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q-Enumeration of Type B and Type D Eulerian Polynomials Based on Parity of Descents

Hiranya Kishore Dey[†], Umesh Shankar[‡], and Sivaramakrishnan Sivasubramanian[⋆]

[†] *Department of Mathematics, Indian Institute of Science, Bangalore 560012, India*
Email: hiranya.dey@gmail.com

[‡] *Department of Mathematics, Indian Institute of Technology Bombay, Mumbai 400076, India.*
Email: 204093001@iitb.ac.in

[⋆] *Department of Mathematics, Indian Institute of Technology Bombay, Mumbai 400076, India*
Email: krishnan@math.iitb.ac.in

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Abstract: Carlitz and Scoville in 1973 considered a four-variable polynomial that enumerates permutations in $\mathfrak{S}_n$ with respect to the parity of its descents and ascents. In recent work, Pan and Zeng proved a q-analogue of Carlitz-Scoville's generating function by enumerating permutations with the above four statistics along with the inversion number. Further, they also proved a type B analogue by enumerating signed permutations with respect to the parity of descents and ascents. In this work, we prove a q-analogue of the type B result of Pan and Zeng by enumerating permutations in $\mathfrak{B}_n$ with the above four statistics and the type B inversion number. We also obtain a q-analogue of the generating function for the type B bivariate alternating descent polynomials. We consider a similar five-variable polynomial in the type D Coxeter groups as well and give their egf. Alternating descents for the type D groups were previously also defined by Remmel, but our definition is slightly different. As a by-product of our proofs, we get bivariate q-analogues of Hyatt's recurrences for the type B and type D Eulerian polynomials. Further corollaries of our results are some symmetry relations for these polynomials and q-analogues of generating functions for snakes of types B and D.

Keywords: Euler-Mahonian polynomials; Inversion number; Type B Coxeter groups; Type D Coxeter groups
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1. Introduction

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$ and let $\mathfrak{S}_n$ be the set of permutations of $[n]$. For a permutation $\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathfrak{S}_n$, an index $i \in [n-1]$ is said to be a descent of π if $\pi_i > \pi_{i+1}$. Define $\mathrm{DES}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$ to be the set of descents of π and let $\mathrm{des}(\pi) = |\mathrm{DES}(\pi)|$. The classical Eulerian polynomial is defined as the generating function of the descent statistic over $\mathfrak{S}_n$, that is,

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)}.$$

These polynomials are very well-studied. The books by Foata and Schutzenberger [4] and by Petersen [9] contain many interesting results on these polynomials. The Eulerian polynomials are further generalized to get $1/k$-Eulerian polynomials, see for example Ma and Mansour [6]. An index $i \in [n]$ is called an ascent of $\pi \in \mathfrak{S}_n$ if $\pi_i < \pi_{i+1}$. Taking parity of the position of the descents, one can define *odd ascents, odd descents, even ascents* and *even descents*. Formally, let $\mathrm{EvenDES}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}, i \text{ is even}\}$, $\mathrm{EvenASC}(\pi) = \{i \in [n-1] : \pi_i < \pi_{i+1}, i \text{ is even}\}$, $\mathrm{OddDES}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}, i \text{ is odd}\}$ and $\mathrm{OddASC}(\pi) = \{i \in [n-1] : \pi_i < \pi_{i+1}, i \text{ is odd}\}$. Carlitz and Scoville in [2] considered the polynomial

$$A_n(s_0, s_1, t_0, t_1) = \sum_{\pi \in \mathfrak{S}_n} s_0^{\mathrm{easc}(\pi)} s_1^{\mathrm{oasc}(\pi)} t_0^{\mathrm{edes}(\pi)} t_1^{\mathrm{odes}(\pi)}.$$

They gave the exponential generating function (egf henceforth) for the above polynomial (see Theorem [2, Theorem 3.1]). Pan and Zeng considered a q-analogue of the above polynomial by adding the inversion number as well. For $\pi \in \mathfrak{S}_n$ define $\mathrm{inv}(\pi) = |\{1 \leq i < j \leq n : \pi_i > \pi_j\}|$. They considered

$$A_n(s_0, s_1, t_0, t_1, q) = \sum_{\pi \in \mathfrak{S}_n} s_0^{\mathrm{easc}(\pi)} s_1^{\mathrm{oasc}(\pi)} t_0^{\mathrm{edes}(\pi)} t_1^{\mathrm{odes}(\pi)} q^{\mathrm{inv}(\pi)}.$$

Pan and Zeng gave the following egf for $A_n(s_0, s_1, t_0, t_1, q)$. For integers $i \geq 0$, define $[i]_q = (1+q+\cdots+q^{i-1})$ and define $n!_q = \prod_{i=1}^n [i]_q$. Recall that $\mathrm{e}_q(u) = \sum_{n\geq 0} \frac{u^n}{[n]_q!}$. Separating the odd and even terms, let

$$\cosh_q(u) = \frac{\mathrm{e}_q(u) + \mathrm{e}_q(-u)}{2} \qquad \text{and} \qquad \sinh_q(u) = \frac{\mathrm{e}_q(u) - \mathrm{e}_q(-u)}{2}.$$

Pan and Zeng in [7, Theorem 1.2] showed the following (they use the variables x, y for what we denote t, s respectively.)

Theorem 1.1 (Pan and Zeng). *Let* $\alpha = \sqrt{(t_0 - s_0)(t_1 - s_1)}$. *Then,*

$$\sum_{n\geq 1} A_n(s_0, s_1, t_0, t_1, q) u^n / n!_q = \frac{(s_1 + t_1)\cosh_q(\alpha u) + \alpha \sinh_q(\alpha u) - t_1(\cosh_q^2(\alpha u) - \sinh_q^2(\alpha u)) - s_1}{s_0 s_1 - (s_0 t_1 + s_1 t_0)\cosh_q(\alpha u) + t_0 t_1(\cosh_q^2(\alpha u) - \sinh_q^2(\alpha u))}.$$

Using the same notation, Theorem 1.1 gives rise to an identity for the bivariate Eulerian polynomial and the bivariate alternating Eulerian polynomial. It is easy to see (and noted by Pan and Zeng [7]) among the four statistics that involve descents and ascents in Theorem 1.1, there are choices of *two* of them which determine the other two statistics. Indeed, using our result, we get type B and type D counterparts of Theorem 1.1. These are presented as Theorem 2.4 and Theorem 3.2 respectively.

Pan and Zeng in [7] also gave a type B counterpart of these identities without the variable q (that is, without taking type B inversions into account). For a positive integer n, let $[\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. $\mathfrak{B}_n$ is the set of permutations π of $[\pm n]$ that satisfy $\pi(-i) = -\pi(i)$. Let $\pi_0 = 0$ for all $\pi \in \mathfrak{B}_n$ and let $[n]_0 = \{0, 1, 2, \ldots, n\}$. Define $\mathrm{EvenDES}_B(\pi) = \{i \in [n-1]_0 : \pi_i > \pi_{i+1}, i \text{ is even}\}$, $\mathrm{EvenASC}_B(\pi) = \{i \in [n-1]_0 : \pi_i < \pi_{i+1}, i \text{ is even}\}$, $\mathrm{OddDES}_B(\pi) = \{i \in [n-1]_0 : \pi_i > \pi_{i+1}, i \text{ is odd}\}$ and $\mathrm{OddASC}_B(\pi) = \{i \in [n-1]_0 : \pi_i < \pi_{i+1}, i \text{ is odd}\}$. Define $\mathrm{odes}_B(\pi) = |\mathrm{OddDES}_B(\pi)|$, $\mathrm{edes}_B(\pi) = |\mathrm{EvenDES}_B(\pi)|$, $\mathrm{oasc}_B(\pi) = |\mathrm{OddASC}_B(\pi)|$ and lastly $\mathrm{easc}_B(\pi) = |\mathrm{EvenASC}_B(\pi)|$. Further, define

$$B_n(s,t) = \sum_{\pi \in \mathfrak{B}_n} s^{\mathrm{edes}_B(\pi)} t^{\mathrm{odes}_B(\pi)} \qquad \text{and} \qquad \hat{B}_n(s,t) = \sum_{\pi \in \mathfrak{B}_n} s^{\mathrm{easc}_B(\pi)} t^{\mathrm{odes}_B(\pi)}. \tag{1}$$

Setting $s = t$ in the polynomial $\hat{B}_n(s,t)$ gives $\hat{B}_n(t)$, the type B alternating Eulerian polynomial which has been studied for example by Ma, Fang, Mansour, and Yeh [3].

Theorem 1.2 (Pan and Zeng). *Let* $\alpha = (1-s)(1-t)$. *Then, we have*

$$\sum_{n\geq 1} B_{2n}(s,t) \frac{u^{2n}}{(2n)!} = \frac{(s+t)\sum_{n\geq 0} \frac{\alpha^n (2u)^{2n}}{(2n)!} + \sum_{n\geq 0} \frac{\alpha^{n+1} u^{2n}}{(2n)!} - (1+st)}{(1+st) - (s+t)\sum_{n\geq 0} \frac{\alpha^n (2u)^{2n}}{(2n)!}},$$

$$\sum_{n\geq 0} B_{2n+1}(s,t) \frac{u^{2n+1}}{(2n+1)!} = \frac{(s^2-1)(t-1)\sum_{n\geq 0} \frac{\alpha^n u^{2n+1}}{(2n+1)!}}{(1+st) - (s+t)\sum_{n\geq 0} \frac{\alpha^n (2u)^{2n}}{(2n)!}}.$$

They also gave similar results about the type B alternating descent polynomials. Their result is as follows.

Theorem 1.3 (Pan and Zeng). *Let* $\alpha = (1-s)(1-t)$. *Then, we have*

$$\sum_{n\geq 1} \hat{B}_{2n}(s,t) u^{2n}/(2n)! = \frac{(1+st)\sum_{n\geq 0} \frac{(-\alpha)^n (2u)^{2n}}{(2n)!} + \sum_{n\geq 0} \frac{(-\alpha)^{n+1} u^{2n}}{(2n)!} - (s+t)}{(s+t) - (1+st)\sum_{n\geq 0} \frac{(-\alpha)^n (2u)^{2n}}{(2n)!}},$$

$$\sum_{n\geq 0} \hat{B}_{2n+1}(s,t) u^{2n+1}/(2n+1)! = \frac{(1+s)\sum_{n\geq 0} \frac{(-\alpha)^{n+1} u^{2n+1}}{(2n+1)!}}{(s+t) - (1+st)\sum_{n\geq 0} \frac{\alpha^n (2u)^{2n}}{(2n)!}}$$

Let $H_0(s,t,u) = \sum_{n\geq 0} B_{2n}(s,t) \frac{u^{2n}}{(2n)!}$ and $H_1(s,t,u) = \sum_{n\geq 0} B_{2n+1}(s,t) \frac{u^{2n+1}}{(2n+1)!}$.

Recall that $\cosh(x) = \frac{1}{2}\Big(\exp(x) + \exp(-x)\Big)$ and $\sinh(x) = \frac{1}{2}\Big(\exp(x) - \exp(-x)\Big)$.

$$\text{Define } M^2 = \alpha. \tag{2}$$

It is easy to see that the following alternate form can be used to state Theorem 1.2.

Theorem 1.4 (Pan and Zeng)**.** *With the above notation,*

$$H_0(s,t,u) = \frac{M^2\cosh(uM)}{M^2\cosh^2(uM) - (s+1)(t+1)\sinh^2(uM)}, \tag{3}$$

$$H_1(s,t,u) = \frac{M(s+1)\sinh(uM)}{M^2\cosh^2(uM) - (s+1)(t+1)\sinh^2(uM)}. \tag{4}$$

Recall that length in Type B Coxeter groups is defined as follows (see [9, Page 294]). For $\pi \in \mathfrak{B}_n$,

$$\mathrm{inv}_B(\pi) = |\{1 \le i < j \le n : \pi_i > \pi_j\}| + |\{1 \le i < j \le n : -\pi_i > \pi_j\}| + |\mathrm{Negs}(\pi)|,$$

where $\mathrm{Negs}(\pi) = \{\pi_i : i > 0, \pi_i < 0\}$. Further, recall the definition of $\mathrm{odes}_B(\pi)$ and $\mathrm{edes}_B(\pi)$ from earlier. Define

$$B_n(t,q) = \sum_{\pi\in\mathfrak{B}_n} t^{\mathrm{des}_B(\pi)} q^{\mathrm{inv}_B(\pi)} \quad\text{and}\quad B_n(s,t,q) = \sum_{\pi\in\mathfrak{B}_n} s^{\mathrm{edes}_B(\pi)} t^{\mathrm{odes}_B(\pi)} q^{\mathrm{inv}_B(\pi)}, \tag{5}$$

$$H_0(s,t,q,u) = \sum_{k\ge 0} B_{2n}(s,t,q)\frac{u^{2n}}{B_{2n}(1,q)} \text{ and } H_1(s,t,q,u) = \sum_{k\ge 0} B_{2n+1}(s,t,q)\frac{u^{2n+1}}{B_{2n+1}(1,q)}. \tag{6}$$

Let $\exp_B(u;q) = \sum_{n\ge 0} \frac{u^n}{B_n(1,q)}$. As before, we separate terms with odd and even exponents and define

$$\cosh_B(u;q) = \frac{\exp_B(u;q) + \exp_B(-u;q)}{2} \qquad\text{and}\qquad \sinh_B(u;q) = \frac{\exp_B(u;q) - \exp_B(-u;q)}{2}.$$

With this notation, our first main result is the following q-analogue of Theorem 1.4.

Theorem 1.5. *We have*

$$H_0(s,t,q,u) = \frac{(1-s)\Big(\big(1 - t\cosh_q(Mu)\big)\cosh_B(Mu;q) + t\sinh_q(Mu)\sinh_B(Mu;q)\Big)}{1 - (s+t)\cosh_q(Mu) + st\mathrm{e}_q(Mu)\mathrm{e}_q(-Mu)}, \tag{7}$$

$$H_1(s,t,q,u) = \frac{M\Big(\big(1 - s\cosh_q(Mu)\big)\sinh_B(Mu;q) + s\sinh_q(Mu)\cosh_B(Mu;q)\Big)}{1 - (s+t)\cosh_q(Mu) + st\mathrm{e}_q(Mu)\mathrm{e}_q(-Mu)}. \tag{8}$$

Theorem 1.5 is proved in Subsection 2.1. Recalling (1), define

$$\hat{H}_0(s,t,u) = \sum_{n\ge 0} \hat{B}_{2n}(s,t)\frac{u^{2n}}{(2n)!} \qquad\text{and}\qquad \hat{H}_1(s,t,u) = \sum_{n\ge 0} \hat{B}_{2n+1}(s,t)\frac{u^{2n+1}}{(2n+1)!}.$$

We have rewritten Theorem 1.2 as Theorem 1.4 and stated our generalization as Theorem 1.5. Similarly, it is easy to see that Theorem 1.3 can be rewritten as follows.

Theorem 1.6 (Pan and Zeng)**.** *With the above notation,*

$$\hat{H}_0(s,t,u) = \frac{-(s-1)(t-1)\cos(Mu)}{s + t - (ts+1)\cos(2Mu)},$$

$$\hat{H}_1(s,t,u) = \frac{-M(s+1)\sin(Mu)}{s + t - (ts+1)\cos(2Mu)}.$$

Define $\hat{B}_n(s,t,q) = \sum_{\pi\in\mathfrak{B}_n} t^{\mathrm{odes}_B(\pi)} s^{\mathrm{easc}_B(\pi)} q^{\mathrm{inv}_B(\pi)}$ and let

$$\hat{H}_1(s,t,q,u) = \sum_{n\ge 0} \hat{B}_{2n+1}(s,t,q)\frac{u^{2n+1}}{B_{2n+1}(1,q)}, \quad\text{and}\quad \widehat{H}_0(s,t,q,u) = \sum_{n\ge 0} \hat{B}_{2n}(s,t,q)\frac{u^{2n}}{B_{2n}(1,q)}.$$

Moreover, let

$$\cos_B(u;q) = \frac{\exp_B(iu;q) + \exp_B(-iu;q)}{2} \qquad\text{and}\qquad \sin_B(u;q) = \frac{\exp_B(iu;q) - \exp_B(-iu;q)}{2}.$$

Another of our main results is the following q-analogue of Theorem 1.6.

Theorem 1.7. *We have*

$$\begin{aligned}\widehat{H}_0(s,t,q,u) &= \frac{(s-1)\Big((1-t\cos_q(Mu))\cos_B(Mu;q)-t\sin_q(Mu)\sin_B(Mu;q)\Big)}{s+t\mathrm{e}_q(iMu)\mathrm{e}_q(-iMu)-(ts+1)\cos_q(Mu)},\\ \widehat{H}_1(s,t,q,u) &= \frac{-M\Big((s-\cos_q(Mu)\sin_B(Mu;q)+\sin_q(Mu)\cos_B(Mu;q)\Big)}{s+t\mathrm{e}_q(iMu)\mathrm{e}_q(-iMu)-(ts+1)\cos_q(Mu)}.\end{aligned}$$

The proof of Theorem 1.7 is also given in Subsection 2.1. We move to our counterpart of this result to type D Coxeter groups $\mathfrak{D}_n$. Recall that $\mathfrak{D}_n$ is the subgroup of $\mathfrak{B}_n$ consisting of the signed permutations which have an even number of negative signs. We denote -1 as $\overline{1}$ and for $\pi=\pi_1,\pi_2,\ldots,\pi_n\in\mathfrak{D}_n$, define $\pi_{\overline{1}}=-\pi_1$ and let $\mathrm{DES}_D(\pi)=\{i\in\{-1,1,\ldots,n-1\}:\pi_i>\pi_{|i|+1}\}$ be its set of descents. Let $\mathrm{des}_D(\pi)=|\mathrm{DES}_D(\pi)|$. Moreover, let $\mathrm{OddDES}_D(\pi)=\{i\in[-1,n-1]-\{0\}:\pi_i>\pi_{|i|+1}$ and i is odd$\}$ be the set of odd indices where descents occur in π and similarly let $\mathrm{EvenDES}_D(\pi)=\{i\in[-1,n-1]-\{0\}:\pi_i>\pi_{i+1}$ and i is even$\}$. Let $\mathrm{odes}_D(\pi)=|\mathrm{OddDES}_D(\pi)|$ and $\mathrm{edes}_D(\pi)=|\mathrm{EvenDES}_D(\pi)|$. Recall that length in Type D Coxeter groups inv_D is defined as follows (see [9, Page 302]):

$$\mathrm{inv}_D(\pi)=|\{1\le i<j\le n:\pi_i>\pi_j\}|+|\{1\le i<j\le n:-\pi_i>\pi_j\}|.$$

Remmel in [12] has given a definition of alternating descent for type D Coxeter groups based on a total order on the elements of $[\pm n]$. The polynomial that Remmel gets is different from the one we have. Remmel's main result is a joint distribution of alternating descents and alternating major index in types B and D Coxeter groups. Below, we consider a slightly different polynomial enumerating alternating descents and type D inversion number in $\mathfrak{D}_n$. Our definition uses the parity of the position of descents as before. Formally, define

$$D_n(t,q)=\sum_{\pi\in\mathfrak{D}_n}t^{\mathrm{des}_D(\pi)}q^{\mathrm{inv}_D(\pi)} \text{ and } D_n(s,t,q)=\sum_{\pi\in\mathfrak{D}_n}s^{\mathrm{edes}_D(\pi)}t^{\mathrm{odes}_D(\pi)}q^{\mathrm{inv}_D(\pi)}. \text{ Define}$$

$$\mathcal{D}_0(s,t,q,u)=\sum_{k>0}D_{2k}(s,t,q)\frac{u^{2k}}{D_{2k}(1,q)},\mathcal{D}_1=\sum_{k>0}D_{2k+1}(s,t,q)\frac{u^{2k+1}}{D_{2k+1}(1,q)}.$$

Define $\hat{D}_n(s,t,q)=\sum_{\pi\in\mathfrak{D}_n}t^{\mathrm{odes}_D(\pi)}s^{\mathrm{easc}_D(\pi)}q^{\mathrm{inv}_D(\pi)}$ and let

$$\widehat{\mathcal{D}}_0(s,t,q,u)=\sum_{n\ge1}\hat{D}_{2n}(s,t,q)\frac{u^{2n}}{D_{2n}(1,q)},\ \widehat{\mathcal{D}}_1(s,t,q,u)=\sum_{n\ge1}\hat{D}_{2n+1}(s,t,q)\frac{u^{2n+1}}{D_{2n+1}(1,q)}. \tag{9}$$

Moreover, let $\exp_D(u;q)=\sum_{n\ge0}\frac{u^n}{D_n(1,q)}$. We split it is odd and even parts and write

$$\cosh_D(u;q)=\frac{\exp_D(u;q)+\exp_D(-u;q)}{2} \quad\text{and}\quad \sinh_D(u;q)=\frac{\exp_D(u;q)-\exp_D(-u;q)}{2}.$$

Recalling M from (2), let

$$\begin{aligned}\mathrm{OD} &= ut^2(\cosh_q(Mu)-1)+\frac{(1-t)M}{(1-s)}(\sinh_D(Mu;q)-Mu)+\frac{2t(1-t)}{M}(\sinh_q(Mu)-Mu),\\ \mathrm{ED} &= 2t(\cosh_q(Mu)-1)+(1-t)(\cosh_D(Mu;q)-1)+\frac{ut^2(1-s)}{M}\sinh_q(Mu).\end{aligned}$$

For type D Coxeter groups, our main results are the following.

Theorem 1.8. *We have the egfs*

$$\begin{aligned}\mathcal{D}_0(s,t,q,u) &= \frac{\mathrm{ED}(1-t\cosh_q(Mu))+\mathrm{OD}(\frac{t(1-s)}{M}\sinh_q(Mu))}{1-(s+t)\cosh_q(Mu)+st\mathrm{e}_q(Mu)\mathrm{e}_q(-Mu)},\\ \mathcal{D}_1(s,t,q,u) &= \frac{\mathrm{OD}(1-s\cosh_q(Mu))+\mathrm{ED}(\frac{s(1-t)}{M}\sinh_q(Mu))}{1-(s+t)\cosh_q(Mu)+st\mathrm{e}_q(Mu)\mathrm{e}_q(-Mu)}.\end{aligned}$$

Theorem 1.9. *We have the egfs*

$$\widehat{\mathcal{D}}_0(s,t,q,u) = \frac{T'(\mathrm{ED})(1-t\cos_q(Mu))-T'(\mathrm{OD})(\frac{t(s-1)\sqrt{s}}{Ms}\sin_q(Mu))}{s-(st+1)\cos_q(Mu)+t\mathrm{e}_q(iMu)\mathrm{e}_q(-iMu)},$$

$$\widehat{\mathcal{D}}_1(s,t,q,u) \quad = \quad \frac{\frac{T'(\mathrm{OD})}{\sqrt{s}}(s-\cos_q(Mu)) - T'(\mathrm{ED})(\frac{(1-t)}{M}\sin_q(Mu))}{s-(st+1)\cos_q(Mu)+t\mathrm{e}_q(iMu)\mathrm{e}_q(-iMu)},$$

where

$$\begin{aligned} T'(\mathrm{OD}) &= \sqrt{s}ut^2(\cos_q(Mu)-1) - \frac{(1-t)M}{(s-1)\sqrt{s}}(\sin_D(Mu;q)-Mu) \\ &\quad + \frac{2t(1-t)\sqrt{s}}{M}(\sin_q(Mu)-Mu), \\ T'(\mathrm{ED}) &= 2t(\cos_q(Mu)-1) + \frac{ut^2(s-1)\sqrt{s}}{sM}\sin_q(Mu) + \frac{(1-t)}{t}(\cosh_D(Mu;q)-1). \end{aligned}$$

The proof of Theorem 1.8 and Theorem 1.9 appear in Subsection 3.1. It can be checked that Theorem 1.8 refines a result of Reiner [10, Corollary 4.5] for type D Euler-Mahonian polynomials. Our proofs in both the type B and type D cases use an inclusion-exclusion-based argument.

1.1 Refining Hyatt's recurrences for the Type B and Type D Eulerian polynomial

As an outcome of our proofs, we get a refinement of Hyatt's recurrences for the type B and type D Eulerian polynomials. Hyatt in [5] gave the following recurrences for type B Eulerian polynomials. We partition $\mathfrak{B}_n$ based on the sign of the last element. Define $\mathfrak{B}_n^+ = \{\pi \in \mathfrak{B}_n : \pi_n > 0\}$ to be the set containing the elements of $\mathfrak{B}_n$ with last element being positive and let $\mathfrak{B}_n^- = \mathfrak{B}_n - \mathfrak{B}_n^+$. Define $B_n^+(t) = \sum_{\pi \in \mathfrak{B}_n^+} t^{\mathrm{des}_B(\pi)}$. The following result is due to Hyatt.

Theorem 1.10 (Hyatt)**.** *For integers $n \geq 1$, we have*

$$B_n^+(t) \quad = \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k(t)(t-1)^{n-k-1}.$$

Our extension of Theorem 1.10 involves the following polynomial. Define

$$B_n^{\pm}(s,t,q) = \sum_{\pi \in \mathfrak{B}_n^{\pm}} s^{\mathrm{edes}_B(\pi)} t^{\mathrm{odes}_B(\pi)} q^{\mathrm{inv}_B(\pi)}. \tag{10}$$

Our type B generalization is the following.

Theorem 1.11. *For even positive integers n, we have*

$$\begin{aligned} B_n^+(s,t,q) &= \sum_{r=0}^{\frac{n}{2}-1} q^{\binom{2r+1}{2}} \binom{n}{2r+1}_q B_{n-2r-1}(s,t,q)(s-1)^r(t-1)^r \\ &\quad + \sum_{r=1}^{\frac{n}{2}} q^{\binom{2r}{2}} \binom{n}{2r}_q B_{n-2r}(s,t,q)(s-1)^{r-1}(t-1)^r. \end{aligned}$$

For odd positive integers n, we have

$$\begin{aligned} B_n^+(s,t,q) &= \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{2r+1}{2}} \binom{n}{2r+1}_q B_{n-2r-1}(s,t,q)(s-1)^r(t-1)^r \\ &\quad + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{2r}{2}} \binom{n}{2r}_q B_{n-2r}(s,t,q)(s-1)^r(t-1)^{r-1}. \end{aligned}$$

It is clear that setting $q = 1$ and $s = t$ in Theorem 1.11 gives us Theorem 1.10. The proof of Theorem 1.11 appears in Subsection 2.2. For Type D Coxeter groups, our analogous result is Theorem 3.3.

1.2 More consequences

Another outcome of our results is some symmetry relations. For the type B case, our results are Theorem 2.5 and Lemma 2.4. For the type D case, our symmetry results are Theorem 3.4 and Corollary 3.6.

From the q-analogue of our generating function, we naturally get a q-analogue of the enumeration of type B and type D snakes. These results are presented in Section 4. Enumeration of type B and D snakes with respect to some statistics and thus q-analogues have been obtained, see for example, Verges [14]. However, to the best of our knowledge, we have not seen q-analogues involving the appropriate length function in these groups.

2. Type B results

Recall that $\mathfrak{B}_n$ is the set of permutations of $[\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}$ satisfying $\pi(-i) = -\pi(i)$. We think of π as a word $\pi = \pi_0, \pi_1, \pi_2, \ldots, \pi_n$ where $\pi_i = \pi(i)$ and $\pi_0 = 0$.

For positive integers n and an integer i with $0 \leq i \leq n$, let $\binom{[n]}{i} = \{A \subseteq [n] : |A| = i\}$ be the set of subsets of $[n]$ with cardinality i. We define a signed subset (A, ϵ) to be a subset $A \subseteq [n]$ and ϵ is a string of signs $\pm$ of length $|A|$. Here, each element $a_i \in A$ has either a positive or a negative sign, encoded by ϵ_i, attached to it. When $a \in A$, we denote a positive signed a just by a and a negative signed a by $\overline{a}$. The set of all signed subsets of size i of $[n]$ will be denoted as $\mathrm{sgn}\binom{[n]}{i}$. Clearly, $|\mathrm{sgn}\binom{[n]}{i}| = 2^i\binom{n}{i}$.

Let $G_{n,i}$ be the set of signed permutations $\pi \in \mathfrak{B}_n$ such that the last $n-i$ elements of π are increasing, that is we have $\pi_{i+1} < \pi_{i+2} < \cdots < \pi_{n-1} < \pi_n$. It is easy to see that $|G_{n,i}| = 2^n\binom{n}{i}i!$. Let $\pi = 0, 1, 2, \ldots, n$ and define $G_{n,-1}$ to be the set containing the single element π. For example, when $n = 3$, $\pi = 0, 1, 2, 3$ and $G_{3,-1} = \{\pi\}$.

Let $\sigma = 0, \sigma_1, \cdots, \sigma_{n-i} \in \mathfrak{B}_{n-i}$ and $(A, \epsilon) \in \mathrm{sgn}\binom{[n]}{i}$ be a signed subset. Moreover, let $[n] - A = \{c_1, c_2, \ldots, c_{n-i}\}$ be written in ascending order, that is with $c_1 < c_2 < \cdots < c_{n-i}$. We define a map $h : \mathfrak{B}_{n-i} \to \mathfrak{B}_{\{c_1, c_2, \ldots, c_{n-i}\}}$ which for $1 \leq k \leq n-i$, maps k to c_k and preserves the sign. Formally,

$$h(\sigma) = 0, \pi_1, \pi_2, \ldots, \pi_{n-i}, \tag{11}$$

where for $1 \leq i \leq n-i$, if $|\sigma_i| = k$ then $|\pi_i| = c_k$ and π_i has the same sign as σ_i. This map h is clearly a bijection and is hence invertible.

By inverting the map h on the elements of $[0, n] - A$ and appending the elements of (A, ϵ) in ascending order, we get a signed permutation in $G_{n,n-i}$. This map is also invertible, and thus we have a bijection $f : \mathfrak{B}_{n-i} \times \mathrm{sgn}\binom{[n]}{i} \mapsto G_{n,n-i}$ defined below. Let $\sigma \in \mathfrak{B}_{n-i}$ and $(A, \epsilon) \in \mathrm{sgn}\binom{[n]}{i}$. For a set S (resp. a signed set (S, ϵ)), by $[S]$ (respectively by $[(S, \epsilon)]$), we denote the string obtained by writing the elements of S (respectively (S, ϵ)) in ascending order in the usual linear order of $\mathbb{Z}$. Define $f(\sigma, (A, \epsilon)) = h(\sigma)[(A, \epsilon)]$ where $h(\sigma)[(A, \epsilon)]$ denotes the juxtaposition of $h(\sigma)$ and $[(A, \epsilon)]$.

Example 2.1. *Let $n = 7, i = 4, \sigma = 0, \overline{2}, 1, 3 \in \mathfrak{B}_3$ and $(A, \epsilon) = \{1, \overline{4}, 5, \overline{6}\}$ be a signed subset of $\mathrm{sgn}\binom{[7]}{4}$. Then, $[0, n] - A = \{0, 2, 3, 7\}$ and thus $h(\sigma) = 0\overline{3}27$. Moreover, we have $[[0, n] - A] = 0, 2, 3, 7$ and $[(A, \epsilon)] = \overline{6}, \overline{4}, 1, 5$. Therefore, $f(\sigma, (A, \epsilon)) = 0, \overline{3}, 2, 7, \overline{6}, \overline{4}, 1, 5$. We also have $f([[0, 7] - A], (A, \epsilon)) = 0, 2, 3, 7, \overline{6}, \overline{4}, 1, 5$.*

Lemma 2.1. *For positive integers n, we have*

$$\sum_{(A,\epsilon) \in \mathrm{sgn}\binom{[n]}{r}} q^{\mathrm{inv}_B(f([[0,n]-A],(A,\epsilon)))} = \binom{n}{r}_q (1 + q^n)(1 + q^{n-1}) \cdots (1 + q^{n-r+1}).$$

Proof. We proceed by induction on n. The base case when $n = 1$ is easy to verify. We assume the result is true for n and want to show it holds for $n + 1$. Thus, we want to show that

$$\sum_{(A,\epsilon) \in \mathrm{sgn}\binom{[n+1]}{r+1}} q^{\mathrm{inv}_B(f([[0,n+1]-A],(A,\epsilon)))} = \binom{n+1}{r+1}_q (1 + q^{n+1})(1 + q^n) \cdots (1 + q^{n-r+1}). \tag{12}$$

Let $\eta(n, r) = (1 + q^n) \cdots (1 + q^{n-r+1})$. We partition $\mathrm{sgn}\binom{[n+1]}{r+1}$ into the disjoint union of the following three subsets and determine the contribution of each of these three sets.

1. $\mathcal{A}_1 = \{(A, \epsilon) \in \mathrm{sgn}\binom{[n+1]}{r+1}; n + 1 \in (A, \epsilon)\}$,
2. $\mathcal{A}_2 = \{(A, \epsilon) \in \mathrm{sgn}\binom{[n+1]}{r+1}; \overline{n + 1} \in (A, \epsilon)\}$,
3. $\mathcal{A}_3 = \{(A, \epsilon) \in \mathrm{sgn}\binom{[n+1]}{r+1}; n + 1 \notin (A, \epsilon)\}$.

If $n + 1 \in (A, \epsilon)$, as $[(A, \epsilon)]$ is in ascending order, it will be the rightmost element of $f([[0, n+1] - A], [(A, \epsilon)])$ and thus it will contribute no extra inversions. Thus

$$\sum_{(A,\epsilon) \in \mathcal{A}_1} q^{\mathrm{inv}_B(f([[0,n+1]-A],[(A,\epsilon)]))} = \eta(n, r)\binom{n}{r}_q. \tag{13}$$

If $\overline{n + 1} \in (A, \epsilon)$, then $\overline{n + 1}$ has to be in the '$n - r + 1$'-th position in $f([[0, n+1] - A], (A, \epsilon))$. Every element of $[[0, n + 1] - A]$ will be to its left and will thus contribute 2 inversions. Further, every element to its right will contribute 1 inversion. Thus, we get $2n - r + 1$ new inversions. Therefore,

$$\sum_{(A,\epsilon) \in \mathcal{A}_2} q^{\mathrm{inv}_B(f([[0,n+1]-A],[(A,\epsilon)]))} = \eta(n, r)q^{2n-r+1}\binom{n}{r}_q. \tag{14}$$

Lastly, when $n+1 \in [0,n+1]-A$, then it has to be the rightmost element in $[0,n+1]-A$. Every element of (A,ϵ) will contribute one inversion and thus we get '$r+1$' extra inversions. Hence,

$$\sum_{(A,\epsilon)\in\mathcal{A}_3} q^{\mathrm{inv}_B(f([[0,n+1]-A],(A,\epsilon)))} = q^{r+1}\eta(n,r+1)\binom{n}{r+1}_q = q^{r+1}(1+q^{n-r})\eta(n,r)\binom{n}{r+1}_q. \tag{15}$$

Summing up (13), (14) and (15), we get

$$\begin{aligned}
&\sum_{(A,\epsilon)\in\mathrm{sgn}\binom{[n+1]}{r+1}} q^{\mathrm{inv}_B(f([[0,n+1]-A],(A,\epsilon)))} \\
= \;& \eta(n,r)\left(\binom{n}{r}_q + q^{2n-r+1}\binom{n}{r}_q + q^{r+1}(1+q^{n-r})\binom{n}{r+1}_q\right) \\
= \;& \eta(n,r)\left((1+q^{n+1})\binom{n+1}{r+1}_q\right) = \eta(n+1,r+1)\binom{n+1}{r+1}_q.
\end{aligned}$$

The last equation follows from the q-Pascal recurrence for the Gaussian binomial coefficients (see [9, Chapter 6]). The proof of (12) and hence of Lemma 2.1 is complete. □

We illustrate the statement of Lemma 2.1 by the following example and we write permutations without comma for brevity.

Example 2.2. *When $n=3$ and $r=1$, the set $\mathrm{sgn}\binom{[n]}{r}$ is clearly $\{\{3\},\{\overline{3}\},\{2\},\{\overline{2}\},\{1\},\{\overline{1}\}\}$ and hence we have*

$$\begin{aligned}
&\sum_{(A,\epsilon)\in\mathrm{sgn}\binom{[3]}{1}} q^{\mathrm{inv}_B(f([[0,n]-A],(A,\epsilon)))} \\
= \;& q^{\mathrm{inv}_B(0123)} + q^{\mathrm{inv}_B(012\overline{3})} + q^{\mathrm{inv}_B(0132)} + q^{\mathrm{inv}_B(013\overline{2})} + q^{\mathrm{inv}_B(0231)} + q^{\mathrm{inv}_B(023\overline{1})} \\
= \;& 1+q^5+q+q^4+q^2+q^3 = (1+q+q^2)(1+q^3) = \binom{3}{1}_q(1+q^3).
\end{aligned}$$

When $n=4$ and $r=2$, the set $\mathrm{sgn}\binom{[n]}{r}$ is clearly $\{\{3,4\},\{\overline{3},4\},\{3,\overline{4}\},\{\overline{3},\overline{4}\},\{2,4\},\{\overline{2},4\},\{2,\overline{4}\},\{\overline{2},\overline{4}\},\{2,3\},$ $\{\overline{2},3\},\{2,\overline{3}\},\{\overline{2},\overline{3}\},\{1,4\},\{\overline{1},4\},\{1,\overline{4}\},\{\overline{1},\overline{4}\},\{1,3\},\{\overline{1},3\},\{1,\overline{3}\},\{\overline{1},\overline{3}\},\{1,2\},\{\overline{1},2\},\{1,\overline{2}\},\{\overline{1},\overline{2}\}\}$. It is easy to verify that

$$\begin{aligned}
&\sum_{(A,\epsilon)\in\mathrm{sgn}\binom{[3]}{1}} q^{\mathrm{inv}_B(f([[0,n]-A],(A,\epsilon)))} \\
= \;& 1+q+2q^2+2q^3+3q^4+3q^5+3q^6+3q^7+2q^8+2q^9+q^{10}+q^{11} \\
= \;& (1+q+2q^2+q^3+q^4)(1+q^4)(1+q^3) = \binom{4}{2}_q(1+q^3).
\end{aligned}$$

Corollary 2.1. *Let $\sigma \in \mathfrak{B}_{n-r}$ be a signed permutation and $(A,\epsilon)\in\mathrm{sgn}\binom{[n]}{r}$ be a signed subset. Then*

$$\sum_{(A,\epsilon)\in\mathrm{sgn}\binom{[n]}{r}} q^{\mathrm{inv}_B(f(\sigma,(A,\epsilon)))} = q^{\mathrm{inv}_B(\sigma)}\binom{n}{r}_q(1+q^n)(1+q^{n-1})\cdots(1+q^{n-r+1}). \tag{16}$$

Proof. For $\sigma\in\mathfrak{B}_{n-r}$ and $(A,\epsilon)\in\mathrm{sgn}\binom{[n]}{r}$, we have

$$\mathrm{inv}_B(f(\sigma,(A,\epsilon))) = \mathrm{inv}_B(h(\sigma),[(A,\epsilon)])) = \mathrm{inv}_B(f([[0,n]-A],(A,\epsilon))) + \mathrm{inv}_B(\sigma).$$

The proof follows as it takes exactly $\mathrm{inv}_B(\sigma)$ inversions to get $h(\sigma)$ from the identity permutation in $\mathfrak{B}_{n-r}$ (recall $h(\sigma)$ is defined in (11)). □

Adding (16) overall $\pi\in\mathfrak{B}_{n-r}$ gives us the following.

Corollary 2.2. *For positive integers n, we have*

$$\sum_{\sigma\in\mathfrak{B}_{n-r}}\sum_{(A,\epsilon)\in\mathrm{sgn}\binom{[n]}{r}} t^{\mathrm{odes}_B(\sigma)}s^{\mathrm{edes}_B(\sigma)}q^{\mathrm{inv}_B(f(\sigma,(A,\epsilon)))} = B_{n-r}(s,t,q)\binom{n}{r}_q(1+q^n)\cdots(1+q^{n-r+1}).$$

Reiner in [11] gave the following egf for the polynomial enumerating descents and length in $\mathfrak{B}_n$.

Theorem 2.1 (Reiner)**.** *We have the following:*

$$\sum_{n\geq 0} B_n(t,q)\frac{u^n}{B_n(1,q)} = \frac{(1-t)\exp_B\left(u(1-t);q\right)}{1-t\exp(u(1-t);q)}.$$

It can be seen that Theorem 2.1 is equivalent to the following.

Theorem 2.2 (Reiner)**.** *For positive integers n, the polynomials $B_n(q,t)$ satisfy the following:*

$$\frac{B_n(t,q)}{B_n(1,q)} = t\sum_{k=0}^{n} \frac{B_{n-k}(t,q)(1-t)^k}{B_{n-k}(1,q)[k]_q!} + \frac{(1-t)^{n+1}}{B_n(1,q)}.$$

We are now interested in proving a trivariate analogue of Theorem 2.2. Towards that, we start with the following lemma.

Lemma 2.2. *Let n be a positive integer and let $0 \leq i \leq n$. When i is odd, we have*

$$\begin{aligned}\sum_{\pi'\in G_{n,i}} t^{\mathrm{odes}_B(\pi')}s^{\mathrm{edes}_B(\pi')}q^{\mathrm{inv}_B(\pi')} &= t\frac{B_i(s,t,q)B_n(1,q)}{B_i(1,q)[n-i]_q!} \\ &+ (1-t)\Big\{\sum_{\pi'\in G_{n,i-1}} t^{\mathrm{odes}_B(\pi')}s^{\mathrm{edes}_B(\pi')}q^{\mathrm{inv}_B(\pi')}\Big\}. \qquad (17)\end{aligned}$$

When i is even, we have

$$\begin{aligned}\sum_{\pi'\in G_{n,i}} t^{\mathrm{odes}_B(\pi')}s^{\mathrm{edes}_B(\pi')}q^{\mathrm{inv}_B(\pi')} &= s\frac{B_i(s,t,q)B_n(1,q)}{B_i(1,q)[n-i]_q!} \\ &+(1-s)\Big\{\sum_{\pi'\in G_{n,i-1}} t^{\mathrm{odes}_B(\pi')}s^{\mathrm{edes}_B(\pi')}q^{\mathrm{inv}_B(\pi')}\Big\}. \qquad (18)\end{aligned}$$

Proof. We prove (17) first and therefore take i to be odd. Let $F_{n,i} = G_{n,i} - G_{n,i-1}$. We have

$$\begin{aligned}
&\sum_{(\pi,(A,\epsilon))\in\mathfrak{B}_i\times\mathrm{sgn}\binom{[n]}{n-i}} t^{\mathrm{odes}_B(\pi)}s^{\mathrm{edes}_B(\pi)}q^{\mathrm{inv}_B(f(\pi,(A,\epsilon)))} \\
=& \sum_{(\pi,(A,\epsilon))\in f^{-1}(G_{n,i})} t^{\mathrm{odes}_B(\pi)}s^{\mathrm{edes}_B(\pi)}q^{\mathrm{inv}_B(f(\pi,(A,\epsilon)))} \\
=& \sum_{(\pi,(A,\epsilon))\in f^{-1}(G_{n,i-1})} t^{\mathrm{odes}_B(\pi)}s^{\mathrm{edes}_B(\pi)}q^{\mathrm{inv}_B(f(\pi,(A,\epsilon)))} \\
&+ \sum_{(\pi,(A,\epsilon))\in f^{-1}(F_{n,i})} t^{\mathrm{odes}_B(\pi)}s^{\mathrm{edes}_B(\pi)}q^{\mathrm{inv}_B(f(\pi,(A,\epsilon)))} \\
=& \sum_{f(\pi,(A,\epsilon))\in G_{n,i-1}} t^{\mathrm{odes}_B(f(\pi,A))}s^{\mathrm{edes}_B(f(\pi,A))}q^{\mathrm{inv}_B(f(\pi,(A,\epsilon)))} \\
&+\frac{1}{t}\Big\{\sum_{f(\pi,(A,\epsilon))\in F_{n,i}} t^{\mathrm{odes}_B(f(\pi,A))}s^{\mathrm{edes}_B(f(\pi,A))}q^{\mathrm{inv}_B(f(\pi,(A,\epsilon)))}\Big\} \\
=& \sum_{\pi'\in G_{n,i-1}} t^{\mathrm{odes}_B(\pi')}s^{\mathrm{edes}_B(\pi')}q^{\mathrm{inv}_B(\pi')} \\
&+\frac{1}{t}\Big\{\sum_{\pi'\in G_{n,i}} t^{\mathrm{odes}_B(\pi')}s^{\mathrm{edes}_B(\pi')}q^{\mathrm{inv}_B(\pi')} - \sum_{\pi'\in G_{n,i-1}} t^{\mathrm{odes}_B(\pi')}s^{\mathrm{edes}_B(\pi')}q^{\mathrm{inv}_B(\pi')}\Big\}.
\end{aligned}$$

The second equality follows because f is a bijection between $\mathfrak{B}_i\times\mathrm{sgn}\binom{[n]}{n-i}$ and $G_{n,i}$. For the fourth equality, we have used that i is odd. In the fifth equality, we are again using that f is a bijection and $F_{n,i} = G_{n,i} - G_{n,i-1}$.

From Corollary 2.2 with $i = n-r$, we have

$$B_i(s,t,q)\binom{n}{n-i}_q (1+q^n)\cdots(1+q^{i+1})$$

$$= (t-1)\sum_{\pi'\in G_{n,i-1}} t^{\mathrm{odes}(\pi')}s^{\mathrm{edes}_B(\pi')}q^{\mathrm{inv}_B(\pi')} + \sum_{\pi'\in G_{n,i}} t^{\mathrm{odes}_B(\pi')}s^{\mathrm{edes}_B(\pi')}q^{\mathrm{inv}_B(\pi')}. \tag{19}$$

The following result is easy to see

$$\binom{n}{n-i}_q (1+q^n)\cdots(1+q^{i+1}) = \frac{B_n(1,q)}{B_i(1,q)[n-i]_q!}. \tag{20}$$

Combining (19) and (20) completes the proof of (17). The proof when i is even is similar and hence is omitted. ☐

In the following example, we illustrate Lemma 2.2.

Example 2.3. *We first provide an example when i is odd. Let $n=3$ and $i=1$. It is easy to verify the following:*

$$\begin{aligned}&\sum_{\pi'\in G_{3,1}} t^{\mathrm{odes}_B(\pi')}s^{\mathrm{edes}_B(\pi')}q^{\mathrm{inv}_B(\pi')}\\ =\;& 1+(s+t)q+(s+2t)q^2+2(t+s)q^3+(ts+2t+s)(q^4+q^5)+(ts+t+s)q^6+(t+ts)q^7+tsq^8.\end{aligned}$$

Moreover, $B_1(s,t,q)=1+sq$, $B_3(1,q)=(1+q)(1+q+q^2+q^3)(1+q+q^2+q^3+q^4+q^5)$, and

$$\sum_{\pi'\in G_{3,0}} t^{\mathrm{odes}_B(\pi')}s^{\mathrm{edes}_B(\pi')}q^{\mathrm{inv}_B(\pi')} = 1+sq+sq^2+2sq^3+sq^4+sq^5+sq^6.$$

Equation (17) clearly holds when $n=3$ and $i=1$ by simple algebraic manipulation. We now give an example when i is even. Let $n=3$ and $i=2$. We have

$$\begin{aligned}&\sum_{\pi'\in G_{3,2}} t^{\mathrm{odes}_B(\pi')}s^{\mathrm{edes}_B(\pi')}q^{\mathrm{inv}_B(\pi')}\\ =\;& 1+(2s+t)q+(s^2+2s+2t)q^2+(s^2+ts+2t+3s)q^3+2(s^2+ts+s+t)(q^4+q^5)\\ &+(2s^2+3ts+t+s)q^6+(t+2ts+2s^2)q^7+(2ts+s^2)q^8+ts^2q^9.\end{aligned}$$

Further, $B_2(s,t,q)=1+(t+s)(q+q^2+q^3)+tsq^4$ and $\frac{B_3(1,q)}{B_2(1,q)}=(1+q+q^2+q^3+q^4+q^5)$. Now it can be easily verified that equation (18) holds when $n=3$ and $i=2$.

We are now in a position to give a refinement of Theorem 2.2.

Theorem 2.3. *Let $B_0(s,t,q)=1$. When $n\geq 1$, the polynomials $B_n(s,t,q)$ satisfy the following recurrence:*

$$\begin{aligned}\frac{B_n(s,t,q)}{B_n(1,q)} &= \frac{(1-t)^k(1-s)^{k+1}}{B_n(1,q)} + \sum_{r=0}^{k-1} t(1-t)^r(1-s)^{r+1}\frac{B_{n-2r-1}(s,t,q)}{B_{n-2r-1}(1,q)[2r+1]_q!}\\ &\quad+\sum_{r=0}^{k} s(1-t)^r(1-s)^r\frac{B_{n-2r}(s,t,q)}{B_{n-2r}(1,q)[2r]_q!} \qquad \textit{if } n=2k \textit{ is even}, \qquad (21)\\ \frac{B_n(s,t,q)}{B_n(1,q)} &= \frac{(1-t)^{k+1}(1-s)^{k+1}}{B_n(1,q)} + \sum_{r=0}^{k} s(1-t)^{r+1}(1-s)^{r}\frac{B_{n-2r-1}(s,t,q)}{B_{n-2r-1}(1,q)[2r+1]_q!}\\ &\quad+\sum_{r=0}^{k} t(1-t)^r(1-s)^r\frac{B_{n-2r}(s,t,q)}{B_{n-2r}(1,q)[2r]_q!} \qquad \textit{if } n=2k+1 \textit{ is odd}. \qquad (22)\end{aligned}$$

Proof. Recall that $G_{n,i}$ is the set of signed permutations whose rightmost $(n-i)$ entries form an increasing run. Thus, $G_{n,n-1}=\mathfrak{B}_n$. Further, $G_{n,-1}$ is just the signed permutation $\pi=0,1,2,\cdots,n$. Let n be even. By repeatedly applying (18) and (17) we have

$$\begin{aligned}B_n(s,t,q) &= \sum_{\pi\in G_{n,n-1}} t^{\mathrm{odes}_B(\pi)}s^{\mathrm{edes}_B(\pi)}q^{\mathrm{inv}_B(\pi)}\\ &= t\frac{B_{n-1}(s,t,q)B_n(1,q)}{B_{n-1}(1,q)[1]_q!} + (1-t)\left(\sum_{\pi\in G_{n,n-2}} t^{\mathrm{odes}_B(\pi)}s^{\mathrm{edes}_B(\pi)}q^{\mathrm{inv}_B(\pi)}\right)\\ &= t\frac{B_{n-1}(s,t,q)B_n(1,q)}{B_{n-1}(1,q)[1]_q!} + (1-t)s\frac{B_{n-2}(s,t,q)B_n(1,q)}{B_{n-2}(1,q)[2]_q!}\end{aligned}$$

$$\begin{aligned}
& +(1-t)(1-s)\Big(\sum_{\pi\in G_{n,n-3}} t^{\mathrm{odes}_B(\pi)}s^{\mathrm{edes}_B(\pi)}q^{\mathrm{inv}_B(\pi)}\Big)\\
= \; & (1-t)^k(1-s)^{k+1}+\sum_{r=0}^{k-1} t(1-t)^r(1-s)^{r+1}\frac{B_{n-2r-1}(s,t,q)B_n(1,q)}{B_{n-2r-1}(1,q)[2r+1]_q!}\\
& +\sum_{r=0}^{k} s(1-t)^r(1-s)^r\frac{B_{n-2r}(s,t,q)B_n(1,q)}{B_{n-2r}(1,q)[2r]_q!}.
\end{aligned}$$

This completes the proof of (21). We now consider the case when n is odd. Here, we will get

$$\begin{aligned}
B_n(s,t,q) \; &= \sum_{\pi\in G_{n,n-1}} t^{\mathrm{odes}_B(\pi)}s^{\mathrm{edes}_B(\pi)}q^{\mathrm{inv}_B(\pi)}\\
&= s\frac{B_{n-1}(s,t,q)}{B_{n-1}(1,q)[1]_q!}+(1-s)\left(\sum_{\pi\in G_{n,n-2}} t^{\mathrm{odes}_B(\pi)}s^{\mathrm{edes}_B(\pi)}q^{\mathrm{inv}_B(\pi)}\right)\\
&= s\frac{B_{n-1}(s,t,q)B_n(1,q)}{B_{n-1}(1,q)[1]_q!}+(1-s)t\frac{B_{n-2}(s,t,q)}{B_{n-2}(1,q)[2]_q!}+\\
&\quad +(1-s)(1-t)\Big(\sum_{\pi\in G_{n,n-3}} t^{\mathrm{odes}_B(\pi)}s^{\mathrm{edes}_B(\pi)}q^{\mathrm{inv}_B(\pi)}\Big).
\end{aligned}$$

Continuing as in the case when n was even, completes the proof of (22) and hence completes the proof of Theorem 2.3. ☐

2.1 Type B Generating Functions

We recast Theorem 2.3 in the language of egfs to prove Theorem 1.5. Recall our definitions from (6).

Proof of Theorem 1.5. For positive integers $n=2k$, we have

$$\begin{aligned}
\frac{B_n(s,t,q)u^{2k}}{B_n(1,q)} \; &= \frac{(1-t)^k(1-s)^{k+1}u^{2k}}{B_n(1,q)}+\sum_{r=0}^{k}\left(\frac{(1-t)^r(1-s)^r u^{2r}}{[2r]_q!}\right)\left(\frac{sB_{n-2r}(s,t,q)u^{n-2r}}{B_{n-2r}(1,q)}\right)\\
&\quad +\sum_{r=0}^{k-1}\left(\frac{(1-t)^r(1-s)^{r+1}u^{2r+1}}{[2r+1]_q!}\right)\left(\frac{tB_{n-2r-1}(s,t,q)u^{n-2r-1}}{B_{n-2r-1}(1,q)}\right).
\end{aligned} \tag{23}$$

When $n=2k+1$, we have

$$\begin{aligned}
\frac{B_n(s,t,q)u^{2k+1}}{B_n(1,q)} \; &= \frac{(1-t)^{k+1}(1-s)^{k+1}u^{2k+1}}{B_n(1,q)}\\
&\quad +\sum_{r=0}^{k}\left(\frac{(1-t)^r(1-s)^r u^{2r}}{[2r]_q!}\right)\left(\frac{tB_{n-2r}(s,t,q)u^{n-2r}}{B_{n-2r}(1,q)}\right)\\
&\quad +\sum_{r=0}^{k}\left(\frac{(1-t)^{r+1}(1-s)^r u^{2r+1}}{[2r+1]_q!}\right)\left(\frac{sB_{n-2r-1}(s,t,q)u^{n-2r-1}}{B_{n-2r-1}(1,q)}\right).
\end{aligned} \tag{24}$$

Summing (23), (24) over $k\geq 0$ yields

$$\begin{aligned}
(1-s)\cosh_B(Mu;q)+M\sinh_B(Mu;q) \; &= \mathcal{B}_0\left(1-s\cosh_q(Mu)-\frac{s\sinh_q(Mu)}{L}\right)\\
&\quad +\mathcal{B}_1(1-t\cosh_q(Mu)-tL\sinh_q(Mu)),
\end{aligned} \tag{25}$$

where $L=\sqrt{(1-s)/(1-t)}$, $\mathcal{B}_0=H_0(s,t,q,u)$ and $\mathcal{B}_1=H_1(s,t,q,u)$. Changing u to $-u$ gives us

$$\begin{aligned}
(1-s)\cosh_B(Mu;q)-M\sinh_B(Mu;q) \; &= \mathcal{B}_0\left(1-s\cosh_q(Mu)+\frac{s\sinh_q(Mu)}{L}\right)\\
&\quad -\mathcal{B}_1(1-t\cosh_q(Mu)+tL\sinh_q(Mu)).
\end{aligned} \tag{26}$$

Solving (25) and (26), completes the proof. ☐

Remark 2.1. *We show that setting $q = 1$ in Theorem 1.5 gives Theorem 1.4. We claim that $H_0(s,t,1,u) = H_0(s,t,\frac{u}{2})$ and likewise $H_1(s,t,1,u) = H_1(s,t,\frac{u}{2})$. As $B_n(1,1) = 2^n n!$, $\cosh_B(u;1) = \cosh(\frac{u}{2})$, $\sinh_B(u;1) = \sinh(\frac{u}{2})$, and $\mathrm{e}_q(u)\mathrm{e}_q(-u)|_{q=1} = 1$, setting $q = 1$ on the right-hand side of (7) gives the right hand side of (3). Similarly, setting $q = 1$ on the right-hand side of (8), we get the right hand side of (4).*

We are now in a position to prove Theorem 1.7.

Proof of Theorem 1.7. If $B_{2k}(s,t,q)$ is the polynomial defined in (5), it is easy to see that $\hat{B}_{2k}(s,t,q) = s^k B_{2k}(1/s,t,q)$. Therefore,

$$\begin{aligned}\hat{H}_0(s,t,q,u) &= H_0\Big(\frac{1}{s},t,q,\sqrt{s}u\Big)\\ &= \frac{(s-1)\bigg((1-t\cos_q(Mu))\cos_B(Mu;q) - t\sin_q(Mu)\sin_B(Mu;q)\bigg)}{s+t\mathrm{e}_q(iMu)\mathrm{e}_q(-iMu)-(ts+1)\cos_q(Mu)}.\end{aligned}$$

As the proof is complete, we move on to the case when $n = 2k+1$ is odd. Clearly, in this case, we have $\hat{B}_{2k+1}(s,t,q) = s^{k+1}B_{2k+1}\bigg(\frac{1}{s},t,q\bigg)$. Therefore,

$$\begin{aligned}\hat{H}_1(s,t,q,u) &= \sqrt{s}H_1(\frac{1}{s},t,q,\sqrt{s}u)\\ &= \frac{-M\bigg((s-\cos_q(Mu)\sin_B(Mu;q)+\sin_q(Mu)\cos_B(Mu;q)\bigg)}{s+t\mathrm{e}_q(iMu)\mathrm{e}_q(-iMu)-(ts+1)\cos_q(Mu)}.\end{aligned}$$

This completes the proof. □

Corollary 2.3. *We have the following egf for the type $\mathfrak{B}$ bivariate alternating descent polynomials:*

$$\sum_{n\geq 0}\hat{B}_n(s,t)\frac{u^n}{n!} = \frac{-(s-1)(t-1)\cos(Mu) - M(s+1)\sin(Mu)}{s+t-(ts+1)\cos(2Mu)}.$$

Corollary 2.4. *We get an alternate proof of the following egf for the type B alternating descent polynomials (see also [3] and [8]):*

$$\sum_{n\geq 0}\hat{B}_n(t)\frac{u^n}{n!} = \frac{-(t-1)^2\cos((1-t)u)+(t^2-1)\sin((1-t)u)}{2s-(t^2+1)\cos(2(1-t)u)}.$$

As mentioned in Section 1 though we consider a two variable enumerator, we can get a four-variable version and hence a type B counterpart of Theorem 1.1.

Define variables s_0, t_0, s_1 and t_1 to keep track of even ascents, even descents, odd ascents, and odd descents respectively. Let $m = \sqrt{(s_0-t_0)(s_1-t_1)}$. Define the five variable distribution

$$B_n(s_0,s_1,t_0,t_1,q) = \sum_{w\in\mathfrak{B}_n} s_0^{\mathrm{easc}_B(w)} s_1^{\mathrm{oasc}_B(w)} t_0^{\mathrm{edes}_B(w)} t_1^{\mathrm{odes}_B(w)} q^{\mathrm{inv}_B(w)}.$$

Further, define the generating functions

$$\begin{aligned}H_0(s_0,s_1,t_0,t_1,q,u) &= \sum_{k\geq 0} B_{2k}(s_0,s_1,t_0,t_1,q)\frac{u^{2k}}{B_{2k}(1,q)},\\ H_1(s_0,s_1,t_0,t_1,q,u) &= \sum_{k\geq 0} B_{2k+1}(s_0,s_1,t_0,t_1,q)\frac{u^{2k+1}}{B_{2k+1}(1,q)}.\end{aligned}$$

Theorem 2.4. *We have the egfs*

$$\begin{aligned}\frac{H_0(s_0,s_1,t_0,t_1,q,u)}{s_0-t_0} &= \frac{\bigg((s_1-t_1\cosh_q(mu))\cosh_B(mu;q)+t_1\sinh_q(mu)\sinh_B(mu;q)\bigg)}{s_0s_1-(t_0s_1+s_0t_1)\cosh_q(mu)+t_0t_1\mathrm{e}_q(mu)\mathrm{e}_q(-mu)},\\ H_1(s_0,s_1,t_0,t_1,q,u) &= \frac{m\bigg((s_0-t_0\cosh_q(mu))\sinh_B(mu;q)+t_0\sinh_q(mu)\cosh_B(mu;q)\bigg)}{s_0s_1-(s_1t_0+t_1s_0)\cosh_q(mu)+t_0t_1\mathrm{e}_q(mu)\mathrm{e}_q(-mu)}.\end{aligned}$$

Proof. Recalling (6), it is easy to see that

$$\begin{aligned} H_0(s_0, s_1, t_0, t_1, q, u) &= H_0\left(\frac{t_0}{s_0}, \frac{t_1}{s_1}, q, \sqrt{s_0 s_1} u\right), \\ H_1(s_0, s_1, t_0, t_1, q, u) &= \frac{\sqrt{s_0}}{\sqrt{s_1}} H_1\left(\frac{t_0}{s_0}, \frac{t_1}{s_1}, q, \sqrt{s_0 s_1} u\right). \end{aligned}$$

The proof is complete. □

2.2 q-analogues of Hyatt's recurrences and symmetries

Hyatt gives a proof of Theorem 1.10 for the polynomials $B_n^+(t)$ by considering a statistic $\mathrm{maxdrop}_B$. We give an inclusion-exclusion argument.

Proof of Theorem 1.11. We prove the two recurrences separately. Let $\hat{A}_k$ be the set of signed permutations in $\mathfrak{B}_n$ such that the last $k+1$ elements are positive and are arranged in descending order. Thus, the set $\hat{A}_{k-1} - \hat{A}_k$ is the set of signed permutations with no descent in the $(n-k)$-th position and with their last k elements being positive and descending. Define

$$A_k(s, t, q) = \sum_{w \in \hat{A}_k} t^{\mathrm{odes}_B(w)} s^{\mathrm{edes}_B(w)} q^{\mathrm{inv}_B(w)}.$$

We abbreviate $A_k(s, t, q)$ as A_k for the rest of this proof for better readability. When n is even, we claim that

$$q^{\binom{2r}{2}} \binom{n}{2r}_q B_{n-2r}(s, t, q) s^r t^r = sA_{2r-1} - (s-1)A_{2r}, \tag{27}$$

$$q^{\binom{2r+1}{2}} \binom{n}{2r+1}_q B_{n-2r-1}(s, t, q) s^r t^{r+1} = tA_{2r} - (t-1)A_{2r+1}. \tag{28}$$

When n is odd, we claim that

$$q^{\binom{2r}{2}} \binom{n}{2r}_q B_{n-2r}(s, t, q) s^r t^r = tA_{2r-1} - (t-1)A_{2r}, \tag{29}$$

$$q^{\binom{2r+1}{2}} \binom{n}{2r+1}_q B_{n-2r-1}(s, t, q) s^{r+1} t^r = sA_{2r} - (s-1)A_{2r+1}. \tag{30}$$

We prove (27) and (29). The proofs of (28) and (30) follow from a very similar argument. Recall that $\binom{[n]}{n-i}$ is the set of all $(n-i)$-sized subsets of $[n]$. Given $A \in \binom{[n]}{n-i}$, we arrange its elements in descending order and list them as $a_1, a_2, \cdots, a_{n-i}$ with $a_1 > a_2 > \cdots > a_{n-i} > 0$. Define a new juxtaposition map $f' : \mathfrak{B}_i \times \binom{[n]}{n-i} \to \hat{A}_{n-i-1}$ that takes (ψ, A) to the signed permutation $\psi_{[n]-A}, a_1, a_2, \cdots, a_{n-i}$, i.e.

$$f'(\psi, A) = \psi_{[n]-A}, a_1, a_2, \cdots, a_{n-i}.$$

It is easy to see that f' is a bijection from $\mathfrak{B}_i \times \binom{[n]}{n-i}$ to $\hat{A}_{n-i-1}$. We define $\mathrm{inv}_B([X],[Y])$ to be the number of inversions that occur between the X and Y. The LHS of (27) is clearly obtained as follows

$$\begin{aligned} & \sum_{((\psi, A) \in \mathfrak{B}_{n-2r} \times \binom{[n]}{2r}} t^{\mathrm{odes}_B(\psi)+\mathrm{odes}_B(A)} s^{\mathrm{edes}_B(\psi)+\mathrm{edes}_B(A)} q^{\mathrm{inv}_B(\psi)+\mathrm{inv}_B(A)+\mathrm{inv}_B([\psi],[A])} \\ = \; & q^{\binom{2r}{2}} s^{r-1} t^r \sum_{(\psi, A) \in \mathfrak{B}_{n-2r} \times \binom{[n]}{2r}} t^{\mathrm{odes}_B(\psi)} s^{\mathrm{edes}_B(\psi)} q^{\mathrm{inv}_B(\psi)+\mathrm{inv}_B([\psi],[A])} \\ = \; & q^{\binom{2r}{2}} s^{r-1} t^r \sum_{\psi \in \mathfrak{B}_{n-2r}} \sum_{A \in \binom{[n]}{2r}} t^{\mathrm{odes}_B(\psi)} s^{\mathrm{edes}_B(\psi)} q^{\mathrm{inv}_B(\psi)+\mathrm{inv}_B([\psi],[A])} \\ = \; & q^{\binom{2r}{2}} s^{r-1} t^r \binom{n}{2r}_q B_{n-2r}(s, t, q). \end{aligned}$$

The expression above does not account for the descent occurring at the $(n-2r)$-th position. Thus, it is off by a factor of $\frac{1}{s}$ on the set $\hat{A}_{2r}$. Further, it counts correctly on the set $\hat{A}_{2r-1} - \hat{A}_{2r}$. This gives us

$$q^{\binom{2r}{2}} s^{r-1} t^r \binom{n}{2r}_q B_{n-2r}(s, t, q) = A_{2r-1} - A_{2r} + \frac{1}{s} A_{2r},$$

which is equivalent to (27). Similarly for $2r+1$, we get (28)

$$q^{\binom{2r+1}{2}} s^r t^{r+1} \binom{n}{2r+1}_q B_{n-2r-1}(s,t,q) = tA_{2r} - (t-1)A_{2r+1}.$$

We now give a short proof of (29). For $A \in \binom{[n]}{n-i}$, we again arrange its elements in descending order and list them as $a_1, a_2, \cdots, a_{n-i}$ with $a_1 > a_2 > \cdots > a_{n-i} > 0$. We now define the juxtaposition map $f'' : \mathfrak{B}_i \times \binom{[n]}{n-i} \to \hat{A}_{n-i-1}$ that takes (ψ, A) to the signed permutation $\psi_{[n]-A}, a_1, a_2, \cdots, a_{n-i}$, i.e.

$$f''(\psi, A) = \psi_{[n]-A}, a_1, a_2, \cdots, a_{n-i}.$$

It is easy to see that f'' is a bijection from $\mathfrak{B}_i \times \binom{[n]}{n-i}$ to $\hat{A}_{n-i-1}$. By a similar argument to that of (27) we can now show that the LHS of (29) is

$$\begin{aligned} &\sum_{((\psi,A)\in\mathfrak{B}_{n-2r}\times\binom{[n]}{2r})} t^{\mathrm{odes}_B(\psi)+\mathrm{odes}_B(A)} s^{\mathrm{edes}_B(\psi)+\mathrm{edes}_B(A)} q^{\mathrm{inv}_B(\psi)+\mathrm{inv}_B(A)+\mathrm{inv}_B([\psi],[A])} \\ = \;& q^{\binom{2r}{2}} t^{r-1} s^r \binom{n}{2r}_q B_{n-2r}(s,t,q). \end{aligned}$$

The expression above does not account for the descent occurring at the $(n-2r)$-th position. Thus, it is off by a factor of $\frac{1}{t}$ on the set $\hat{A}_{2r}$. Further, it counts correctly on the set $\hat{A}_{2r-1} - \hat{A}_{2r}$. This gives us

$$q^{\binom{2r}{2}} t^{r-1} s^r \binom{n}{2r}_q B_{n-2r}(s,t,q) = A_{2r-1} - A_{2r} + \frac{1}{t} A_{2r},$$

which is equivalent to (29). Equations (27) and (28) gives

$$\begin{aligned} & q^{\binom{2r}{2}} \binom{n}{2r}_q B_{n-2r}(s,t,q)(s-1)^{r-1}(t-1)^r \\ & = \left(\frac{s-1}{s}\right)^{r-1} \left(\frac{t-1}{t}\right)^r A_{2r-1} - \left(\frac{s-1}{s}\right)^r \left(\frac{t-1}{t}\right)^r A_{2r}, \end{aligned} \tag{31}$$

$$\begin{aligned} & q^{\binom{2r+1}{2}} \binom{n}{2r+1}_q B_{n-2r-1}(s,t,q)(s-1)^r(t-1)^r \\ & = \left(\frac{s-1}{s}\right)^r \left(\frac{t-1}{t}\right)^r A_{2r} - \left(\frac{s-1}{s}\right)^r \left(\frac{t-1}{t}\right)^{r+1} A_{2r+1}. \end{aligned} \tag{32}$$

Summing (31), over the indices $1 \leq r \leq \frac{n}{2}$ and (32) over the indices $0 \leq r \leq \frac{n-2}{2}$, we get

$$\begin{aligned} A_0 \;=\; & \sum_{r=0}^{\frac{n}{2}-1} q^{\binom{2r+1}{2}} \binom{n}{2r+1}_q B_{n-2r-1}(s,t,q)(s-1)^r(t-1)^r \\ & + \sum_{r=1}^{\frac{n}{2}} q^{\binom{2r}{2}} \binom{n}{2r}_q B_{n-2r}(s,t,q)(s-1)^{r-1}(t-1)^r. \end{aligned}$$

As $\hat{A}_0$ is the set of signed permutations with elements having a positive last element (ie $\mathfrak{B}_n^+$), this completes our proof. ☐

We recall the polynomials $B_n^+(s,t,q)$ and $B_n^-(s,t,q)$ from (10). We consider the map that flips the sign of all elements below and give a few properties.

Lemma 2.3. *Let $f : \mathfrak{B}_n \to \mathfrak{B}_n$ be the involution that sends $w = w_1, \cdots, w_n$ to $\overline{w} = \overline{w_1}, \cdots, \overline{w_n}$. Then, we have the following.*

1. *When $n = 2k+1$, we have $\mathrm{odes}_B(w) + \mathrm{odes}_B(f(w)) = k$ and when $n = 2k$, we have $\mathrm{odes}_B(w) + \mathrm{odes}_B(f(w)) = k$.*
2. *When $n = 2k+1$, we have $\mathrm{edes}_B(w) + \mathrm{edes}_B(f(w)) = k+1$ and when $n = 2k$, we have $\mathrm{edes}_B(w) + \mathrm{edes}_B(f(w)) = k$.*
3. *The sum $\mathrm{inv}_B(w) + \mathrm{inv}_B(f(w)) = n^2$.*

Proof. The proof of the first two assertions are straightforward and hence omitted. For the third part, we recall that $\mathrm{inv}_B(w) = \mathrm{inv}(w) + \mathrm{NegSum}(w)$ where $\mathrm{inv}(w) = |\{(i,j) : 1 \leq i < j \leq n : w_i > w_j\}|$ and $\mathrm{NegSum}(w) = \sum_{i \in \mathrm{Negs}(w)} i$. Thus, we have

$$\begin{aligned}
\mathrm{inv}_B(w) + \mathrm{inv}_B(f(w)) &= \mathrm{inv}(w) + \mathrm{NegSum}(w) + \mathrm{inv}(\overline{w}) + \mathrm{NegSum}(\overline{w}) \\
&= \mathrm{inv}(w) + \mathrm{inv}(\overline{w}) + \mathrm{NegSum}(w) + \mathrm{NegSum}(\overline{w}) \\
&= \binom{n+1}{2} + \binom{n}{2} = n^2.
\end{aligned}$$

The proof is complete. □

2.3 Symmetry results

Theorem 2.5. *For positive integers n, we have*

$$\begin{aligned}
B_n^-(s,t,q) &= q^{n^2} s^{k+1} t^k B_n^+(s^{-1}, t^{-1}, q^{-1}) && \text{when } n = 2k+1, \\
B_n^-(s,t,q) &= q^{n^2} s^{k} t^k B_n^+(s^{-1}, t^{-1}, q^{-1}) && \text{when } n = 2k.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
B_n(s,t,q) &= B_n^+(s,t,q) + q^{n^2} s^{k+1} t^k B_n^+(s^{-1}, t^{-1}, q^{-1}) && \text{when } n = 2k+1, \\
B_n(s,t,q) &= B_n^+(s,t,q) + q^{n^2} s^{k} t^k B_n^+(s^{-1}, t^{-1}, q^{-1}) && \text{when } n = 2k.
\end{aligned}$$

Proof. Let $f : \mathfrak{B}_n^+ \to \mathfrak{B}_n^-$ be the map that sends $w = w_1, \cdots, w_n$ to $\overline{w} = \overline{w_1}, \cdots, \overline{w_n}$. By Lemma 2.3, when $n = 2k$, we have

$$\begin{aligned}
\sum_{w \in \mathfrak{B}_n^-} t^{\mathrm{odes}_B(w)} s^{\mathrm{edes}_B(w)} q^{\mathrm{inv}_B(w)} &= \sum_{w \in \mathfrak{B}_n^+} t^{\mathrm{odes}_B(f(w))} s^{\mathrm{edes}_B(f(w))} q^{\mathrm{inv}_B(f(w))} \\
&= \sum_{w \in \mathfrak{B}_n^+} t^{k-\mathrm{odes}_B(w)} s^{k-\mathrm{edes}_B(w)} q^{n^2-\mathrm{inv}_B(w)} \\
&= q^{n^2} s^k t^k \sum_{w \in \mathfrak{B}_n^+} t^{-\mathrm{odes}_B(w)} s^{-\mathrm{edes}_B(w)} q^{-\mathrm{inv}_B(w)}.
\end{aligned}$$

When $n = 2k+1$, we have

$$\begin{aligned}
\sum_{w \in \mathfrak{B}_n^-} t^{\mathrm{odes}_B(w)} s^{\mathrm{edes}_B(w)} q^{\mathrm{inv}_B(w)} &= \sum_{w \in \mathfrak{B}_n^+} t^{\mathrm{odes}_B(f(w))} s^{\mathrm{edes}_B(f(w))} q^{\mathrm{inv}_B(f(w))} \\
&= \sum_{w \in \mathfrak{B}_n^+} t^{k-\mathrm{odes}_B(w)} s^{k+1-\mathrm{edes}_B(w)} q^{n^2-\mathrm{inv}_B(w)} \\
&= q^{n^2} s^{k+1} t^k \sum_{w \in \mathfrak{B}_n^+} t^{-\mathrm{odes}_B(w)} s^{-\mathrm{edes}_B(w)} q^{-\mathrm{inv}_B(w)}.
\end{aligned}$$

completing the proof. □

In a similar manner, the following result also follows.

Lemma 2.4. *We have*

$$\begin{aligned}
B_{2k}(s,t,q) &= q^{n^2} s^k t^k B_{2k}\left(\frac{1}{s}, \frac{1}{t}, \frac{1}{q}\right) && \text{when } n = 2k, \\
B_{2k+1}(s,t,q) &= q^{n^2} s^{k+1} t^k B_{2k+1}\left(\frac{1}{s}, \frac{1}{t}, \frac{1}{q}\right) && \text{when } n = 2k+1.
\end{aligned}$$

3. Type D analogues

Let $H_{n,i}$ be the set of signed permutations $\pi \in \mathfrak{D}_n$ such that the last $n-i$ elements of π are increasing, that is we have $\pi_{i+1} < \pi_{i+2} < \cdots < \pi_{n-1} < \pi_n$. Clearly, $|H_{n,i}| = 2^{n-1}\binom{n}{i} i!$.

Let $\sigma = \sigma_1, \cdots, \sigma_{n-i} \in \mathfrak{D}_{n-i}$ and $(A, \epsilon) \in \mathrm{sgn}\binom{[n]}{i}$ be a signed subset. Further, let $[n]-A = \{c_1, c_2, \ldots, c_{n-i}\}$ be written in ascending order. Thus, $c_1 < c_2 < \cdots < c_{n-i}$. We define two maps $h : \mathfrak{D}_{n-i} \to \mathfrak{D}_{\{c_1,c_2,\ldots,c_{n-i}\}}$ and $h_D : \mathfrak{D}_{n-i} \to \mathfrak{B}_{c_1,c_2,\ldots,c_{n-i}} - \mathfrak{D}_{\{c_1,c_2,\ldots,c_{n-i}\}}$ as follows:

$$h(\sigma) = \pi_1, \pi_2, \ldots, \pi_{n-i} \quad \text{and} \quad h_D(\sigma) = \overline{\pi_1}, \pi_2, \ldots, \pi_{n-i},$$

where for $1 \leq i \leq n-i$, if $|\sigma_i| = k$ then $|\pi_i| = c_k$ and π_i has the same sign as σ_i. Both maps h, h_D are clearly bijections and hence invertible.

If (A, ϵ) has an even number of negative elements, then by inverting the map h on the elements of $[0, n] - A$ and appending the elements of (A, ϵ) in ascending order, we get a signed permutation in $H_{n,n-i}$. Similarly, if (A, ϵ) has odd number of negative elements, then by inverting the map h_D on the elements of $[0, n] - A$ and appending the elements of (A, ϵ) in ascending order, we get a signed permutation in $H_{n,n-i}$.

These maps are also invertible, so we have a bijection $f_{\mathfrak{D}} : \mathfrak{D}_{n-i} \times \mathrm{sgn}\binom{[n]}{i} \mapsto H_{n,n-i}$ defined as follows. Let $\sigma \in \mathfrak{B}_{n-i}$ and $(A, \epsilon) \in \mathrm{sgn}\binom{[n]}{i}$.

Define

$$f_{\mathfrak{D}}(\sigma, (A, \epsilon)) = \begin{cases} h(\sigma)[(A, \epsilon)] & \text{if } (A, \epsilon) \text{ has even no. of negatives} \\ h_D(\sigma)[(A, \epsilon)] & \text{if } (A, \epsilon) \text{ has odd no. of negatives} \end{cases}$$

where (A, ϵ) is juxtaposed at the end of the $h(\sigma)$ or $h_D(\sigma)$.

We start with the following type D counterpart of Lemma 2.1.

Lemma 3.1. *Let $(A, \epsilon) \in \mathrm{sgn}\binom{[n]}{r}$ be a signed subset of $[n]$. Then,*

$$\sum_{(A,\epsilon) \in \mathrm{sgn}\binom{[n]}{r}} q^{\mathrm{inv}_D(f_{\mathfrak{D}}([[n]-A],[(A,\epsilon)]))} = \binom{n}{r}_q (1+q^{n-1})(1+q^{n-2}) \cdots (1+q^{n-r}).$$

Proof. We proceed by induction on n. The base case when $n = 1$ is easy. We assume that our Lemma is true for n and show that it holds for $n+1$. Thus, we need to show the following:

$$\sum_{(A,\epsilon) \in \mathrm{sgn}\binom{[n+1]}{r+1}} q^{\mathrm{inv}_D(f_{\mathcal{D}}([[n+1]-A],(A,\epsilon)))} = \binom{n+1}{r+1}_q (1+q^{n})(1+q^{n-1}) \cdots (1+q^{n-r}).$$

Let $\eta(n, r) = (1+q^{n-1}) \cdots (1+q^{n-r})$. We partition $\mathrm{sgn}\binom{[n+1]}{r+1}$ into the disjoint union of the following three subsets:

1. $\mathcal{A}_1 = \{(A, \epsilon) \in \mathrm{sgn}\binom{[n+1]}{r+1} : n+1 \in (A, \epsilon)\}$,
2. $\mathcal{A}_2 = \{(A, \epsilon) \in \mathrm{sgn}\binom{[n+1]}{r+1} : \overline{n+1} \in (A, \epsilon)\}$,
3. $\mathcal{A}_3 = \{(A, \epsilon) \in \mathrm{sgn}\binom{[n+1]}{r+1} : n+1 \notin (A, \epsilon)\}$.

We next determine the contribution to $\sum_{(A,\epsilon) \in \mathrm{sgn}\binom{[n+1]}{r+1}} q^{\mathrm{inv}_D(f_{\mathfrak{D}}([[n+1]-A],(A,\epsilon)))}$ from each of the above sets. If $n+1 \in (A, \epsilon)$, as $[(A, \epsilon)]$ is in ascending order, it will be the rightmost element of $f([[n+1]-A], [(A, \epsilon)])$ and thus it will contribute no extra inversions. Thus

$$\sum_{(A,\epsilon) \in \mathcal{A}_1} q^{\mathrm{inv}_D(f_{\mathfrak{D}}([[n+1]-A],[(A,\epsilon)]))} = \eta(n, r)\binom{n}{r}_q. \tag{33}$$

If $\overline{n+1} \in (A, \epsilon)$, then $\overline{n+1}$ has to be in the $(n-r+1)$-th position in $f([[n+1]-A], (A, \epsilon))$. Every element of $[[n+1]-A]$ will be to its left and will thus contribute 2 inversions. Further, every element to its right will contribute 1 inversion. Thus, we get $2n-r$ new inversions. Therefore,

$$\sum_{(A,\epsilon) \in \mathcal{A}_2} q^{\mathrm{inv}_D(f_{\mathfrak{D}}([[n+1]-A],[(A,\epsilon)]))} = \eta(n, r) q^{2n-r}\binom{n}{r}_q. \tag{34}$$

Lastly, when $n+1 \in [n+1]-A$, then it has to be the rightmost element in $[n+1]-A$. Every element of (A, ϵ) will contribute one inversion and thus we get '$r+1$' extra inversions. Hence,

$$\sum_{(A,\epsilon) \in \mathcal{A}_3} q^{\mathrm{inv}_D(f_{\mathfrak{D}}([[n+1]-A],(A,\epsilon)))} = q^{r+1}\eta(n, r+1)\binom{n}{r+1}_q = q^{r+1}(1+q^{n-r-1})\eta(n, r)\binom{n}{r+1}_q. \tag{35}$$

Summing up (33), (34) and (35), we get

$$\begin{aligned}
&\sum_{(A,\epsilon)\in \mathrm{sgn}\binom{[n+1]}{r+1}} q^{\mathrm{inv}_D(f_{\mathcal{D}}([[n+1]-A],(A,\epsilon)))} \\
= \quad & \eta(n,r)\left(\binom{n}{r}_q + q^{2n-r}\binom{n}{r}_q + q^{r+1}(1+q^{n-r-1})\binom{n}{r+1}_q\right) = \eta(n+1,r+1)\binom{n+1}{r+1}_q.
\end{aligned}$$

The last equation follows from the q-Pascal recurrence for the Gaussian binomial coefficients. This completes the proof. □

We now illustrate the statement of Lemma 3.1 by the following example. Recall that we abbreviate permutations by dropping commas between elements.

Example 3.1. *For $n = 3$ and $r = 1$, the set $\mathrm{sgn}\binom{[n]}{r}$ is clearly $\{\{3\},\{\overline{3}\},\{2\},\{\overline{2}\},\{1\},\{\overline{1}\}\}$ and hence we have*

$$\begin{aligned}
&\sum_{(A,\epsilon)\in \mathrm{sgn}\binom{[3]}{1}} q^{\mathrm{inv}_D(f_{\mathfrak{D}}([[n]-A],(A,\epsilon)))} \\
= \quad & q^{\mathrm{inv}_D(123)} + q^{\mathrm{inv}_D(231)} + q^{\mathrm{inv}_D(132)} + q^{\mathrm{inv}_D(\overline{1}2\overline{3})} + q^{\mathrm{inv}_D(\overline{2}3\overline{1})} + q^{\mathrm{inv}_D(\overline{1}3\overline{2})} \\
= \quad & 1+q^2+q+q^4+q^2+q^3 = (1+q+q^2)(1+q^2) = \binom{3}{1}_q(1+q^2).
\end{aligned}$$

Corollary 3.1. *Let $\sigma \in \mathfrak{D}_{n-r}$ be a signed permutation, $(A,\epsilon) \in \mathrm{sgn}\binom{[n]}{r}$ be a signed subset.*

$$\sum_{(A,\epsilon)\in \mathrm{sgn}\binom{[n]}{r}} q^{\mathrm{inv}_D(f_{\mathfrak{D}}(\sigma,[(A,\epsilon)]))} = q^{\mathrm{inv}_D(\sigma)}\binom{n}{r}_q(1+q^{n-1})(1+q^{n-2})\cdots(1+q^{n-r}).$$

Proof. The proof follows exactly as the proof of Corollary 2.1. The result follows by noting that changing the sign of the first element does not affect the type D inversion statistic. □

Corollary 3.2. *We have*

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{D}_{n-r}} \sum_{(A,\epsilon)\in \mathrm{sgn}\binom{[n]}{r}} t^{\mathrm{odes}_D(\sigma)} s^{\mathrm{edes}_D(\sigma)} q^{\mathrm{inv}_D(f_{\mathfrak{D}}(\sigma,[(A,\epsilon)]))} \\
&= D_{n-r}(s,t,q)\binom{n}{r}_q(1+q^{n-1})(1+q^{n-2})\cdots(1+q^{n-r}).
\end{aligned}$$

Proof. For a particular $\sigma \in \mathfrak{D}_{n-r}$, we have

$$\begin{aligned}
&t^{\mathrm{odes}_D(\sigma)} s^{\mathrm{edes}_D(\sigma)} \sum_{(A,\epsilon)\in \mathrm{sgn}\binom{[n]}{r}} q^{\mathrm{inv}_D(f_{\mathfrak{D}}(\sigma,[(A,\epsilon)]))} \\
= \quad & t^{\mathrm{odes}_D(\sigma)} s^{\mathrm{edes}_D(\sigma)} q^{\mathrm{inv}_D(\sigma)}\binom{n}{r}_q(1+q^{n-1})(1+q^{n-2})\cdots(1+q^{n-r}).
\end{aligned}$$

Summing over all possible $\sigma \in \mathfrak{D}_{n-r}$ finishes the proof. □

Lemma 3.2. *Let $X_{\{1,\overline{1}\}}$ be the set of signed permutations in $\mathfrak{D}_n$ such that the descent set is a subset of $\{1,\overline{1}\}$. Then,*

$$\sum_{w\in X_{\{1,\overline{1}\}}} t^{\mathrm{odes}_D(w)} s^{\mathrm{edes}_D(w)} q^{\mathrm{inv}_D(w)} = t^2\frac{D_n(1,q)}{[n-1]_q!} + t(1-t)\left(\frac{2D_n(1,q)}{[n]_q!}-1\right) + (1-t).$$

Proof. Let $Y_1 = X_{\{1,\overline{1}\}}$ be the set of signed permutations of $\mathfrak{D}_n$ with the last $n-1$ elements in ascending order and Y_0 be the set of signed permutations of $\mathfrak{D}_n$ such that the descent set is a subset of $\{1\}$ or $\{\overline{1}\}$. Then, by inclusion-exclusion, we can say that

$$\sum_{w\in X_{\{1,\overline{1}\}}} t^{\mathrm{odes}_D(w)} s^{\mathrm{edes}_D(w)} q^{\mathrm{inv}_D(w)} = t^2\left(\sum_{w\in Y_1} q^{\mathrm{inv}_D(w)}\right) + t(1-t)\left(\sum_{w\in Y_0} q^{\mathrm{inv}_D(w)}\right) + (1-t). \quad (36)$$

From (3.1), we get $\sum_{w\in Y_1} q^{\mathrm{inv}_D(w)} = \frac{D_n(1,q)}{[n-1]_q!}$. We just need to show that $\sum_{w\in Y_0} q^{\mathrm{inv}_D(w)} = \frac{2D_n(1,q)}{[n]_q!} - 1$.

If we want a descent at $\{1\}$ but not at $\{\overline{1}\}$ or vice versa, then we need $|\pi(1)| > |\pi(2)|$. This can be done in the following way. We assign signs to the elements of $[n]$ and arrange them in ascending order. Then, choose the sign of the first element accordingly to make it an element of $\mathfrak{D}_n$ (i.e. to make the total number of negative signs even). An element i will either contribute 1 if it is positive or q^{i-1} if it is negative, giving the term $(1+q^{i-1})$. Therefore, the total contribution would be $(1+q^0)(1+q)\cdots(1+q^{n-1})$. However, this procedure also produces $1, 2, \ldots, n$ and $\overline{1}, 2, \cdots, n$, out of which we only need the former. The latter has a length of 1 which we subtract to complete the proof. □

Example 3.2. *For $n = 2$, the set of permutations of $\mathfrak{D}_2$ with descent set being a subset of $\{1, \overline{1}\}$ is the whole $\mathfrak{D}_2$ itself. For $n = 3$, the permutations of $\mathfrak{D}_3$ with descent set being a subset of $\{1, \overline{1}\}$ is*

$$X_{1,\overline{1}} = \{123, \overline{12}3, \overline{1}3\overline{2}... \}$$

It is easy to verify that

$$\begin{aligned}&\sum_{w \in X_{\{1,\overline{1}\}}} t^{\mathrm{odes}_D(w)} s^{\mathrm{edes}_D(w)} q^{\mathrm{inv}_D(w)} \\ =\ & 1 + t^2q^2 + t^2q^3 + 2t^2q^4 + 2tq + 2tq^2 + 2tq^3 + t^2q^5 \\ =\ & t^2[3]_q[4]_q + t(1-t)(2[4]_q - 1) + (1-t).\end{aligned}$$

Lemma 3.3. *With the above notations, when i is odd, we have*

$$\begin{aligned}\sum_{\pi' \in H_{n,i}} t^{\mathrm{odes}_D(\pi')} s^{\mathrm{edes}_D(\pi')} q^{\mathrm{inv}_D(\pi')} = \ & t\frac{D_i(s,t,q)D_n(1,q)}{D_i(1,q)[n-i]_q!} \\ + \ & (1-t)\left\{\sum_{\pi' \in H_{n,i-1}} t^{\mathrm{odes}_D(\pi')} s^{\mathrm{edes}_D(\pi')} q^{\mathrm{inv}_D(\pi')}\right\}. \end{aligned} \tag{37}$$

When i is even, we have

$$\begin{aligned}\sum_{\pi' \in H_{n,i}} t^{\mathrm{odes}_D(\pi')} s^{\mathrm{edes}_D(\pi')} q^{\mathrm{inv}_D(\pi')} = \ & s\frac{D_i(s,t,q)D_n(1,q)}{D_i(1,q)[n-i]_q!} \\ + \ & (1-s)\left\{\sum_{\pi' \in H_{n,i-1}} t^{\mathrm{odes}_D(\pi')} s^{\mathrm{edes}_D(\pi')} q^{\mathrm{inv}_D(\pi')}\right\}. \end{aligned} \tag{38}$$

Proof. We at first prove (37) and therefore take i to be odd. We evaluate $\sum_{(\pi,(A,\epsilon)) \in \mathfrak{D}_i \times \mathrm{sgn}\binom{[n]}{n-i}} t^{\mathrm{odes}_D(\pi)} s^{\mathrm{edes}_D(\pi)} q^{\mathrm{inv}_D(f_{\mathfrak{D}}(\pi,(A,\epsilon)))}$ in a different way as compared to (3.2).

$$\begin{aligned}&\sum_{(\pi,(A,\epsilon)) \in \mathfrak{D}_i \times \mathrm{sgn}\binom{[n]}{n-i}} t^{\mathrm{odes}_D(\pi)} s^{\mathrm{edes}_D(\pi)} q^{\mathrm{inv}_D(f_{\mathfrak{D}}(\pi,(A,\epsilon)))} \\ =\ & \sum_{(\pi,(A,\epsilon)) \in f_{\mathfrak{D}}^{-1}(H_{n,i})} t^{\mathrm{odes}_D(\pi)} s^{\mathrm{edes}_D(\pi)} q^{\mathrm{inv}_D(f_{\mathfrak{D}}(\pi,(A,\epsilon)))} \\ =\ & \sum_{(\pi,(A,\epsilon)) \in f_{\mathfrak{D}}^{-1}(H_{n,i-1})} t^{\mathrm{odes}_D(\pi)} s^{\mathrm{edes}_D(\pi)} q^{\mathrm{inv}_D(f_{\mathfrak{D}}(\pi,(A,\epsilon)))} \\ & + \sum_{(\pi,(A,\epsilon)) \in f_{\mathfrak{D}}^{-1}(H'_{n,i})} t^{\mathrm{odes}_D(\pi)} s^{\mathrm{edes}_D(\pi)} q^{\mathrm{inv}_D(f_{\mathfrak{D}}(\pi,(A,\epsilon)))} \\ =\ & \sum_{f_{\mathfrak{D}}(\pi,(A,\epsilon)) \in H_{n,i-1}} t^{\mathrm{odes}_D(f_{\mathfrak{D}}(\pi,(A,\epsilon)))} s^{\mathrm{edes}_D(f_{\mathfrak{D}}(\pi,(A,\epsilon)))} q^{\mathrm{inv}_D(f_{\mathfrak{D}}(\pi,(A,\epsilon)))} \\ & + \frac{1}{t}\left\{\sum_{f_{\mathfrak{D}}(\pi,(A,\epsilon)) \in H'_{n,i}} t^{\mathrm{odes}_D(f_{\mathfrak{D}}(\pi,(A,\epsilon)))} s^{\mathrm{edes}_D(f_{\mathfrak{D}}(\pi,(A,\epsilon)))} q^{\mathrm{inv}_D(f_{\mathfrak{D}}(\pi,(A,\epsilon)))}\right\} \\ =\ & \sum_{\pi' \in H_{n,i-1}} t^{\mathrm{odes}_D(\pi')} s^{\mathrm{edes}_D(\pi')} q^{\mathrm{inv}_D(\pi')} \\ & + \frac{1}{t}\left\{\sum_{\pi' \in H_{n,i}} t^{\mathrm{odes}_D(\pi')} s^{\mathrm{edes}_D(\pi')} q^{\mathrm{inv}_D(\pi')} - \sum_{\pi' \in H_{n,i-1}} t^{\mathrm{odes}_D(\pi')} s^{\mathrm{edes}_D(\pi')} q^{\mathrm{inv}_D(\pi')}\right\}. \end{aligned} \tag{39}$$

The second equality follows because $f_{\mathfrak{D}}$ is a bijection between $\mathfrak{D}_i \times \mathrm{sgn}\binom{[n]}{n-i}$ to $H_{n,i}$. For the fourth equality, we have used that i is odd. For the fifth equality, we are again using that $f_{\mathfrak{D}}$ is a bijection and $H'_{n,i} = H_{n,i} - H_{n,i-1}$. We determine the contribution of each of these three sets. From (3.2) and (39), we have

$$\begin{aligned} t\frac{D_i(s,t,q)D_n(1,q)}{D_i(1,q)[n-i]_q!} &= (t-1)\Bigg\{\sum_{\pi' \in H_{n,i-1}} t^{\mathrm{odes}_D(\pi')} s^{\mathrm{edes}_D(\pi')} q^{\mathrm{inv}_D(\pi')}\Bigg\} \\ &\quad + \sum_{\pi' \in H_{n,i}} t^{\mathrm{odes}_D(\pi')} s^{\mathrm{edes}_D(\pi')} q^{\mathrm{inv}_D(\pi')} \end{aligned}$$

This completes the proof of (37). The proof when i is even is similar and hence is omitted. ☐

Example 3.3. *When $n = 3, i = 2$, the set $H_{3,2}$ is $\mathfrak{D}_3$. The set $H_{3,1}$ is*

$$\{123, \overline{12}3, \overline{13}2, 1\overline{32}, 213, \overline{21}3, \overline{23}1, 2\overline{31}, 312, \overline{31}2, \overline{32}1, 3\overline{21}\}.$$

Now,

$$\begin{aligned} & s\frac{D_2(s,t,q)D_2(1,q)}{D_3(1,q)[1]_q!} + (1-s)\Bigg\{\sum_{\pi' \in H_{3,1}} t^{\mathrm{odes}_D(\pi')} s^{\mathrm{edes}_D(\pi')} q^{\mathrm{inv}_D(\pi')}\Bigg\} \\ =\ & s(1+q^2)(1+q+q^2)(1+2tq+t^2q^2) + (1-s)(1+t^2q^2+t^2q^3+2t^2q^4+2tq+2tq^2+2tq^3+t^2q^5) \\ =\ & t^2sq^6 + t^2(q^2+q^3+2q^4+q^5) + ts(2q^3+2q^4+2q^5) + t(2q+2q^2+2q^3) + s(q+2q^2+q^3+q^4) + 1 \\ =\ & D_3(s,t,q) \end{aligned}$$

Thus, equation (38) clearly holds when $n = 3$ and $i = 2$.

Our next result is a type D counterpart of the recurrence given in Theorem 2.3.

Theorem 3.1. *Define $D_2(s,t,q) = (1+tq)^2$. When $n \geq 3$, the polynomials $D_n(s,t,q)$ satisfy the following recurrence.*

$$\begin{aligned} \frac{D_n(s,t,q)}{D_n(1,q)} &= \frac{(1-t)^{k+1}(1-s)^k}{D_n(1,q)} + \frac{2t(1-t)^k(1-s)^k}{[n]_q!} + \frac{t^2(1-t)^{k-1}(1-s)^k}{[n-1]_q!} \\ &\quad + \sum_{r=0}^{k-1} t(1-t)^r(1-s)^{r+1}\frac{D_{n-2r-1}(s,t,q)}{D_{n-2r-1}(1,q)[2r+1]_q!} \\ &\quad + \sum_{r=0}^{k-1} s(1-t)^r(1-s)^r\frac{D_{n-2r}(s,t,q)}{D_{n-2r}(1,q)[2r]_q!} \qquad \text{if } n = 2k \text{ is even,} \qquad (40) \\ \frac{D_n(s,t,q)}{D_n(1,q)} &= \frac{(1-t)^{k+2}(1-s)^k}{D_n(1,q)} + \frac{2t(1-t)^{k+1}(1-s)^k}{[n]_q!} + \frac{t^2(1-t)^k(1-s)^k}{[n-1]_q!} \\ &\quad + \sum_{r=0}^{k-1} s(1-t)^{r+1}(1-s)^r\frac{D_{n-2r-1}(s,t,q)}{D_{n-2r-1}(1,q)[2r+1]_q!} \\ &\quad + \sum_{r=0}^{k-1} t(1-t)^r(1-s)^r\frac{D_{n-2r}(s,t,q)}{D_{n-2r}(1,q)[2r]_q!} \qquad \text{if } n = 2k+1 \text{ is odd.} \qquad (41) \end{aligned}$$

Proof. As $H_{n,i}$ is the set of signed permutations in $\mathfrak{D}_n$ whose rightmost $(n-i)$ entries form an increasing run, we see that $H_{n,n-1}$ must be the whole of $\mathfrak{D}_n$. We first consider the case when n is even. By repeatedly applying (37) and (38), we have

$$\begin{aligned} D_n(s,t,q) &= \sum_{\pi \in H_{n,n-1}} t^{\mathrm{odes}_D(\pi)} s^{\mathrm{edes}_D(\pi)} q^{\mathrm{inv}_D(\pi)} \\ &= t\frac{D_{n-1}(s,t,q)D_n(1,q)}{D_{n-1}(1,q)[1]_q!} + (1-t)\left(\sum_{\pi \in H_{n,n-2}} t^{\mathrm{odes}_D(\pi)} s^{\mathrm{edes}_D(\pi)} q^{\mathrm{inv}_D(\pi)}\right) \\ &= t\frac{D_{n-1}(s,t,q)D_n(1,q)}{D_{n-1}(1,q)[1]_q!} + s(1-t)\frac{D_{n-2}(s,t,q)D_n(1,q)}{D_{n-2}(1,q)[2]_q!} \\ &\quad + (1-s)(1-t)\sum_{\pi \in H_{n,n-3}} t^{\mathrm{odes}_D(\pi)} s^{\mathrm{edes}_D(\pi)} q^{\mathrm{inv}_D(\pi)} \end{aligned}$$

$$\begin{aligned}
&= t\frac{D_{n-1}(s,t,q)D_n(1,q)}{D_{n-1}(1,q)[1]_q!} + s(1-t)\frac{D_{n-2}(s,t,q)D_n(1,q)}{D_{n-2}(1,q)[2]_q!} + \cdots \\
&\quad + s(1-t)^{\frac{n}{2}-1}(1-s)^{\frac{n}{2}-2}\frac{D_2(s,t,q)D_n(1,q)}{D_2(1,q)[n-2]_q!} \\
&\quad + (1-t)^{\frac{n}{2}-1}(1-s)^{\frac{n}{2}-1}\left(\sum_{\pi \in X_{\{1,\bar{1}\}}} t^{\mathrm{odes}_D(\pi)} s^{\mathrm{edes}_D(\pi)} q^{\mathrm{inv}_D(\pi)}\right).
\end{aligned}$$

This completes the proof of (40). We now consider the case when n is odd. Here, we have

$$\begin{aligned}
D_n(s,t,q) &= \sum_{\pi \in H_{n,n-1}} t^{\mathrm{odes}_D(\pi)} s^{\mathrm{edes}_D(\pi)} q^{\mathrm{inv}_D(\pi)} \\
&= s\frac{D_{n-1}(s,t,q)D_n(1,q)}{D_{n-1}(1,q)[1]_q!} + (1-s)\left(\sum_{\pi \in H_{n,n-2}} t^{\mathrm{odes}_D(\pi)} s^{\mathrm{edes}_D(\pi)} q^{\mathrm{inv}_D(\pi)}\right) \\
&= s\frac{D_{n-1}(s,t,q)D_n(1,q)}{D_{n-1}(1,q)[1]_q!} \\
&\quad + (1-s)\left(t\frac{D_{n-2}(s,t,q)D_n(1,q)}{D_{n-2}(1,q)[2]_q!} + (1-t)\Big(\sum_{\pi \in H_{n,n-3}} t^{\mathrm{odes}_D(\pi)} s^{\mathrm{edes}_D(\pi)} q^{\mathrm{inv}_D(\pi)}\Big)\right)
\end{aligned}$$

and we can continue as in the case for n being even, to complete the proof of (41). This completes the proof. ☐

Setting $s = t$ in Theorem 3.1, we have the following corollary.

Corollary 3.3 (Reiner)**.** *Define $D_2(t,q) = (1+tq)^2$. When $n \geq 3$, the polynomials $D_n(t,q)$ satisfy the following recurrence:*

$$\begin{aligned}
\frac{D_n(t,q)}{D_n(1,q)} &= \frac{(1-t)^{2k+1}}{D_n(1,q)} + \frac{2t(1-t)^{2k}}{[n]_q!} + \frac{t^2(1-t)^{2k-1}}{[n-1]_q!} + \sum_{r=0}^{k-1} t(1-t)^{2r+1}\frac{D_{n-2r-1}(t,q)}{D_{n-2r-1}(1,q)[2r+1]_q!} \\
&\quad + \sum_{r=0}^{k-1} t(1-t)^{2r}\frac{D_{n-2r}(t,q)}{D_{n-2r}(1,q)[2r]_q!} \qquad \text{if } n = 2k \text{ is even,} \\
\frac{D_n(t,q)}{D_n(1,q)} &= \frac{(1-t)^{2k+2}}{D_n(1,q)} + \frac{2t(1-t)^{2k+1}}{[n]_q!} + \frac{t^2(1-t)^{2k}}{[n-1]_q!} + \sum_{r=0}^{k-1} t(1-t)^{2r+1}\frac{D_{n-2r-1}(t,q)}{D_{n-2r-1}(1,q)[2r+1]_q!} \\
&\quad + \sum_{r=0}^{k-1} t(1-t)^{2r}\frac{D_{n-2r}(t,q)}{D_{n-2r}(1,q)[2r]_q!} \qquad \text{if } n = 2k+1 \text{ is odd.}
\end{aligned}$$

Setting $q = 1$ in Theorem 3.1, we also have the following.

Corollary 3.4. *Define $D_2(s,t) = (1+t)^2$. When $n \geq 3$, the polynomials $D_n(s,t)$ satisfy the following recurrence:*

$$\begin{aligned}
\frac{D_n(s,t)}{2^{n-1}n!} &= \frac{(1-t)^{k+1}(1-s)^k}{2^{n-1}n!} + \frac{2t(1-t)^k(1-s)^k}{n!} + \frac{t^2(1-t)^{k-1}(1-s)^k}{(n-1)!} \\
&\quad + \sum_{r=0}^{k-1} t(1-t)^r(1-s)^{r+1}\frac{D_{n-2r-1}(s,t)}{2^{n-2r-2}(n-2r-1)!(2r+1)!} \\
&\quad + \sum_{r=0}^{k-1} s(1-t)^r(1-s)^r\frac{D_{n-2r}(s,t)}{2^{n-2r-1}(n-2r)!(2r)!} \qquad \text{if } n = 2k \text{ is even,} \\
\frac{D_n(s,t)}{2^{n-1}n!} &= \frac{(1-t)^{k+2}(1-s)^k}{2^{n-1}n!} + \frac{2t(1-t)^{k+1}(1-s)^k}{n!} + \frac{t^2(1-t)^k(1-s)^k}{(n-1)!} \\
&\quad + \sum_{r=0}^{k-1} s(1-t)^{r+1}(1-s)^r\frac{D_{n-2r-1}(s,t)}{2^{n-2r-2}(n-2r-1)!(2r+1)!} \\
&\quad + \sum_{r=0}^{k-1} t(1-t)^r(1-s)^r\frac{D_{n-2r}(s,t)}{2^{n-2r-1}(n-2r)!(2r)!} \qquad \text{if } n = 2k+1 \text{ is odd.}
\end{aligned}$$

3.1 Type D generating functions

We again cast the recurrences in egf language to get generating functions. We begin with our proof of Theorem 1.8.

Proof of Theorem 1.8. Recurrences (40) and (41) give rise to the following:

$$\begin{aligned}&\mathcal{D}_0\left(1 - s\cosh_q(Mu) - \frac{s(1-t)\sinh_q(Mu)}{M}\right) + \mathcal{D}_1\left(1 - t\cosh_q(Mu) - \frac{t(1-s)\sinh_q(Mu)}{M}\right)\\ =\ & \mathrm{OD} + \mathrm{ED}.\end{aligned}$$

Changing u to $-u$ gives us

$$\begin{aligned}&\mathcal{D}_0\left(1 - s\cosh_q(Mu) + \frac{s(1-t)\sinh_q(Mu)}{M}\right) - \mathcal{D}_1\left(1 - t\cosh_q(Mu) + \frac{t(1-s)\sinh_q(Mu)}{M}\right)\\ =\ & -\mathrm{OD} + \mathrm{ED}\end{aligned}$$

Solving the above two equations for $\mathcal{D}_0$ and $\mathcal{D}_1$ completes the proof. □

We can now prove Theorem 1.9.

Proof of Theorem 1.9. As done in the proof of Theorem 1.7, one can check when $n = 2k$ is even, that $\hat{D}_{2k}(s,t,q) = s^k D_{2k}(1/s,t,q)$ and when $n = 2k+1$ is odd, that $\hat{D}_{2k+1}(s,t,q) = s^{k+1}D_{2k+1}\left(\frac{1}{s},t,q\right)$. The other details follow as in the proof of Theorem 1.7, completing the proof. □

Using Theorem 1.8 we get a type D counterpart of Theorem 1.1. Define

$$D_n(s_0,s_1,t_0,t_1,q) = \sum_{w\in\mathfrak{D}_n} s_0^{\mathrm{easc}_D(w)} s_1^{\mathrm{oasc}_D(w)} t_0^{\mathrm{edes}_D(w)} t_1^{\mathrm{odes}_D(w)} q^{\mathrm{inv}_D(w)}.$$

Further, define the generating functions

$$\begin{aligned}\mathcal{D}_0(s_0,s_1,t_0,t_1,q,u) &= \sum_{k\geq 1} D_{2k}(s_0,s_1,t_0,t_1,q)\frac{u^{2k}}{D_{2k}(1,q)},\\ \mathcal{D}_1(s_0,s_1,t_0,t_1,q,u) &= \sum_{k\geq 1} D_{2k+1}(s_0,s_1,t_0,t_1,q)\frac{u^{2k+1}}{D_{2k+1}(1,q)}.\end{aligned}$$

We move to our type D counterpart of Theorem 1.1. Recall $\mathcal{D}_0(s,t,q,u)$ and $\mathcal{D}_1(s,t,q,u)$ from (9).

Theorem 3.2. *We have the egf*

$$\mathcal{D}_0(s_0,s_1,t_0,t_1,q,u) = \frac{\frac{1}{s_0}[T(\mathrm{ED})(s_1 - t_1\cosh_q(mu))] + T(\mathrm{OD})(\frac{t_1(s_0-t_0)\sqrt{s_0s_1}}{ms_0}\sinh_q(mu))}{s_0s_1 - (s_0t_1+s_1t_0)\cosh_q(mu) + t_0t_1\mathrm{e}_q(mu)\mathrm{e}_q(-mu)},$$

$$\mathcal{D}_1(s_0,s_1,t_0,t_1,q,u) = \frac{\frac{s_1\sqrt{s_1}T(\mathrm{OD})}{\sqrt{s_0}}(s_0 - t_0\cosh_q(mu)) + T(\mathrm{ED})(\frac{s_1t_0(s_1-t_1)}{m}\sinh_q(mu))}{s_0s_1 - (s_0t_1+s_1t_0)\cosh_q(mu) + t_0t_1\mathrm{e}_q(mu)\mathrm{e}_q(-mu)}.$$

where

$$\begin{aligned}T(\mathrm{OD}) &= \frac{\sqrt{s_0s_1}ut_1^2}{s_1^2}(\cosh_q(mu)-1) + \frac{(s_1-t_1)m}{(s_0-t_0)\sqrt{s_0s_1}}(\sinh_D(mu;q) - mu)\\ &\quad + \frac{2t_1(s_1-t_1)\sqrt{s_0s_1}}{s_1^2m}(\sinh_q(mu) - mu),\\ T(\mathrm{ED}) &= \frac{2t_1}{s_1}(\cosh_q(mu)-1) + \frac{ut_1^2(s_0-t_0)\sqrt{s_0s_1}}{s_1^2s_0m}\sinh_q(mu) + \frac{(s_1-t_1)}{t_1}(\cosh_D(mu;q)-1).\end{aligned}$$

Proof. We proceed as we did in the proof of Theorem 2.4. It is easy to see that

$$\begin{aligned}\mathcal{D}_0(s_0,s_1,t_0,t_1,q,u) &= \frac{s_1}{s_0}\mathcal{D}_0(\frac{t_0}{s_0},\frac{t_1}{s_1},q,\sqrt{s_0s_1}u),\\ \mathcal{D}_1(s_0,s_1,t_0,t_1,q,u) &= \frac{\sqrt{s_1}}{\sqrt{s_0}}\mathcal{D}_1(\frac{t_0}{s_0},\frac{t_1}{s_1},q,\sqrt{s_0s_1}u).\end{aligned}$$

We denote by T the transformation that sends s to $\frac{t_0}{s_0}$, t to $\frac{t_1}{s_1}$ and u to $\sqrt{s_0s_1}u$. It is easy to see that $T(OD)$ and $T(ED)$ are as given above, completing the proof. □

3.2 Type D q-analogue of Hyatt's recurrences

We give our q-analogue of Hyatt-type recurrences in this subsection.

Theorem 3.3. *For even n,*

$$\sum_{w\in\mathfrak{D}_n^+} t^{\mathrm{odes}_D(w)} s^{\mathrm{edes}_D(w)} q^{\mathrm{inv}_D(w)} = \sum_{r=0}^{\frac{n}{2}-1} q^{\binom{2r+1}{2}} \binom{n}{2r+1}_q D_{n-2r-1}(s,t,q)(s-1)^r(t-1)^r + \sum_{r=1}^{\frac{n}{2}} q^{\binom{2r}{2}} \binom{n}{2r}_q D_{n-2r}(s,t,q)(s-1)^{r-1}(t-1)^r.$$

For odd n,

$$\sum_{w\in\mathfrak{D}_n^+} t^{\mathrm{odes}_D(w)} s^{\mathrm{edes}_D(w)} q^{\mathrm{inv}_D(w)} = \sum_{r=0}^{\lfloor\frac{n}{2}\rfloor} q^{\binom{2r+1}{2}} \binom{n}{2r+1}_q D_{n-2r-1}(s,t,q)(s-1)^r(t-1)^r + \sum_{r=1}^{\lfloor\frac{n}{2}\rfloor} q^{\binom{2r}{2}} \binom{n}{2r}_q D_{n-2r}(s,t,q)(s-1)^{r}(t-1)^{r-1}.$$

Proof. Define ${}_{\mathfrak{D}}\hat{A}_k$ to be the signed permutations in $\mathfrak{D}_n^+$ that have their rightmost $k+1$ elements being positive and arranged in descending order. Thus, the first $n-k-1$ elements must have an even number of negative signs. With this observation note that the map $f'' : \mathfrak{D}_k \times \binom{[n]}{n-k} \to {}_{\mathfrak{D}}\hat{A}_{n-k-1}$ that carries (ψ, A) to the signed permutation $\psi_{[n]-A} a_1 a_2 \cdots a_{n-k}$ $(a_1 > a_2 > \cdots a_{n-k} > 0)$, that is,

$$f''(\psi, A) = \psi_{[n]-A}, a_1, a_2, \cdots, a_{n-k}.$$

is a bijection from $\mathfrak{D}_k \times \binom{[n]}{n-k}$ onto ${}_{\mathfrak{D}}\hat{A}_{n-k-1}$.

Write ${}_{\mathfrak{D}}A_k(s,t,q) = \sum_{w\in{}_{\mathfrak{D}}\hat{A}_k} t^{\mathrm{odes}_D(w)} s^{\mathrm{edes}_D(w)} q^{\mathrm{inv}_D(w)}$. We will abbreviate the LHS as ${}_{\mathfrak{D}}A_k$ for brevity. The following recurrences are then easy to prove. For even n, we have

$$\begin{aligned} q^{\binom{2r}{2}} \binom{n}{2r}_q D_{n-2r}(s,t,q)s^r t^r &= s{}_{\mathfrak{D}}A_{2r-1} - (s-1){}_{\mathfrak{D}}A_{2r}. \\ q^{\binom{2r+1}{2}} \binom{n}{2r+1}_q D_{n-2r-1}(s,t)s^r t^{r+1} &= t{}_{\mathfrak{D}}A_{2r} - (t-1){}_{\mathfrak{D}}A_{2r+1}. \end{aligned}$$

For odd n, we have

$$\begin{aligned} q^{\binom{2r}{2}} \binom{n}{2r}_q D_{n-2r}(s,t)s^r t^r &= t{}_{\mathfrak{D}}A_{2r-1} - (t-1){}_{\mathfrak{D}}A_{2r}. \\ q^{\binom{2r+1}{2}} \binom{n}{2r+1}_q D_{n-2r-1}(s,t)s^{r+1} t^{r} &= s{}_{\mathfrak{D}}A_{2r} - (s-1){}_{\mathfrak{D}}A_{2r+1}. \end{aligned}$$

The proofs of these recurrences are along the same lines as the proofs of (27),(28),(29) and (30). The only ambiguity might be when $r = \lfloor\frac{n}{2}\rfloor - 1$, but this is easily resolved as in ${}_{\mathfrak{D}}\hat{A}_{2r}$ the rightmost $n-1$ elements are positive and descending for even n or ${}_{\mathfrak{D}}\hat{A}_{2r+1}$ when n is odd, the first element has to be positive due to the constraint that there are an even number of negative signs. Therefore, there is no possibility of $w_1 + w_2$ being lesser than 0. ☐

Setting $s = t$ in Theorem 3.3, we have the following corollary.

Corollary 3.5. *For even n,*

$$\sum_{w\in\mathfrak{D}_n^+} t^{\mathrm{des}_D(w)} q^{\mathrm{inv}_D(w)} = \sum_{r=0}^{\frac{n}{2}-1} q^{\binom{2r+1}{2}} \binom{n}{2r+1}_q D_{n-2r-1}(t,q)(t-1)^{2r} + \sum_{r=1}^{\frac{n}{2}} q^{\binom{2r}{2}} \binom{n}{2r}_q D_{n-2r}(t,q)(t-1)^{2r-1}.$$

For odd n,

$$\sum_{w\in\mathfrak{D}_n^+} t^{\mathrm{des}_D(w)}q^{\mathrm{inv}_D(w)} = \sum_{r=0}^{\lfloor \frac{n}{2}\rfloor} q^{\binom{2r+1}{2}}\binom{n}{2r+1}_q D_{n-2r-1}(t,q)(t-1)^{2r} + \sum_{r=1}^{\lfloor \frac{n}{2}\rfloor} q^{\binom{2r}{2}}\binom{n}{2r}_q D_{n-2r}(t,q)(t-1)^{2r-1}.$$

Setting $q=1$ in Corollary 3.5, we get Hyatt's recurrences [5, Lemma 3.2].

To preserve elements being in $\mathfrak{D}_n$, we consider the map that flips the sign of all elements when n is even and the map that flips the sign of all elements except the first when n is odd.

Lemma 3.4. *Let $f_D:\mathfrak{D}_n\to\mathfrak{D}_n$ be the involution that sends $w=w_1,\cdots,w_n$ to $\overline{w}=\overline{w_1},\ldots,\overline{w_n}$ if n is even and $w=w_1,\cdots,w_n$ to $\overline{w}=w_1,\overline{w_2}\ldots,\overline{w_n}$ if n is odd. Then, we have the following:*

1. *When $n=2k+1$, we have $\mathrm{odes}_D(w)+\mathrm{odes}_D(f_D(w))=k+1$ and when $n=2k$, we have $\mathrm{odes}_D(w)+\mathrm{odes}_D(f_D(w))=k+1$.*
2. *When $n=2k+1$, we have $\mathrm{edes}_D(w)+\mathrm{edes}_D(f_D(w))=k$ and when $n=2k$, we have $\mathrm{edes}_D(w)+\mathrm{edes}_D(f_D(w))=k-1$.*
3. *$\mathrm{inv}_D(w)+\mathrm{inv}_D(f_D(w))=n(n-1)$.*

Proof. The proof of the first two assertions is straightforward and hence omitted. For the third part, recall that $\mathrm{inv}_D(w)=\mathrm{inv}_B(w)-|\mathrm{Negs}(w)|$. Thus, we have, for even n,

$$\begin{aligned}\mathrm{inv}_D(w)+\mathrm{inv}_D(\overline{w}) &= \mathrm{inv}_B(w)+\mathrm{inv}_B(\overline{w})-|\mathrm{Negs}(w)|-|\mathrm{Negs}(\overline{w})|\\ &= n^2-n=n(n-1).\end{aligned}$$

When n is odd, it is easy to see that $\mathrm{inv}_D(w_1,\overline{w_2},\ldots,\overline{w_n})=\mathrm{inv}_D(\overline{w_1},\ldots,\overline{w_n})$. The rest follows from the previous argument. The proof is complete. $\square$

3.3 Symmetry results

In this Subsection, we give our type D counterparts of our symmetry results.

Theorem 3.4. *We have*

$$D_n^-(s,t,q)=\begin{cases} q^{n(n-1)}s^kt^{k+1}D_n^+(s^{-1},t^{-1},q^{-1}) & \text{when } n=2k+1,\\ q^{n(n-1)}s^{k-1}t^{k+1}D_n^+(s^{-1},t^{-1},q^{-1}) & \text{when } n=2k.\end{cases}$$

Therefore, we have

$$D_n(s,t,q)=\begin{cases} D_n^+(s,t,q)+q^{n(n-1)}s^kt^{k+1}D_n^+(s^{-1},t^{-1},q^{-1}) & \text{when } n=2k+1,\\ D_n^+(s,t,q)+q^{n(n-1)}s^{k-1}t^{k+1}D_n^+(s^{-1},t^{-1},q^{-1}) & \text{when } n=2k.\end{cases}$$

Proof. Let $f_D:\mathfrak{D}_n^+\to\mathfrak{D}_n^-$ be the map described earlier. By Lemma 3.4, when $n=2k$, we have

$$\begin{aligned}\sum_{w\in\mathfrak{D}_n^-} t^{\mathrm{odes}_D(w)}s^{\mathrm{edes}_D(w)}q^{\mathrm{inv}_D(w)} &= \sum_{w\in\mathfrak{D}_n^+} t^{\mathrm{odes}_D(f_D(w))}s^{\mathrm{edes}_D(f_D(w))}q^{\mathrm{inv}_D(f_D(w))}\\ &= \sum_{w\in\mathfrak{D}_n^+} t^{k+1-\mathrm{odes}_D(w)}s^{k-1-\mathrm{edes}_D(w)}q^{n(n-1)-\mathrm{inv}_D(w)}\\ &= q^{n(n-1)}s^{k-1}t^{k+1}\sum_{w\in\mathfrak{D}_n^+} t^{-\mathrm{odes}_D(w)}s^{-\mathrm{edes}_D(w)}q^{-\mathrm{inv}_D(w)}.\end{aligned}$$

When $n=2k+1$, we have

$$\begin{aligned}\sum_{w\in\mathfrak{D}_n^-} t^{\mathrm{odes}_D(w)}s^{\mathrm{edes}_D(w)}q^{\mathrm{inv}_D(w)} &= \sum_{w\in\mathfrak{D}_n^+} t^{\mathrm{odes}_D(f_D(w))}s^{\mathrm{edes}_D(f_D(w))}q^{\mathrm{inv}_D(f_D(w))}\\ &= \sum_{w\in\mathfrak{D}_n^+} t^{k+1-\mathrm{odes}_D(w)}s^{k-\mathrm{edes}_D(w)}q^{n(n-1)-\mathrm{inv}_D(w)}\\ &= q^{n(n-1)}s^{k}t^{k+1}\sum_{w\in\mathfrak{D}_n^+} t^{-\mathrm{odes}_D(w)}s^{-\mathrm{edes}_D(w)}q^{-\mathrm{inv}_D(w)}.\end{aligned}$$

completing the proof. $\square$

Since the following corollary is straightforward, we only state it and omit its proof.

Corollary 3.6 (Type-D Symmetry)**.** *We have*

$$D_n(s,t,q) = \begin{cases} q^{n(n-1)} s^k t^{k+1} D_n(s^{-1}, t^{-1}, q^{-1}) & \text{when } n = 2k+1, \\ q^{n(n-1)} s^{k-1} t^{k+1} D_n(s^{-1}, t^{-1}, q^{-1}) & \text{when } n = 2k. \end{cases}$$

4. Snakes

A snake in $\mathfrak{B}_n$ is a signed permutation $w \in \mathfrak{B}_n$ satisfying $0 < w_1 > w_2 < \cdots$. Let Snake_n^B be the set of snakes in $\mathfrak{B}_n$ and denote $|\mathrm{Snake}_n^B|$ by S_n^B. The paper by Arnol'd [1] is a good reference for this topic. Let $S_n^B(q) = \sum_{w \in \mathrm{Snake}_n^B} q^{\mathrm{inv}_B(w)}$. Springer in [13] showed the following.

Theorem 4.1 (Springer)**.** *The following is the egf for the numbers S_n^B:*

$$\sum_{n\geq 0} S_n^B \frac{u^n}{n!} = \frac{1}{\cos(u) - \sin(u)}.$$

The following corollary of Theorem 2.4 is now easy.

Corollary 4.1. *We have the following egf of the $S_n^B(q)$ polynomials:*

$$\sum_{n\geq 0} S_n^B(q) \frac{u^n}{B_n(1,q)} = \frac{\cos_q(u)\cos_B(u;q) + (\sin_q(u) - 1)\sin_B(u;q)}{\cos_q(u)}.$$

Proof. Setting $s_1 = t_0 = 0$ and $t_1 = s_0 = 1$ in both $H_0(s_0, s_1, t_0, t_1, q, u)$ and $H_1(s_0, s_1, t_0, t_1, q, u)$ from Theorem 2.4 and adding completes the proof. □

It is easy to see that setting $q = 1$ in Corollary 4.1 gives us Theorem 4.1.

4.1 D-snakes

A d-snake in $\mathfrak{D}_n$ is a signed permutation w in $\mathfrak{D}_n$ that satisfies $-w_2 > w_1 > w_2 < w_3 > \ldots w_n$. Let Snake_n^D be the set of all d-snakes in $\mathfrak{D}_n$. Denote $|\mathrm{Snake}_n^D|$ by S_n^D. Let $S_n^D(q) = \sum_{w \in \mathrm{Snake}_n^D} q^{\mathrm{inv}_D(w)}$. Define $\mathcal{SD}_0(q,u) = \sum_{n\geq 1} S_{2n}^D(q) \frac{u^{2n}}{D_{2n}(1,q)}$ and $\mathcal{SD}_1(q,u) = \sum_{n\geq 1} S_{2n+1}^D(q) \frac{u^{2n+1}}{D_{2n+1}(1,q)}$.

Corollary 4.2. *We have the following egf of the $S_n^D(q)$ polynomials:*

$$\mathcal{SD}_0(q,u) = \frac{-2\cos_q^2(u) + \cos_q(u)(\cos_D(u;q) - 1) - 2\sin_q^2(u) + \sin_q(u)\sin_D(u;q)}{-\cos_q(u)}, \tag{42}$$

$$\mathcal{SD}_1(q,u) = \frac{-2\sin_q(u) + u\cos_q(u) + \sin_D(u;q)}{-\cos_q(u)}. \tag{43}$$

Proof. Set $t = 1/t$, $u = u\sqrt{t}$, multiply by t and setting $s = t = 0$ in Theorem 1.8 gives us (42). Set $t = 1/t$, $u = u\sqrt{t}$, multiply by $\sqrt{t}$ and setting $s = t = 0$ in Theorem 1.8 gives us (43). □

Setting $q = 1$ in Corollary 4.2 gives us the following egf which is given by Springer.

Corollary 4.3 (Springer)**.** *The egf for the S_n^D is:*

$$\sum_{n\geq 1} S_{2n}^D \frac{u^{2n}}{(2n)!} = \frac{\cos(u) - \cos(2u) - 1}{-\cos(2u)},$$

$$\sum_{n\geq 1} S_{2n+1}^D \frac{u^{2n+1}}{(2n+1)!} = \frac{-\sin(2u) + u\cos(2u) + \sin(u)}{-\cos(2u)}.$$

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