I \mathcal{S}(v)^{alg} \cap \mathcal{S}(u)^{alg}$. Then $d \in K \cap \mathcal{S}^*(u)^e \prod_{I_u} \mathcal{S}^*(v)^{alg}$. By elementarity, we internalise all parameters into the base A and we recover the desired result. $\square$

The next proposition will provide the regular partitions in our hypergraph regularity results.

Proposition 2.2.6. *Let $\mathcal{S}$ be an I-system of difference fields on V over A. Then $\mathcal{S}$ has a ∂I-refinement $\mathcal{S}'$ over some finitely generated extension A' of A which is regular.*

Proof. Moving to field extensions if necessary, we can assume that A is algebraically closed, inversive, and existentially closed in K. The field A' in the statement is then recovered by choosing finitely many parameters in A over which $\mathcal{S}'$ is defined. We use the following claim:

Claim 2.2.6.1. $\mathcal{S}(V)^e \cap \prod_{v \in P(V)^-} \mathcal{S}(v)^{alg} = \prod_{v \in P(V)^-} \mathcal{S}(v)^e$.

Proof of claim. We show that for every downward-closed collection J of subsets of V, we have $\mathcal{S}(V)^e \cap \prod_J \mathcal{S}(v)^{alg} = \prod_J \mathcal{S}(v)^e$. We proceed by induction on $|J|$. The case $|J| = 1$ is an instance of Lemma 2.2.5.

Let $J = J' \cup \{w\}$ where $P(w)^- \subseteq J'$ and suppose the claim holds for J'. Then we know that $\mathcal{S}(V)^e$ is linearly disjoint from $\mathcal{S}(w)^e \prod_{J'} \mathcal{S}(v)^{alg}$ over $\prod_J \mathcal{S}(v)^e$. By the Towers property, it is enough to show that $\mathcal{S}(V)^e \prod_{J'} \mathcal{S}(v)^{alg}$ is linearly disjoint from $\prod_J \mathcal{S}(v)^{alg}$ over $\mathcal{S}(w)^e \prod_{J'} \mathcal{S}(v)^{alg}$.

By Lemma 2.2.5, $\mathcal{S}(V)^e \prod_{J'} \mathcal{S}(v)^{alg}$ is linearly disjoint from $\mathcal{S}(w)^{alg}$ over $\mathcal{S}(w)^e \prod_{P(w)^-} \mathcal{S}(v)^{alg}$. By the Towers property, $\mathcal{S}(V)^e \prod_{J'} \mathcal{S}(v)^{alg}$ is linearly disjoint from $\mathcal{S}(w)^{alg} \prod_{J'} \mathcal{S}(v)^{alg} = \prod_J \mathcal{S}(v)^{alg}$ over $\mathcal{S}(w)^e \prod_{J'} \mathcal{S}(v)^{alg}$, as desired. ☐

Now let (u_n) be an enumeration of I such that for all n, u_n is a maximal element of $\{u_m \mid m \geq n\}$. Fix $k \geq 0$ and assume we have constructed a ∂I-refinement $\mathcal{S}'$ of $\mathcal{S}$ such that for all $n < k$, $\mathcal{S}'(u_n) \cap \prod_{P(u_n)^-} \mathcal{S}(v)^{alg} = \prod_{P(u_n)^-} \mathcal{S}'(v)$.

If $\mathcal{S}'(u_k) \cap \prod_{P(u_k)^-} \mathcal{S}(v)^{alg} \neq \prod_{P(u_k)^-} \mathcal{S}'(v)$, then by applying the claim to $\mathcal{S}' \restriction P(u_k)$, we construct finite extensions $\mathcal{S}''(v) \subseteq \mathcal{S}'(v)^e$ for every $v \in P(u_k)^-$ such that

$$\mathcal{S}'(u_k) \cap \prod_{P(u_k)^-} \mathcal{S}(v)^{alg} \subseteq \prod_{P(u_k)^-} \mathcal{S}''(v).$$

By making coherent choices of field extensions, we can assume that the extensions $\mathcal{S}''(v)$ define a $P(u_k)^-$-refinement of $\mathcal{S}'$.

$\mathcal{S}''(u_k)$ is clearly linearly disjoint from $\prod_{P(u_k)^-} \mathcal{S}(v)^{alg}$ over $\prod_{P(u_k)^-} \mathcal{S}''(v)$. For $n < k$, $\mathcal{S}'' \restriction P(u_n)$ is a $P(u_n)^-$-refinement of $\mathcal{S}' \restriction P(u_n)$, so $\mathcal{S}''(u)$ is also linearly disjoint from $\prod_{P(u_n)^-} \mathcal{S}(v)^{alg}$ over $\prod_{P(u_n)^-} \mathcal{S}''(v)$. The proposition follows inductively. ☐

We will now prove some properties of regular systems which will be useful in the rest of this chapter. The key technical fact behind these results is Lemma 2.2.7. Its proof is close to the argument which underpins the Generalised Independence Theorem in ACFA (see [CH99b]).

Lemma 2.2.7. *Let $\mathcal{S}$ be an I-system of perfect difference fields on V over A. Fix $u \subseteq V$ and take J any collection of subsets of V. Then $\mathcal{S}(V) \prod_{v \in J \cup I_u} \mathcal{S}(v)^{alg}$ is linearly disjoint from $\mathcal{S}(u)^{alg}$ over $\mathcal{S}(u) \prod_{v \in J_u \cup I_u} \mathcal{S}(v)^{alg}$.*

Proof. Let (a_i) be a finite collection of elements of $\mathcal{S}(u)^{alg}$ and suppose that (a_i) is linearly dependent over $\mathcal{S}(V) \prod_{J \cup I_u} \mathcal{S}(v)^{alg}$ as witnessed by elements (b_i). We can find

1. a tuple $c \subseteq \prod_{I_u} \mathcal{S}(v)^{alg}$
2. tuples d_v where $d_v \subseteq \mathcal{S}(v)^{alg}$ for $v \in J$
3. tuples e_v where $e_v \subseteq \mathcal{S}(v)$ for $v \in I$
4. rational maps $f_i(x, y, z)$ for all i

such that $b_i = f_i(c, (d_v)_J, (e_v)_I)$ for all i.

For every $v \in J$ choose tuples $\alpha_v \subseteq \mathcal{S}(v \cap u)$ and $\beta_v \subseteq \mathcal{S}(v \setminus u)$ such that d_v is algebraic over $A(\alpha_v, \beta_v)$. Similarly, for every $v \in I$, choose tuples $\alpha'_v \subseteq \mathcal{S}(u \cap v)$ and $\beta'_v \subseteq \mathcal{S}(u \setminus v)$ such that e_v is algebraic over $A(\alpha'_v, \beta'_v)$.

Then (a_i) satisfies some formula

$$\phi((x_i), c, (\alpha_v), (\alpha'_v), (\beta_v), (\beta'_v))$$

which says that (a_i) is linearly dependent over $\mathcal{S}(V) \prod_{J \cup I_u} \mathcal{S}(v)^{alg}$. This formula is in the language of fields. By elementary stability theory, we find tuples (δ_v) and (δ'_v) in A^{alg} such that (a_i) satisfies $\phi((x_i), c, (\alpha_v), (\alpha'_v), (\delta_v), (\delta'_v))$ and this entails that (a_i) is linearly dependent over $\mathcal{S}(u) \prod_{J_u \cup I_u} \mathcal{S}(v)^{alg}$ □

We now prove some useful properties of regular systems. We fix a system $\mathcal{S}$ of difference fields on V over A.

Lemma 2.2.8. *Suppose $\mathcal{S}$ is regular. For all $u \subseteq V$, $\mathcal{S}(u)$ is linearly disjoint from $\prod_{v \in P(V,u,|u|-1)} \mathcal{S}(v)^{alg}$ over $\prod_{v \in P(u)^-} \mathcal{S}(v)$.*

Proof. Let $\mathcal{S}' = \mathcal{S} \restriction P_{|u|-1}(V)$. By Lemma 2.2.7, $\mathcal{S}(u) \prod_{P(u)^-} \mathcal{S}(v)^{alg}$ is linearly disjoint from $\prod_{P(V,u,|u|-1)} \mathcal{S}(v)^{alg}$ over $\mathcal{S}'(u) \prod_{P(u)^-} \mathcal{S}(v)^{alg} = \prod_{P(u)^-} \mathcal{S}(v)^{alg}$. The lemma follows from the definition of regularity and the Towers property. □

Lemma 2.2.9. *Suppose $\mathcal{S}$ is regular. Let I be a collection of subsets of V. Then $\mathcal{S}(V)$ is linearly disjoint from $\prod_{v \in I} \mathcal{S}(v)^{alg}$ over $\prod_{v \in I} \mathcal{S}(v)$.*

Proof. We proceed by induction to show that for any downward-closed collection J of subsets of V, $\prod_{J \cup I} \mathcal{S}(v)$ is linearly disjoint from $\prod_I \mathcal{S}(v)^{alg}$ over $\prod_I \mathcal{S}(v)$. The case $J = \emptyset$ is trivial, so we assume the claim holds for some J' where $J = J' \cup \{u\}$ and $P(u)^- \subseteq J'$.

By the Towers property, it is enough to show that $\mathcal{S}(u)\prod_{J'\cup I}\mathcal{S}(v)$ is linearly disjoint from $\prod_{J'}\mathcal{S}(v)\prod_I\mathcal{S}(v)^{alg}$ over $\prod_{J'\cup I}\mathcal{S}(v)$. By Lemma 2.2.8, $\mathcal{S}(u)$ is linearly disjoint from $\prod_{P(V,u,|u|-1)}\mathcal{S}(v)^{alg}$ over $\prod_{P(u)^-}\mathcal{S}(v)$ and the result follows by the Towers property. $\square$

Lemma 2.2.10. *Suppose $\mathcal{S}$ is regular. Let $u \subseteq V$ and let I be a collection of subsets of V. Then $\mathcal{S}(u)$ is linearly disjoint from $\prod_{v\in I}\mathcal{S}(v)^{alg}$ over $\prod_{v\in I_u}\mathcal{S}(v)$.*

Proof. We can assume that I is a simplicial complex of V and that $u \notin I$. Let $\mathcal{S}'$ be the restriction of $\mathcal{S}$ to I, so that $\mathcal{S}'(V) = \prod_I \mathcal{S}(v)$ and $\mathcal{S}'(u) = \prod_{I_u}\mathcal{S}(v)$. Applying Lemma 2.2.7, $\mathcal{S}(u)\prod_{I_u}\mathcal{S}(v)^{alg}$ is linearly disjoint from $\prod_I\mathcal{S}(v)^{alg}$ over $\prod_{I_u}\mathcal{S}(v)^{alg}$.

By Lemma 2.2.9, $\mathcal{S}(u)$ is linearly disjoint from $\prod_{I_u}\mathcal{S}(v)^{alg}$ over $\prod_{I_u}\mathcal{S}(v)$ and the result follows by the Towers property. $\square$

2.2.2 Regular systems of varieties

We introduce systems of varieties. These are closely related to systems of difference fields and in fact all the key theorems about systems of varieties could be formulated in terms of systems of difference fields. However systems of varieties are more natural from a combinatorial point of view.

In what follows, we will work with projections between varieties. If X, Y are difference varieties in affine space, we say that a map $\pi : X \to Y$ is a projection if $X \subseteq K^m$, $Y \subseteq K^n$ and π restricts X to certain coordinates in affine space. We say that π is dominant if for every top-dimensional component Z of Y, there is a top-dimensional component Z' of X such that π sends Z' to Z and $\pi(Z')$ is not contained in a subvariety of Z. [2]

Let V be a finite set and let A be a perfect inversive difference field. Let Ω be a contravariant functor from $P(V)$ to *difference varieties over A of finite total dimension with dominant projections $\Omega(u) \to \Omega(v)$ when $v \subseteq u$.* We always assume that $\Omega(\emptyset) = 0$, viewed as a variety in K^0, and that $\Omega(u)$ has positive total dimension over A when $u \neq \emptyset$.

For any downward closed collection I of subsets of V, write $\prod(\Omega(v), v \in I)$ for the fibre product of the family $(\Omega(v))_{v\in I}$. This is a difference variety with dominant projections to each $\Omega(v)$ such that for any $v \subseteq v' \in I$, the projection $\prod(\Omega(v), v \in I) \to \Omega(v') \to \Omega(v)$ equals the projection $\prod(\Omega(v), v \in I) \to \Omega(v)$. This variety has the usual universal property of fibre products.

[2] We use projections because there are a few places in this chapter where it is useful to refer to the implicit syntactical structure of Galois formulas. By default, we take Galois formulas to be the result of projecting one variety onto another. Another advantage of working with projections is that it is easy to see when commutative diagrams arise. This will be essential in our arguments. However all our results generalise easily to systems of varieties with general morphisms of difference varieties instead of projections.

Definition 2.2.11. *Let Ω be a functor on $P(V)$ as above. We say that Ω is a system of varieties on V over A if for all $u \subseteq V$ with $|u| \geq 2$,*

1. *there is a finite family of polynomials $\mathcal{P}_u$ over $A(x)_\sigma$ such that points of $\Omega(u)$ can be expressed as pairs (a, b) with $a \in \prod(\Omega(v), v \in P(u)^-)$ and $A(a, b)_\sigma$ is the splitting field of $\mathcal{P}_u$ over $A(a)_\sigma$*

2. *the projection $\Omega(u) \to \prod(\Omega(v), v \in P(u)^-)$ is dominant. Equivalently, $dim(\Omega(u)) = dim(\prod(\Omega(v), v \in P(u)^-))$.*

When $I \subseteq P(V)$ is a simplicial complex, we say that Ω is an I-system of difference varieties if for all $u \notin I$, $\Omega(u) = \prod(\Omega(v), v \in P(u)^-)$ (equivalently, $\mathcal{P}_u$ is empty).

We say that Ω is irreducible if $\Omega(V)$ is absolutely irreducible over A.

Remarks: (1) If Ω is irreducible, then for all u with $|u| \geq 2$, every projection from $\Omega(u)$ to the fibre product $\prod(\Omega(v), v \in P(u)^-)$ has generically constant multiplicity. This will be useful for decomposing systems of varieties into disjoint sections in Section 2.4.

(2) The connection between Galois formulas and systems of varieties is clear: any Galois formula ϕ arises as the image of a projection $\Omega(u) \to \prod(\Omega(v), v \in P(u)^-)$ in some appropriate system of varieties Ω. Hence systems of varieties are a useful framework for studying definable hypergraphs.

(3) We remarked after Definition 2.2.4 that the concept of a $P(V)^-$-system of difference fields should be viewed as analogous to the *solution* of an amalgamation problem. Accordingly, a $P(V)^-$-system of varieties can be viewed as being analogous to the amalgamation problem itself.

We will rely heavily on the notation introduced in the next definition.

Definition 2.2.12. *Let Ω be a system of varieties on V over A.*

1. *For every $u \subseteq V$ with $|u| \geq 2$, we write $\Omega(u)^-$ for the fibre product $\prod(\Omega(v), v \in P(u)^-)$.*

2. *For every $u \subseteq V$, write $\rho_u\Omega(u)$ for the projection of $\Omega(u)$ onto $\Omega(u)^-$.*

We define analogously $\mathcal{S}(u)^- = \prod_{v \in P(u)^-} \mathcal{S}(v)$ when $\mathcal{S}$ is a system of difference fields on V over A.

Definition 2.2.13. *Let Ω be a system of varieties on V over A. Let A' be a perfect inversive difference field containing A and Ω' a system of varieties on V over A'.*

We say that Ω' is a refinement of Ω if for every $u \subseteq V$ there is a finite dominant projection $\Omega'(u) \to \Omega(u)$ and for $u \subseteq v$, the projections $\Omega'(u) \to \Omega(u) \to \Omega(v)$ and $\Omega'(u) \to \Omega'(v) \to \Omega(v)$ commute.

Let I be a collection of subsets of V. We say that Ω' is an I-refinement of Ω if Ω' is a refinement and for every $u \notin I$, $\Omega'(u)$ is the fibre product of the varieties $\Omega(u), (\Omega'(v))_{v \in I_u}$.

We say that Ω' is a surjective refinement of Ω if the projection $\Omega'(V) \to \Omega(V)$ is generically surjective.

Remark: Taking an extension of A if necessary, every system of varieties over A admits a partition into irreducible systems. If Ω' is a surjective refinement of Ω over the extension A', the partition of Ω' into irreducible components will give us a notion of étale-partition of Ω.

Definition 2.2.14. *Let Ω be a system of varieties on V over A. Let a be a generic point of $\Omega(V)$ and write a_u for the image of a under the projection $\Omega(V) \to \Omega(u)$. We define the system of difference fields associated to a to be the system $\mathcal{S}(u) = A(a_u)_\sigma^{insep}$.*

We also say that the system $\mathcal{S}$ above is a system of difference fields associated to Ω. When Ω is irreducible, systems of difference fields associated to Ω are unique up to isomorphism.

We say that Ω is a regular system of varieties if Ω is irreducible and the system of difference fields associated to Ω is regular.

Proposition 2.2.15. *Let Ω be an I-system of varieties on V over A. Then there is a surjective ∂I-refinement Ω' of Ω over an extension A' of A such that each irreducible component of Ω' is regular.*

Proof. We can assume that Ω is irreducible over A. Let a be a generic point of $\Omega(V)$ over A and let $\mathcal{S}$ be the associated system of difference fields on V over A. Write a_u for the projection of a to $\Omega(u)$.

Let $\mathcal{S}'$ be a regular ∂I-refinement of $\mathcal{S}$ over a finitely generated extension A' of A, as given by Proposition 2.2.6. For every $u \in \partial I$, let $\mathcal{P}_u$ be a family of polynomials over $\mathcal{S}(u)$ such that $\mathcal{S}'(u)$ is the splitting field of $\mathcal{P}_u$ over $\mathcal{S}(u)$. We can assume that $\mathcal{P}_u \subseteq \mathcal{P}_v$ when $u \subseteq v$ and we can take $\mathcal{P}_u$ to be over $A'(a_u)_\sigma$.

We can define a system Ω' over A' refining Ω such that the projection $\Omega'(V) \to \Omega(V)$ is generically surjective and for every $u \in \partial I$, $\Omega'(u)$ is a variety of points of the form (c, d) where $c \in \Omega(u)$ and $A'(c, d)_\sigma$ is the splitting field of the polynomials $\mathcal{P}_u$ over $A'(c)_\sigma$.

Every irreducible component of Ω' is regular: for any generic point b of $\Omega'(V)$, even though the system of difference fields associated to b may not be isomorphic to $\mathcal{S}'$ as a system of difference fields, it is isomorphic to $\mathcal{S}'$ in the sense of pure fields, since the field extensions are just the splitting fields of the families $\mathcal{P}_v$. This guarantees regularity of the system associated to b. $\square$

Example: We give a basic example of a $P(V)$-system Ω with a regular $P(V)^-$-refinement. Working over the prime field of K and taking $V = \{0, 1\}$, let $\Omega(0)$

and $\Omega(1)$ be copies of the fixed field. Let $\Omega(V)$ be the variety of points of the form (x, y, z, t) where $x \in \Omega(0)$, $y \in \Omega(1)$, z is a square root of y with $\sigma(z) = z$ and t is a square root of $x + z$ with $\sigma(t) = t$. Take $\Omega'(0) = \Omega(0)$, $\Omega'(1)$ the variety of points of the form (y, z) where z is a square root of y with $\sigma(z) = z$, and take $\Omega'(V) = \Omega(V)$. Then Ω' is a regular $P(V)^-$-refinement of Ω. Note that the projection $\Omega'(V) \to \Omega(V)$ is surjective (in fact it is the identity) and the the projection $\Omega'(1) \to \Omega(1)$ is only dominant. This is consistent with Definition 2.2.13.

While the definition of regularity for systems of difference fields and varieties is natural and easy to state, we note that it is slightly stronger than what will be needed in the rest of this paper. We make this precise in Proposition 2.2.16. First, we recall some classical notions of difference algebra. If K is a difference field, recall that a difference field extension L/K is *monadic* if for any difference field extension M/K, there is at most one difference homomorphism $L \to M$ over K. Equivalently, if L/K is Galois, then L/K is monadic if and only if $\{g \in Gal(L/K) \mid g\sigma = \sigma g \text{ on } L\} = \{e\}$.

Recall also that if L/K is a finite Galois extension and is monadic, then L/K is compatible with every difference field extension of K, meaning that for any M/K, there is N/K and difference homomorphisms $L \to N$ and $M \to N$ over K. It follows that if L/K is a finite Galois extension and is monadic, then σ is the unique extension of $\sigma \restriction K$ to L. See [Coh65] or [Lev08] for details.

Let ϕ be a Galois formula over an inversive difference field A corresponding to a projection of irreducible varieties $X \to Y$. Let a be a generic point of Y and L the Galois extension of $A(a)_\sigma^{insep}$ associated to ϕ. By inspecting the Galois information present in the proof of Proposition 2.1.9, we see that if $L/A(a)_\sigma^{insep}$ gives a monadic extension, then the projection $X \to Y$ has multiplicity 1 and hence $\mu(\phi) = \mu(Y)$. Therefore, from the geometric or combinatorial point of view, ϕ is trivial and information related to monadic field extensions can be ignored. However, in Definitions 2.2.4 and 2.2.14 we have defined regularity with respect to the effective algebraic closure of difference fields, which means that our notion of regularity is stronger than what we need in this paper. The next proposition makes this precise.

Proposition 2.2.16. *Let Ω be an irreducible system of varieties on V over A. The following are equivalent:*

1. *for any system $\mathcal{S}$ of difference fields associated to Ω, for every $u \subseteq V$, $\mathcal{S}(u) \cap \prod_{P(u)^-} \mathcal{S}(v)^{alg}$ is a monadic extension of $\mathcal{S}(u)^-$*

2. *for any $u \subseteq V$, let Ω' be an irreducible $P(u)^-$-refinement of $\Omega \restriction P(u)^-$. Let $\Omega^+(u)$ be the fibre product of $\Omega(u)$ and $\Omega'(u)$ over $\Omega(u)^-$. Then $dim(\Omega^+(u)) = dim(\Omega(u))$.*

Proof. (1) $\Rightarrow$ (2): We check (2) for $u = V$. Let Ω' and Ω^+ be as in the statement. Let $\mathcal{S}$ be a system of difference fields associated to Ω and $\mathcal{S}'$ a system of difference fields associated to Ω'. Write $\mathcal{S}''$ for the restriction of $\mathcal{S}'$ to a system associated to $\Omega \restriction P(V)^-$. By irreducibility of Ω, there is a difference field isomorphism $f : \mathcal{S}''(V) \to \mathcal{S}(V)^-$. We can extend f to a field homomorphism $f : \mathcal{S}'(V) \to \mathcal{S}(V)^{alg}$. Then $f(\mathcal{S}'(V)) \subseteq \prod \mathcal{S}(v)^{alg}$ so $f(\mathcal{S}'(V))$ is linearly disjoint from $\mathcal{S}(V)$ over $f(\mathcal{S}'(V)) \cap \mathcal{S}(V)$ and $f(\mathcal{S}'(V)) \cap \mathcal{S}(V)/\mathcal{S}(V)^-$ is a monadic extension.

By universal compatibility of monadic extensions and taking a Galois conjugate of f if necessary, we can assume that f is a difference field homomorphism on $f^{-1}(f(\mathcal{S}'(V)) \cap \mathcal{S}(V))$. Then by linear disjointness, we can construct a difference operator τ on $\mathcal{S}(V)^{alg}$ such that $\tau = \sigma$ on $\mathcal{S}(V)$ and $\tau = f \circ \sigma \circ f^{-1}$ on $f(\mathcal{S}'(V))$. Now $(\mathcal{S}(V) f(\mathcal{S}'(V)), \tau)$ is a difference field associated to $\Omega^+(V)$ and hence $dim(\Omega^+(V)) = dim(\Omega(V)) = dim(\Omega'(V))$.

(2) $\Rightarrow$ (1): Suppose (1) fails. Let $\mathcal{S}$ be a system of difference fields associated to Ω and assume that $L := \mathcal{S}(V) \cap \prod_{P(V)^-} \mathcal{S}(v)^{alg}$ is not monadic over $\mathcal{S}(V)^-$. By properties of monadic extensions, there is an extension τ of $\sigma \restriction \mathcal{S}(V)^-$ to L such that (L, τ) is not isomorphic to (L, σ).

We can find an irreducible $P(V)^-$-refinement Ω' of $\Omega \restriction P(V)^-$ such that for any system $\mathcal{S}'$ associated to Ω', writing $\mathcal{S}''$ for the subfield of $\mathcal{S}'(V)$ corresponding to a generic point of $\Omega(V)^-$, $\mathcal{S}'(V)/\mathcal{S}''(V)$ contains a subextension $L'/\mathcal{S}''(V)$ isomorphic to $(L, \tau)/\mathcal{S}(V)^-$. It is clear that the fibre product of Ω' and Ω has dimension smaller than $dim(\Omega') = dim(\Omega)$, as $\mathcal{S}'(V)$ and $\mathcal{S}(V)$ are incompatible. $\square$

Remarks: (1) In the remainder of this paper, we will be interested in geometric and combinatorial properties of systems of varieties, so property (2) in Proposition 2.2.16 will be sufficient in all applications of regularity. However, we believe that the notion of regularity given in Definition 2.2.4 and 2.2.14 is more natural, so we will refer to that one for simplicity.

(2) Take $\Omega, \Omega', \Omega(u)^-, \Omega^+(u)$ as in Proposition 2.2.16(2) and suppose $K \models ACFA$. Writing $d = dim(\Omega(u)^-)$, we find that Ω is regular if and only if $\mu_d(\Omega(u)) = \mu_d(\Omega^+(u))$ for every $u \subseteq V$. This property is essentially the notion of definable étale-edge-uniformity which we will study in the next section.

We prove some technical lemmas about regular systems of varieties. The next two lemmas show that systems of difference varieties behave as expected with respect to the definable measure. We fix V a finite set and I an abstract simplicial complex on V.

Lemma 2.2.17. *Suppose that $K \models ACFA$ and let $A \leq K$ be perfect and inversive. Let Ω be a regular I-system of varieties on V over A where $V \notin I$. For $i \in V$, write $d_i = dim(\Omega(i))$.*

Then $\prod(\Omega(v), v \in P(V)^-)$ is irreducible, so that $\Omega(V) = \prod(\Omega(v), v \in P(V)^-)$. $\Omega(V)$ has dimension $d = \sum_{i \in V} d_i$, and $\mu_d(\Omega(V)) = \prod_{i \in V} \mu_{d_i}(\Omega(i))$.

Proof. We argue inductively on $|I|$, the base case being $I = \{\{i\} \mid i \in V\}$, which is trivial. Write $I = I' \cup \{u\}$ where u is a maximal element of I, so that $P(u)^- \subseteq I'$. Let $\Omega' = \Omega \upharpoonright I'$. Then Ω' is a regular system. We assume that the lemma holds for Ω' and $\Omega \upharpoonright P(u)^-$.

We argue that $X := \prod(\Omega(v), v \in P(V)^-)$ is irreducible, which will entail that $\Omega(V) = X$. Let x_0, x_1 be generic points of X in K. We need to show that $A(x_0)_\sigma$ is isomorphic to $A(x_1)_\sigma$ over A as a difference field. Write x'_0, x'_1 for the projections of x_0, x_1 to $\Omega'(V)$. Let $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}'_0, \mathcal{S}'_1$ be the systems of difference fields associated to x_0, x_1, x'_0, x'_1 respectively.

By induction hypothesis, $\prod_{P(u)^-} \mathcal{S}_0(v)$ is isomorphic to $\prod_{P(u)^-} \mathcal{S}_1(v)$ over A, and so are $\mathcal{S}'_0(V)$ and $\mathcal{S}'_1(V)$. $\Omega(u)$ is irreducible so it is enough to show that $\mathcal{S}_0(u)$ is linearly disjoint from $\mathcal{S}'_0(V)$ over $\prod_{P(u)^-} \mathcal{S}_0(v)$. By Lemma 2.2.9, $\mathcal{S}_0(u)$ is linearly disjoint from $\prod_{P(u)^-} \mathcal{S}_0(v)^{alg}$ over $\prod_{P(u)^-} \mathcal{S}_0(v)$. Therefore it is enough to show that $\mathcal{S}_0(u) \prod_{P(u)^-} \mathcal{S}_0(v)^{alg}$ is linearly disjoint from $\mathcal{S}'_0(V) \prod_{P(u)^-} \mathcal{S}_0(v)^{alg}$. This is a direct application of Lemma 2.2.7.

Hence $\Omega(V) = X$. The statements about $dim(\Omega(V))$ and $\mu_d(\Omega(V))$ follow inductively by direct applications of Lemma 2.1.6(4). $\square$

Lemma 2.2.18. *Suppose that $K \models ACFA$ and let $A \leq K$ be perfect and inversive. Let Ω be a regular I-system of varieties on V over A. Fix $u \subseteq V$ and for any $b \in \Omega(u)$ write $\Omega(V, b)$ for the pullback of b to $\Omega(V)$. Then for any generic $b \in \Omega(u)$, $\Omega(V, b)$ is an irreducible variety over $(A(b)_\sigma)^{alg}$ of dimension $d = dim(\Omega(V \setminus u))$ and $\mu_d(\Omega(V, b)) = \mu_d(\Omega(V \setminus u))$.*

Proof. First we show that $\Omega(V, b)$ has dimension d. Let a be a generic point of $\Omega(V)$ and let $\mathcal{S}$ be the associated system of perfect difference fields. By Lemma 2.2.9, $\mathcal{S}(V)$ is linearly disjoint from $\mathcal{S}(u)^{alg}$ over $\mathcal{S}(u)$. Since $\Omega(u)$ is irreducible, $\mathcal{S}(u)$ is isomorphic to $A(b)_\sigma^{insep}$ over A.

We define a new automorphism τ of $\mathcal{S}(V)^{alg}$ as follows. On $\mathcal{S}(V)$, τ is equal to σ. Let $f : \mathcal{S}(u)^{alg} \to A(b)_\sigma^{alg}$ be an isomorphism of pure fields extending the difference fied isomorphism $\mathcal{S}(u) \to A(b)_\sigma^{insep}$. Then define τ on $\mathcal{S}(u)^{alg}$ to be $f^{-1} \circ \sigma \circ f$. Now extend τ arbitrarily to an automorphism of $\mathcal{S}(V)^{alg}$.

As $\mathcal{S}(V)$ has finite total dimension over A, $\mathcal{S}(V)^{alg}$ is inversive. By model completeness, we can find an embedding of $(\mathcal{S}(V)^{alg}, \tau)$ in our model K of ACFA. By the description of types of Lemma 2.1.1, we know that the copy of b in $\mathcal{S}(V)^{alg}$ satisfies the same complete type as b over A. It follows that we can embed $(\mathcal{S}(V)^{alg}, \tau)$ in K over $(A(b)_\sigma)^{alg}$ and hence $\Omega(V, b)$ has dimension d.

Now let a_1, a_2 be generic points of $\Omega(V, b)$ and let $\mathcal{S}_1, \mathcal{S}_2$ be the corresponding systems of difference fields. Note that $\mathcal{S}_1(u) = \mathcal{S}_2(u) = (A(b)_\sigma)^{insep}$. Write $B = (A(b)_\sigma)^{alg}$. To show that $\Omega(V, b)$ is irreducible over B, it is enough to show that $B(a_1)_\sigma \cong_B B(a_2)_\sigma$. By regularity, $\mathcal{S}_i(V)$ is linearly disjoint from B over $\mathcal{S}_i(u)$ for $i = 1, 2$. By irreducibility, we have $\mathcal{S}_1(V) \cong_{\mathcal{S}_1(u)} \mathcal{S}_2(V)$. It follows that we can construct a difference field isomorphism $B(a_1)_\sigma \to B(a_2)_\sigma$ over B so $\Omega(V, b)$ is irreducible.

The statement about μ_d follows from the dimension of $\Omega(V, b)$ being d, $\Omega(V, b)$ being irreducible, and Lemma 2.1.6(4). □

2.2.3 Definable étale-edge-uniformity and a first hypergraph regularity lemma

In this section, we work inside a model K of ACFA and we fix $A \leq K$ a finitely generated perfect inversive difference subfield. V is a finite set.

In this section, we return to the study of Galois formulas and we show that our lemmas about regular systems of difference fields already prove an interesting form of hypergraph regularity: given a definable hypergraph ϕ with edges indexed by the simplicial complex $P(V)^-$, we find an étale partition of the domain of ϕ such that each induced sub-hypergraph is edge-uniform with respect to definable $P(V)^-$-refinements of ϕ. We make our terminology precise in the following definition.

Definition 2.2.19. *Let Ω be a system of varieties on V over A. For every $u \subseteq V$, write $d_u = dim(\Omega(u))$ and write $\phi_u = \rho_u \Omega(u)$ for the projection of $\Omega(u)$ to $\Omega(u)^-$.*

We say that Ω is definably étale-edge-uniform if for every $u \subseteq V$ with $|u| \geq 2$, the following holds: let Ω' be a $P(u)^-$-refinement of $\Omega \upharpoonright P(u)^-$ and write π for the projection $\Omega'(u) \to \Omega(u)^-$. Then

$$\frac{\mu_{d_u}(\phi_u \wedge \pi(\Omega'(u)))}{\mu_{d_u}(\phi_u)} = \frac{\mu_{d_u}(\pi(\Omega'(u)))}{\mu_{d_u}(\Omega(u)^-)}.$$

Remarks: (1) With the notation above, we emphasise that $\Omega'(u)$ is the fibre product $\prod(\Omega'(v), v \in P(u)^-)$ and that $\pi(\Omega'(u))$ is contained in $\Omega(u)^-$. Therefore $\Omega'(u)$ lives inside the boundary of $\Omega(u)$, modulo moving to an étale cover.

(2) If Ω is an I-system of varieties, then it is enough to check the property of definition 2.2.19 for $u \in I$.

The following proposition is an elaboration of the comment following Proposition 2.2.16.

Proposition 2.2.20. *Let Ω be a system of varieties on V over A. Then Ω is definably étale-edge-uniform if and only if the irreducible components of Ω satisfy either of the equivalent conditions in Proposition 2.2.16. In particular, if Ω is regular, then Ω is definably étale-edge-uniform.*

Proof. We can assume that Ω is irreducible. Suppose that Ω satisfies (1) in Proposition 2.2.16. Let $\phi = \rho_V \Omega(V)$ and let Ω' be a $P(V)^-$-refinement of

$\Omega \restriction P(V)^-$. Write π for the projection $\Omega' \to \Omega$. We need to check that

$$\frac{\mu_{d_V}(\phi \wedge \pi(\Omega'(V)))}{\mu_{d_V}(\phi)} = \frac{\mu_{d_V}(\pi(\Omega'(V)))}{\mu_{d_V}(\Omega(V)^-)}.$$

Let a be a generic point of $\Omega(V)$, let $\mathcal{S}$ be the associated system of difference fields. Let L' be the Galois extension of $\mathcal{S}(V)^-$ associated to $\Omega'(V)$. In the notation of Proposition 2.1.9, we only need to check that

$$N(\phi \wedge \pi(\Omega'(V))) = N(\phi)N(\pi(\Omega'(V))).$$

Since $\mathcal{S}(V) \cap L'/\mathcal{S}(V)^-$ is monadic, $N(\phi)$ is equal to the number of extensions τ of σ from $\mathcal{S}(V) \cap L'$ to $\mathcal{S}(V)$ such that ϕ is satisfied in (L', τ). Similar equalities hold for $N(\pi(\Omega'(V))$ and $N(\phi \wedge \pi(\Omega'(V)))$. Now L' and $\mathcal{S}(V)$ are linearly disjoint over $\mathcal{S}(V) \cap L'$, so the result follows by counting extensions of σ in the obvious way.

Conversely, suppose there is a $P(V)^-$-refinement Ω' of $\Omega \restriction P(V)^-$ such that (2) of Proposition 2.2.16 fails. Then $\phi \cap \pi(\Omega'(V))$ is the projection to $\Omega(V)^-$ of the fibre product of $\Omega'(V)$ and $\Omega(V)$ over $\Omega(V)^-$. Since $\mu_{d_V}(\Omega'(V) \times_{\Omega(V)^-} \Omega(V)) = 0$, it is clear that definable étale-edge-uniformity fails. □

We state a first hypergraph regularity lemma at the level of ACFA, in terms of systems of varieties and definable edge-uniformity. We repeat all our notation and background assumptions.

Theorem 2.2.21. *Let V be a finite set and I a simplicial complex on V. Let A be an inversive difference field and let Ω be an I-system of varieties on V over A.*

Then there is a surjective ∂I-refinement Ω' of Ω over a finitely generated extension A' of A such that each irreducible component of Ω' is definably étale-edge-uniform.

Proof. We obtain Ω' by Proposition 2.2.15. Definable étale-edge-uniformity is given by Proposition 2.2.20. □

2.3 The Stochastic Independence Theorem, the Stationarity Theorem, and Quasirandomness

2.3.1 The Stochastic Independence Theorem

We fix $K \models ACFA$ and $A \leq K$ a perfect inversive subfield. V is a finite set.

The next definition sets up some useful notation for moving between étale covers.

Definition 2.3.1. *Let Ω be a system of difference varieties on V over A. Let ϕ be a definable set contained in $\Omega(u)$ for some $u \subseteq V$.*

1. *If $v \subseteq u$ and $b \in \Omega(v)$, then we write $\phi(\Omega(u), b)$ for the set of elements $x \in \phi$ which map to b under the projection $\Omega(u) \to \Omega(v)$.*
2. *When $u \subseteq w \subseteq V$, we write $\phi(\Omega(w))$ for the pullback of ϕ under the projection $\Omega(w) \to \Omega(u)$.*
3. *When Ω' is a refinement of Ω, we also write $\phi(\Omega'(u))$ for the pullback of ϕ to $\Omega'(u)$ by the projection $\Omega' \to \Omega$.*

In the next theorem, an *antichain J on V* is a collection of subsets of V of size at least 2 such that for any two $u \neq v \in J$, u does not contain v. We ask that elements of J have size at least 2 because it makes the statement of Theorem 2.3.2 easier and there will be results later on which break down if we allow for antichains with singletons.

Theorem 2.3.2 (The Stochastic Independence Theorem)**.** *Let J be an antichain of V with $\bigcup J = V$. Let I be the simplicial complex on V generated by J and let Ω be a regular I-system of varieties on V over A. For $u \in J$, write $\phi_u = \rho_u \Omega(u)$.*

Let $\Omega_0 = \Omega \restriction \partial J$. For every $u \subseteq V$, write $d_u = dim(\Omega_0(u))$ and $\mathbb{P}_u = \mu_{d_u}/(\mu_{d_u}(\Omega_0(u)))$, so that $\mathbb{P}_u$ is a probability measure on $\Omega(u)$. Then

$$\mathbb{P}_V(\bigwedge_{u\in J} \phi_u(\Omega_0(V))) = \prod_{u\in J} \mathbb{P}_u(\phi_u). \tag{2.1}$$

Proof. Let $\mathcal{S}$ be a system of difference fields on V over A associated to Ω and let $\mathcal{S}_0 = \mathcal{S} \restriction \partial J$. For $u \in J$, we will view $\mathcal{S}(u)$ as a pure field extending $\mathcal{S}_0(u)$ (in fact we could start with $\mathcal{S}_0$ and consider abstract field extensions isomorphism to $\mathcal{S}$). For every $u \in J$, let τ_u be an arbitrary extension of σ from $\mathcal{S}_0(u)$ to $\mathcal{S}(u)$ such that ϕ_u is satisfied in the structure $(\mathcal{S}(u), \tau_u)$.

Claim 2.3.2.1. *The difference operators τ_u have a common extension τ to $\mathcal{S}(V)$ and $(\mathcal{S}(V), \tau) \models \bigwedge_{u\in J} \phi_u(\Omega_0(V))$*

Proof of claim. By regularity and Lemma 2.2.10, we know that $\mathcal{S}(u)$ is linearly disjoint from $\prod_I \mathcal{S}(v)^{alg}$ over $\mathcal{S}_0(u)$. Therefore, we extend each τ_u to the field $\mathcal{S}(u)\prod_I \mathcal{S}(v)^{alg}$ so that $\tau_u \restriction \prod_I \mathcal{S}(v)^{alg} = \sigma$.

Fix an enumeration (u_k) of J and suppose we have found a common extension τ of $\tau_{u_0}, \ldots, \tau_{u_k}$ to $\prod_{i=0}^k \mathcal{S}(u_i)\prod_{v\in I} \mathcal{S}(v)^{alg}$. By Lemma 2.2.10, $\mathcal{S}(u_{k+1})$ is linearly disjoint from $\prod_{i=0}^k \mathcal{S}(u_i)^{alg}\prod_{v\in I} \mathcal{S}(v)^{alg}$ over $\mathcal{S}_0(u_{k+1})$. By the Towers property, $\mathcal{S}(u_{k+1})\prod_I \mathcal{S}(v)^{alg}$ is linearly disjoint from $\prod_{i=0}^k \mathcal{S}(u_i)\prod_I \mathcal{S}(v)^{alg}$ over $\prod_I \mathcal{S}(v)^{alg}$. Moreover, τ and τ_{k+1} are both equal to σ on $\prod_I \mathcal{S}(v)^{alg}$. This proves that we can find a common extension of the τ_u to $\prod_J \mathcal{S}(v)\prod_I \mathcal{S}(v)^{alg}$, and this field contains $\mathcal{S}(V)$. □

$\bigwedge \phi_u(\Omega_0(V))$ is equal to the projection of $\Omega(V)$ to $\Omega_0(V)$. By Lemma 2.2.17, $\Omega(V)$ has dimension d_V, so $\bigwedge \phi_u(\Omega_0(V))$ has dimension d_V. By Proposition 2.1.9 and in the notation of Definition 2.1.7, it is enough to prove

$$\nu(\bigwedge_J \phi_u(\Omega_0(V)) = \prod_J \nu(\phi_u).$$

The claim proves that $N(\bigwedge_{u\in J} \phi_u) = \prod_{u\in J} N(\phi_u)$, since our choices of τ_u were arbitrary. Therefore, we only need to prove that $[\mathcal{S}(V) : \mathcal{S}_0(V)] = \prod_J [\mathcal{S}(v) : \mathcal{S}_0(v)]$. We have already seen that $\mathcal{S}(u)$ is linearly disjoint from $B := \prod_I \mathcal{S}(v)^{alg}$ over $\mathcal{S}_0(u)$ for every $u \in J$ and that for every $k \geq 0$, $\mathcal{S}(u_{k+1})B$ is linearly disjoint from $\prod_{i=0}^k \mathcal{S}(u_i)B$ over B, where (u_k) is an enumeration of J as in the proof of the claim. Therefore, working by induction:

$$\begin{aligned}[\prod_{i=0}^{k+1} \mathcal{S}(u_i)B : B] &= [\prod_{i=0}^{k+1} \mathcal{S}(u_i)B : \prod_{i=0}^{k} \mathcal{S}(u_i)B][\prod_{i=0}^{k} \mathcal{S}(u_i)B : B] \\ &= [\mathcal{S}(u_{k+1})B : B] \prod_{i=0}^{k} [\mathcal{S}(u_i)B : B] = \prod_{i=0}^{k+1} [\mathcal{S}(u_i) : \mathcal{S}_0(u_i)]\end{aligned}$$

Therefore $[\prod_{u\in J} \mathcal{S}(u)B : B] = \prod_{u\in J} [\mathcal{S}(u) : \mathcal{S}_0(u)]$. Now

$$\prod_{u\in J} [\mathcal{S}(u) : \mathcal{S}_0(u)] = [\prod_{u\in J} \mathcal{S}(u)B : B] \leq [\mathcal{S}(V) : \mathcal{S}_0(V)] \leq \prod_{u\in J} [\mathcal{S}(u) : \mathcal{S}_0(u)]$$

so we deduce $[\mathcal{S}(V) : \mathcal{S}_0(V)] = \prod_{u\in J} [\mathcal{S}(u) : \mathcal{S}_0(u)]$. □

Remarks: (1) The statement of the stochastic independence theorem may appear technical at first but we stress that any collection of definable sets $(\phi_u)_{u\in J}$ with variables suitably indexed by V can always be lifted to a ∂J-system Ω_0 so that the setting of the stochastic independence theorem holds.

(2) The stochastic independence theorem can be seen as an analogue of the counting lemma usually associated to the Szemerédi counting lemma. See [Gow06] 3.4 or 4.4 for example.

We now prove the stationarity theorem. This is a higher-order and a quantitative version of the stationarity theorem used in [Tao12]. We show that this theorem follows from the Stochastic Independence Theorem and from additional facts about regular systems of varieties.

We will not use this theorem in our proof of the algebraic hypergraph regularity lemmas to follow. However, it is striking that it always holds in regular systems and the theorem is of wider interest for model theory and algebraic geometry.

Corollary 2.3.3 (The Stationarity Theorem)**.** *Let J be an antichain of V with $\bigcup J = V$. Let I be the simplicial complex on V generated by ∂J and let Ω be a regular I-system of varieties on V over A. For $u \in J$ write $\phi_u = \rho_u \Omega(u)$.*

Let $\Omega_0 = \Omega \restriction \partial J$. For every $u \subseteq V$, write $d_u = dim(\Omega_0(u))$ and $\mathbb{P}_u = \mu_{d_u}/(\mu_{d_u}(\Omega_0(u)))$.

Fix any nonempty $u \subseteq V$ and a generic point a_u in $\Omega_0(u)$. For any $v \subseteq u$, write a_v for the projection of a_u to $\Omega_0(v)$. Then we have

$$\mathbb{P}_{V\setminus u}(\bigwedge_{v\in J} \phi_v(\Omega_0(V), a_u)) = \prod_{v\in J} \mathbb{P}_{v\setminus u}(\phi_v(\Omega_0(v), a_{u\cap v})). \tag{2.2}$$

Proof. Fix a_u generic in $\Omega_0(u)$. Write $J' = \{v \in J \mid v \subseteq u\}$. We can assume that for any $v \in J'$, a_v satisfies ϕ_v (otherwise both sides of (2.2) are 0). In this case, (2.2) becomes

$$\mathbb{P}_{V\setminus u}(\bigwedge_{J\setminus J'} \phi_v(\Omega_0(V), a_u)) = \prod_{J\setminus J'} \mathbb{P}_{v\setminus u}(\phi_v(\Omega_0(v), a_{u\cap v}))).$$

For every $v \in J\setminus J'$ and generic $b \in \Omega_0(u\cap v)$, the measure of $\phi_v(\Omega_0(u\cap v), b)$ is constant. This follows from Proposition 2.1.9 and regularity. Therefore, we can apply Fubini to get

$$\begin{aligned}\mu_v(\phi_v(\Omega_0(V)) &= \int_{b\in\Omega_0(v\cap u)} \mu_{v\setminus u}(\phi_v(\Omega(V, b))d\mu_{v\cap u} \\ &= \mu_{u\cap v}(\Omega_0(u \cap v))\mu_{v\setminus u}(\phi_v(\Omega_0(v), a_{u\cap v})).\end{aligned}$$

By Lemma 2.2.17, $\mu_V(\Omega_0(V)) = \mu_u(\Omega_0(u))\mu_{V\setminus u}(\Omega_0(V \setminus u))$. Therefore

$$\begin{aligned}\mathbb{P}_v(\phi_v(\Omega_0(v))) &= \frac{\mu_{u\cap v}(\Omega_0(u \cap v))\mu_{v\setminus u}(\Omega_0(v \setminus u))}{\mu_v(\Omega_0(v))}\mathbb{P}_{v\setminus u}(\phi_v(\Omega_0(v), a_{v\cap u})) \\ &= \mathbb{P}_{v\setminus u}(\phi_v(\Omega_0(v), a_{u\cap v})).\end{aligned}$$

Similarly, for generic $b \in \Omega_0(u)$, $\mu_{V\setminus u}(\bigwedge_{v\in J\setminus J'} \phi_v(\Omega_0(V), b))$ is constant. Therefore, for generic $b \in \bigwedge_{v\in J'} \phi_v(\Omega_0(u))$, $\mu_{V\setminus u}(\bigwedge_J \phi_v(\Omega_0(V), b))$ is constant and we deduce by Fubini

$$\mu_V(\bigwedge_{v\in J} \phi_v(\Omega_0(V))) = \mu_u(\bigwedge_{v\in J'} \phi_v(\Omega_0(u)))\mu_{V\setminus u}(\bigwedge_{v\in J} \phi_v(\Omega_0(V), a_u))$$

By Lemma 2.2.17, we have

$$\mathbb{P}_V(\bigwedge_J \phi_v(\Omega_0(V))) = \mathbb{P}_u(\bigwedge_{J'} \phi_v(\Omega_0(u)))\mathbb{P}_{V\setminus u}(\bigwedge_{J\setminus J'} \phi_v(\Omega_0(V), a_u)).$$

By the stochastic independence theorem applied to Φ, we conclude:

$$\begin{aligned}\mathbb{P}_{V\setminus u}(\bigwedge_{J\setminus J'} \phi_v(\Omega_0(V), a_u)) &= \prod_{J\setminus J'} \mathbb{P}_v(\phi_v(\Omega_0(v))) \\ &= \prod_{J\setminus J'} \mathbb{P}_{v\setminus u}(\phi_v(\Omega_0(v), a_{u\cap v})).\end{aligned}$$

$\square$

2.3.2 Quasirandomness

As before K is a model of ACFA and $A \leq K$ is a perfect inversive subfield. V is a finite set.

In this section, we show that quasirandomness in the sense of [Gow06] follows easily from the stochastic independence theorem.

Let Ω be a system of varieties on V over A. Definition 2.3.4 introduces the doubling of Ω, written $D(\Omega)$. In the case where Ω is the Cartesian product $\prod_{i \in V} \Omega(i)$, $D(\Omega)$ is obtained by taking copies $\Omega(i, 0)$ and $\Omega(i, 1)$ of each $\Omega(i)$. $D(\Omega)$ is the hypergraph on $V \times \{0, 1\}$ consisting of those tuples $(x_{i,j})_{i \in V, j=0,1}$ such that for every choice function $\iota : V \to \{0, 1\}$, the tuple $(x_{i,\iota(i)})$ is in $\Omega(V)$. When Ω is a system of varieties, $D(\Omega)$ takes the form of a fibre product construction. Definition 2.3.4 also defines $D(\Omega, u)$, the doubling of Ω outside a designated subset u of V. This will be useful in the next section.

In Definition 2.3.4, if u is a set, we write 2^u for the set of functions $u \to [2]$ where $[2] = \{0, 1\}$.

Definition 2.3.4. *Let Ω be an I-system of varieties on V over A.*

1. *Let $u \subseteq V$ and $\iota \in 2^u$. We identify the pair (u, ι) with the subset of $V \times [2]$ given by $\{(i, \iota(i)) \mid i \in u\}$.*
2. *Define $D(I)$, the doubling of I, to be the collection of subsets of $V \times [2]$ of the form (u, ι) where $u \in I$ and $\iota \in 2^u$. We view I as a partial order with the obvious inclusions. Note that $D(I)$ is a simplicial complex on $V \times [2]$.*
3. *For every $(u, \iota) \in D(I)$, define $\Omega(u, \iota)$ to be a copy of the variety $\Omega(u)$. When $(v, \iota') \subseteq (u, \iota)$, we have a natural projection $\Omega(u, \iota) \to \Omega(v, \iota')$*
4. *Define $D(\Omega)$, the doubling of Ω, to be the $D(I)$-system of varieties on $V \times [2]$ taking (u, ι) to $\Omega(u, \iota)$.*
5. *To make notation lighter, we usually write $D(\Omega)$ for the variety in $D(\Omega)$ associated to $V \times [2]$ (instead of writing $D(\Omega)(V \times [2])$).*

Remark: Let Ω be a regular I-system of varieties on V over A. Lemma 2.2.17 generalises easily to prove that $D(\Omega)$ is irreducible and hence $D(\Omega)$ is regular.

If ϕ is a definable set contained in $\Omega(u)$ and $\iota \in 2^u$, write $\phi(\Omega(u, \iota))$ for the definable set contained in $\Omega(u, \iota)$ obtained by carrying over ϕ to $\Omega(u, \iota)$. As a set in K, $\phi(\Omega(u, \iota))$ is equal to $\phi(\Omega(u))$ but it is useful to distinguish them in notation. If we were to rephrase this in terms of formulas with free variables, $\phi(\Omega(u, \iota))$ would be the result of substituting the free variables of $\Omega(u, \iota)$ for the free variables of $\Omega(u)$ in ϕ.

We make here some comments which clarify the technical background of Definition 2.3.5. In that definition, we consider a system of varieties Ω with $dim(\Omega(V)) = d$, the definable set $\phi = \rho_V\Omega(V)$ and the function f on $\Omega(V)^-$ defined by $\mathbb{1}_\phi - \frac{\mu_d(\phi)}{\mu_d(\Omega(V)^-)}\mathbb{1}_{\Omega(V)^-}$. f is definable, in the sense that it can be viewed as a function from a type space over K in the appropriate variables to the interval $[-1, 1]$ which is continuous with respect to the logic topology. Since the measure μ_d extends to a Borel measure on the types over K contained in $\Omega(V)^-$, we can integrate f on $\Omega(V)^-$. When K is ω_1-saturated, it is possible to move away from the type space and integrate f directly over the set $\Omega(V)^-$ in K, with respect to the σ-algebra of K-definable sets. In any case, the identity $\int_{D(\Omega(V)^-)} f d\mu_d = 0$ expresses an identity which does not depend on any model.

When f is a definable function on $\Omega(V)^-$ and $\iota \in 2^V$, we will write f^ι for the result of carrying f over to $\Omega(V, \iota)^-$. Thus, for f defined above, $f^\iota = \mathbb{1}_{\phi(V,\iota)} - \frac{\mu_d(\phi)}{\mu_d(\Omega(V)^-)}\mathbb{1}_{\Omega(V,\iota)^-}$. We also write f^ι for the result of pullback back from $\Omega(V, \iota)$ to $D(\Omega)$ when the context is clear. Therefore we can make sense of the integral $\int_{D(\Omega)} \prod_{\iota\in 2^V} f^\iota d\mu_{2d}$.

Definition 2.3.5 follows Definition 6.3 in [Gow06].

Definition 2.3.5. *Let Ω be a system of varieties on V over A. We say that Ω is quasirandom if the following holds:*

For every $u \subseteq V$ with $|u| \geq 2$, write $d_u = dim(\Omega(u))$, let $\phi_u = \rho_u\Omega(u)$ and define the function $f_u : \Omega(u)^- \to [-1, 1]$ by $f_u = \mathbb{1}_{\phi_u} - \frac{\mu_{d_u}(\phi_u)}{\mu_{d_u}(\Omega(u)^-}\mathbb{1}_{\Omega(u)^-}$. Then

$$\int_{D(\Omega\restriction P(u)^-)} \prod_{\iota\in 2^u} f_u^\iota d\mu_{2d_u} = 0.$$

The following proposition is essential for the next section, but it follows easily from the stochastic independence theorem.

Proposition 2.3.6. *Let Ω be a regular system of varieties on V over A. Then Ω is quasirandom.*

Proof. We check quasirandomness at level V. Suppose $dim(\Omega(V)) = d$. Write $\mathbb{P}_d = \mu_d/(\mu_d(\Omega(V))$ and $\mathbb{P}_{2d} = \mu_{2d}/(\mu_d(\Omega(V))^2$. Let $\phi = \rho_V\Omega(V)$.

Let $f = \mathbb{1}_\phi - \frac{\mu_d(\phi)}{\mu_d(\Omega(V)^-)}\mathbb{1}_{\Omega(V)^-}$ on $\Omega(V)^-$. By the stochastic independence theorem, the set of functions $\{f^\iota \mid \iota \in 2^V\}$ on $D(\Omega \restriction P(V)^-)$ is independent in the probabilistic sense. Therefore

$$\int_{D(\Omega\restriction P(V)^-)} \prod_{\iota\in 2^V} f^\iota d\mathbb{P}_{2d} = \prod_{\iota\in 2^V} \int_{\Omega(V,\iota)^-} f^\iota d\mathbb{P}_d = 0.$$

$\square$

In Section 2.4.2, we will see that if Ω is quasirandom, then Ω is definably étale-edge-uniform, and hence Ω is regular. This proof goes via the fundamental equivalence of Gowers discussed in the next section. It would be interesting to find a proof of this result at the algebraic level.

2.4 A Combinatorial Approach to Algebraic Hypergraph Regularity

In this section, we leave models of ACFA behind and we apply our theorems to derive combinatorial results concerning finite definable sets in the difference fields K_q where q is a power of a prime p, K_q is the algebraic closure of the field containing p elements and the difference operator on K_q is the q-th power $x \mapsto x^q$.

2.4.1 Combinatorial notions and asymptotics

We fix A a perfect inversive finitely generated difference field. V is a finite set.

We recall the notion of Frobenius specialisation from [Hru22], section 12. Suppose that we have some finite definable data D defined over a finitely generated perfect inversive difference field A (e.g. D is a variety, a definable set, or a system of varieties). We will say that some property P holds of D for *almost all* q to mean the following: there is a sufficiently large finitely generated difference domain R in A such that D is defined over R and for any sufficiently large prime power q and for any difference ring homomorphism $h : R \to K_q$, P holds of D^h in K_q, where D^h is the definable data in K_q obtained by applying h to the parameters in D and interpreting σ as the q-th power.

Equivalently, if D is some data definable over a difference domain R, we say P holds of D for almost all q if there is $c \in R$ such that for any sufficiently large prime power q and any homomorphism $h : R \to K_q$ with $h(c) \neq 0$, P holds of D^h in K_q. See [Hru22] for an in-depth discussion of Frobenius specialisations.

In an effort to make notation lighter, we always consider data D over an inversive difference field A and we suppress the reference to the difference domain R and to the homomorphism h when discussing specialisations. It will always be clear from context whether we are working with D over A or if we are working with specialisations.

The fundamental theorem about Frobenius specialisations is the twisted Lang-Weil estimates due to Hrushovski:

Theorem 2.4.1 ([Hru22][3]). *Let X be a variety over A and suppose that X has finite total dimension d. Then for almost all q, specialising X to K_q, we have*

$$|X(K_q) - \mu_d(X)q^d| = O(q^{d-1/2}).$$

where $O(\cdot)$ depends only on the degree of X.

[3] While the manuscript we reference is dated from 2022, the twisted Lang-Weil estimates were discovered by Hrushovski in the early 2000s.

Theorem 2.4.1 extends to Galois formulas over A in the obvious way. Since ACFA is the asymptotic theory of the difference fields K_q, Theorem 2.4.1 gives asymptotics for all definable sets of finite total dimension.

We now extend the notions of quasirandomness and edge-uniformity introduced in the previous section so that we can apply them inside the structures K_q.

Definition 2.4.2. *let Ω be a system of varieties on V over A. For every $u \subseteq V$ with $|u| \geq 2$, write $\phi_u = \rho_u \Omega(u)$ and let ϵ_u be a function $\mathbb{N} \to [0, \infty)$. Write $\epsilon = (\epsilon_u)_{|u| \geq 2}$.*

We say that Ω is ϵ-quasirandom if for almost all q and all $u \subseteq V$ with $|u| \geq 2$, specialising Ω to K_q and writing $f_u = \mathbb{1}_{\phi_u} - \frac{|\phi_u|}{|\Omega(u)^-|} \mathbb{1}_{\Omega(u)^-}$, we have

$$\sum_{D(\Omega \restriction P(u)^-)} \prod_{\iota \in 2^u} f_u^\iota = O(\epsilon_u(q)) |\Omega(u)^-|^2$$

where $O(\cdot)$ only depends on the degrees of the varieties in Ω.

In the next definition, we introduce chains which are a purely combinatorial notion, since they exist in finite sets inside K_q and we are not interested in identifying them as definable sets. This terminology comes from [Gow06].

Definition 2.4.3. *Let Ω be a system of varieties on V over A. Specialise Ω to some K_q.*

1. *A chain $W = (W(v))_{v \in P(V)}$ in Ω is a collection of sets such that for all $v \subseteq V$, $W(v) \subseteq \Omega(v)$ and for any $v \subseteq u$, $W(v)$ contains the image of $W(u)$ under the projection $\Omega(u) \to \Omega(v)$.*

2. *Let W be a chain in Ω. Let I be a simplicial complex on V. We say that W is an I-chain if for all $u \notin I$, $W(u)$ is the fibre product of the sets $(W(v))_{v \in P(u)^-}$.*

3. *Let W be a chain in Ω. For every $u \subseteq V$, write $W(u)^-$ for the fibre product $\prod(W(v), v \in P(u)^-)$ and $\rho_u W(u)$ for the projection $W(u) \to W(u)^-$.*

4. *Let I be a simplicial complex on V. An I-chain decomposition of Ω is a collection $(\mathcal{W}_v)_{v \in I}$ such that each $\mathcal{W}_v$ is a partition of $\Omega(v)$ and for every $u \in I$ and every $X \in \mathcal{W}_u$, there is a $P(u)^-$-chain W such that $X \subseteq W(u)$ and for every $v \in P(u)^-$, $W(v) \in \mathcal{W}_v$.*

   We say that an I-chain W is contained in the I-chain decomposition $(\mathcal{W}_v)_{v \in I}$ if for all $v \in I$, $W(v) \in \mathcal{W}_v$.

The next definition makes precise the notion of ϵ-edge-uniformity. We also define ϵ-étale edge-uniformity in order to relate ϵ-edge-uniformity to definable edge-uniformity, but we will soon see that ϵ-étale edge-uniformity is equivalent to ϵ-edge-uniformity. The notion of definable ϵ-étale-edge-uniformity will be used only in Corollary 2.4.5.

Definition 2.4.4. *Let Ω be a system of varieties on V over A. For every $u \subseteq V$ with $|u| \geq 2$, let $\phi_u = \rho_u \Omega(u)$ and let ϵ_u be a function $\mathbb{N} \to [0, \infty)$. Write $\epsilon = (\epsilon_u)_{|u| \geq 2}$.*

1. *We say that Ω is ϵ-edge-uniform if for almost all q, specialising Ω and ϕ to K_q, for all $u \subseteq V$ with $|u| \geq 2$, if W is $P(u)^-$-chain contained in $\Omega \restriction P(u)^-$, then*
$$|\frac{|\phi_u \cap W(u)|}{|\phi_u|} - \frac{|W(u)|}{|\Omega(u)^-|}| = O(\epsilon_u(q))$$
where $O(\cdot)$ only depends on the degrees of the varieties in Ω.

2. *We say that Ω is definably ϵ-étale-edge-uniform if for all $u \subseteq V$ with $|u| \geq 2$, for any $P(u)^-$-refinement Ω' of $\Omega \restriction P(u)^-$ over a finitely generated algebraic extension A' of A with projection $\pi : \Omega'(u) \to \Omega(u)^-$, for almost all q, we have*
$$|\frac{|\phi_u \cap \pi(\Omega'(u))|}{|\phi_u|} - \frac{|\pi(\Omega'(u))|}{|\Omega(u)^-|}| = O(\epsilon_u(q))$$
where $O(\cdot)$ only depends on the degrees of the varieties in Ω.

3. *We say that Ω is ϵ-étale-edge-uniform if for all $u \subseteq V$ with $|u| \geq 2$, for any $P(u)^-$-refinement Ω' of $\Omega \restriction P(u)^-$ over a finitely generated algebraic extension A' of A with projection $\pi : \Omega'(u) \to \Omega(u)^-$, for almost all q and for any $P(u)^-$-chain W in Ω', we have*
$$|\frac{|\phi_u \cap \pi(W(u))|}{|\phi_u|} - \frac{|\pi(W(u))|}{|\Omega(u)^-|}| = O(\epsilon_u(q))$$
where $O(\cdot)$ only depends on the degrees of the varieties in Ω.

The following corollary is just a reformulation of Theorem 2.2.21 and a direct application of Theorem 2.4.1.

Corollary 2.4.5. *Let Ω be a system of varieties on V over A. For every $u \subseteq V$ with $|u| \geq 2$, set $\epsilon_u(q) = q^{-1/2}$ and let $\epsilon = (\epsilon_u)_{|u| \geq 2}$.*

Then there is a surjective $P(V)^-$-refinement Ω' of Ω over some finitely generated extension A' of A with regular components $\Omega_1, \ldots, \Omega_N$ such that for every i, Ω_i is definably ϵ-étale-edge-uniform and ϵ-quasirandom. N depends only on the degrees of the varieties in Ω.

2.4.2 Equivalences between defined notions

We fix A a perfect inversive finitely generated difference field. V is a finite set.

We show that the notions of ϵ-edge-uniformity and ϵ-étale-edge-uniformity coincide. We state the lemma in the language of systems of varieties but this is really a purely combinatorial result. We use the following technical lemma:

Lemma 2.4.6. *Let I be a simplicial complex on V. Let Ω be a system of varieties on V over A and let Ω' be an I- refinement of Ω with projection $\pi : \Omega' \to \Omega$. For every $u \subseteq V$, write $d_u = dim(\Omega(u))$.*

Then there is some N such that for almost all q, specialising Ω and Ω' to K_q, the following holds: there is an I-chain decomposition $(\mathcal{W}_v)_{v\in I}$ of Ω' containing at most N chains such that for every $u \in I$ and every set $X \in \mathcal{W}_u$, the map $\pi : X \to \Omega(u)$ is injective.

Proof. We construct $(\mathcal{W}_v)_I$ inductively. The base case where I is the set of singletons of V is clear. Suppose that $I = I' \cup \{u\}$ where $P(u)^- \subseteq I'$ and $(\mathcal{W}_v)_{I'}$ have been constructed.

Writing $W_1, \dots, W_n$ for the I'-chains contained in $(\mathcal{W}_v)_{I'}$, we find that for every $i \le n$, the projection $W_i(u) \to \Omega(u)^-$ is injective and that the sets $W_1(u), \dots, W_n(u)$ partition $\Omega'(u)^-$. Now take $\mathcal{W}_u$ to be the partition of $\Omega'(u)$ obtained by taking sections of each projection $\rho_u : \rho_u^{-1}(W_i(u)) \to W_i(u)$. The size of $\mathcal{W}(u)$ depends only on n and the multiplicity of the projection ρ_u.

Take $X \in \mathcal{W}_u$. To see that $\pi : X \to \Omega(u)$ is injective, note that the maps $\Omega'(u) \to \Omega(u) \to \Omega(u)^-$ and $\Omega'(u) \to \Omega'(u)^- \to \Omega(u)^-$ form a commutative diagram. We know that $X \to \Omega'(u)^- \to \Omega(u)^-$ is injective, so the result follows. $\square$

Lemma 2.4.7. *Let Ω be a system of difference varieties on V over A. For $u \subseteq V$ with $|u| \ge 2$, let $\epsilon_u : \mathbb{N} \to [0, \infty)$ and write $\epsilon = (\epsilon_u)$. Then Ω is ϵ-étale-edge-uniform if and only if Ω is ϵ-edge-uniform.*

Proof. Let $\phi = \rho_V \Omega(V)$ and let Ω' be a $P(V)^-$-refinement of $\Omega \restriction P(V)^-$. Write π for the projection $\Omega' \to \Omega$. It is enough to show that if Ω is ϵ-edge-uniform, then, specialising to some K_q, for every $P(V)^-$-chain W in Ω',

$$\left|\frac{|\phi \cap \pi(W(V))|}{|\phi|} - \frac{|\pi(W(V))|}{|\Omega(V)|}\right| = O(\epsilon_V(q)).$$

Fix W a chain in Ω'. Let $(\mathcal{W}_u)_{P(V)^-}$ be a chain decomposition of Ω' given by Lemma 2.4.6. Let (W_i) be an enumeration of the chains in $(\mathcal{W}_u)_{P(V)^-}$. Define $W_i' = W \cap W_i$.

For every i, $\pi(W_i'(V))$ is equal to the fibre product $\prod(\pi(W_i'(v)), v \in P(V)^-)$, since π is injective on each $W_i'(v)$. Write $Z_i(u) = \pi(W_i'(u))$ so that each Z_i is a $P(V)^-$-chain in Ω. It follows that $\pi(W(V)) = \pi(\bigcup_i W_i'(V)) = \bigcup_i Z_i(V)$. Now ϵ-edge-uniformity applied to each chain Z_i gives the result. $\square$

Remark: Back in definition 2.4.4, we could have defined the notion of *definable ϵ-edge-uniformity* in the expected way, but we chose not to discuss this notion. The reason for this is that Lemma 2.4.7 does not allow us to show that definable ϵ-edge-uniformity is equivalent to definable ϵ-étale-edge-uniformity. This is because the proof of Lemma 2.4.7 goes via the sections constructed in Lemma 2.4.6, which are not definable. It would be interesting to know if Lemma 2.4.7 can be extended to the definable setting.

Combining Proposition 2.2.20 and Proposition 2.3.6, we find that we have already proved that if ϕ is definably $q^{-1/2}$-étale-edge-uniform with respect to Ω then ϕ is $q^{-1/2}$-quasirandom with respect to Ω. Suitably transposed to our setting, this is close to one direction of the fundamental Theorem 4.1 in [Gow06]. Proposition 2.4.8 proves this in full generality, with general error bounds. We follow the proof of [Gow06]. This result will be needed for Theorem 2.4.13.

Proposition 2.4.8. *Let Ω be a system of varieties on V over A. For every $u \subseteq V$ with $|u| \geq 2$, let $\epsilon_u : \mathbb{N} \to [0, \infty)$. Write $\epsilon = (\epsilon_u)_{u \in P(V)}$. If Ω is ϵ-étale-edge-uniform then Ω is ϵ-quasirandom.*

Proof. It is enough to check quasirandomness at the top level V. Let $\phi = \rho_V \Omega(V)$ and write $\Omega_0 = \Omega \restriction P(V)^-$ (so that $\Omega_0(V) = \Omega(V)^-$). We specialise Ω to some K_q and write $f = \mathbb{1}_\phi - \frac{|\phi|}{|\Omega(V)^-|}\mathbb{1}_{\Omega(V)^-}$.

Write $\iota_0, \iota_1 \in 2^V$ for the constant functions equal to 0 and 1 respectively. Then $D(\Omega_0)$ is a finite cover of the Cartesian product of $\Omega_0(V, \iota_0)$ and $\Omega_0(V, \iota_1)$. For any $x \in \Omega_0(V, \iota_0)$ and $y \in \Omega_0(V, \iota_1)$, we write $D(\Omega_0)(xy)$ for the finite set of elements $z \in D(\Omega_0)$ which project onto x and onto y. The cardinality of $D(\Omega_0)(xy)$ is bounded across q. By the triangle inequality we obtain

$$|\sum_{D(\Omega_0)} \prod_{\iota \in 2^V} f^\iota| \leq \sum_{y \in \Omega_0(V,\iota_1)} \Big(|f^{\iota_1}(y)| \cdot | \sum_{x \in \Omega_0(V,\iota_0)} f^{\iota_0}(x) \sum_{D(\Omega_0)(xy)} \prod_{\iota \neq \iota_0, \iota_1} f^\iota | \Big). \quad (2.3)$$

Fix $b \in \Omega_0(V, \iota_1)$, and for every $u \in P(V)^-$, write b_u for the projection of b to $\Omega(u, \iota_1 \restriction u)$. For every $u \in P(V)^-$, define the function $\iota_u \in 2^V$ by $\iota_u \restriction u = 0$ and $\iota_u \restriction u^c = 1$.

We define a $P(V)^-$-refinement Ω_b of $\Omega_0(V, \iota_0)$ as follows: for every $u \in P(V)^-$ and every $a \in \Omega_0(u, \iota_0 \restriction u)$, the pullback of a to $\Omega_b(u)$ is the set of elements c such that there is $v \subseteq u$ with $c \in \Omega_0(V, \iota_v)$ and the projection of c to v equals a_v and the projection of c to v^c equals b_{v^c}. It is clear how to set up projections $\Omega_b(u) \to \Omega_b(v)$ for $v \subseteq u$. Write π for the projections $\Omega_b \to \Omega_0$.[4]

Observe that $\sum_{D(\Omega_0)(xb)} \prod_{\iota \neq \iota_0, \iota_1} f^\iota$ is a function of $x \in \Omega_0(V, \iota_0)$ which takes finitely many values, according to the number of elements above x in $\Omega_b(V)$

[4] Note that Ω_b is defined over Ab as a system of difference varieties, but we could equally well define a refinement of Ω over A with multiplicities matching those of Ω_b and set up identification maps with Ω_b. This would amount to exactly the same, from a combinatorial point of view.

which belong to ϕ^ι or not. Therefore, there is some N (which does not depend on q), some scalars $\lambda_1, \ldots, \lambda_N$, and some chains $W_1, \ldots, W_N$ contained in Ω_b such that

$$\sum_{D(\Omega_0)(xb)} \prod_{\iota \neq \iota_0, \iota_1} f^\iota = \sum_{j \leq N} \lambda_j \mathbb{1}_{\pi(W_j(V))}$$

By ϵ-étale-edge-uniformity, we have

$$|\sum_{x \in \Omega_0(V, \iota_0)} f^{\iota_0}(x) \sum_{D(\Omega_0)(xb)} \prod_{\iota \neq \iota_0, \iota_1} f^\iota| = O(\epsilon_V(q))|\Omega(V)| \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$|\sum_{D(\Omega_0)} \prod_{\iota \in 2^V} f^\iota| = O(\epsilon_V(q))|\Omega(V)|^2.$$

□

We now prove the near-converse of Proposition 2.4.8. The proof essentially follows [Gow06] but requires an application of regularity of the system of varieties.

Proposition 2.4.9. *Let Ω be a regular system of difference varieties on V over A. For every $u \subseteq V$ with $|u| \geq 2$, let $\epsilon_u : \mathbb{N} \to [0, \infty)$ and let $\epsilon = (\epsilon_u)_{|u| \geq 2}$. If Ω is ϵ-quasirandom, then Ω is $(\epsilon_u^{2^{-|u|}})$-edge-uniform.*

Proof. It is enough to check edge-uniformity at the top level V. We specialise all data to some K_q. Let $\phi = \rho_V \Omega(V)$ and let W be a $P(V)^-$-chain contained in $\Omega \restriction P(V)^-$. Write $\Omega_0 = \Omega \restriction P(V)^-$ so that $\phi \subseteq \Omega_0(V)$. In the next expression we write $\mathbb{1}_{W(v)}$ for the indicator function of the set of elements of $\Omega_0(V)$ which project to $W(v)$ under the map $\Omega_0(V) \to \Omega(v)$. Define

$$\Delta = ||\phi \cap W(V)| - \frac{|\phi|}{|\Omega_0(V)|}|W(V)|| = |\sum_{\Omega_0(V)} f \prod_{v \in P(V)^-} \mathbb{1}_{W(v)}|.$$

We want to show that when $|V| = n$,

$$\Delta^{2^n} = O(|\Omega_0(V)|^{2^n - 2})|\sum_{D(\Omega_0)} \prod_{\iota \in 2^V} f^\iota|.$$

We proceed by induction on n. The induction holds for an arbitrary function f but requires the underlying system to be the specialisation of a regular system. The base case for the induction is $n = 2$, which is exactly Theorem 3.1 in [Gow06], since $P(2)^-$-systems are just Cartesian products.

We will use the following notation: for $v \subseteq u \subseteq V$ and $x \in \Omega(v)$, write $\Omega(u)(x)$ for the pullback of x to $\Omega(u)$ by the map $\Omega(u) \to \Omega(v)$.

Let $*$ be an element of V and write $V' = V \setminus \{*\}$ and $V^+ = V' \cup \{(*,0),(*,1)\}$. Let Ω_1 be the system of varieties on V^+ obtained from Ω_0 by "doubling $*$": $\Omega_1((*,0))$ and $\Omega_1((*,1))$ are two copies of $\Omega_0(*)$ and restricting Ω_1 to $V^+ \setminus \{(*,0)\}$ or $V^+ \setminus \{(*,1)\}$ gives isomorphic copies of $\Omega_0(V)$ which project to $\Omega_0(V')$. For $v \subseteq V'$, we write $v0 = v \cup \{(*,0)\}$, $v1 = v \cup \{(*,1)\}$ and $v01 = v \cup \{(*,0),(*,1)\}$. For $i = 0,1$ we write $W(vi)$ for the copy of $W(v \cup \{*\})$ inside $\Omega_1(vi)$.

By Cauchy-Schwartz and then expanding, we have

$$\begin{aligned}
\Delta^{2^n} &= \Big(\sum_{x \in \Omega_0(V')} \mathbb{1}_{W(V')}(x) \sum_{\Omega_0(V)(x)} f \prod_{\substack{v \in P(V)^- \\ v \neq V'}} \mathbb{1}_{W(v)} \Big)^{2^n} \\
&\leq \Big(\sum_{x \in \Omega_0(V')} \mathbb{1}_{W(V')}(x) \Big)^{2^{n-1}} \Big(\sum_{x \in \Omega_0(V')} \big(\sum_{\Omega_0(V)(x)} f \prod_{\substack{v \in P(V)^- \\ v \neq V'}} \mathbb{1}_{W(v)} \big)^2 \Big)^{2^{n-1}} \\
&\leq |\Omega_0(V')|^{2^{n-1}} \Big(\sum_{\Omega_1(V^+)} g \prod_{v \in P(V')^-} \mathbb{1}_{W(v0)} \mathbb{1}_{W(v1)} \Big)^{2^{n-1}} \qquad (2.5)
\end{aligned}$$

where g is defined to be the product of the two copies of f inside $\Omega_1(V^+)$.

Now we decompose $\Omega_1(V^+)$ along the two copies of $\Omega_0(*)$. For any $(x,y) \in (\Omega_0(*))^2$, write Λ_{xy} for the pullback of (x,y) to $\Omega_1(V^+)$. There is a finite projection $\Lambda_{xy} \to \Omega_0(V')$ so $|\Lambda_{xy}| = O(|\Omega_0(V')|)$ where the constant in O does not depend on x or y.

Λ_{xy} is a $P(V')$-system when we define $\Lambda_{xy}(u)$ for $u \subseteq V'$ as the pullback of (x,y) to $\Omega_1(u01)$. For every $v \subseteq V'$, define $W(v01) \subseteq \Lambda_{xy}(v)$ to be the fibre product of the copies of $W(v0)$ and $W(v1)$.

Let Γ_{xy} be the restriction of Λ_{xy} to $P(V')^-$. Observe that Γ_{xy} "forgets" $\Omega_0(V')$ and hence there is a finite projection $\Lambda_{xy}(V') \to \Gamma_{xy}(V')$ corresponding to the projection $\Omega_0(V') \to \Omega_0(V')^-$. For $a \in \Gamma_{xy}(V')$, write $\Lambda_{xy}(V')(a)$ for the pullback of a to $\Lambda_{xy}(V')$ and define

$$h_{xy}(a) = \sum_{b \in \Lambda_{xy}(V')(a)} g(b).$$

Fixing $(x,y) \in \Omega\{*\} \times \Omega\{*\}$ we have

$$\sum_{b \in \Lambda_{xy}(V')} g(b) \prod_{v \in P(V')^-} \mathbb{1}_{W(v0)} \mathbb{1}_{W(v1)}(b) = \sum_{a \in \Gamma_{xy}(V')} h_{xy}(a) \prod_{v \in P(V')^-} \mathbb{1}_{W(v01)}(a) \qquad (2.6)$$

Combining (2.5) and (2.6) and applying Jensen's inequality, we have:

$$\begin{aligned}
\Delta^{2^n} &\leq |\Omega(V')|^{2^{n-1}} \Big(\sum_{(x,y) \in (\Omega(*))^2} \sum_{\Gamma_{xy}(V')} h_{xy} \prod_{v \in P(V')^-} \mathbb{1}_{W(v01)} \Big)^{2^{n-1}} \\
&\leq |\Omega(V')|^{2^{n-1}} |\Omega(*)|^{2^n - 2} \sum_{(x,y) \in (\Omega(*))^2} \Big(\sum_{\Gamma_{xy}(V')} h_{xy} \prod_{v \in P(V')^-} \mathbb{1}_{W(v01)} \Big)^{2^{n-1}} \qquad (2.7)
\end{aligned}$$

By induction hypothesis, we have for every sufficiently generic pair x, y in $\Omega(*)$

$$\Big(\sum_{\Gamma_{xy}(V')} h_{xy} \prod_{v\in P(V')^-} \mathbb{1}_{W(v01)}\Big)^{2^{n-1}} = O(|\Gamma_{xy}(V')|^{2^{n-1}-2}) \sum_{D(\Gamma_{xy})} \prod_{\iota\in 2^{V'}} h^{\iota}_{xy} \tag{2.8}$$

Claim 2.4.9.1. *For* $(x, y) \in (\Omega\{*\})^2$, $\sum_{D(\Gamma_{xy})} \prod_{\iota\in 2^{V'}} h^{\iota}_{xy} = \sum_{D(\Lambda_{xy})} \prod_{\iota\in 2^{V'}} g^{\iota}$.

Proof of Claim. This proof is essentially a definition chase. For $c \in D(\Gamma_{xy})$ and $\iota \in 2^{V'}$, write c_ι for the projection of c to $\Gamma_{xy}(V', \iota)$. By definition,

$$\sum_{c\in D(\Gamma_{xy})} \prod_{\iota\in 2^{V'}} h^{\iota}_{xy}(c) = \sum_{c\in D(\Gamma_{xy})} \prod_{\iota\in 2^{V'}} \sum_{b\in\Lambda_{xy}(V',\iota)(c_\iota)} g^{\iota}(b)$$

where $\Lambda_{xy}(V', \iota)$ is a copy of Λ_{xy} inside $D(\Lambda_{xy})$. Fix $c \in D(\Gamma_{xy})$. Let Ξ be the set of choice functions $2^{V'} \to \bigsqcup_{\iota\in 2^{V'}} \Lambda_{xy}(V', \iota)(c_\iota)$. Expanding, we have

$$\prod_{\iota\in 2^{V'}} h^{\iota}_{xy}(c) = \sum_{\xi\in\Xi} \prod_{\iota\in 2^{V'}} g^{\iota}(\xi(\iota)).$$

Since $D(\Lambda_{xy})$ is the fibre product of the varieties $\Lambda_{xy}(V', \iota)$, any $d \in D(\Lambda_{xy})$ is uniquely determined by its projections to each $\Lambda_{xy}(V', \iota)$. Since $D(\Lambda_{xy})$ is a $D(P(V'))$-system, each set $(V', \iota) \in D(P(V'))$ is maximal and hence we can define a bijection $\Xi \to D(\Gamma_{xy})(c)$ such that $\xi \mapsto (d_\iota)_{\iota\in 2^{V'}}$. This implies that

$$\prod_{\iota\in 2^{V'}} h^{\iota}_{xy}(c) = \sum_{d\in D(\Gamma_{xy})(c)} \prod_{\iota\in 2^{V'}} g^{\iota}(d_\iota).$$

The claim follows. □

By the claim and (2.8), we have

$$\Big(\sum_{\Gamma_{xy}} h_{xy} \prod_{v\in P(V')^-} \mathbb{1}_{W(v01)}\Big)^{2^{n-1}} = O(|\Lambda_{xy}(V')|^{2^{n-1}-2}) \sum_{D(\Lambda_{xy})} \prod_{\iota\in 2^{V'}} g^{\iota}. \tag{2.9}$$

By Lemma 2.2.18, Theorem 2.4.1, and Lemma 2.1.6, $|\Lambda_{xy}(V')| = O(|\Omega(V')|)$ for sufficiently generic x, y. Therefore, we combine (2.7) and (2.9) to obtain

$$\begin{aligned}\Delta^{2^n} &\leq O(|\Omega(V)|^{2^n-2}) \sum_{(x,y)\in\Omega(*)^2} \sum_{D(\Lambda_{xy})} \prod_{\iota\in 2^{V'}} g^{\iota} \\ &= O(|\Omega(V)|^{2^n-2}) \sum_{D(\Omega)} \prod_{\iota\in 2^{V}} f^{\iota}.\end{aligned}$$

This completes the induction. Now the lemma follows from ϵ-quasirandomness. □

We deduce the following hypergraph regularity lemma in the étale setting:

Theorem 2.4.10. *Let Ω an irreducible system of difference varieties on V over A. For every $u \subseteq V$ with $|u| \geq 2$, let $\epsilon_u(q) = q^{-2^{-|u|-1}}$ and let $\delta_u(q) = q^{-1/2}$. Let $\epsilon = (\epsilon_u)_{|u|\geq 2}$ and $\delta = (\delta_u)_{|u|\geq 2}$.*

Then there is a surjective refinement Ω' of Ω over a finitely generated extension A' of A with irreducible components $\Omega_1, \ldots, \Omega_n$ such that each Ω_i is δ-quasirandom and ϵ-edge-uniform. n depends only on the degrees of the varieties in Ω.

Remark: Theorem 2.4.10 gives weaker bounds for edge-uniformity in the case $n = 2$ than the original result of [Tao12], which proves $q^{-1/4}$-edge-uniformity, rather than $q^{-1/8}$-edge-uniformity. Tao's proof of the algebraic regularity lemma does not seem to generalise easily to the higher dimensional setting, even with the stationarity theorem. This is why we have taken the route of quasirandomness, which involves a loss in the error bounds. It would be interesting to know if our bound can be improved.

2.4.3 A classical algebraic hypergraph regularity lemma

We fix A a perfect inversive finitely generated difference field. V is a finite set.

In this section, we use chain decompositions of systems of varieties to deduce an algebraic hypergraph regularity lemma which does not use the étale point of view. This algebraic hypergraph regularity lemma retains the main combinatorial features of Theorem 2.4.10 while eliminating references to refinements, but the trade-off is that we lose definability of the hypergraph partitions. For this reason, the definitions that we have already set up do not apply in this setting exactly, so we refrain from using any prior notions except chain decompositions.

We prove two technical lemmas. The first lemma revisits Proposition 2.2.15. In that proposition, we constructed a surjective refinement of Ω with regular components. However, our definition of surjective refinements only requires surjectivity at the top level V. Here we need to construct a chain decomposition of Ω, so we need a slightly different notion.

Lemma 2.4.11. *Let Ω be an irreducible system of varieties on V over A. There is a system of varieties Ω_0 on V over A and a surjective $P(V)^-$-refinement Ω_1 of Ω_0 over a finitely generated extension A' of A satisfying the following:*

1. *for every $u \subseteq V$, $\Omega(u) \subseteq \Omega_0(u)$*
2. *for every $u \subseteq V$, the projections $\pi : \Omega_1(u) \to \Omega_0(u)$, $\Omega_1(u) \to \Omega_1(u)^-$ and $\Omega_0(u) \to \Omega_0(u)^-$ are generically surjective.*

3. for every $u \subseteq V$ and every section S of the projection $\pi : \Omega_1(u)^- \to \Omega_0(u)^-$, the projection $\rho_u^{-1}(S) \to \Omega_0(u)$ is generically surjective

4. every irreducible component of Ω_1 is regular

Proof. Let $\mathcal{S}$ be a system of difference fields associated to Ω. $\mathcal{S}$ is unique up to isomorphism since Ω is irreducible. For every $u \subseteq V$, let $\mathcal{P}_u$ be a family of polynomials such that $\mathcal{S}(u)$ is the splitting field of $\mathcal{P}_u$ over $\mathcal{S}(u)^-$. Adding purely inseparable polynomials to $\mathcal{P}_u$ if necessary, we can assume that $\mathcal{P}_u$ is a family of polynomials over $A(x_u)_\sigma$, where x_u is the coordinate of a point of $\Omega(u)$.

For $|u| = 1$, set $\Omega_0(u) = \Omega(u)$. For general u, define $\Omega_0(u)$ to be the algebraic cover of $\Omega_0(u)^-$ consisting of points (a, b) where $a \in \Omega_0(u)^-$ and b generates the splitting field of $\mathcal{P}_u$ over $A(a)_\sigma$. $\Omega_0(u)$ adds only algebraic information to $\Omega_0(u)^-$ and does not choose between difference field structures on its generic points. We can assume that $\Omega(u)$ is a subvariety of $\Omega_0(u)$ by using some appropriate syntactical coding. It is clear that the projections $\rho_u : \Omega_0(u) \to \Omega_0(u)^-$ and $\Omega_0(u) \to \Omega_0(v)$ are all surjective.

By Proposition 2.2.6, find a regular $P(V)^-$-refinement $\mathcal{S}'$ of $\mathcal{S}$. For every $u \subseteq V$, let $\mathcal{Q}_u$ be a family of polynomials such that $\mathcal{S}'(u)$ is the splitting field of $\mathcal{Q}_u$ over $\mathcal{S}(u)$. Similarly to the construction sketched above, define $\Omega_1(u)$ to be a variety projecting onto $\Omega_0(u)$ such that the generic points of $\Omega_1(u)$ generate the roots of $\mathcal{Q}_u$ over the generic points of $\Omega_0(u)$ but $\Omega_1(u)$ does not impose any difference structure on these roots. As in Proposition 2.2.15, since the field extensions all lie in the invariant algebraic closure, regularity does not depend on the difference field structure and every component of Ω_1 is regular.

The surjective properties between Ω_1 and Ω_0 follow by construction. □

Our second lemma constructs a chain decomposition of the refinement obtained in Lemma 2.4.11 with useful combinatorial properties. In the following, with Ω_0 and Ω_1 as in Lemma 2.4.11, we assume that the projections $\Omega_1(u) \to \Omega_0(u)$, $\Omega_1(u) \to \Omega_1(u)^-$ and $\Omega_0(u) \to \Omega_0(u)^-$ are exactly surjective, as this amounts to changes in the varieties $\Omega_1(u), \Omega_0(u)$ of size $O(q^{dim(\Omega(u))-1})$. These changes will eventually be absorbed in the error terms.

Lemma 2.4.12. *Let Ω be an irreducible system of varieties on V over A and take $\Omega \subseteq \Omega_0$ and Ω_1 a surjective refinement of Ω_0 as in Lemma 2.4.11. Specialise $\Omega, \Omega_0, \Omega_1$ to some K_q.*

There is a $P(V)$-chain decomposition $(\mathcal{W}_v)_{v \in P(V)}$ of Ω with size depending only on the degrees of the varieties in Ω such that for every $v \subseteq V$ and every $X \in \mathcal{W}_v$, there is a set $X^* \subseteq \Omega_1(v)$ such that the following properties are satisfied:

1. For every $u \subseteq V$ and $X \in \mathcal{W}_u$, $|X| \geq \lambda_u |\Omega(u)|$ for some scalar $\lambda_u > 0$ which does not depend on q

2. *For every $u \subseteq V$, the sets X^* for $X \in \mathcal{W}_u$ are all contained in different irreducible components of $\Omega_1(u)$ and for every $X \in \mathcal{W}_u$, the projection $\pi : X^* \to X$ is bijective*

3. *For every $u \subseteq V$, if W is a $P(u)$-chain in $(\mathcal{W}_v)_{v \in P(u)}$, the sets $W(v)^*$ form a $P(u)$-chain in $\Omega_1 \restriction P(u)$ (and we write $W^*(v)$ instead of $W(v)^*$)*

4. *For every $u \subseteq V$, if W is a $P(u)$-chain in $(\mathcal{W}_v)_{v \in P(u)}$, writing $Z(u)$ for the irreducible component of $\Omega_1(u)$ containing $W^*(u)$, we have $\rho_u W^*(u) = \rho_u Z(u) \cap W^*(u)^-$*

5. *For every $u \subseteq V$ and $X, X' \in \mathcal{W}_u$, the sets $\rho_u X$ and $\rho_u X'$ are either equal or disjoint)*

Proof. We construct the families $(\mathcal{W}_v)_{P(V)^-}$ and $(\mathcal{W}_v^*)_{P(V)^-}$ inductively. For $|v| = 1$, we choose components $Z_1(v), \dots, Z_p(v)$ of $\Omega_1(v)$ such that the sets $\pi(Z_i(v))$ partition $\Omega(v)$. Let $\mathcal{W}_v = \{\pi(Z_i(v))\}$ and define $\mathcal{W}_v^*$ to be a set of sections of the projections $\pi : Z_i(v) \to \Omega(v)$.

Suppose $(\mathcal{W}_v)_I$ and $(\mathcal{W}_v^*)_I$ have been constructed where I is downward closed and take $u \subseteq V$ such that $P(u)^- \subseteq I$. Let $W_1, \dots, W_n$ be an enumeration of the $P(u)^-$-chains contained in $(\mathcal{W}_v)_{P(u)^-}$ which intersect $\rho_u \Omega(u)$. Note that the sets $W_i^*(u)$ all lie in different irreducible components of $\Omega_1(u)^-$ by Lemma 2.2.17.

We consider any $P(u)^-$-chain W among the chains $W_1, \dots, W_n$. Then $\pi : W^*(u) \to W(u)$ is bijective and by property (3) of Lemma 2.4.11, π lifts to a surjection $\pi : \rho_u^{-1}(W^*(u)) \to \rho_u^{-1}(W(u))$. Let $Z_1(u), \dots, Z_k(u)$ be irreducible subvarieties of $\Omega_1(u)$ such that $\rho_u^{-1}(W^*(u)) \cap \pi^{-1}(\Omega(u)) \subseteq \bigcup Z_i(u)$ and each $Z_i(u)$ projects to a subset of $\Omega(u)$ under π.

For every i, since $Z_i(u)$ is irreducible, both projections $\rho_u : Z_i(u) \to \Omega_1(u)^-$ and $\pi : Z_i(u) \to \Omega(u)$ have constant multiplicity. Moreover, for every fibre F of $\rho_u : Z_i(u) \to \Omega_1(u)^-$, the image $\pi(F)$ has constant size.

Starting with $i = 1$, we choose sections $X_1^*, \dots, X_m^*$ of the projection

$$Z_1(u) \cap \rho_u^{-1}(W^*(u)) \to W^*(u)$$

such that $\pi(\bigcup X_j^*) = \pi\big(Z_1(u) \cap \rho_u^{-1}(W^*(u))\big)$ and the sets $\pi(X_j^*)$ are pairwise disjoint. We add the sets X_j^* to $\mathcal{W}_u^*$ and the sets $\pi(X_j^*)$ to $\mathcal{W}_u$. Property (5) can be obtained by the same argument.

For $i = 2$, suppose that some element of $Z_2(u) \cap \rho_u^{-1} W^*(u)$ projects into the set $\pi(Z_1 \cap \rho_u^{-1} W^*(u))$. By irreducibility of $Z_2(u)$, it follows that

$$\pi(Z_2(u) \cap \rho_u^{-1} W^*(u)) \subseteq \pi(Z_1(u) \cap \rho_u^{-1} W^*(u))$$

so that we can ignore $Z_2(u)$ and move on to $Z_3(u)$. If this does not happen, then we can choose sections of $Z_2(u)$ as for $Z_1(u)$. Property (5) is guaranteed by the same argument.

We iterate in this way through all of $Z_1(u), \ldots, Z_k(u)$, thus constructing a partition $\mathcal{W}_u$ of $\rho_u^{-1}(W(u))$ with the desired properties. We then iterate this construction through all of $W_1(u), \ldots, W_n(u)$ and this defines $\mathcal{W}_u$ and $\mathcal{W}_u^*$ as desired. $\square$

We can now prove an algebraic hypergraph regularity lemma which retains the strong combinatorial properties of the étale setting. The theorem is stated for arbitrary systems of varieties, but the main case of interest is for systems Ω on V where $\Omega(V)^-$ is the Cartesian product $\prod_{i \in V} \Omega(i)$ and $\Omega(V)$ is a cover of $\Omega(V)^-$ giving rise to an arbitrary definable set.

We state the theorem in a slightly redundant way: condition (2) directly implies condition (1), as the proof will show. Nevertheless, condition (1) is probably the main import of the theorem.

Theorem 2.4.13. *Let Ω an irreducible system of difference varieties on V over A. Specialise Ω to some K_q and let $(\mathcal{W}_u)_{u \in P(V)}$ be a $P(V)$-chain decomposition constructed as in Lemma 2.4.12. Then the following properties hold:*

1. *For every $P(V)^-$-chain W contained in $(\mathcal{W}_v)_{v \in P(V)^-}$, either $W(V) \cap \rho_V \Omega(V)$ is small, i.e.*

$$|W(V) \cap \rho_V \Omega(V)| = O(q^{|V|-1})$$

   or, for every chain W' contained in W,

$$|\frac{|\rho_V \Omega(V) \cap W'(V)|}{|\rho_V \Omega(V) \cap W(V)|} - \frac{|W'(V)|}{|W(V)|}| = O(q^{-2^{-|V|-1}})$$

2. *for every $u \subseteq V$, for every $P(u)$-chain W contained in $(\mathcal{W}_v)_{v \in P(u)}$, and for every $P(u)^-$-chain W' contained in $W \restriction P(u)^-$,*

$$|\frac{|\rho_u W(u) \cap W'(u)|}{|\rho_u W(u)|} - \frac{|W'(u)|}{|W(u)^-|}| = O(q^{-2^{-|u|-1}})$$

3. *For every $u \subseteq V$, for every $P(u)$-chain W contained in $(\mathcal{W}_v)_{v \in P(u)}$, writing $f = \mathbb{1}_{\rho_u W(u)} - \frac{|\rho_u W(u)|}{|W(u)^-|} \mathbb{1}_{W(u)^-}$ on $W(u)^-$,*

$$\sum_{D(\Omega \restriction P(u)^-)} \prod_{\iota \in 2^u} f^\iota = O(q^{-2^{-|u|-1}}) |\Omega(u)|^2$$

where the functions $O(\cdot)$ depend only on the degrees of the polynomials in Ω.

Proof. Let Ω_0 and Ω_1 be as in Lemma 2.4.11. For every $X \in \mathcal{W}_v$, we write X^* and $\pi : X^* \to X$ as in Lemma 2.4.12.

Take $u \subseteq V$ and W a $P(u)$-chain in $(\mathcal{W}_v)_{P(u)}$. For every $v \subseteq u$, write $Z(v)$ for the irreducible subvariety of $\Omega_1(v)$ containing $W^*(v)$. Then Z is a regular system of varieties. Since $\rho_u W^*(u) = \rho_u Z(u) \cap W^*(u)^-$ and by Theorem 2.4.10, $\rho_u W^*(u)$ has the obvious edge-uniformity properties with respect to chains contained in $W^* \restriction P(u)^-$.

Let W' be a $P(u)^-$-chain contained in $W \restriction P(u)^-$. Write $W''(v)$ for the preimage of $W'(v)$ in $W^*(v)$, for every $v \in P(u)^-$, so that $W'(u) = \pi(W''(u))$. Then we have

$$|\frac{|\rho_u W^*(u) \cap W''(u)|}{|\rho_u W^*(u)|} - \frac{|W''(u)|}{|W^*(u)^-|}| = O(q^{-2^{-|u|-1}})$$

Moreover, since π restricts to bijections $W^*(u)^- \to W(u)^-$ and $W^*(u) \to W(u)$, it follows easily that

$$|\frac{|\rho_u W(u) \cap W'(u)|}{|\rho_u W(u)|} - \frac{|W'(u)|}{|W(u)^-|}| = O(q^{-2^{-|u|-1}})$$

giving property (2). Property (3) follows by a similar argument and an application of Theorem 2.4.10.

Finally, by Property (5) of Lemma 2.4.12, $\rho_V(\Omega(V))$ is the disjoint union of a family of sets $\rho_V W_1(V), \ldots, \rho_V W_n(V)$ where the $W_i(V)$ are elements of $\mathcal{W}_V$. Applying property (2) to each of the sets $\rho_V W_i(V)$ gives property (1). $\square$

It is an open question whether we can strengthen Theorems 2.4.10 and 2.4.13 to obtain an algebraic hypergraph regularity lemma such that the chain decomposition of Theorem 2.4.13 comes from definable sets.

The results of [BH12] about the model theory of pseudofinite fields show that an algebraic hypergraph regularity of this kind is available in many pseudofinite fields. [BH12] shows that for almost all completions T' of the theory T of pseudofinite fields, for $K \models T'$, $M \prec K$ and $M \subseteq A \subseteq K$, we have $\mathrm{acl}(A) = \mathrm{dcl}(A)$. Here, "almost all" is meant with respect to the Haar measure on the absolute Galois group of $\mathbb{Q}$ or $\mathbb{F}_p$. In this setting, the sections of Lemma 2.4.12 become uniformly definable sets and hence the chain decomposition of 2.4.13 is definable in the classical sense.

An algebraic hypergraph regularity lemma in the setting of [BH12] can be found in the PhD thesis of Elad Levi. The proof of the algebraic regularity lemma given in that thesis is rather different from the proof given here. Levi's argument used a weak version of the stationarity theorem and the derivation of the algebraic regularity lemma relied more heavily on combinatorial arguments. The argument in Levi's thesis also gave weaker error bounds on regularity than the ones we obtain here.

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