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Research article

Exact solutions and asymptotic behaviors for the reflected Wiener, Ornstein-Uhlenbeck and Feller diffusion processes

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Abstract: We analyze the transition probability density functions in the presence of a zero-flux condition in the zero-state and their asymptotic behaviors for the Wiener, Ornstein Uhlenbeck and Feller diffusion processes. Particular attention is paid to the time-inhomogeneous proportional cases and to the time-homogeneous cases. A detailed study of the moments of first-passage time and of their asymptotic behaviors is carried out for the time-homogeneous cases. Some relationships between the transition probability density functions for the restricted Wiener, Ornstein-Uhlenbeck and Feller processes are proved. Specific applications of the results to queueing systems are provided.

Keywords: reflected diffusion processes; transition probability density functions; steady-state densities; first-passage time moments; asymptotic behaviors

1. Introduction

Reflected diffusion processes are used extensively in various applied fields, such as mathematical biology, queueing theory, finance and neuroscience. In particular, diffusion processes with a reflection condition at the origin arise as diffusion limits of a number of classical birth-death processes in population dynamics and as a heavy-traffic approximation for queueing systems (see, for instance, Giorno et al. [1], Ward and Glynn [2], Di Crescenzo et al. [3, 4]). Reflected diffusion processes are also applied in economics and finance for modeling regulated markets, interest rates and stochastic volatility (cf., for instance, Linetsky [5], Veestraeten [6]). Moreover, in neuronal models, the membrane potential evolution can be described by focusing the attention on the diffusion processes confined by a lower reflecting boundary that can be interpreted as the neuronal reversal hyperpolarization potential (cf. Lánský and Ditlevsen [7], Buonocore et al. [8, 9], D'Onofrio et al. [10]). References to other applications of reflected diffusion processes in neuroscience, in population dynamics, in economics, in finance and in queueing systems can be found in Di Crescenzo et al. [11], Giorno and Nobile [12, 13], Mishura and Yurchenko-Tytarenko [14]).

In various types of instances, first-passage time (FPT) problems are invoked to describe events such as extinction in population dynamics, busy period in queueing systems and firing times in neuronal modeling (cf., for instance, Ricciardi et a. [15], Masoliver and Perelló [16], Bo et al. [17], Abundo and Pirozzi [18], Giorno and Nobile [19]).

In several applications, it is useful to consider diffusion processes with linear infinitesimal drift and linear infinitesimal variance, having state-space $[0, +\infty)$ with a zero-flux condition at the zero-state. This class incorporates the Wiener, Ornstein-Uhlenbeck and Feller diffusion processes with reflection in the zero-state. Such processes are widely used in the modeling queueing systems under assumptions of heavy-traffic and in the description of populations growth in a random environment. In these contexts, the number of customers or individuals is bound to take non negative values, so that a reflection condition in the zero-state must thus be imposed. In particular, in queueing systems a reflecting condition in the zero-state is required because new customers can access the system if it is empty; moreover, in the population dynamics the reflection in zero allows to include immigration effects. In other contexts, the reflecting boundary can refer to a non-zero state and can be also time-dependent (cf., for instance, [8, 9]). In the present paper, we restrict the attention to the Wiener, Ornstein-Uhlenbeck and Feller diffusion processes in $[0, +\infty)$ with a reflection in the zero-state.

1.1. Plan of the paper

In Section 2, we briefly review some background results on the time-inhomogeneous and time-homogeneous diffusion processes restricted to the interval $[0, +\infty)$, with a reflecting or a zero-flux condition in the zero-state. In this case, a time-inhomogeneous diffusion process is characterized by time-dependent infinitesimal moments, whereas the infinitesimal moments of a time-homogeneous diffusion process are not time-dependent.

In Sections 3–5, we consider the time-inhomogeneous Wiener, Ornstein-Uhlenbeck and Feller diffusion processes in $[0, +\infty)$ (cf. Table 1).

Table 1. Time-inhomogeneous diffusion processes considered in Sections 3–5.

Time-inhomogeneous process	Infinitesimal drift and infinitesimal variance
X(t) - Reflected Wiener	$A_1(t) = \beta(t), \quad A_2(t) = \sigma^2(t)$
(TNH-RW)	$(\beta(t) \in \mathbb{R}, \sigma(t) > 0)$
Y(t) - Reflected Ornstein-Uhlenbeck	$B_1(x, t) = \alpha(t)\, x + \beta(t), \quad B_2(t) = \sigma^2(t)$
(TNH-ROU)	$(\alpha(t) \in \mathbb{R}, \beta(t) \in \mathbb{R}, \sigma(t) > 0)$
Z(t) - Reflected Feller	$C_1(x, t) = \alpha(t)\, x + \beta(t), \quad C_2(x, t) = 2\, r(t)\, x$
(TNH-RF)	$(\alpha(t) \in \mathbb{R}, \beta(t) > 0, r(t) > 0)$

We determine closed form expressions for the transition probability density function (pdf) and for the related conditional moments of the first and the second order in the following cases:

- for the TNH-RW process $X(t)$ with $\beta(t) = \gamma\, \sigma^2(t)$, where $\gamma \in \mathbb{R}$;
- for the TNH-ROU process $Y(t)$ with $\beta(t) = \gamma\, \sigma^2(t)\, e^{-A(t|0)}$, where $\gamma \in \mathbb{R}$ and $A(t|0) = \int_0^t \alpha(u)\, du$;
- for the TNH-RF process $Z(t)$ with $\beta(t) = \xi\, r(t)$, where $\xi > 0$.

For these processes, we analyze the asymptotic behavior of the transition densities. Moreover, in Section 5, for $\beta(t) = r(t)/2$ and for $\beta(t) = 3r(t)/2$ some relationships between the transition pdf for Wiener, Ornstein-Uhlenbeck and Feller processes are proved.

In Sections 3–5, we also take into account the time-homogeneous Wiener, Ornstein-Uhlenbeck and Feller diffusion processes in $[0, +\infty)$ (see Table 2). For these processes, we study:

Table 2. Time-homogeneous diffusion processes considered in Sections 3–5.

Time-homogeneous process	Infinitesimal drift and infinitesimal variance
X(t) - Reflected Wiener	$A_1 = \beta, \quad A_2 = \sigma^2$
(TH-RW)	$(\beta \in \mathbb{R}, \sigma > 0)$
Y(t) - Reflected Ornstein-Uhlenbeck	$B_1(x) = \alpha\, x + \beta, \quad B_2 = \sigma^2$
(TH-ROU)	$(\alpha \neq 0, \beta \in \mathbb{R}, \sigma > 0)$
Z(t) - Reflected Feller	$C_1(x) = \alpha\, x + \beta, \quad C_2(x) = 2\, r\, x$
(TH-RF)	$(\alpha \in \mathbb{R}, \beta > 0, r > 0)$

- the asymptotic transition pdf (steady-state density);
- the transition pdf for the TH-RW and for the TH-RF processes;
- the Laplace transform (LT) of the transition pdf ($\beta \in \mathbb{R}$) and the transition pdf ($\beta = 0$) for the TH-ROU process;
- the asymptotic behavior of the FPT moments for boundaries near to zero-state and for large boundaries.

Moreover, the mean, the coefficient of variation and the skewness of FPT for the TH-RW, TH-ROU and TH-RF processes are analyzed for various choices of parameters making use of Mathematica.

In Section 6, we consider some examples of the TNH-RW, TNH-ROU and TNH-RF diffusion processes useful to modeling queueing systems in heavy-traffic conditions; for these processes, we assume that the infinitesimal drift and the infinitesimal variance are time-dependent and include periodic functions. Several numerical computations with Mathematica are performed to analyze the conditional averages and the conditional variances and their asymptotic behaviors for some choices of the periodic functions and of the parameters.

This paper is dedicated to the memory of Patricia Román Román. Her untimely death leaves a great hole in our scientific community and an even greater hole in our hearts.

2. Background results

In this section, we briefly review some results on the diffusion processes that will be used in the next sections to analyze Wiener, Ornstein-Uhlenbeck and Feller diffusion processes in $[0, +\infty)$, with a zero-flux condition at the zero-state.

Let $D(t)$ be a time-inhomogeneous diffusion (TNH-RD) process with infinitesimal drift $\zeta_1(x, t)$ and the infinitesimal variance $\zeta_2(x, t)$, restricted to interval $[0, +\infty)$, with $+\infty$ unattainable end-point and a zero-flux condition in the zero-state. The transition pdf $r_D(x, t|x_0, t_0)$ of $D(t)$ can be obtained as the solution of the Fokker-Planck equation (cf. Dynkin [20])

$$\frac{\partial r_D(x,t|x_0,t_0)}{\partial t} = -\frac{\partial}{\partial x}\Big\{\zeta_1(x,t)\, r_D(x,t|x_0,t_0)\Big\} + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\Big\{\zeta_2(x,t)\, r_D(x,t|x_0,t_0)\Big\}, \tag{2.1}$$

with the initial delta condition $\lim_{t\downarrow t_0} r_D(x,t|x_0,t_0) = \delta(x-x_0)$ and the condition:

$$\lim_{x\downarrow 0}\Big\{\zeta_1(x,t)\, r_D(x,t|x_0,t_0) - \frac{1}{2}\,\frac{\partial}{\partial x}\Big[\zeta_2(x,t)r_D(x,t|x_0,t_0)\Big]\Big\} = 0. \tag{2.2}$$

We note that (2.1) can be re-written as

$$\frac{\partial r_D(x,t|x_0,t_0)}{\partial t} = -\frac{\partial j_D(x,t|x_0,t_0)}{\partial x},$$

where

$$j_D(x,t|x_0,t_0) = \zeta_1(x,t)\, r_D(x,t|x_0,t_0) - \frac{1}{2}\,\frac{\partial}{\partial x}\Big\{\zeta_2(x,t)\, r_D(x,t|x_0,t_0)\Big\}$$

represents probability flux (or current) of $D(t)$. Eq. (2.2) is the zero-flux or reflecting condition in the zero-state and corresponds to requiring that $\int_0^{+\infty} r_D(x,t|x_0,t_0)\,dx = 1$ for all $t \geq t_0$.

Expressions in closed form for $r_D(x,t|x_0,t_0)$ can be obtained only for some choices of the infinitesimal moments. For instance, if $D(t)$ is a TNH-RD process with space-state $[0,+\infty)$, having infinitesimal drift and infinitesimal variance

$$\begin{aligned} \zeta_1(x,t) &= d\,h_1'(t) + [x - d\,h_1(t)]\,\frac{h_2'(t)}{h_2(t)},\\ \zeta_2(x,t) &\equiv \zeta_2(t) = h_1'(t)\,h_2(t) - h_1(t)\,h_2'(t), \end{aligned} \tag{2.3}$$

with the "prime" symbol denoting derivative with respect to the argument, $d \in \mathbb{R}$, $h_2(t) \neq 0$ and $h_1(t)/h_2(t)$ is a non-negative and monotonically increasing function, then $r_D(x,t|x_0,t_0)$ can be determined in closed form (cf. Buonocore et al. [8]):

$$r_D(x,t|x_0,t_0) = f_D(x,t|x_0,t_0) - \frac{\partial}{\partial x}\Big[\exp\Big\{\frac{2\,d\,x}{h_2(t)}\Big\}F_D(-x,t|x_0,t_0)\Big], \qquad x \geq 0, x_0 \geq 0, \tag{2.4}$$

where $f_D(x,t|x_0,t_0)$ and $F_D(x,t|x_0,t_0) = \int_{-\infty}^{x} f_D(z,t|x_0,t_0)\,dz$ are the transition pdf and probability distribution function of the corresponding unrestricted diffusion process with space-state $\mathbb{R}$, respectively. Specifically, for the diffusion process (2.3), $f_D(x,t|x_0,t_0)$ is a normal density with mean and variance

$$\begin{aligned} M_D(t|x_0,t_0) &= \frac{h_2(t)}{h_2(t_0)}\,x_0 + \frac{d}{h_2(t_0)}\,\Big[h_1(t)\,h_2(t_0) - h_1(t_0)\,h_2(t)\Big],\\ V_D(t|t_0) &= \frac{h_2(t)}{h_2(t_0)}\,\Big[h_1(t)\,h_2(t_0) - h_1(t_0)\,h_2(t)\Big]. \end{aligned}$$

We remark that $r_D(x,t|x_0,t_0)$, given in (2.4), satisfies the Fokker-Planck equation (2.1), the initial delta condition, the zero-flux condition (2.2) and also $\int_0^{+\infty} r_D(x,t|x_0,t_0)\,dx = 1$. Moreover, under suitable conditions, the infinitesimal moments of the Wiener process and of the Ornstein-Uhlenbeck process satisfy (2.3); in these cases, the transition pdf $r_D(x,t|x_0,t_0)$ is obtainable from (2.4). Instead, for the Feller process the infinitesimal variance depends on x, so that (2.4) does not hold.

The knowledge of density $r_D(x,t|x_0,t_0)$ allows to evaluate the conditional moments:

$$E[D^k(t)|D(t_0) = x_0] = \int_0^{+\infty} x^k\, r_D(x,t|x_0,t_0)\,dx, \qquad k = 1,2,\ldots \tag{2.5}$$

2.1. Time-homogeneous reflected diffusion process

For a time-homogeneous reflected diffusion (TH-RD) process $D(t)$, with state-space $[0,+\infty)$, one has $\zeta_1(x,t) = \zeta_1(x)$ and $\zeta_2(x,t) = \zeta_2(x)$ for all t. For the TH-RD process, the transition pdf $r_D(x,t|x_0,t_0) = r_D(x,t-t_0|x_0,0) = r_D(x,t-t_0|x_0)$ and, in this case, we assume that $t_0 = 0$.

We denote by

$$h_D(x) = \exp\Big\{-2\int^x \frac{\zeta_1(u)}{\zeta_2(u)}\,du\Big\}, \qquad s_D(x) = \frac{2}{\zeta_2(u)\,h_D(u)}, \tag{2.6}$$

the scale function and the speed density. These functions allow to classify the end-points of a diffusion process into regular, natural, exit, entrance and in attracting or nonattracting boundaries (cf. Karlin and Taylor [21]). Let

$$T_D(S|x_0) = \begin{cases} \inf_{t\geq 0}\{t : D(t) \geq S\}, & 0 \leq D(0) = x_0 < S, \\ \inf_{t\geq 0}\{t : D(t) \leq S\}, & D(0) = x_0 > S \geq 0, \end{cases}$$

be the random variable that describes the FPT of $D(t)$ through S starting from $D(0) = x_0 \neq S$. We denote by $g_D(S,t|x_0) = dP\{T_D(S|x_0) \leq t\}/dt$ the FPT density and by $P_D(S|x_0) = \int_0^{+\infty} g_D(S,t|x_0)\,dt$ the ultimate FPT probability. When $P_D(S|x_0) = 1$, let

$$t_n^{(D)}(S|x_0) = \int_0^{+\infty} t^n g_D(S,t|x_0)\,dt, \qquad n = 1,2,\dots$$

be the n-th FPT moment. The functions (2.6) allow to determine the FPT moments thanks to the Siegert formula (cf. Siegert [22]). Specifically, if $D(t)$ is a TH-RD process in $[0,+\infty)$, with $+\infty$ nonattracting end-point, for $n = 1,2,\dots$ it results:

- If $x_0 > S > 0$, one has $P_D(S|x_0) = 1$ and, if $\int_z^{+\infty} s_D(u)\,du$ converges, the FPT moments can be evaluated as:

$$t_n^{(D)}(S|x_0) = n\int_S^{x_0} dz\, h_D(z)\int_z^{+\infty} s_D(u)\, t_{n-1}^{(D)}(S|u)\,du, \quad x_0 > S > 0. \tag{2.7}$$

  Equation (2.7) holds also for $S = 0$ provided that the zero-state is a regular boundary.
- If $0 \leq x_0 < S$, one has $P_D(S|x_0) = 1$ and the FPT moments can be iteratively computed as:

$$t_n^{(D)}(S|x_0) = n\int_{x_0}^{S} dz\, h_D(z)\int_0^{z} s_D(u)\, t_{n-1}^{(D)}(S|u)\,du, \quad 0 \leq x_0 < S. \tag{2.8}$$

Making use of (2.8), in Giorno et al. [23] the following asymptotic results for the FPT moments are proved when the boundary S moves indefinitely away from the zero-state and when S is in the neighborhood of zero.

Remark 2.1. *For the TH-RD process $D(t)$, we denote by*

$$k_1(x) = \sqrt{\zeta_2(x)}\,h_D(x)\int_0^x s_D(u)\,du, \qquad k_2(x) = \Big[\zeta_1(x) - \frac{\zeta_2'(x)}{4}\Big]h_D(x)\int_0^x s_D(u)\,du.$$

One has:

1. *If* $\lim_{x\uparrow+\infty} k_1(x) = +\infty$ *and* $\lim_{x\uparrow+\infty} k_2(x) = -\infty$, *then*

$$\lim_{S\uparrow+\infty} \frac{t_n^{(D)}(S\,|0)}{[t_1^{(D)}(S\,|0)]^n} = n!, \qquad n = 0, 1, 2, \dots$$

2. *If* $\lim_{x\downarrow 0} k_1(x) = 0$ *and* $\lim_{x\downarrow 0} k_2(x) = \nu$, *with* $-\infty < \nu < 1$, *then*

$$\lim_{S\downarrow 0} \frac{t_n^{(D)}(S\,|0)}{[t_1^{(D)}(S\,|0)]^n} = u_n, \qquad n = 0, 1, 2, \dots$$

where

$$u_0 = 1, \quad u_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \Big\{\prod_{i=0}^{k-1} [1 + 2\,i\,(1-\nu)]\Big\}^{-1} u_{n-k}.$$

3. Reflected Wiener process

Let $\{X(t), t \ge t_0\}$, $t_0 \ge 0$, be a TNH-RW process, having infinitesimal drift and infinitesimal variance

$$A_1(t) = \beta(t), \qquad A_2(t) = \sigma^2(t), \tag{3.1}$$

respectively, with $\beta(t) \in \mathbb{R}$ and $\sigma(t) > 0$ continuous functions for all t. For the TNH-RW process, the results of Section 2 hold by choosing $\zeta_1(x,t) = \beta(t)$ and $\zeta_2(x,t) = \sigma^2(t)$. We denote by $r_X(x,t|x_0,t_0)$ the transition pdf of $X(t)$.

The reflected Wiener process can be used as the diffusion approximation of the queueing system $M/M/1$ in heavy traffic conditions (see Giorno et al. [1], Kingman [24], Harrison [25]). It plays also an important role in economics and in finance (cf. Veestraeten [6], Linetsky [26]).

Making use of the Fokker-Planck equation (2.1) with the boundary condition (2.2), for $x_0 \ge 0$ one obtains the first two conditional moments of $X(t)$:

$$E[X(t)|X(t_0) = x_0] = x_0 + \int_{t_0}^{t} \beta(u)\,du + \frac{1}{2}\int_{t_0}^{t} \sigma^2(u)\,r_X(0,u|x_0,t_0)\,du,$$
$$E[X^2(t)|X(t_0) = x_0] = x_0^2 + \int_{t_0}^{t} \sigma^2(u)\,du + 2\int_{t_0}^{t} \beta(u)\,E[X(u)|X(t_0) = x_0]\,du.$$

In the following proposition, we determine the transition pdf $r_X(x,t|x_0,t_0)$ in a special case.

Proposition 3.1. *Let* $X(t)$ *be a TNH-RW process, having* $\beta(t) = \gamma\,\sigma^2(t)$, *with* $\gamma \in \mathbb{R}$, *and* $\sigma(t) > 0$ *in (3.1). One has:*

$$r_X(x,t|x_0,t_0) = f_X(x,t|x_0,t_0) - \frac{\partial}{\partial x}\Big[e^{2\gamma x}\,F_X(-x,t|x_0,t_0)\Big], \quad x \ge 0, x_0 \ge 0, \tag{3.2}$$

where

$$\begin{aligned} f_X(x,t|x_0,t_0) &= \frac{1}{\sqrt{2\pi V_X(t|t_0)}}\,\exp\Big\{-\frac{[x - M_X(t|x_0,t_0)]^2}{2\,V_X(t|t_0)}\Big\},\\ F_X(x,t|x_0,t_0) &= \frac{1}{2}\Big[1 + \mathrm{Erf}\Big(\frac{x - M_X(t|x_0,t_0)}{\sqrt{2\,V_X(t|t_0)}}\Big)\Big], \end{aligned} \tag{3.3}$$

with

$$M_X(t|x_0,t_0) = x_0 + \int_{t_0}^{t} \beta(u)\, du, \qquad V_X(t|t_0) = \int_{t_0}^{t} \sigma^2(u)\, du \tag{3.4}$$

and $\mathrm{Erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-z^2} dz$ *denoting the error function.*

Proof. Comparing (2.3) with (3.1), one has $d\,h_1'(t) = \beta(t)$, $h_2'(t) = 0$ and $h_1'(t)h_2(t) = \sigma^2(t)$ for all t, from which

$$h_2(t) = c \neq 0, \qquad h_1(t) = \frac{1}{c}\int_0^t \sigma^2(u)\, du, \qquad \beta(t) = \frac{d}{c}\,\sigma^2(t).$$

Hence, for $\beta(t) = \gamma\,\sigma^2(t)$, Eq. (3.2) follows from (2.4) by setting $\gamma = d/c$. □

Under the assumption of Proposition 3.1, making use of (3.3) in (3.2), we have:

$$\begin{aligned} r_X(x,t|x_0,t_0) = {} & \frac{1}{\sqrt{2\pi V_X(t|t_0)}}\Big[\exp\Big\{-\frac{[x - x_0 - \gamma\,V_X(t|t_0)]^2}{2\,V_X(t|t_0)}\Big\} + e^{2\gamma x}\exp\Big\{-\frac{[x + x_0 + \gamma\,V_X(t|t_0)]^2}{2\,V_X(t|t_0)}\Big\}\Big] \\ & -\gamma\, e^{2\gamma x}\,\mathrm{Erfc}\Big(\frac{x + x_0 + \gamma\,V_X(t|t_0)}{\sqrt{2\,V_X(t|t_0)}}\Big), \qquad x \geq 0, x_0 \geq 0, \end{aligned} \tag{3.5}$$

where $\mathrm{Erfc}(x) = 1 - \mathrm{Erf}(x)$. We note that Eq. (3.5) for $t_0 = 0$ is in agreement with Eq. (29) in Molini et al. [27].

Corollary 3.1. *Under the assumptions of Proposition 3.1, the following results hold:*

- *If* $\beta(t) = 0$, *one has:*

$$\begin{aligned} E[X(t)|X(t_0) = x_0] &= \sqrt{\frac{2V_X(t|t_0)}{\pi}}\,\exp\Big\{-\frac{x_0^2}{2V_X(t|t_0)}\Big\} + x_0\,\mathrm{Erf}\Big(\frac{x_0}{\sqrt{2\,V_X(t|t_0)}}\Big), \\ E[X^2(t)|X(t_0) = x_0] &= x_0^2 + V_X(t|t_0). \end{aligned} \tag{3.6}$$

- *If* $\beta(t) = \gamma\,\sigma^2(t)$, *with* $\gamma \neq 0$, *one obtains:*

$$\begin{aligned} E[X(t)|X(t_0) = x_0] = {} & \sqrt{\frac{V_X(t|t_0)}{2\pi}}\,\exp\Big\{-\frac{[x_0 + \gamma\,V_X(t|t_0)]^2}{2V_X(t|t_0)}\Big\} + \frac{x_0 + \gamma\,V_X(t|t_0)}{2}\Big[1 + \mathrm{Erf}\Big(\frac{x_0 + \gamma\,V_X(t|t_0)}{\sqrt{2V_X(t|t_0)}}\Big)\Big] \\ & -\frac{1}{4\gamma}\,\mathrm{Erfc}\Big(\frac{x_0 + \gamma\,V_X(t|t_0)}{\sqrt{2V_X(t|t_0)}}\Big) + \frac{e^{-2\gamma x_0}}{4\gamma}\,\mathrm{Erfc}\Big(\frac{x_0 - \gamma V_X(t|t_0)}{\sqrt{2V_X(t|t_0)}}\Big), \\ E[X^2(t)|X(t_0) = x_0] = {} & \frac{[x_0 + \gamma V_X(t|t_0)]^2 + V_X(t|t_0)}{2}\Big[1 + \mathrm{Erf}\Big(\frac{x_0 + \gamma V_X(t|t_0)}{\sqrt{2V_X(t|t_0)}}\Big)\Big] \\ & + \frac{1}{4\gamma^2}\,\mathrm{Erfc}\Big(\frac{x_0 + \gamma V_X(t|t_0)}{\sqrt{2V_X(t|t_0)}}\Big) + \sqrt{\frac{V_X(t|t_0)}{2\pi}}\Big(x_0 + \gamma V_X(t|t_0) + \frac{1}{\gamma}\Big) \\ & \times\exp\Big\{-\frac{[x_0 + \gamma V_X(t|t_0)]^2}{2V_X(t|t_0)}\Big\} - \frac{e^{-2\gamma x_0}}{2\gamma}\Big(x_0 - \gamma V_X(t|t_0) + \frac{1}{2\gamma}\Big)\,\mathrm{Erfc}\Big(\frac{x_0 - \gamma V_X(t|t_0)}{\sqrt{2V_X(t|t_0)}}\Big). \end{aligned} \tag{3.7}$$

Proof. It follows making use of (3.5) in (2.5) with $k = 1, 2$, respectively. □

Corollary 3.2. *The TNH-RW process, having $A_1(t) = \gamma\,\sigma^2(t)$ and $A_2(t) = \sigma^2(t)$, with $\gamma \in \mathbb{R}$ and $\sigma(t) > 0$, admits the following asymptotic behaviors:*

1. *when $\gamma < 0$ and $\lim_{t\to+\infty} \sigma^2(t) = \sigma^2$ one has:*

$$W_X(x) = \lim_{t\to+\infty} r_X(x, t|x_0, t_0) = 2\,|\gamma|\, e^{-2|\gamma| x}, \qquad x \geq 0. \tag{3.8}$$

2. *when $\gamma < 0$ and $\sigma^2(t)$ is a positive periodic function of period Q, one obtains:*

$$\lim_{n\to+\infty} r_X(x, t + nQ|x_0, t_0) = 2\,|\gamma|\, e^{-2|\gamma| x}, \qquad x \geq 0. \tag{3.9}$$

Proof. Eqs. (3.8) and (3.9) follow taking the appropriate limits in (3.5). □

Under the assumptions of Corollary 3.2, the TNH-RW process $X(t)$ exhibits an exponential asymptotic behavior with mean and variance given by $(2\,|\gamma|)^{-1}$ and $(2\,|\gamma|)^{-2}$, respectively.

3.1. Time-homogeneous reflected Wiener process

For the TH-RW process $X(t)$, in (3.1) we set $\beta(t) = \beta$ and $\sigma^2(t) = \sigma^2$, with $\beta \in \mathbb{R}$ and $\sigma > 0$. The scale function and the speed density, defined in (2.6), are:

$$h_X(x) = \exp\Big\{-\frac{2\beta}{\sigma^2}\,x\Big\}, \qquad s_X(x) = \frac{2}{\sigma^2}\,\exp\Big\{\frac{2\beta}{\sigma^2}\,x\Big\}, \tag{3.10}$$

respectively. When $\beta > 0$ ($\beta \leq 0$) the end-point $+\infty$ is an attracting (nonattracting) natural boundary. By choosing $\beta(t) = \beta$, $\sigma^2(t) = \sigma^2$ and $\gamma = \beta/\sigma^2$, from Eq. (3.5) one has:

$$\begin{aligned} r_X(x, t|x_0) = {} & \frac{1}{\sigma\sqrt{2\pi\,t}}\Big[\exp\Big\{-\frac{(x - x_0 - \beta\,t)^2}{2\sigma^2 t}\Big\} + \exp\Big\{\frac{2\beta\,x}{\sigma^2}\Big\}\exp\Big\{-\frac{(x + x_0 + \beta\,t)^2}{2\sigma^2 t}\Big\}\Big] \\ & -\frac{\beta}{\sigma^2}\,\exp\Big\{\frac{2\beta\,x}{\sigma^2}\Big\}\,\mathrm{Erfc}\Big(\frac{x + x_0 + \beta\,t}{\sigma\sqrt{2t}}\Big), \qquad x \geq 0, x_0 \geq 0, \end{aligned} \tag{3.11}$$

and from (3.6) and (3.7) one obtains the first two conditional moments. Eq. (3.11) is in agreement with the expression given in Cox and Miller [28] obtained by using the method of images. We note that, if $\beta < 0$, the TH-RW process $X(t)$ admits an asymptotic behavior and the steady-state density is given in (3.8) with $\gamma = \beta/\sigma^2$. Hence, for $\beta < 0$ the steady-state density of the TH-RW process $X(t)$ is exponential with mean $E(X) = \sigma^2/(2\,|\beta|)$ and variance $\mathrm{Var}(X) = \sigma^4/(4\beta^2)$.

In Figure 1, we consider a TH-RW process $X(t)$ in $[0, +\infty)$, having $A_1 = \beta$ and $A_2 = \sigma^2$, with $\beta = -0.6$, $\sigma = 1$, $x_0 = 5$ and $t_0 = 0$. Making use of Algorithm 4.2 in Buonocore et al. [29], we obtain a random sample of $N = 5 \cdot 10^4$ observations of $X(t)$. Then, we compare the histogram of the random sample with the transition pdf (3.11) as function of x ($x \geq 0$) for $t = 3$ on the left and $t = 5$ on the right. As shown in Figure 1, the histograms fit the exact transition densities (3.11) reasonably well.

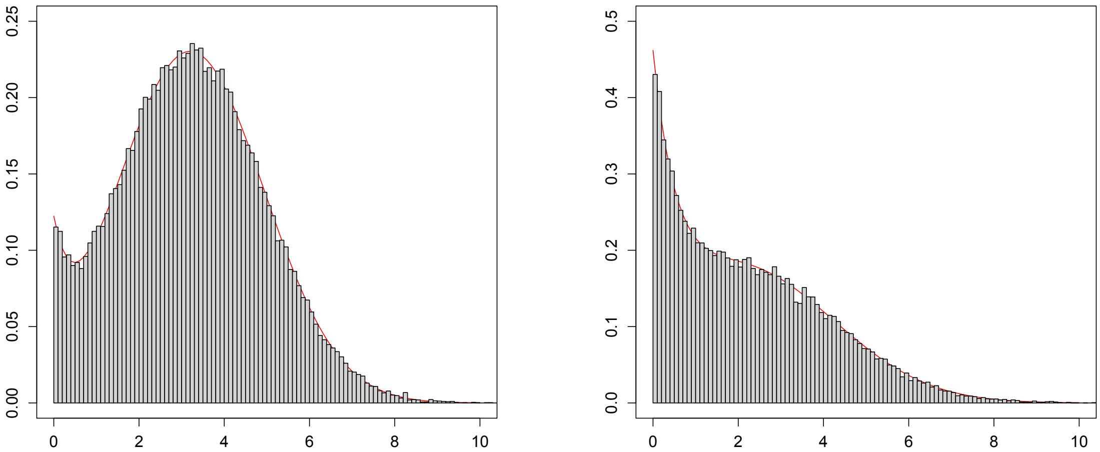

Figure 1. For the TH-RW process $X(t)$, with $\beta = -0.6$, $\sigma = 1$, $x_0 = 5$, the transition pdf (3.11) is plotted (red curve) as function of x, for $t = 3$ (on the left) and $t = 5$ (on the right) and the histogram, obtained via the simulation of the sample paths, is superimposed over the density.

3.2. *FPT for TH-RW process*

For the TH-RW process, if $0 \leq S < x_0$ the FPT through S starting from x_0 is not affected to reflecting boundary in the zero-state; in this case, if $\beta < 0$ the ultimate FPT probability $P_X(S|x_0) = 1$ and from (2.7) the FPT mean is $t_1^{(X)}(S|x_0) = (S - x_0)/\beta$. Moreover, if $0 \leq x_0 < S$ the probability $P_X(S|x_0) = 1$ and making use of (2.8) one obtains:

$$t_1^{(X)}(S|x_0) = \begin{cases} \frac{S^2-x_0^2}{\sigma^2}, & \beta = 0, \\ \frac{S-x_0}{\beta} + \frac{\sigma^2}{2\beta^2}\Big[\exp\Big\{-\frac{2\beta S}{\sigma^2}\Big\} - \exp\Big\{-\frac{2\beta x_0}{\sigma^2}\Big\}\Big], & \beta \neq 0. \end{cases} \tag{3.12}$$

When $0 \leq x_0 < S$, from (3.12) one has:

- $t_1^{(X)}(S|x_0)$ decreases as β increases and

$$\lim_{\beta\to-\infty} t_1^{(X)}(S|x_0) = +\infty, \qquad \lim_{\beta\to+\infty} t_1^{(X)}(S|x_0) = 0;$$

- for $\beta \leq 0$, $t_1^{(X)}(S|x_0)$ decreases as σ^2 increases; moreover, one has

$$\lim_{\sigma^2\to 0} t_1^{(X)}(S|x_0) = \begin{cases} +\infty, & \beta \leq 0, \\ (S - x_0)/\beta, & \beta > 0, \end{cases} \qquad \lim_{\sigma^2\to+\infty} t_1^{(X)}(S|x_0) = 0.$$

In Figure 2, the FPT mean (3.12) of the TH-RW process is plotted for $x_0 = 5$, $S = 10$ and for different choices of β and σ^2.

Making use of (3.10) in Remark 2.1, one can derive some asymptotic behaviors for the FPT moments of the TH-RW process.

Remark 3.1. *For the TH-RW process $X(t)$, when $n = 1, 2, \ldots$ one has:*

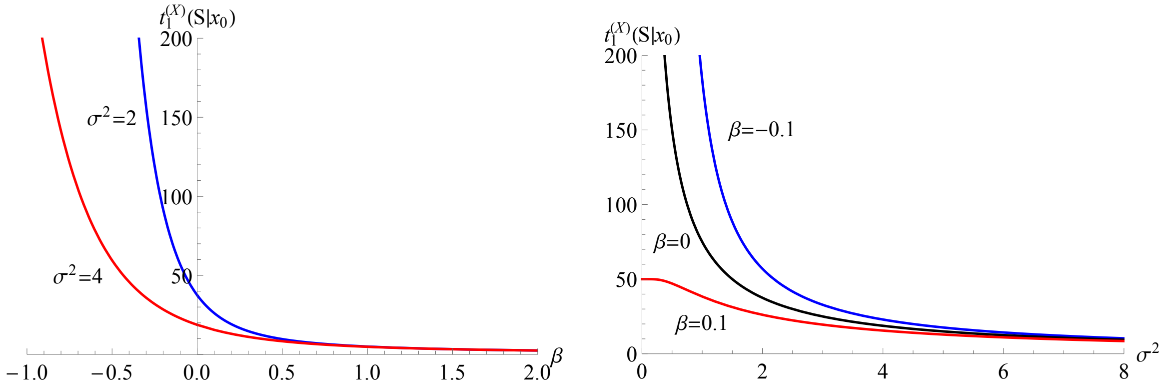

Figure 2. The FPT mean (3.12) of the TH-RW process is plotted for $x_0 = 5$ and $S = 10$ as function of β on the left and as function of σ^2 on the right.

1. $\lim_{S\uparrow+\infty} \frac{t_n^{(X)}(S\,|0)}{[t_1^{(X)}(S\,|0)]^n} = n!, \qquad \beta < 0,$

2. $\lim_{S\downarrow 0} \frac{t_n^{(X)}(S\,|0)}{[t_1^{(X)}(S\,|0)]^n} = \frac{(-1)^n\,E_{2n}}{(2n-1)!!},$

where

$$E_0 = 1, \qquad E_{2n} = -\sum_{j=0}^{n-1} \binom{2n}{2j} E_{2j}, \qquad n = 1, 2\ldots$$

denote the Euler numbers.

Then, from Remark 3.1 it follows:

- for $S \uparrow +\infty$ one has:

$$t_n^{(X)}(S\,|0) \simeq n!\,[t_1^{(X)}(S\,|0)]^n, \qquad \beta < 0, \tag{3.13}$$

  so that for $\beta < 0$ the FPT density $g_X(S, t|0)$ of the TH-RW process exhibits an exponential asymptotic behavior for large boundaries;
- for $S \downarrow 0$ results:

$$t_n^{(X)}(S\,|0) \simeq \frac{(-1)^n\,E_{2n}}{(2n-1)!!}\,[t_1^{(X)}(S\,|0)]^n = m_n^{(X)}(S\,|0). \tag{3.14}$$

  In particular, one has:

$$m_2^{(X)}(S\,|0) = \frac{5}{3}\,[t_1^{(X)}(S\,|0)]^2, \qquad m_3^{(X)}(S\,|0) = \frac{61}{15}\,[t_1^{(X)}(S\,|0)]^3.$$

In Tables 3–5, $t_1^{(X)}(S\,|0)$ is computed via (3.12), whereas $t_2^{(X)}(S\,|0)$ and $t_3^{(X)}(S\,|0)$ are numerically evaluated making use of the Siegert formula (2.8), with the scale function and the speed density given in (3.10). In particular, in Table 3, the mean $t_1^{(X)}(S\,|0)$, the variance $\mathrm{Var}^{(X)}(S\,|0)$, the coefficient of variation $\mathrm{Cv}^{(X)}(S\,|0)$ and the skewness $\Sigma^{(X)}(S\,|0)$ of the FPT are listed for $\beta = -0.1$, $\sigma = 1$ and some choices of $S > 0$. In agreement with the exponential asymptotic behavior, being $\beta < 0$, the coefficient of variation and the skewness approach to 1 and to 2, respectively, as S increases.

Moreover, in Tables 4 and 5, for the TH-RW process with $\beta = -0.1, 0.1$ and $\sigma = 1$, we compare the FPT moments $t_2^{(X)}(S\,|0)$ and $t_3^{(X)}(S\,|0)$ with the approximate values $m_2^{(X)}(S\,|0)$ and $m_3^{(X)}(S\,|0)$, respectively.

Table 3. For the TH-RW process, with $\beta = -0.1$ and $\sigma = 1$, $t_1^{(X)}(S|0)$, $\mathrm{Var}^{(X)}(S|0)$, $\mathrm{Cv}^{(X)}(S|0)$ and $\Sigma^{(X)}(S|0)$ are listed for increasing values of $S > 0$.

S	$t_1^{(X)}(S\|0)$	$\mathrm{Var}^{(X)}(S\|0)$	$\mathrm{Cv}^{(X)}(S\|0)$	$\Sigma^{(X)}(S\|0)$
10.	219.453	40104.8	0.912551	1.99238
20.	2479.91	5.78195×10^{6}	0.96962	1.99921
30.	19821.4	3.86673×10^{8}	0.992057	1.99995
40.	148598	2.20066×10^{10}	0.998305	2.0
50.	1.10077×10^{6}	1.21093×10^{12}	0.999681	2.00001
60.	8.13709×10^{6}	6.62049×10^{13}	0.999945	1.99999
70.	6.01295×10^{7}	3.61549×10^{15}	0.999991	2.00001
80.	4.44305×10^{8}	1.97406×10^{17}	0.999998	2.0
90.	3.28300×10^{9}	1.07780×10^{19}	0.999999	1.99995
100.	2.42583×10^{10}	5.88463×10^{20}	1.0	1.99999

Table 4. For the TH-RW process, with $\beta = -0.1$ and $\sigma = 1$, $t_1^{(X)}(S|0)$, $t_2^{(X)}(S|0)$ and $t_3^{(X)}(S|0)$ and their approximate values $m_2^{(X)}(S|0)$ and $m_3^{(X)}(S|0)$ are listed for decreasing values of S.

S	$t_1^{(X)}(S\|0)$	$t_2^{(X)}(S\|0)$	$m_2^{(X)}(S\|0)$	$t_3^{(X)}(S\|0)$	$m_3^{(X)}(S\|0)$
1.0	1.070138	1.929005	1.908659	5.105382	4.983768
0.9	8.608682×10^{-1}	1.247008	1.235157	2.651441	2.594469
0.8	6.755435×10^{-1}	7.670863×10^{-1}	7.605985×10^{-1}	1.278182	1.253714
0.7	5.136899×10^{-1}	4.430784×10^{-1}	4.397956×10^{-1}	5.606533×10^{-1}	5.512413×10^{-1}
0.6	3.748426×10^{-1}	2.356767×10^{-1}	2.341783×10^{-1}	2.173172×10^{-1}	2.141832×10^{-1}
0.5	2.585459×10^{-1}	1.120041×10^{-1}	1.114100×10^{-1}	7.114007×10^{-2}	7.028321×10^{-2}
0.4	1.643534×10^{-1}	4.521212×10^{-2}	4.502006×10^{-2}	1.823010×10^{-2}	1.805405×10^{-2}
0.3	9.182733×10^{-2}	1.409873×10^{-2}	1.405376×10^{-2}	3.171892×10^{-3}	3.148868×10^{-3}
0.2	4.053871×10^{-2}	2.744821×10^{-3}	2.738978×10^{-3}	2.722449×10^{-4}	2.709245×10^{-4}
0.1	1.006700×10^{-2}	1.690877×10^{-4}	1.689075×10^{-4}	4.159065×10^{-6}	4.148957×10^{-6}

We note that the goodness of the approximations improves as the boundary S approaches the reflecting zero-state.

4. Reflected Ornstein-Uhlenbeck process

Let $\{Y(t), t \geq t_0\}$, $t_0 \geq 0$, be a TNH-ROU process, having infinitesimal drift and infinitesimal variance

$$B_1(x,t) = \alpha(t)\,x + \beta(t), \qquad B_2(t) = \sigma^2(t), \qquad x \in \mathbb{R}, \tag{4.1}$$

with $\alpha(t) \in \mathbb{R}$, $\beta(t) \in \mathbb{R}$ and $\sigma(t) > 0$ continuous functions for all t. For the TNH-ROU process, the results of Section 2 hold by choosing $\zeta_1(x,t) = \alpha(t)\,x + \beta(t)$ and $\zeta_2(x,t) = \sigma^2(t)$. Note that when $\alpha(t) = 0$ for all t, the process $Y(t)$ identifies with the TNH-RW process $X(t)$ with infinitesimal moments (3.1). We denote by $r_Y(x,t|x_0,t_0)$ the transition pdf of $Y(t)$.

The reflected Ornstein-Uhlenbeck process arises as a diffusion approximation for population dynamics and for queueing systems (cf. Giorno et al. [1, 30], Ward and Glynn [31]). Furthermore, the membrane potential evolution in neuronal diffusion models can be described by focusing the attention on the Ornstein-Uhlenbeck process confined by a lower reflecting boundary (cf., for instance, Buonocore et al. [9]). The reflected Ornstein-Uhlenbeck process can be also applied to the regulated financial

Table 5. As in Table 4, for $\beta = 0.1$ and $\sigma = 1$

S	$t_1^{(X)}(S\|0)$	$t_2^{(X)}(S\|0)$	$m_2^{(X)}(S\|0)$	$t_3^{(X)}(S\|0)$	$m_3^{(X)}(S\|0)$
1.0	9.365377×10^{-1}	1.446255	1.461838	3.259422	3.340522
0.9	7.635106×10^{-1}	9.622582×10^{-1}	9.715807×10^{-1}	1.770458	1.810022
0.8	6.071894×10^{-1}	6.092237×10^{-1}	6.144650×10^{-1}	8.926616×10^{-1}	9.103559×10^{-1}
0.7	4.679118×10^{-1}	3.621786×10^{-1}	3.649024×10^{-1}	4.095229×10^{-1}	4.166108×10^{-1}
0.6	3.460218×10^{-1}	1.982750×10^{-1}	1.995519×10^{-1}	1.660226×10^{-1}	1.684803×10^{-1}
0.5	2.418709×10^{-1}	9.698262×10^{-2}	9.750256×10^{-2}	5.684286×10^{-2}	5.754260×10^{-2}
0.4	1.558173×10^{-1}	4.029243×10^{-2}	4.046506×10^{-2}	1.523487×10^{-2}	1.538458×10^{-2}
0.3	8.822668×10^{-2}	1.293173×10^{-2}	1.297324×10^{-2}	2.772401×10^{-3}	2.792791×10^{-3}
0.2	3.947196×10^{-2}	2.591186×10^{-3}	2.596726×10^{-3}	2.488771×10^{-4}	2.500947×10^{-4}
0.1	9.933665×10^{-3}	1.642874×10^{-4}	1.644628×10^{-4}	3.976568×10^{-6}	3.986274×10^{-6}

market (see, Linetsky [26], Nie and Linetsky [32]).

Recalling the Fokker-Planck equation (2.1), with the boundary condition (2.2), for $x_0 \geq 0$ one has the first two conditional moments of $Y(t)$:

$$E[Y(t)|Y(t_0) = x_0] = x_0\, e^{A(t|t_0)} + \int_{t_0}^{t} e^{A(t|u)}\beta(u)\, du + \frac{1}{2}\int_{t_0}^{t} e^{A(t|u)}\sigma^2(u)\, r_Y(0, u|x_0, t_0)\, du,$$

$$E[Y^2(t)|Y(t_0) = x_0] = x_0^2\, e^{2A(t|t_0)} + \int_{t_0}^{t} \sigma^2(u)\, e^{2A(t|u)}\, du + 2\int_{t_0}^{t} \beta(u)\, e^{2A(t|u)}\, E[Y(u)|Y(t_0) = x_0]\, du,$$

being

$$A(t|t_0) = \int_{t_0}^{t} \alpha(z)\, dz. \tag{4.2}$$

We now determine the transition pdf $r_Y(x, t|x_0, t_0)$ in a special case.

Proposition 4.1. *Let $Y(t)$ be a TNH-ROU process, having $\beta(t) = \gamma\, \sigma^2(t)\, e^{-A(t|0)}$, with $\gamma \in \mathbb{R}$, $\sigma(t) > 0$ and $A(t|t_0)$ given in (4.2). One has:*

$$r_Y(x, t|x_0, t_0) = f_Y(x, t|x_0, t_0) - \frac{\partial}{\partial x}\Big[\exp\Big\{2\gamma\, x\, e^{-A(t|0)}\Big\}\, F_Y(-x, t|x_0, t_0)\Big], \qquad x \geq 0, x_0 \geq 0, \tag{4.3}$$

where

$$\begin{aligned} f_Y(x, t|x_0, t_0) &= \frac{1}{\sqrt{2\pi V_Y(t|t_0)}}\, \exp\Big\{-\frac{[x - M_Y(t|x_0, t_0)]^2}{2\, V_Y(t|t_0)}\Big\}, \\ F_Y(x, t|x_0, t_0) &= \frac{1}{2}\Big[1 + \mathrm{Erf}\Big(\frac{x - M_Y(t|x_0, t_0)}{\sqrt{2\, V_Y(t|t_0)}}\Big)\Big] \end{aligned} \tag{4.4}$$

with

$$M_Y(t|x_0, t_0) = x_0\, e^{A(t|t_0)} + \int_{t_0}^{t} \beta(u)\, e^{A(t|u)}\, du, \qquad V_Y(t|t_0) = \int_{t_0}^{t} \sigma^2(u)\, e^{2A(t|u)}\, du. \tag{4.5}$$

Proof. Comparing (2.3) with (4.1), one has

$$\frac{h_2'(t)}{h_2(t)} = \alpha(t), \qquad h_1'(t)\, h_2(t) - h_1(t)\, h_2'(t) = \sigma^2(t), \qquad \frac{d\,\sigma^2(t)}{h_2(t)} = \beta(t)$$

for all $t \geq 0$, from which

$$h_2(t) = c\, e^{A(t|0)}, \quad h_1(t) = \frac{e^{A(t|0)}}{c} \int_0^t \sigma^2(u) e^{-2A(u|0)}\, du, \quad \beta(t) = \frac{d\,\sigma^2(t)}{c}\, e^{-A(t|0)},$$

with $c \neq 0$. Then, if $\beta(t) = \gamma\,\sigma^2(t)\, e^{-A(t|0)}$, Eq. (4.3) follows from (2.4) by setting $\gamma = d/c$. □

Under the assumptions of Proposition 4.1, making use of (4.4) in (4.3), for $x \geq 0, x_0 \geq 0$ we have:

$$r_Y(x,t|x_0,t_0) = \frac{1}{\sqrt{2\pi V_Y(t|t_0)}} \Big[\exp\Big\{-\frac{[x - H(t|x_0,t_0)]^2}{2\,V_Y(t|t_0)}\Big\} + \exp\big\{2\gamma\, x\, e^{-A(t|0)}\big\} \exp\Big\{-\frac{[x + H(t|x_0,t_0)]^2}{2\,V_Y(t|t_0)}\Big\}\Big] \\ -\gamma\, e^{-A(t|0)} \exp\big\{2\gamma\, x\, e^{-A(t|0)}\big\} \,\mathrm{Erfc}\Big(\frac{x + H(t|x_0,t_0)}{\sqrt{2\,V_Y(t|t_0)}}\Big), \tag{4.6}$$

where we have set:

$$H(t|x_0,t_0) = x_0\, e^{A(t|t_0)} + \gamma\, e^{-A(t|0)} V_Y(t|t_0). \tag{4.7}$$

Corollary 4.1. *Under the assumptions of Proposition 4.1, the following results hold:*

- *when $\beta(t) = 0$, one obtains*

$$E[Y(t)|Y(t_0) = x_0] = \sqrt{\frac{2V_Y(t|t_0)}{\pi}}\, \exp\Big\{-\frac{x_0^2\, e^{2A(t|t_0)}}{2V_Y(t|t_0)}\Big\} + x_0\, e^{A(t|t_0)}\, \mathrm{Erf}\Big(\frac{x_0\, e^{A(t|t_0)}}{\sqrt{2\,V_Y(t|t_0)}}\Big), \\ E[Y^2(t)|Y(t_0) = x_0] = x_0^2\, e^{2\,A(t|t_0)} + V_Y(t|t_0). \tag{4.8}$$

- *when $\beta(t) = \gamma\,\sigma^2(t)\, e^{-A(t|0)}$, with $\gamma \neq 0$, one has*

$$E[Y(t)|Y(t_0) = x_0] = \sqrt{\frac{V_Y(t|t_0)}{2\,\pi}}\, \exp\Big\{-\frac{H^2(t|x_0,t_0)}{2V_Y(t|t_0)}\Big\} + \frac{H(t|x_0,t_0)}{2}\Big[1 + \mathrm{Erf}\Big(\frac{H(t|x_0,t_0)}{\sqrt{2V_Y(t|t_0)}}\Big)\Big] \\ -\frac{e^{A(t|0)}}{4\gamma}\, \mathrm{Erfc}\Big(\frac{H(t|x_0,t_0)}{\sqrt{2V_Y(t|t_0)}}\Big) + \frac{e^{A(t|0)}}{4\gamma}\, \exp\big\{-2\gamma\, x_0\, e^{-A(t_0|0)}\big\}\, \mathrm{Erfc}\Big(\frac{H(t|x_0,t_0) - 2\gamma\, e^{-A(t|0)}\, V_Y(t|t_0)}{\sqrt{2V_Y(t|t_0)}}\Big),$$

$$E[Y^2(t)|Y(t_0) = x_0] = \frac{H^2(t|x_0,t_0) + V_Y(t|t_0)}{2}\Big[1 + \mathrm{Erf}\Big(\frac{H(t|x_0,t_0)}{\sqrt{2V_Y(t|t_0)}}\Big)\Big] + \frac{e^{2A(t|0)}}{4\gamma^2}\, \mathrm{Erfc}\Big(\frac{H(t|x_0,t_0)}{\sqrt{2V_Y(t|t_0)}}\Big) \\ + \sqrt{\frac{V_Y(t|t_0)}{2\pi}}\Big(H(t|x_0,t_0) + \frac{e^{A(t|0)}}{\gamma}\Big) \exp\Big\{-\frac{H^2(t|x_0,t_0)}{2V_Y(t|t_0)}\Big\} - \frac{e^{2A(t|0)}}{4\,\gamma^2}\, \exp\big\{-2\gamma\, x_0\, e^{-A(t_0|0)}\big\} \\ \times\big\{1 - 4\gamma^2\, e^{-2A(t|0)}\, V_Y(t|t_0) + 2\,\gamma\, e^{-A(t|0)}\, H(t|x_0,t_0)\big\}\, \mathrm{Erfc}\Big(\frac{H(t|x_0,t_0) - 2\gamma\, e^{-A(t|0)}\, V_Y(t|t_0)}{\sqrt{2V_Y(t|t_0)}}\Big), \tag{4.9}$$

with $H(t|x_0,t_0)$ given in (4.7).

Proof. It follows making use of (4.6) in (2.5) for $k = 1, 2$, respectively. □

We note that if $\alpha(t) = 0$ for all t, the conditional moments (4.8) and (4.9) identify with conditional moments (3.6) and (3.7) of the TNH-RW process with $\beta(t) = \gamma\,\sigma^2(t)$, being $H(t|x_0,t_0) = x_0 + \gamma\, V_X(t|t_0)$ in (4.7).

Corollary 4.2. *For the TNH-ROU process, having $B_1(x) = \alpha\, x$ and $B_2(t) = \sigma^2(t)$, with $\alpha \in \mathbb{R}$ and $\sigma(t) > 0$, the following asymptotic behaviors hold:*

1. *when $\alpha < 0$ and $\lim_{t\to+\infty} \sigma^2(t) = \sigma^2$ one has:*

$$W_Y(x) = \lim_{t\to+\infty} r_Y(x,t|x_0,t_0) = 2\sqrt{\frac{|\alpha|}{\pi\sigma^2}}\,\exp\Big\{-\frac{|\alpha|\,x^2}{\sigma^2}\Big\}, \qquad x \ge 0 \tag{4.10}$$

and the first two asymptotic moments are

$$E(Y) = \lim_{t\to+\infty} E[Y(t)|Y(t_0) = x_0] = \frac{\sigma}{\sqrt{\pi\,|\alpha|}}, \qquad E(Y^2) = \lim_{t\to+\infty} E[Y^2(t)|Y(t_0) = x_0] = \frac{\sigma^2}{2\,|\alpha|}\cdot$$

2. *when $\alpha < 0$ and $\sigma^2(t)$ is a positive periodic function of period Q, one obtains:*

$$\lim_{n\to+\infty} r_Y(x, t+nQ|x_0,t_0) = \sqrt{\frac{2}{\pi\,\omega_1(t)}}\,\exp\Big\{-\frac{x^2}{2\,\omega_1(t)}\Big\}, \qquad x \ge 0, \tag{4.11}$$

with

$$\omega_1(t) = \frac{V_Y(t+Q|t)}{1-e^{-2|\alpha|Q}}, \qquad t \ge 0,$$

and the first two asymptotic moments are

$$\lim_{n\to+\infty} E[Y(t+nQ)|Y(t_0) = x_0] = \sqrt{\frac{2\,\omega_1(t)}{\pi}}, \qquad \lim_{n\to+\infty} E[Y^2(t+nQ)|Y(t_0) = x_0] = \omega_1(t).$$

Proof. Eqs. (4.10) and (4.11) follow from (4.6) by setting $\alpha(t) = \alpha$ and $\gamma = 0$. When $\alpha < 0$, in the case 1. one has $\lim_{t\to+\infty} V_Y(t|t_0) = \sigma^2/(2\,|\alpha|)$, whereas in the case 2. it results $\lim_{n\to+\infty} V_Y(t+nQ|t_0) = \omega_1(t)$. □

4.1. Time-homogeneous reflected Ornstein-Uhlenbeck process

For the TH-ROU process $Y(t)$, in (4.1) we set $\alpha(t) = \alpha, \beta(t) = \beta, \sigma^2(t) = \sigma^2$, with $\alpha \neq 0, \beta \in \mathbb{R}$ and $\sigma > 0$. The scale function and the speed density, defined in (2.6), are:

$$h_Y(x) = \exp\Big\{-\frac{\alpha}{\sigma^2}\Big(x^2 + \frac{2\beta}{\alpha}\,x\Big)\Big\}, \quad s_Y(x) = \frac{2}{\sigma^2}\,\exp\Big\{\frac{\alpha}{\sigma^2}\Big(x^2 + \frac{2\beta}{\alpha}\,x\Big)\Big\}, \tag{4.12}$$

respectively. When $\alpha > 0$ ($\alpha < 0$) the end-point $+\infty$ is an attracting (nonattracting) natural boundary.

Proposition 4.2. *Let $r_\lambda^{(Y)}(x|x_0) = \int_0^{+\infty} e^{-\lambda t} r_Y(x,t|x_0)\,dt$ ($\mathrm{Re}\,\lambda > 0$) be the LT of the transition pdf of the*

TH-ROU process $Y(t)$. One has

$$r_\lambda^{(Y)}(x|x_0) = \begin{cases} \frac{2^{\frac{\lambda}{|\alpha|}-1}}{\sigma\pi\sqrt{|\alpha|}}\Gamma\Big(\frac{\lambda}{2|\alpha|}\Big)\Gamma\Big(\frac{1}{2}+\frac{\lambda}{2|\alpha|}\Big)\exp\Big\{-\frac{|\alpha|}{2\sigma^2}\Big[\Big(x+\frac{\beta}{\alpha}\Big)^2-\Big(x_0+\frac{\beta}{\alpha}\Big)^2\Big]\Big\} \\ \quad\times D_{-\frac{\lambda}{|\alpha|}}\Big(\frac{\sqrt{2|\alpha|}}{\sigma}\Big[x_0\vee x+\frac{\beta}{\alpha}\Big]\Big)\Big[D_{-\frac{\lambda}{|\alpha|}}\Big(-\frac{\sqrt{2|\alpha|}}{\sigma}\Big[x_0\wedge x+\frac{\beta}{\alpha}\Big]\Big) \\ \quad+\frac{D_{-\frac{\lambda}{|\alpha|}-1}\Big(-\frac{\sqrt{2|\alpha|}}{\sigma}\frac{\beta}{\alpha}\Big)}{D_{-\frac{\lambda}{|\alpha|}-1}\Big(\frac{\sqrt{2|\alpha|}}{\sigma}\frac{\beta}{\alpha}\Big)}D_{-\frac{\lambda}{|\alpha|}}\Big(\frac{\sqrt{2|\alpha|}}{\sigma}\Big[x_0\wedge x+\frac{\beta}{\alpha}\Big]\Big)\Big], & \alpha<0, \\ \frac{2^{\frac{\lambda}{\alpha}}}{\sigma\pi\sqrt{\alpha}}\Gamma\Big(1+\frac{\lambda}{2\alpha}\Big)\Gamma\Big(\frac{1}{2}+\frac{\lambda}{2\alpha}\Big)\exp\Big\{-\frac{\alpha}{2\sigma^2}\Big[\Big(x_0+\frac{\beta}{\alpha}\Big)^2-\Big(x+\frac{\beta}{\alpha}\Big)^2\Big]\Big\} \\ \quad\times D_{-\frac{\lambda}{\alpha}-1}\Big(\frac{\sqrt{2\alpha}}{\sigma}\Big[x_0\vee x+\frac{\beta}{\alpha}\Big]\Big)\Big[D_{-\frac{\lambda}{\alpha}-1}\Big(-\frac{\sqrt{2\alpha}}{\sigma}\Big[x_0\wedge x+\frac{\beta}{\alpha}\Big]\Big) \\ \quad+\frac{D_{-\frac{\lambda}{\alpha}}\Big(-\frac{\sqrt{2\alpha}}{\sigma}\frac{\beta}{\alpha}\Big)}{D_{-\frac{\lambda}{\alpha}}\Big(\frac{\sqrt{2\alpha}}{\sigma}\frac{\beta}{\alpha}\Big)}D_{-\frac{\lambda}{\alpha}-1}\Big(\frac{\sqrt{2\alpha}}{\sigma}\Big[x_0\wedge x+\frac{\beta}{\alpha}\Big]\Big)\Big], & \alpha>0, \end{cases} \tag{4.13}$$

where $x_0 \vee x = \max(x_0, x)$, $x_0 \wedge x = \min(x_0, x)$, $D_\nu(z)$ is the parabolic cylinder function defined as

$$D_\nu(z) = 2^{\nu/2}e^{-z^2/4}\Big\{\frac{\sqrt{\pi}}{\Gamma\Big(\frac{1-\nu}{2}\Big)}\Phi\Big(-\frac{\nu}{2},\frac{1}{2};\frac{z^2}{2}\Big)-\frac{z\sqrt{2\pi}}{\Gamma\Big(-\frac{\nu}{2}\Big)}\Phi\Big(\frac{1-\nu}{2},\frac{3}{2};\frac{z^2}{2}\Big)\Big\}$$

and

$$\Phi(a,c;x) = 1+\sum_{n=1}^{+\infty}\frac{(a)_n}{(c)_n}\frac{x^n}{n!},$$

is the Kummer's confluent hypergeometric function, with $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n = 1, 2, \dots$

Proof. The proof is given in A. □

Eq. (4.13) can be used to analyze the asymptotic behavior of the TH-ROU process.

Corollary 4.3. *For $\alpha < 0$, the TH-ROU process $Y(t)$ admits an asymptotic behavior. The steady-state density is*

$$W_Y(x) = \lim_{t\to+\infty} r_Y(x,t|x_0) = \frac{2}{\sigma}\sqrt{\frac{|\alpha|}{\pi}}\exp\Big\{-\frac{|\alpha|}{\sigma^2}\Big(x+\frac{\beta}{\alpha}\Big)^2\Big\}\frac{1}{\mathrm{Erfc}\Big(\frac{\sqrt{|\alpha|}}{\sigma}\frac{\beta}{\alpha}\Big)}, \qquad x\geq 0, \tag{4.14}$$

and the asymptotic moments of the first and second order are:

$$\begin{aligned} E(Y) &= \frac{\beta}{|\alpha|}+\frac{\sigma}{\sqrt{\pi|\alpha|}}\exp\Big\{-\frac{|\alpha|}{\sigma^2}\Big(\frac{\beta}{\alpha}\Big)^2\Big\}\frac{1}{\mathrm{Erfc}\Big(\frac{\sqrt{|\alpha|}}{\sigma}\frac{\beta}{\alpha}\Big)}, \\ E(Y^2) &= \Big(\frac{\beta}{\alpha}\Big)^2+\frac{\sigma^2}{2|\alpha|}+\frac{\beta}{|\alpha|}\frac{\sigma}{\sqrt{\pi|\alpha|}}\exp\Big\{-\frac{|\alpha|}{\sigma^2}\Big(\frac{\beta}{\alpha}\Big)^2\Big\}\frac{1}{\mathrm{Erfc}\Big(\frac{\sqrt{|\alpha|}}{\sigma}\frac{\beta}{\alpha}\Big)}. \end{aligned} \tag{4.15}$$

Proof. If $\alpha < 0$, from (4.13) one has:

$$W_Y(x) = \lim_{\lambda \downarrow 0} \lambda\, r_\lambda^{(Y)}(x|x_0) = \frac{1}{\sigma}\sqrt{\frac{|\alpha|}{\pi}} \exp\Big\{-\frac{|\alpha|}{2\sigma^2}\Big[\Big(x+\frac{\beta}{\alpha}\Big)^2 - \Big(x_0+\frac{\beta}{\alpha}\Big)^2\Big]\Big\} D_0\Big(\frac{\sqrt{2|\alpha|}}{\sigma}\Big[x_0 \vee x + \frac{\beta}{\alpha}\Big]\Big)$$
$$\times\Big[D_0\Big(-\frac{\sqrt{2|\alpha|}}{\sigma}\Big[x_0 \wedge x + \frac{\beta}{\alpha}\Big]\Big) + \frac{D_{-1}\Big(-\frac{\sqrt{2|\alpha|}}{\sigma}\frac{\beta}{\alpha}\Big)}{D_{-1}\Big(\frac{\sqrt{2|\alpha|}}{\sigma}\frac{\beta}{\alpha}\Big)} D_0\Big(\frac{\sqrt{2|\alpha|}}{\sigma}\Big[x_0 \wedge x + \frac{\beta}{\alpha}\Big]\Big)\Big], \qquad x \geq 0. \qquad (4.16)$$

Since (cf. Gradshteyn and Ryzhik [33], p. 1030, no. 9.251 and no. 9.254)

$$D_0(x) = e^{-x^2/4}, \quad D_{-1}(x) = \sqrt{\frac{\pi}{2}}\, e^{x^2/4}\, \mathrm{Erfc}\Big(\frac{x}{\sqrt{2}}\Big), \qquad (4.17)$$

from (4.16) one obtains (4.14). The asymptotic moments (4.15) follow from (4.14). □

Eq. (4.14) is in agreement with the result of the Proposition 1 in Ward and Glinn [31]; moreover, for $\beta = 0$ Eq. (4.14) identifies with (4.10).

The inverse LT of (4.13) is obtainable only for $\beta = 0$; in this case, the following result holds.

Proposition 4.3. *For the TH-ROU process $Y(t)$, having $B_1(x) = \alpha x$ and $B_2 = \sigma^2$, with $\alpha \neq 0$ and $\sigma > 0$, one has:*

$$r_Y(x,t|x_0) = \sqrt{\frac{\alpha}{\pi\,\sigma^2(e^{2\alpha t}-1)}}\Big[\exp\Big\{-\frac{\alpha\,(x - x_0\, e^{\alpha t})^2}{\sigma^2(e^{2\alpha t}-1)}\Big\} + \exp\Big\{-\frac{\alpha\,(x + x_0\, e^{\alpha t})^2}{\sigma^2(e^{2\alpha t}-1)}\Big\}\Big], \qquad x \geq 0, x_0 \geq 0. \qquad (4.18)$$

Proof. It follows from (4.6) with $\gamma = 0$, by setting $t_0 = 0$, $A(t|0) = e^{\alpha t}$ and $V_Y(t|0) = \sigma^2(e^{2\alpha t} - 1)/\alpha$. Alternatively, Eq. (4.18) can be obtained from (4.13) and (A.4) with $\beta = 0$, by noting that $r_\lambda^{(Y)}(x|x_0) = f_\lambda^{(Y)}(x|x_0) + f_\lambda^{(Y)}(x| - x_0)$. □

In Figure 3, we consider a TH-ROU process $Y(t)$ in $[0, +\infty)$, having $B_1(x) = \alpha x + \beta$ and $B_2 = \sigma^2$, with $\alpha = -0.5, \beta = 0, \sigma = 1, x_0 = 5$ and $t_0 = 0$. Using the Algorithm 4.1 in Buonocore et al. [29], we obtain a random sample of $N = 5 \cdot 10^4$ observations of $Y(t)$. Then, we compare the histogram of the random sample with the transition pdf (4.18) as function of x ($x \geq 0$) for $t = 2$ on the left and $t = 4$ on the right. Figure 3 shows the good agreement between the histograms obtained via simulation and the exact density (4.18).

Under the assumption of Proposition 4.3, the first two conditional moments of the TH-ROU process can be obtained from (4.8).

4.2. FPT for TH-ROU process

For the TH-ROU process, if $0 \leq S < x_0$ the FPT through S starting from x_0 is not affected to reflecting boundary in the zero-state; in this case, for $\alpha < 0$ the ultimate FPT probability $P_Y(S|x_0) = 1$ and from (2.7) the FPT mean is

$$t_1^{(Y)}(S|x_0) = \frac{1}{|\alpha|}\Big\{\psi_1\Big(\frac{\sqrt{|\alpha|}}{\sigma}\Big(S + \frac{\beta}{\alpha}\Big)\Big) - \psi_1\Big(\frac{\sqrt{|\alpha|}}{\sigma}\Big(x_0 + \frac{\beta}{\alpha}\Big)\Big)$$

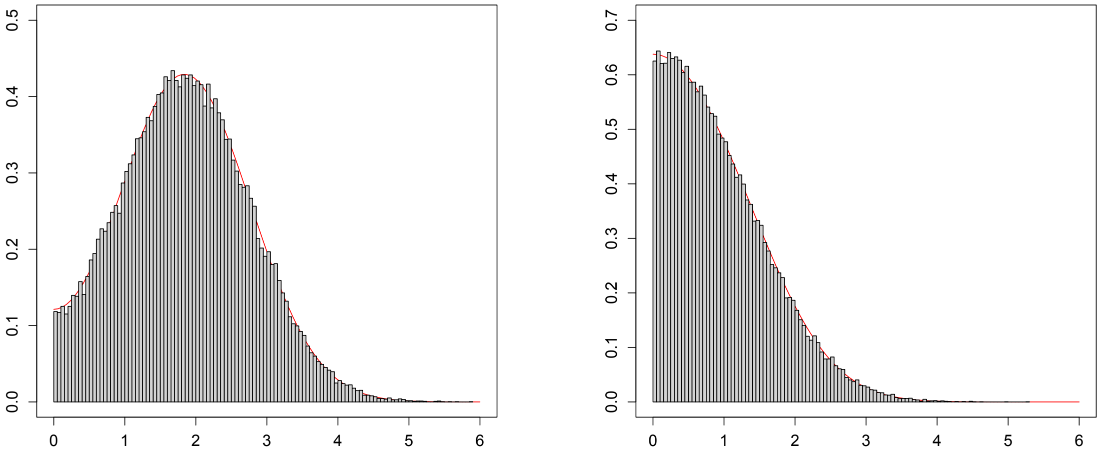

Figure 3. For the TH-ROU process $Y(t)$, with $\alpha = -0.5, \beta = 0, \sigma = 1, x_0 = 5$, the transition pdf (4.18) is plotted (red curve) as function of x, for $t = 2$ (on the left) and $t = 4$ (on the right) and the histogram, obtained via the simulation of the sample paths, is superimposed over the density.

$$+\frac{\pi}{2}\left[\mathrm{Erfi}\Big(\frac{\sqrt{|\alpha|}}{\sigma}\Big(x_0+\frac{\beta}{\alpha}\Big)\Big)-\mathrm{Erfi}\Big(\frac{\sqrt{|\alpha|}}{\sigma}\Big(S+\frac{\beta}{\alpha}\Big)\Big)\right]\Bigg\}, \quad 0 \le S < x_0, \tag{4.19}$$

where

$$\psi_1(x) = \sum_{k=0}^{+\infty} \frac{2^k\, x^{2k+2}}{(k+1)\,(2k+1)!!}, \quad \mathrm{Erfi}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{z^2}\, dz = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{+\infty}\frac{x^{2k+1}}{(2k+1)\,k!}.$$

Moreover, if $0 \le x_0 < S$ the probability $P_Y(S\,|x_0) = 1$ and making use of (2.8) one obtains:

$$t_1^{(Y)}(S\,|x_0) = \begin{cases} \frac{1}{|\alpha|}\Big\{\psi_1\Big(\frac{\sqrt{|\alpha|}}{\sigma}\Big(S+\frac{\beta}{\alpha}\Big)\Big)-\psi_1\Big(\frac{\sqrt{|\alpha|}}{\sigma}\Big(x_0+\frac{\beta}{\alpha}\Big)\Big)-\frac{\pi}{2}\,\mathrm{Erf}\Big(\frac{\beta}{\sigma\sqrt{|\alpha|}}\Big) \\ \quad\times\Big[\mathrm{Erfi}\Big(\frac{\sqrt{|\alpha|}}{\sigma}\Big(S+\frac{\beta}{\alpha}\Big)\Big)-\mathrm{Erfi}\Big(\frac{\sqrt{|\alpha|}}{\sigma}\Big(x_0+\frac{\beta}{\alpha}\Big)\Big)\Big]\Big\}, & \alpha < 0, \\ \\ \frac{1}{\alpha}\Big\{\psi_2\Big(\frac{\sqrt{\alpha}}{\sigma}\Big(S+\frac{\beta}{\alpha}\Big)\Big)-\psi_2\Big(\frac{\sqrt{\alpha}}{\sigma}\Big(x_0+\frac{\beta}{\alpha}\Big)\Big)-\frac{\pi}{2}\,\mathrm{Erfi}\Big(\frac{\beta}{\sigma\sqrt{\alpha}}\Big) \\ \quad\times\Big[\mathrm{Erf}\Big(\frac{\sqrt{\alpha}}{\sigma}\Big(S+\frac{\beta}{\alpha}\Big)\Big)-\mathrm{Erf}\Big(\frac{\sqrt{\alpha}}{\sigma}\Big(x_0+\frac{\beta}{\alpha}\Big)\Big)\Big]\Big\}, & \alpha > 0, \end{cases} \tag{4.20}$$

where

$$\psi_2(z) = \sum_{k=0}^{+\infty}\frac{(-1)^k\, 2^k\, z^{2k+2}}{(k+1)\,(2k+1)!!}.$$

When $0 \le x_0 < S$, from (4.20) one has:

- $t_1^{(Y)}(S\,|x_0)$ decreases as β increases and

$$\lim_{\beta\to-\infty} t_1^{(Y)}(S\,|x_0) = +\infty, \qquad \lim_{\beta\to+\infty} t_1^{(X)}(S\,|x_0) = 0;$$

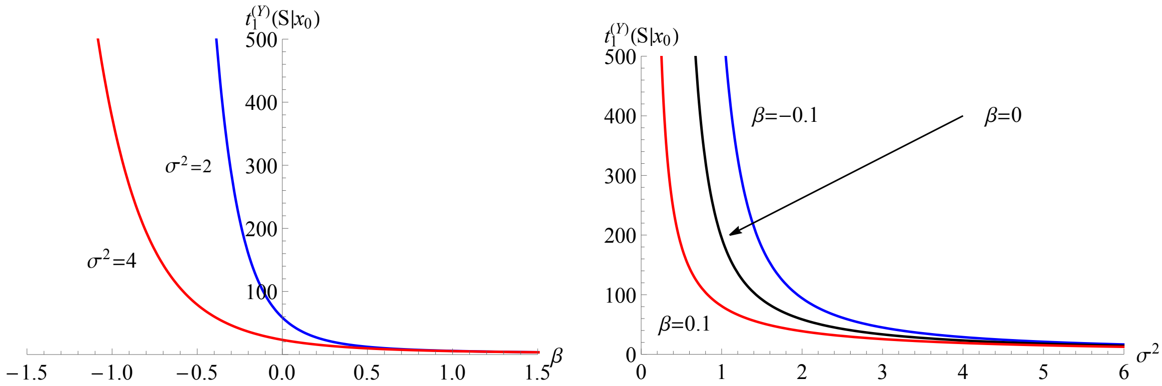

Figure 4. The FPT mean (4.20) is plotted for $\alpha = -0.02$, $x_0 = 5$ and $S = 10$ as function of β on the left and as function of σ^2 on the right.

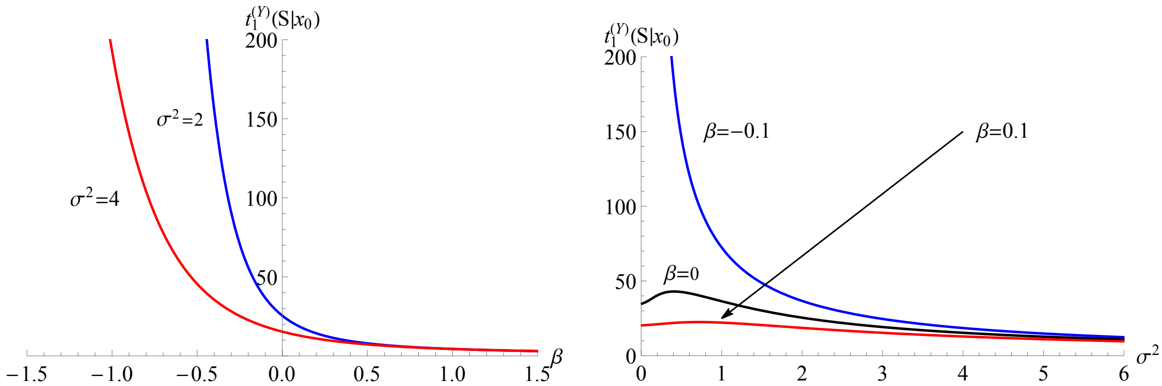

Figure 5. As in Figure 4 with $\alpha = 0.02$.

- for $\alpha < 0$, $t_1^{(Y)}(S|x_0)$ decreases as σ^2 increases; moreover, one has:

$$\lim_{\sigma^2 \to 0} t_1^{(Y)}(S|x_0) = \begin{cases} +\infty, & [\alpha < 0, \beta \in \mathbb{R}] \text{ or } [\alpha > 0, \beta < 0], \\ \frac{1}{\alpha} \ln\left(\frac{\alpha S + \beta}{\alpha x_0 + \beta}\right), & [\alpha > 0, \beta \geq 0], \end{cases} \qquad \lim_{\sigma^2 \to +\infty} t_1^{(Y)}(S|x_0) = 0.$$

In Figures 4 and 5, the FPT mean (4.20) of the TH-ROU process is plotted for $x_0 = 5$, $S = 10$ and for different choices of β and σ^2, with $\alpha = -0.02, 0.02$, respectively.

Making use of (4.12) in Remark 2.1, one can derive some asymptotic behaviors for the FPT moments of the TH-ROU process.

Remark 4.1. *For the TH-ROU process $Y(t)$, when $n = 1, 2, \dots$ one has:*

1. $\displaystyle \lim_{S \uparrow +\infty} \frac{t_n^{(Y)}(S|0)}{[t_1^{(Y)}(S|0)]^n} = n!, \qquad \alpha < 0,$
2. $\displaystyle \lim_{S \downarrow 0} \frac{t_n^{(Y)}(S|0)}{[t_1^{(Y)}(S|0)]^n} = \frac{(-1)^n E_{2n}}{(2n-1)!!},$

   where $E_0, E_1, \dots$ are the Euler numbers.

Then, from Remark 4.1, it follows:

Table 6. For the TH-ROU process, with $\alpha = -0.02$, $\beta = -0.1$ and $\sigma = 1$, $t_1^{(Y)}(S|0)$, $\mathrm{Var}^{(Y)}(S|0)$, $\mathrm{Cv}^{(Y)}(S|0)$ and $\Sigma^{(Y)}(S|0)$ are listed for increasing values of $S > 0$.

S	$t_1^{(Y)}(S\|0)$	$\mathrm{Var}^{(Y)}(S\|0)$	$\mathrm{Cv}^{(Y)}(S\|0)$	$\Sigma^{(Y)}(S\|0)$
10.	635.964	380988.	0.970562	1.99926
20.	1.11639×10^{6}	1.24622×10^{12}	0.999956	1.99998
30.	1.26765×10^{11}	1.60694×10^{22}	1.0	1.99997
40.	8.68434×10^{17}	7.54177×10^{35}	1.0	1.99991
50.	3.43245×10^{26}	1.17817×10^{53}	1.0	1.99998
60.	7.67474×10^{36}	5.89016×10^{73}	1.0	1.99996
70.	9.60516×10^{48}	9.22590×10^{97}	1.0	1.99992
80.	6.68550×10^{62}	4.46959×10^{125}	1.0	1.99984
90.	2.57707×10^{78}	6.64129×10^{156}	1.0	2.00008
100.	5.48553×10^{95}	3.00911×10^{191}	1.0	1.99998

Table 7. As in Table 6 for $\alpha = -0.02, \beta = 0.1$ and $\sigma = 1$.

S	$t_1^{(Y)}(S\|0)$	$\mathrm{Var}^{(Y)}(S\|0)$	$\mathrm{Cv}^{(Y)}(S\|0)$	$\Sigma^{(Y)}(S\|0)$
10.	102.243	7056.41	0.821592	1.96257
20.	3664.17	1.29250×10^{7}	0.981159	1.99973
30.	5.92059×10^{6}	3.50521×10^{13}	0.999982	1.99990
40.	6.72233×10^{11}	4.51897×10^{23}	1.0	1.99990
50.	4.60528×10^{18}	2.12086×10^{37}	1.0	1.99993
60.	1.82022×10^{27}	3.31320×10^{54}	1.0	2.00001
70.	4.06990×10^{37}	1.65640×10^{75}	1.0	2.00009
80.	5.09359×10^{49}	2.59446×10^{99}	1.0	1.99994
90.	3.54531×10^{63}	1.25692×10^{127}	1.0	1.99995
100.	1.36661×10^{79}	1.86763×10^{158}	1.0	1.99993

- for $S \uparrow +\infty$ one has:
$$t_n^{(Y)}(S|0) \simeq n!\,[t_1^{(Y)}(S|0)]^n, \qquad \alpha < 0, \tag{4.21}$$
so that for $\alpha < 0$ the FPT density $g_Y(S, t|0)$ of the TH-ROU process exhibits an exponential trend for large boundary;
- for $S \downarrow 0$ it results:
$$t_n^{(Y)}(S|0) \simeq \frac{(-1)^n E_{2n}}{(2n-1)!!}\,[t_1^{(Y)}(S|0)]^n = m_n^{(Y)}(S|0). \tag{4.22}$$
In particular, one has:
$$m_2^{(Y)}(S|0) = \frac{5}{3}\,[t_1^{(Y)}(S|0)]^2, \qquad m_3^{(Y)}(S|0) = \frac{61}{15}\,[t_1^{(Y)}(S|0)]^3.$$

In Tables 6–9, $t_1^{(Y)}(S|0)$ is computed via (4.20), whereas $t_2^{(Y)}(S|0)$ and $t_3^{(Y)}(S|0)$ are numerically evaluated making use of (2.8) and (4.12). In particular, in Tables 6 and 7, the mean $t_1^{(Y)}(S|0)$, the variance $\mathrm{Var}^{(Y)}(S|0)$, the coefficient of variation $\mathrm{Cv}^{(Y)}(S|0)$ and the skewness $\Sigma^{(Y)}(S|0)$ of the FPT are listed for $\alpha = -0.02, \beta = -0.1, 0.1, \sigma = 1$ and some choices of $S > 0$. Being $\alpha < 0$, the coefficient of variation and the skewness approach to 1 and to 2, respectively, as S increases.

Moreover, in Tables 8 and 9, for the TH-ROU process with $\alpha = -0.02, \beta = -0.1, 0.1$ and $\sigma = 1$ we compare the FPT moments $t_2^{(Y)}(S|0)$ and $t_3^{(Y)}(S|0)$ with the approximate values $m_2^{(Y)}(S|0)$ and $m_3^{(Y)}(S|0)$, respectively. From Tables 8 and 9, we note that the goodness of the approximations improves as the boundary S approaches the reflecting zero-state.

Table 8. For the TH-ROU with $\alpha = -0.02$, $\beta = -0.1$ and $\sigma = 1$, $t_1^{(Y)}(S|0)$, $t_2^{(Y)}(S|0)$ and $t_3^{(Y)}(S|0)$ and their approximate values $m_2^{(Y)}(S|0)$ and $m_3^{(Y)}(S|0)$ are listed for decreasing values of S.

S	$t_1^{(Y)}(S\|0)$	$t_2^{(Y)}(S\|0)$	$m_2^{(Y)}(S\|0)$	$t_3^{(Y)}(S\|0)$	$m_3^{(Y)}(S\|0)$
1.0	1.077553	1.957892	1.935202	5.224688	5.088091
0.9	8.656790×10^{-1}	1.262062	1.249	2.701365	2.63821
0.8	6.785138×10^{-1}	7.743699×10^{-1}	7.673017×10^{-1}	1.297103	1.270324
0.7	5.154121×10^{-1}	4.462854×10^{-1}	4.427493×10^{-1}	5.669774×10^{-1}	5.568040×10^{-1}
0.6	3.757621×10^{-1}	2.369247×10^{-1}	2.353286×10^{-1}	2.191101×10^{-1}	2.157633×10^{-1}
0.5	2.589846×10^{-1}	1.124143×10^{-1}	1.117884×10^{-1}	7.154595×10^{-2}	7.064161×10^{-2}
0.4	1.645312×10^{-1}	4.531770×10^{-2}	4.511752×10^{-2}	1.829640×10^{-2}	1.811271×10^{-2}
0.3	9.188300×10^{-2}	1.411719×10^{-2}	1.407081×10^{-2}	3.178358×10^{-3}	3.154599×10^{-3}
0.2	4.054959×10^{-2}	2.746412×10^{-3}	2.740449×10^{-3}	2.724907×10^{-4}	2.711428×10^{-4}
0.1	1.006767×10^{-2}	1.691121×10^{-4}	1.689301×10^{-4}	4.160001×10^{-6}	4.149790×10^{-6}

Table 9. As in Table 8 for $\alpha = -0.02, \beta = 0.1$ and $\sigma = 1$.

S	$t_1^{(Y)}(S\|0)$	$t_2^{(Y)}(S\|0)$	$m_2^{(Y)}(S\|0)$	$t_3^{(Y)}(S\|0)$	$m_3^{(Y)}(S\|0)$
1.0	9.426077×10^{-1}	1.466641	1.480849	3.331454	3.405897
0.9	7.675284×10^{-1}	9.732598×10^{-1}	9.818331×10^{-1}	1.802163	1.838747
0.8	6.097203×10^{-1}	6.147356×10^{-1}	6.195981×10^{-1}	9.053004×10^{-1}	9.217869×10^{-1}
0.7	4.694088×10^{-1}	3.646917×10^{-1}	3.672411×10^{-1}	4.139661×10^{-1}	4.206223×10^{-1}
0.6	3.468374×10^{-1}	1.992877×10^{-1}	2.004936×10^{-1}	1.673475×10^{-1}	1.696744×10^{-1}
0.5	2.422679×10^{-1}	9.732731×10^{-2}	9.782287×10^{-2}	5.715830×10^{-2}	5.782639×10^{-2}
0.4	1.559815×10^{-1}	4.038428×10^{-2}	4.055036×10^{-2}	1.528907×10^{-2}	1.543325×10^{-2}
0.3	8.827911×10^{-2}	1.294836×10^{-2}	1.298867×10^{-2}	2.777959×10^{-3}	2.797773×10^{-3}
0.2	3.948242×10^{-2}	2.592670×10^{-3}	2.598102×10^{-3}	2.490994×10^{-4}	2.502936×10^{-4}
0.1	9.934325×10^{-3}	1.643110×10^{-4}	1.644847×10^{-4}	3.977457×10^{-6}	3.987069×10^{-6}

5. Feller process with a zero-flux condition in the zero-state

Let $\{Z(t), t \geq t_0\}$, $t_0 \geq 0$, be a TNH-RF process, having the following infinitesimal drift and infinitesimal variance

$$C_1(x,t) = \alpha(t)\,x + \beta(t), \qquad C_2(x,t) = 2\,r(t)\,x, \tag{5.1}$$

with $\alpha(t) \in \mathbb{R}$, $\beta(t) > 0$ and $r(t) > 0$ continuous functions for all t, defined in the state-space $[0, +\infty)$ with a zero-flux condition in the zero-state. For the TNH-RF process, the results of Section 2 hold by choosing $\zeta_1(x,t) = \alpha(t)\,x + \beta(t)$ and $\zeta_2(x,t) = 2\,r(t)\,x$. We denote with $r_Z(x,t|x_0,t_0)$ the transition pdf of $Z(t)$.

Feller diffusion process is applied in population dynamics to model the growth of a population (cf. Ricciardi et al. [15], Giorno and Nobile [34]). This process is also used in queueing systems to describe the number of customers in a queue (cf. Di Crescenzo and Nobile [35]), in neurobiology to analyze the input-output behavior of single neurons (see, for instance, Giorno et al. [36], Ditlevsen and Lánský [37]), in mathematical finance to model asset prices, market indices, interest rates and stochastic volatility (see, Tian and Zhang [38], Cox et al. [39], Di Nardo and D'Onofrio [40]). We emphasize that in the mathematical finance, the Feller process is also known as Cox-Ingersoll-Ross (CIR) model.

Making use of Eq. (2.1), with the zero-flux condition in the zero-state (2.2), for $x_0 \geq 0$ one obtains the conditional mean and the conditional variance of $Z(t)$:

$$\begin{aligned} &\mathrm{E}[Z(t)|Z(t_0)=x_0] = x_0\, e^{A(t|t_0)} + \int_{t_0}^{t} \beta(u)\, e^{A(t|u)}\, du, \\ &\mathrm{Var}[Z(t)|Z(t_0)=x_0] = 2\, x_0\, e^{2\,A(t|t_0)}\, R(t|t_0) + 2\, e^{A(t|t_0)} \int_{t_0}^{t} \beta(u)\, e^{A(t|u)} [R(t|t_0) - R(u|t_0)]\, du, \end{aligned} \tag{5.2}$$

with $A(t|t_0)$ defined in (4.2) and

$$R(t|t_0) = \int_{t_0}^{t} r(\tau)\, e^{-A(\tau|t_0)}\, d\tau, \qquad t \geq t_0. \tag{5.3}$$

We note that when $\alpha(t) \neq 0$, the conditional mean $M_Y(t|x_0, t_0)$ of the unrestricted Ornstein-Uhlenbeck process, given in (4.5), identifies with the conditional average of the Feller process $Z(t)$, given in (5.2). Similarly, when $\alpha(t) = 0$ for all t, the conditional average of the Feller process is equal to the conditional mean $M_X(t|x_0, t_0)$ of the unrestricted Wiener process, given in (3.4).

In the following proposition, we consider the transition pdf $r_Z(x, t|x_0, t_0)$ in a special case.

Proposition 5.1. *Let $Z(t)$ be a TNH-RF process, with $\alpha(t) \in \mathbb{R}$, $r(t) > 0$ and $\beta(t) = \xi\, r(t)$, with $\xi > 0$, in (5.1). The following results hold:*

$$r_Z(x, t|x_0, t_0) = \begin{cases} \frac{1}{x\,\Gamma(\xi)} \Big[\frac{x\, e^{-A(t|t_0)}}{R(t|t_0)}\Big]^{\xi} \exp\Big\{-\frac{x\, e^{-A(t|t_0)}}{R(t|t_0)}\Big\}, & x_0 = 0, \\ \frac{e^{-A(t|t_0)}}{R(t|t_0)} \Big[\frac{x\, e^{-A(t|t_0)}}{x_0}\Big]^{(\xi-1)/2} \exp\Big\{-\frac{x_0 + x\, e^{-A(t|t_0)}}{R(t|t_0)}\Big\}\, I_{\xi-1}\Big[\frac{2\sqrt{x\, x_0\, e^{-A(t|t_0)}}}{R(t|t_0)}\Big], & x_0 > 0, \end{cases} \tag{5.4}$$

where

$$I_\nu(z) = \sum_{k=0}^{+\infty} \frac{1}{k!\,\Gamma(\nu + k + 1)} \Big(\frac{z}{2}\Big)^{2k+\nu},$$

is the modified Bessel function of the first kind, with $\Gamma(\nu)$ denoting the Euler gamma function. Moreover, one has

$$\begin{aligned} &E[Z(t)|Z(t_0) = x_0] = e^{A(t|t_0)} \Big[x_0 + \xi\, R(t|t_0)\Big], \\ &E[Z^2(t)|Z(t_0) = x_0] = e^{2A(t|t_0)} \Big[x_0^2 + 2(\xi + 1) R(t|t_0) x_0 + \xi(\xi + 1) R^2(t|t_0)\Big], \end{aligned} \tag{5.5}$$

where $A(t|t_0)$ and $R(t|t_0)$ are defined in (4.2) and (5.3), respectively.

Proof. Eq. (5.4) follows as in Giorno and Nobile [34] and Masoliver [41]. Moreover, relations (5.5) are obtainable from (5.2). □

Since, for fixed ν, when $z \to 0$ one has

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu + 1)} \Big(\frac{z}{2}\Big)^{\nu}, \qquad \nu \neq -1, -2, \ldots, \tag{5.6}$$

the first formula in (5.4) follows from the second expression as $x_0 \downarrow 0$.

Corollary 5.1. *The TNH-RF process $Z(t)$, having $C_1(x,t) = \alpha\, x + \xi\, r(t)$ and $C_2(x,t) = 2\, r(t)\, x$, with $\alpha \in \mathbb{R}$, $\xi > 0$ and $r(t) > 0$, admits the following asymptotic behaviors:*

1. *when $\alpha < 0$ and $\lim_{t\to+\infty} r(t) = r$, with $r > 0$, one has:*

$$W_Z(x) = \lim_{t\to+\infty} r_Z(x,t|x_0,t_0) = \frac{1}{x\,\Gamma(\xi)}\Big(\frac{|\alpha|\,x}{r}\Big)^{\xi} \exp\Big\{-\frac{|\alpha|\,x}{r}\Big\}, \quad x > 0, \tag{5.7}$$

and the asymptotic moments are

$$E(Z^k) = \lim_{t\to+\infty} E[Z^k(t)|Z(t_0) = x_0] = \Big(\frac{r}{|\alpha|}\Big)^k \frac{\Gamma(k+\xi)}{\Gamma(\xi)}, \quad k = 1, 2, \dots$$

2. *when $\alpha < 0$ and $r(t)$ is a positive periodic function of period Q, one obtains:*

$$\lim_{n\to+\infty} r_Z(x, t+nQ|x_0,t_0) = \frac{1}{x\,\Gamma(\xi)}\Big[\omega_2(t)\,x\Big]^{\xi} \exp\Big\{-\omega_2(t)\,x\Big\}, \quad x > 0, \tag{5.8}$$

with

$$\omega_2(t) = \frac{e^{|\alpha|\,Q} - 1}{R(t+Q|t)}, \qquad t \geq 0,$$

and the asymptotic moments are

$$\lim_{n\to+\infty} E[Z^k(t+nQ)|Z(t_0) = x_0] = [\omega_2(t)]^{-k}\,\frac{\Gamma(k+\xi)}{\Gamma(\xi)}, \quad k = 1, 2, \dots$$

Proof. Eq. (5.7) follows from (5.4) making use of (5.6) and by noting that

$$\lim_{t\to+\infty} R(t|t_0) = +\infty, \quad \lim_{t\to+\infty} [e^{A(t|t_0)} R(t|t_0)] = \frac{r}{|\alpha|}, \quad \lim_{t\to+\infty} \frac{e^{-A(t|t_0)/2}}{R(t|t_0)} = 0.$$

Similarly, since $\lim_{n\to+\infty}\Big[e^{-A(t+nQ|t_0)}/R(t+nQ|t_0)\Big] = \omega_2(t)$, one can be obtain (5.8). □

We note that (5.7) is a gamma density of parameters ξ and $r/|\alpha|$, that is a decreasing function of x when $0 < \xi \leq 1$, whereas it has a single maximum in $x = r(\xi - 1)/|\alpha|$ for $\xi > 1$. Similarly, (5.8) is a non-homogeneous gamma density.

In Propositions 5.2 and 5.3, we prove some relations between the transition pdf of TNH-RF process with $\beta(t) = r(t)/2$ and $\beta(t) = 3\,r(t)/2$ in (5.1) and the transition pdf of Wiener and of Ornstein-Uhlenbeck processes under suitable conditions on the zero-state and for specific choices of the infinitesimal moments.

Proposition 5.2. *Let $Z(t)$ be a TNH-RF process with $C_1(x,t) = \alpha(t)\,x + r(t)/2$ and $C_2(x,t) = 2\,r(t)\,x$, where $\alpha(t) \in \mathbb{R}$ and $r(t) > 0$.*

1. *If $\alpha(t) = 0$ for all t, one has*

$$r_Z(x,t|x_0,t_0) = \frac{1}{2\sqrt{x}}\, r_X(\sqrt{x}, t|\sqrt{x_0}, t_0), \qquad x_0 \geq 0, x > 0, \tag{5.9}$$

where $r_X(x,t|x_0,t_0)$ denotes the transition pdf of the TNH-RW process with $A_1 = 0$ and $A_2(t) = r(t)/2$.

2. *If $\alpha(t)$ is not always zero, it follows:*

$$r_Z(x,t|x_0,t_0) = \frac{1}{2\sqrt{x}}\, r_Y(\sqrt{x},t|\sqrt{x_0},t_0), \qquad x_0 \ge 0, x > 0, \tag{5.10}$$

where $r_Y(x,t|x_0,t_0)$ denotes the transition pdf of the TNH-ROU process with $B_1(x,t) = \alpha(t)\,x/2$ and $B_2(t) = r(t)/2$.

Proof. For the TNH-RF process $Z(t)$, by setting $\beta(t) = r(t)/2$ in (5.4) and recalling that

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi}}\,\frac{\cosh(x)}{\sqrt{x}},$$

for $x_0 \ge 0, x > 0$ one has:

$$r_Z(x,t|x_0,t_0) = \begin{cases} \dfrac{1}{2\sqrt{\pi\,x\,\widetilde{R}(t|t_0)}}\Bigg[\exp\Bigg\{-\dfrac{(\sqrt{x}-\sqrt{x_0})^2}{\widetilde{R}(t|t_0)}\Bigg\} + \exp\Bigg\{-\dfrac{(\sqrt{x}+\sqrt{x_0})^2}{\widetilde{R}(t|t_0)}\Bigg\}\Bigg], & \alpha(t) = 0, \\ \dfrac{e^{-A(t|t_0)/2}}{2\sqrt{\pi\,x\,R(t|t_0)}}\Bigg[\exp\Bigg\{-\dfrac{(\sqrt{x\,e^{-A(t|t_0)}}-\sqrt{x_0})^2}{R(t|t_0)}\Bigg\} + \exp\Bigg\{-\dfrac{(\sqrt{x\,e^{-A(t|t_0)}}+\sqrt{x_0})^2}{R(t|t_0)}\Bigg\}\Bigg], & \alpha(t) \ne 0, \end{cases} \tag{5.11}$$

with $\widetilde{R}(t|t_0) = \int_{t_0}^t r(\theta)\,d\theta$ and with $A(t|t_0)$ and $R(t|t_0)$ given in (4.2) and (5.3), respectively. We now analyze separately the cases 1. and 2.

1. For the TNH-RW process $X(t)$, defined in (3.1), with $\beta(t) = 0$ and $\sigma^2(t) = r(t)/2$, one has $V_X(t|t_0) = \widetilde{R}(t|t_0)/2$, so that from (3.5) with $\gamma = 0$ one obtains:

$$r_X(\sqrt{x},t|\sqrt{x_0},t_0) = \frac{1}{\sqrt{\pi\widetilde{R}(t|t_0)}}\Bigg[\exp\Bigg\{-\frac{(\sqrt{x}-\sqrt{x_0})^2}{\widetilde{R}(t|t_0)}\Bigg\} + \exp\Bigg\{-\frac{(\sqrt{x}+\sqrt{x_0})^2}{\widetilde{R}(t|t_0)}\Bigg\}\Bigg] \tag{5.12}$$

for $x_0 \ge 0$ and $x \ge 0$. Then, by comparing the first of (5.11) with (5.12), for $x_0 \ge 0$ and $x > 0$ relation (5.9) follows.

2. In the TNH-ROU process $Y(t)$, defined in (4.1), we set $\beta(t) = 0$, $\sigma^2(t) = r(t)/2$ and we change $\alpha(t)$ into $\alpha(t)/2$; by virtue of (4.5) and (5.3), one has $V_Y(t|t_0) = R(t|t_0)e^{A(t|t_0)}/2$, so that from (4.6) with $\gamma = 0$ one obtains:

$$r_Y(\sqrt{x},t|\sqrt{x_0},t_0) = \frac{e^{-A(t|t_0)/2}}{\sqrt{\pi\,R(t|t_0)}}\Bigg[\exp\Bigg\{-\frac{(\sqrt{x\,e^{-A(t|t_0)}}-\sqrt{x_0})^2}{R(t|t_0)}\Bigg\} + \exp\Bigg\{-\frac{(\sqrt{x\,e^{-A(t|t_0)}}+\sqrt{x_0})^2}{R(t|t_0)}\Bigg\}\Bigg], \quad x_0 \ge 0, x \ge 0. \tag{5.13}$$

Hence, for $x_0 \ge 0$ and $x > 0$, Eq. (5.10) follows by comparing the second of (5.11) with (5.13).

□

Proposition 5.3. *Let $Z(t)$ be a TNH-RF process with $C_1(t) = \alpha(t)\,x + 3r(t)/2$ and $C_2(x,t) = 2\,r(t)\,x$, where $\alpha(t) \in \mathbb{R}$ and $r(t) > 0$.*

1. *If $\alpha(t) = 0$ for all t, one has*

$$r_Z(x, t|x_0, t_0) = \frac{1}{2\sqrt{x_0}}\, a_X(\sqrt{x}, t|\sqrt{x_0}, t_0), \qquad x_0 > 0, x > 0, \tag{5.14}$$

where $a_X(x, t|x_0, t_0)$ denotes the transition pdf of the inhomogeneous Wiener process with $A_1 = 0$ and $A_2(t) = r(t)/2$, restricted to $(0, +\infty)$ with an absorbing boundary in the zero-state.

2. *If $\alpha(t)$ is not always zero, it follows:*

$$r_Z(x, t|x_0, t_0) = \frac{e^{-A(t|t_0)/2}}{2\sqrt{x_0}}\, a_Y(\sqrt{x}, t|\sqrt{x_0}, t_0), \qquad x_0 > 0, x > 0, \tag{5.15}$$

where $a_Y(x, t|x_0, t_0)$ denotes the transition pdf of the inhomogeneous Ornstein-Uhlenbeck process with $B_1(x, t) = \alpha(t)\, x/2$ and $B_2(t) = r(t)/2$, restricted to $(0, +\infty)$ with an absorbing boundary in the zero-state.

Proof. For the TNH-RF process $Z(t)$, by setting $\beta(t) = 3r(t)/2$ in (5.4) and recalling that

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sinh(x)}{\sqrt{x}},$$

for $x_0 > 0, x > 0$ one has:

$$r_Z(x, t|x_0, t_0) = \begin{cases} \dfrac{1}{2\sqrt{\pi\, x_0\, \widetilde{R}(t|t_0)}}\Big[\exp\Big\{-\dfrac{(\sqrt{x}-\sqrt{x_0})^2}{\widetilde{R}(t|t_0)}\Big\} - \exp\Big\{-\dfrac{(\sqrt{x}+\sqrt{x_0})^2}{\widetilde{R}(t|t_0)}\Big\}\Big], & \alpha(t) = 0, \\[2ex] \dfrac{e^{-A(t|t_0)}}{2\sqrt{\pi\, x_0\, R(t|t_0)}}\Big[\exp\Big\{-\dfrac{(\sqrt{x\, e^{-A(t|t_0)}}-\sqrt{x_0})^2}{R(t|t_0)}\Big\} + \exp\Big\{-\dfrac{(\sqrt{x\, e^{-A(t|t_0)}}+\sqrt{x_0})^2}{R(t|t_0)}\Big\}\Big], & \alpha(t) \neq 0, \end{cases} \tag{5.16}$$

with $\widetilde{R}(t|t_0) = \int_{t_0}^{t} r(\theta)\, d\theta$ and with $A(t|t_0)$ and $R(t|t_0)$ given in (4.2) and (5.3), respectively. We now take into account the cases 1. and 2.

1. For a time-inhomogeneous Wiener process with $A_1 = 0$ and $A_2(t) = r(t)/2$, for $x_0 > 0$ and $x > 0$ one has (cf. Giorno and Nobile [19])

$$a_X(\sqrt{x}, t|\sqrt{x_0}, t_0) = \frac{1}{\sqrt{\pi\, \widetilde{R}(t|t_0)}}\Big[\exp\Big\{-\frac{(\sqrt{x}-\sqrt{x_0})^2}{\widetilde{R}(t|t_0)}\Big\} - \exp\Big\{-\frac{(\sqrt{x}+\sqrt{x_0})^2}{\widetilde{R}(t|t_0)}\Big\}\Big]. \tag{5.17}$$

Hence, Eq. (5.14) follows by comparing the first of (5.16) with (5.17).

2. For a time-inhomogeneous Ornstein-Uhlenbeck process with $B_1(x, t) = \alpha(t)\, x/2$ and $B_2(t) = r(t)/2$, one has (cf. Giorno and Nobile [19]):

$$a_Y(\sqrt{x}, t|\sqrt{x_0}, t_0) = \frac{e^{-A(t|t_0)/2}}{\sqrt{\pi R(t|t_0)}}\Big[\exp\Big\{-\frac{(\sqrt{x\, e^{-A(t|t_0)}}-\sqrt{x_0})^2}{R(t|t_0)}\Big\} - \exp\Big\{-\frac{(\sqrt{x\, e^{-A(t|t_0)}}+\sqrt{x_0})^2}{R(t|t_0)}\Big\}\Big], \quad x > 0, x_0 > 0. \tag{5.18}$$

Then, by comparing the second of (5.16) with (5.18), Eq. (5.15) follows.

□

5.1. Time-homogeneous Feller process

We consider the TH-RF process $Z(t)$, obtained from (5.1) by setting $\alpha(t) = \alpha$, $\beta(t) = \beta$, $r(t) = r$, with $\alpha \in \mathbb{R}, \beta > 0$, $r > 0$ and a zero-flux condition in the zero-state. From (2.6), for the TH-RF process $Z(t)$ one has:

$$h_Z(x) = x^{-\beta/r} \exp\Big\{-\frac{\alpha x}{r}\Big\}, \qquad s_Z(x) = \frac{x^{\beta/r-1}}{r} \exp\Big\{\frac{\alpha x}{r}\Big\}. \tag{5.19}$$

As proved by Feller [42], the boundary 0 is regular for $0 < \beta < r$ and entrance for $\beta \geq r$. Furthermore, the end-point $+\infty$ is a nonattracting natural boundary for $\alpha \leq 0$ and an attracting natural boundary for $\alpha > 0$.

The transition pdf of the TH-RF process is obtainable from (5.4) by setting $\alpha(t) = \alpha$, $\beta(t) = \beta$, $r(t) = r$ and $\xi = \beta/r$. In particular,

- if $\alpha = 0$, one has:

$$r_Z(x,t|x_0) = \begin{cases} \frac{1}{x\,\Gamma(\beta/r)} \Big[\frac{x}{r\,t}\Big]^{\beta/r} \exp\Big\{-\frac{x}{r\,t}\Big\}, & x_0 = 0, \\ \frac{1}{r\,t} \Big(\frac{x}{x_0}\Big)^{(\beta-r)/(2r)} \exp\Big\{-\frac{x_0+x}{r\,t}\Big\} I_{\beta/r-1}\Big[\frac{2\sqrt{x\,x_0}}{r\,t}\Big], & x_0 > 0, \end{cases} \tag{5.20}$$

- if $\alpha \neq 0$, one obtains:

$$r_Z(x,t|x_0) = \begin{cases} \frac{1}{x\,\Gamma(\beta/r)} \Big[\frac{\alpha\,x}{r(e^{\alpha\,t}-1)}\Big]^{\beta/r} \exp\Big\{-\frac{\alpha\,x}{r(e^{\alpha\,t}-1)}\Big\}, & x_0 = 0, \\ \frac{\alpha}{r(e^{\alpha\,t}-1)} \Big[\frac{x\,e^{-\alpha\,t}}{x_0}\Big]^{(\beta-r)/(2r)} \exp\Big\{-\frac{\alpha\,[x+x_0\,e^{\alpha\,t}]}{r(e^{\alpha\,t}-1)}\Big\} I_{\beta/r-1}\Big[\frac{2\alpha\sqrt{x\,x_0\,e^{\alpha\,t}}}{r(e^{\alpha\,t}-1)}\Big], & x_0 > 0. \end{cases} \tag{5.21}$$

The conditional mean and the conditional variance of the TH-RF process can be obtained from (5.5) with $\xi = \beta/r$. We note that if $\alpha < 0, \beta > 0$ and $r > 0$, the TH-RF process admits the steady-state density given in (5.7) with $\xi = \beta/r$.

In Figure 6, we consider a TH-RF process $Z(t)$ in $[0, +\infty)$, having $C_1(x) = \alpha\,x + \beta$ and $C_2(x) = 2\,r\,x$, with $\alpha = 0, \beta = 0.25$, $r = 0.5$, $x_0 = 5$ and $t_0 = 0$. We compare the histogram of the random sample of $N = 5 \cdot 10^4$ observations of $Z(t)$ with the transition pdf (5.20) as function of x ($x \geq 0$) for $t = 0.5$ on the left and $t = 1$ on the right. Instead, in Figure 7, we consider a TH-RF process $Z(t)$ in $[0, +\infty)$, with $\alpha = -0.5, \beta = 0.25$, $r = 0.5$, $x_0 = 5$ and $t_0 = 0$ and we compare the histogram of the random sample of $N = 5 \cdot 10^4$ observations of $Z(t)$ with the transition pdf (5.21) as function of x ($x \geq 0$) for $t = 0.5$ on the left and $t = 1$ on the right. Figures 6 and 7 show the good agreement between the exact and simulated results.

5.2. FPT moments for the TH-RF process

In Giorno and Nobile [43] the FPT problem through a state S starting from x_0 for the TH-RF process has been considered. When $0 < S < x_0$ the ultimate FPT probability $P_Z(S|x_0) = 1$ if and only if $[\alpha < 0, \beta > 0]$ or $[\alpha = 0, 0 < \beta \leq r]$. Making use of (2.7), for $\alpha = 0$ and $0 < \beta \leq r$ one has that $t_1^{(Z)}(S|x_0)$ diverges, whereas if $\alpha < 0$ and $\beta > 0$ one obtains:

$$t_1^{(Z)}(S|x_0) = \frac{1}{|\alpha|}\,\Gamma\Big(\frac{\beta}{r}\Big) \int_{|\alpha|S/r}^{|\alpha|x_0/r} z^{-\beta/r}\,e^z\,dz - \frac{1}{\beta} \sum_{n=0}^{+\infty} \frac{1}{(1+\beta/r)_n} \Big(\frac{|\alpha|}{r}\Big)^n \frac{x_0^{n+1} - S^{n+1}}{n+1}, \quad 0 < S < x_0. \tag{5.22}$$

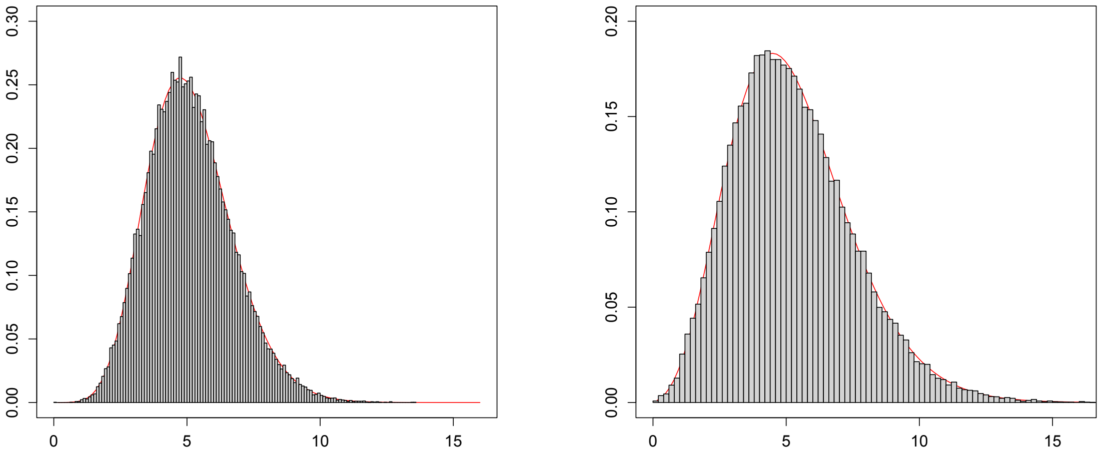

Figure 6. For the TH-RF process $Z(t)$, with $\alpha = 0, \beta = 0.25, r = 0.5, x_0 = 5$, the transition pdf (5.20) is plotted (red curve) as function of x, for $t = 0.5$ (on the left) and $t = 1$ (on the right) and the histogram, obtained via the simulation of the sample paths, is superimposed over the density.

Moreover, if $x_0 > 0$ the ultimate FPT probability $P_Z(0|x_0) = 1$ if and only if $\alpha \leq 0$ and $0 < \beta < r$. From (2.7), for $\alpha = 0$ and $0 < \beta < r$ one has that $t_1^{(Z)}(0|x_0)$ diverges, whereas for $\alpha < 0$ and $0 < \beta < r$ (5.22) holds with $S = 0$.

Instead, when $0 \leq x_0 < S$, one obtains $P_Z(S|x_0) = 1$ and from (2.8) it results

$$t_1^{(Z)}(S|x_0) = \frac{1}{\beta} \int_{x_0}^{S} \Phi\Big(1, \frac{\beta}{r} + 1; -\frac{\alpha z}{r}\Big)\, dz = \frac{1}{\beta} \sum_{n=0}^{+\infty} \frac{1}{(1+\beta/r)_n} \Big(-\frac{\alpha}{r}\Big)^n \frac{S^{n+1} - x_0^{n+1}}{n+1}, \quad 0 \leq x_0 < S, \quad (5.23)$$

so that for $\alpha = 0$ one has $t_1^{(Z)}(S|x_0) = (S - x_0)/\beta$. Moreover, when $0 \leq x_0 < S$, from (5.23) it follows:

- $t_1^{(Z)}(S|x_0)$ decreases as β increases and

$$\lim_{\beta \to 0} t_1^{(Z)}(S|x_0) = +\infty, \qquad \lim_{\beta \to +\infty} t_1^{(Z)}(S|x_0) = 0;$$

- $t_1^{(Z)}(S|x_0)$ decreases as r increases and

$$\lim_{r \to 0} t_1^{(Z)}(S|x_0) = \begin{cases} +\infty, & \alpha < 0, \\ \frac{S - x_0}{\beta}, & \alpha = 0, \\ \frac{1}{\alpha} \ln\Big(\frac{\alpha S + \beta}{\alpha x_0 + \beta}\Big), & \alpha > 0, \end{cases} \qquad \lim_{r \to +\infty} t_1^{(Z)}(S|x_0) = \frac{S - x_0}{\beta}.$$

In Figures 8 and 9, the FPT mean (5.23) of the TH-RF process is plotted for $x_0 = 5$, $S = 10$ and for different choices of β and r, with $\alpha = -0.02, 0.02$, respectively.

Making use of (5.19) in Remark 2.1, one can derive the following asymptotic results for the FPT moments of the TH-RF process.

Remark 5.1. *For the TH-RF process $Z(t)$, when $n = 1, 2, \ldots$ one has:*

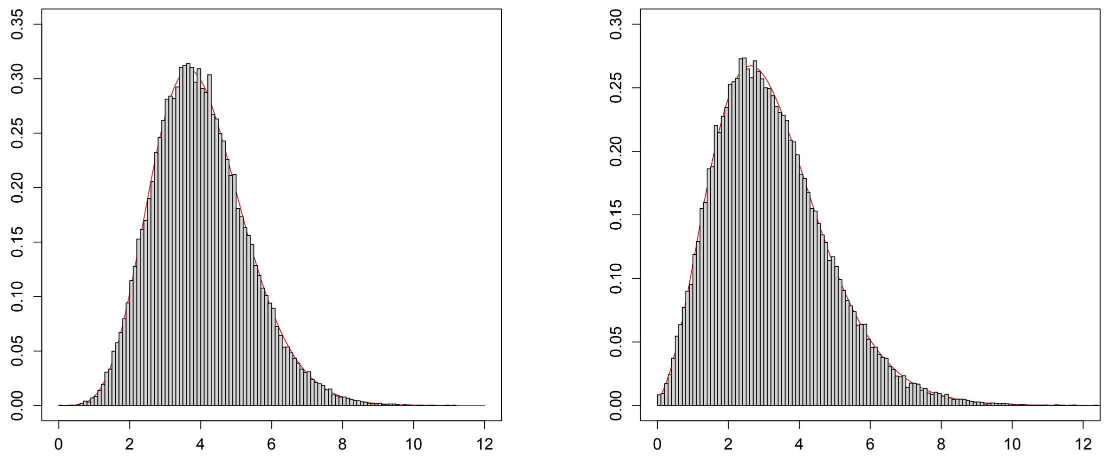

Figure 7. As in Figure 6 with $\alpha = -0.5, \beta = 0.25, r = 0.5$ and $x_0 = 5$.

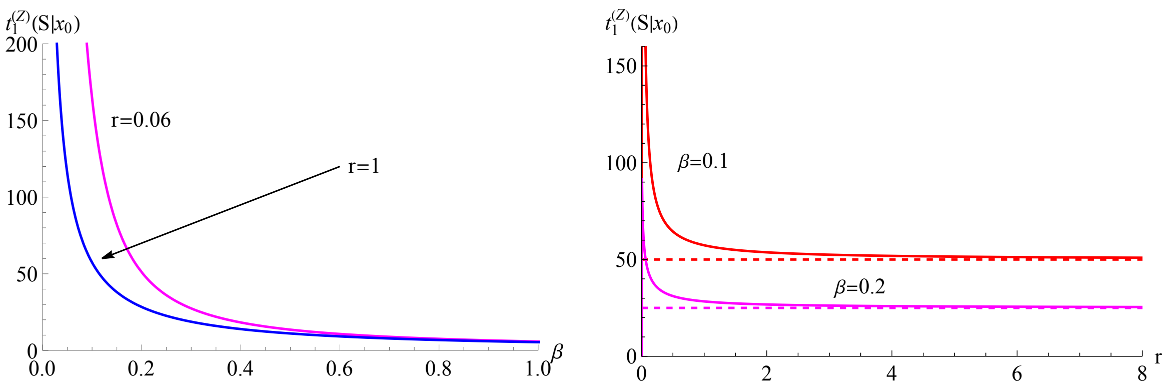

Figure 8. The FPT mean (5.23) is plotted for $\alpha = -0.02$, $x_0 = 5$ and $S = 10$ as function of β on the left and as function of r on the right.

1. $\lim_{S\uparrow+\infty} \frac{t_n^{(Z)}(S\,|0)}{[t_1^{(Z)}(S\,|0)]^n} = n!, \qquad \alpha < 0,$

2. $\lim_{S\downarrow 0} \frac{t_n^{(Z)}(S\,|0)}{[t_1^{(Z)}(S\,|0)]^n} = u_n, \qquad 0 < \beta < r$

where

$$u_0 = 1, \quad u_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{\prod_{i=0}^{k-1}\left(1 + i\,\frac{r}{\beta}\right)}\, u_{n-k}, \quad n = 1, 2, \ldots$$

From Remark 5.1, the following asymptotic behaviors hold:

- for $S \uparrow +\infty$ one has
$$t_n^{(Z)}(S\,|0) \simeq n!\,[t_1^{(Z)}(S\,|0)]^n, \qquad \alpha < 0,$$
so that for $\alpha < 0$ the FPT density of TH-RF process exhibits an exponential trend for large boundary;

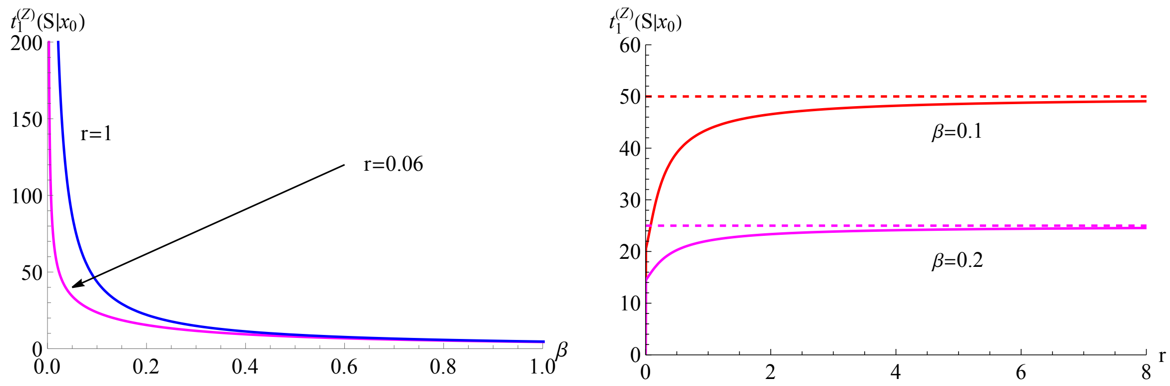

Figure 9. As in Figure 8 with $\alpha = 0.02$.

Table 10. For the TH-RF process, with $\alpha = -0.02, \beta = 0.1$ and $r = 0.5$, $t_1^{(Z)}(S|0)$, $\mathrm{Var}^{(Z)}(S|0)$, $\mathrm{Cv}^{(Z)}(S|0)$ and $\Sigma^{(Z)}(S|0)$ are listed for increasing values of $S > 0$.

S	$t_1^{(Z)}(S\|0)$	$\mathrm{Var}^{(Z)}(S\|0)$	$\mathrm{Cv}^{(Y)}(Z\|0)$	$\Sigma^{(Z)}(S\|0)$
10.	118.892	12127.6	0.926266	1.99421
20.	286.387	72259.4	0.938630	1.99609
30.	524.198	247872.	0.949771	1.99746
40.	864.044	687422.	0.959568	1.99839
50.	1352.34	1.71357×10^6	0.967980	1.99902
60.	2057.09	4.02295×10^6	0.975033	1.99942
70.	3078.07	9.11448×10^6	0.980817	1.99966
80.	4561.82	2.02095×10^7	0.985460	1.99981
90.	6723.80	4.42308×10^7	0.989117	1.99988
100.	9881.02	9.60682×10^7	0.991946	1.99993
300.	2.31543×10^7	5.36117×10^{14}	0.999993	1.99952
600.	3.24856×10^{12}	1.05531×10^{25}	1.0	2.00237
900.	4.86065×10^{17}	2.36259×10^{35}	1.0	1.99494
1200.	7.45766×10^{22}	5.56166×10^{45}	1.0	2.00006

- for $S \downarrow 0$ it results:

$$t_n^{(Z)}(S|0) \simeq u_n \, [t_1^{(Z)}(S|0)]^n = m_n^{(Z)}(S|0), \qquad 0 < \beta < r.$$

In particular, one has:

$$m_2^{(Z)}(S|0) = \frac{\beta + 2r}{\beta + r} \, [t_1^{(Z)}(S|0)]^2, \quad m_3^{(Z)}(S|0) = \frac{\beta^2 + 6r\beta + 12r^2}{(\beta + r)(\beta + 2r)} \, [t_1^{(Z)}(S|0)]^3.$$

In Tables 10–11, $t_1^{(Z)}(S|0)$ is computed via (5.23), whereas $t_2^{(Z)}(S|0)$ and $t_3^{(Z)}(S|0)$ are numerically evaluated making use of (2.8) and (5.19). In Table 10, the mean $t_1^{(Z)}(S|0)$, the variance $\mathrm{Var}^{(Z)}(S|0)$, the coefficient of variation $\mathrm{Cv}^{(Z)}(S|0)$ and the skewness $\Sigma^{(Z)}(S|0)$ of the FPT are listed for $\alpha = -0.02$, $\beta = 0.1$, $r = 0.5$ and some choices of $S > 0$. Being $\alpha < 0$, the coefficient of variation and the skewness approach to 1 and to 2, respectively, as S increases.

Moreover, in Table 11, for the TH-RF process with $\alpha = -0.02, \beta = 0.1$ and $r = 0.5$, we compare the FPT moments $t_2^{(Z)}(S|0)$ and $t_3^{(Z)}(S|0)$ with the approximate values $m_2^{(Z)}(S|0)$ and $m_3^{(Z)}(S|0)$, respectively. We note that the goodness of the approximations improves as S approaches zero.

Table 11. For the TH-RF process with $\alpha = -0.02, \beta = 0.1$ and $r = 0.5$, $t_1^{(Z)}(S\,|0)$, $t_2^{(Z)}(S\,|0)$ and $t_3^{(Z)}(S\,|0)$ and their approximate values $m_2^{(Z)}(S\,|0)$ and $m_3^{(Z)}(S\,|0)$ are listed for decreasing values of S.

S	$t_1^{(Z)}(S\,|0)$	$t_2^{(Z)}(S\,|0)$	$m_2^{(Z)}(S\,|0)$	$t_3^{(Z)}(S\,|0)$	$m_3^{(Z)}(S\,|0)$
3.0	3.155612×10^1	1.833106×10^3	1.825612×10^3	1.589679×10^5	1.575922×10^5
2.5	2.607399×10^1	1.250665×10^3	1.246397×10^3	8.954832×10^4	8.890094×10^4
2.0	2.068314×10^1	7.864368×10^2	7.842855×10^2	4.463324×10^4	4.437447×10^4
1.5	1.538192×10^1	4.346661×10^2	4.337727×10^2	1.833211×10^4	1.825220×10^4
1.0	1.016871×10^1	1.898320×10^2	1.895714×10^2	5.288689×10^3	5.273284×10^3
0.9	9.136485	1.532275×10^2	1.530382×10^2	3.834971×10^3	3.824913×10^3
0.8	8.107709	1.206466×10^2	1.205141×10^2	2.679123×10^3	2.672874×10^3
0.7	7.082364	9.204832×10^1	9.195978×10^1	1.785281×10^3	1.781635×10^3
0.6	6.060439	6.739193×10^1	6.733635×10^1	1.118298×10^3	1.116339×10^3
0.5	5.041920	4.663716×10^1	4.660510×10^1	6.437341×10^2	6.427943×10^2
0.4	4.026796	2.974403×10^1	2.972766×10^1	3.278467×10^2	3.274636×10^2
0.3	3.015055	1.667290×10^1	1.666602×10^1	1.375785×10^2	1.374579×10^2
0.2	2.006683	7.384456	7.382423	4.054846×10^1	4.052474×10^1
0.1	1.001669	1.839710	1.839457	5.041775	5.040300

6. Applications of the results to queueing systems

The TNH-RW, TNH-ROU and TNH-RF diffusion processes can be seen as the continuous approximations of some time-inhomogeneous birth-death processes that modeling queueing systems in heavy-traffic conditions. Referring to the queueing systems, in this section we consider some examples in which the infinitesimal drifts and the infinitesimal variances are time-dependent and include periodic functions. The presence of periodicity in the infinitesimal moments express the existence of rush hours occurring on a daily basis. For the considered reflected processes, we apply the exact and asymptotic results obtained in Sections 3–5 to analyze the conditional averages and the conditional variances for some choices of the periodic functions and of the parameters.

Example 6.1. *(Wiener model)* We consider the TNH-RW process $X(t)$, having infinitesimal drift and infinitesimal variance $A_1(t) = \gamma\,\sigma^2(t)$ and $A_2(t) = \sigma^2(t)$, with $\gamma \in \mathbb{R}$ and $\sigma(t) > 0$. This process can be seen as the continuous approximation of the birth-death queueing system $N_1(t)$ with arrival and departure intensity functions $\lambda_n(t) = \lambda\sigma^2(t)/\varepsilon + \sigma^2(t)/(2\varepsilon^2)$ $(n = 0, 1, \ldots)$ and $\mu_n(t) = \mu\sigma^2(t)/\varepsilon + \sigma^2(t)/(2\varepsilon^2)$ $(n = 1, 2, \ldots)$, where $\lambda > 0$, $\mu > 0$ and ε is a positive scaling parameter. Indeed, the scaled process $N_1(t)\,\varepsilon$ converges weakly to the diffusion process $X(t)$, having state-space $[0, +\infty)$, with infinitesimal moments (see, for instance, [1]):

$$A_1(t) = \lim_{\varepsilon\downarrow 0} \varepsilon \Big[\lambda_n(t) - \mu_n(t)\Big] = \gamma\,\sigma^2(t), \qquad A_2(t) = \lim_{\varepsilon\downarrow 0} \varepsilon^2 \Big[\lambda_n(t) + \mu_n(t)\Big] = \sigma^2(t),$$

with $\gamma = \lambda - \mu$. We assume that

$$\sigma^2(t) = \nu\Big[1 + c\,\sin\Big(\frac{2\pi\,t}{Q}\Big)\Big], \qquad t \geq 0, \tag{6.1}$$

where $\nu > 0$ is the average of the period function $\sigma^2(t)$ of period Q and c is the amplitude of the oscillations, with $0 \leq c < 1$. These choices of parameters ensure that the infinitesimal variance is a positive function.

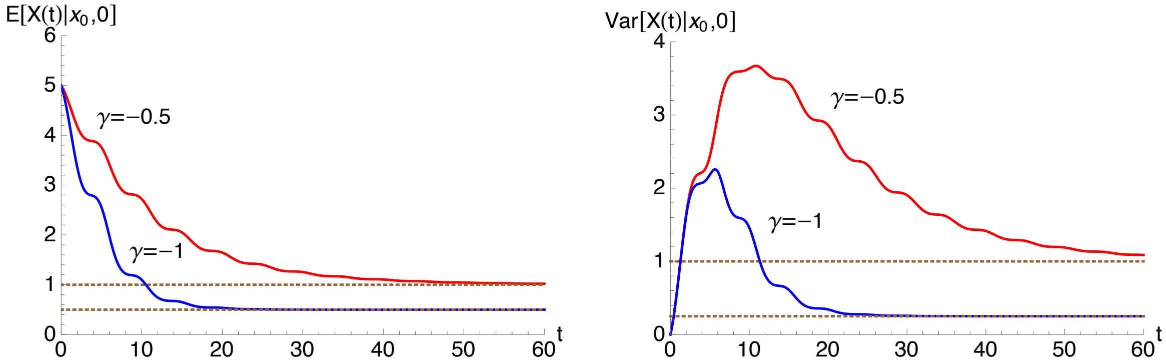

Figure 10. For the TNH-RW process, the conditional mean and the conditional variance are plotted as function of t for $t_0 = 0$, $x_0 = 5$, $\sigma^2(t) = 0.5\,[1 + 0.9\,\sin(2\pi\,t/5)]$; the dotted lines indicate the corresponding asymptotic means and variances.

In Figure 10, we suppose that $\sigma^2(t) = 0.5\,[1 + 0.9\,\sin(2\pi\,t/5)]$ and that at the initial time $t_0 = 0$ the number of customers is $X(t_0) = x_0 = 5$. As proved in Corollary 3.2, the transition pdf of $X(t)$ admits the asymptotic exponential behavior (3.9) for $\gamma < 0$. In Figure 10, the conditional mean and the conditional variance, obtained from (3.7), and the related asymptotic behaviors are shown as function of t for $\gamma = -0.5, -1$; the dotted lines indicate the corresponding asymptotic means and variances $(2\,|\gamma|)^{-1}$ and $(2\,|\gamma|)^{-2}$.

Example 6.2. *(Ornstein-Uhlenbeck model)* We consider the TNH-ROU process $Y(t)$, having infinitesimal drift and infinitesimal variance $B_1(x,t) = \alpha\,x$ and $B_2(t) = \sigma^2(t)$, with $\alpha \in \mathbb{R}$ and $\sigma(t) > 0$. This process can be seen as the continuous approximation of the birth-death queueing system $N_2(t)$ with arrival and departure intensity functions $\lambda_n(t) = \lambda\,n + \sigma^2(t)/(2\varepsilon^2)$ $(n = 0, 1, \ldots)$ and $\mu_n(t) = \mu\,n + \sigma^2(t)/(2\varepsilon^2)$ $(n = 1, 2, \ldots)$, both depending on the number of customers in the systems, where $\lambda > 0$, $\mu > 0$ and ε is a positive scaling parameter. Indeed, the scaled process $N_2(t)\,\varepsilon$ converges weakly to the diffusion process $X(t)$, having state-space $[0, +\infty)$, with infinitesimal moments (see, for instance, [1]):

$$B_1(x,t) = \lim_{\substack{\varepsilon\downarrow 0\\ n=x/\varepsilon}} \varepsilon\Big[\lambda_n(t) - \mu_n(t)\Big] = \alpha\,x, \qquad B_2(t) = \lim_{\substack{\varepsilon\downarrow 0\\ n=x/\varepsilon}} \varepsilon^2\Big[\lambda_n(t) + \mu_n(t)\Big] = \sigma^2(t),$$

where $\alpha = \lambda - \mu$. We assume that (6.1) holds.

As proved in Corollary 4.2, when $\alpha < 0$ the transition pdf of $Y(t)$ admits the asymptotic behavior (4.11) with

$$\omega_1(t) = \frac{\nu}{2\,|\alpha|}\left\{1 + \frac{c\,Q\,\alpha}{\pi^2 + Q^2\alpha^2}\Big[\pi\,\cos\Big(\frac{2\pi\,t}{Q}\Big) + Q\,\alpha\,\sin\Big(\frac{2\pi\,t}{Q}\Big)\Big]\right\}, \qquad t \geq 0.$$

In Figure 11, the conditional mean and the conditional variance, obtained from (4.8), and the related asymptotic behaviors are shown as function of t for $\sigma^2(t) = 0.5\,[1 + 0.9\,\sin(2\pi\,t/5)]$, $t_0 = 0$, $x_0 = 5$ and $\alpha = -0.05, -0.1$; the dotted functions indicate the corresponding asymptotic mean $[2\omega_1(t)/\pi]^{1/2}$ and asymptotic variances $\omega_1(t)\,(1 - 2/\pi)$.

Example 6.3. *(Feller model)* We consider the TNH-RF process $Z(t)$, having infinitesimal drift and infinitesimal variance $C_1(x,t) = \alpha\,x + \xi\,r(t)$ and $C_2(x,t) = 2\,r(t)\,x$, with $\alpha \in \mathbb{R}$, $\xi > 0$ and $r(t) > 0$.

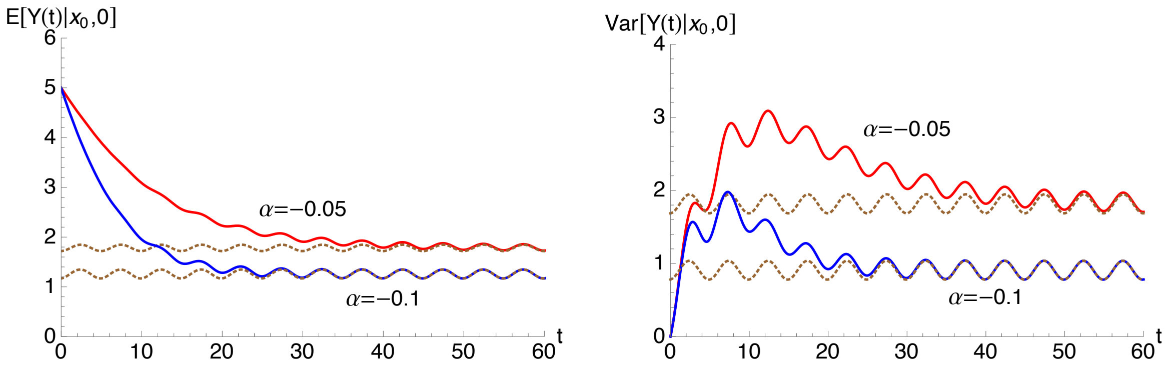

Figure 11. For the TNH-ROU process, the conditional mean and the conditional variance are plotted as function of t for $t_0 = 0$, $x_0 = 5$, $\sigma^2(t) = 0.5\,[1 + 0.9\,\sin(2\pi\, t/5)]$; the dotted functions indicate the corresponding asymptotic means and variances.

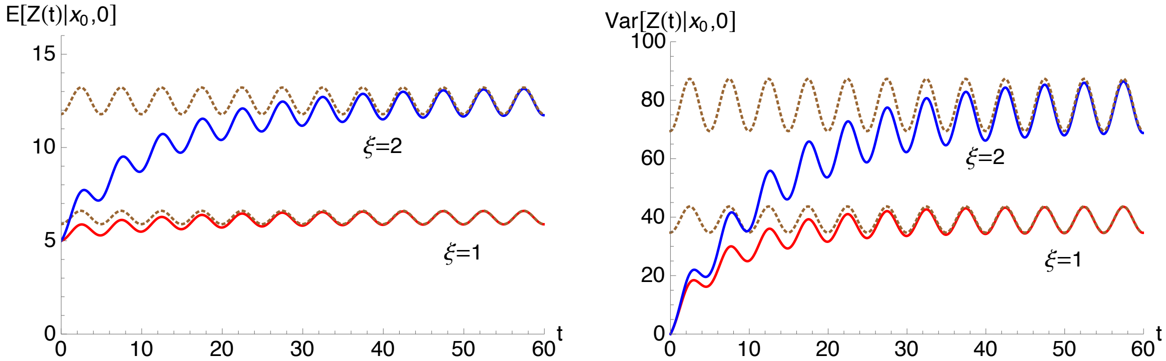

Figure 12. For the TNH-RF process, the conditional mean and the conditional variance are plotted as function of t for $t_0 = 0$, $x_0 = 5$, $\alpha = -0.08$, $r(t) = 0.5\,[1 + 0.9\,\sin(2\pi\, t/5)]$; the dotted functions indicate the corresponding asymptotic means and variances.

This process can be seen as the continuous approximation of the birth-death queueing system $N_3(t)$ with arrival and departure intensity functions $\lambda_n(t) = [\lambda + r(t)/\varepsilon]\, n + \xi\, r(t)/\varepsilon$ $(n = 0, 1, \ldots)$ and $\mu_n(t) = [\mu + r(t)/\varepsilon]\, n$ $(n = 1, 2, \ldots)$, both depending on the number of customers in the systems and on the positive scaling parameter ε. Indeed, the scaled process $N_3(t)\,\varepsilon$ converges weakly to a diffusion process, having state-space $[0, +\infty)$ (see, for instance, [34]) and one has:

$$C_1(x,t) = \lim_{\substack{\varepsilon\downarrow 0\\ n=x/\varepsilon}} \varepsilon\Big[\lambda_n(t) - \mu_n(t)\Big] = \alpha\, x + \xi\, r(t), \qquad C_2(x,t) = \lim_{\substack{\varepsilon\downarrow 0\\ n=x/\varepsilon}} \varepsilon^2\Big[\lambda_n(t) + \mu_n(t)\Big] = 2\, r(t)\, x,$$

where $\alpha = \lambda - \mu$. We assume that $r(t) = \sigma^2(t)$, with $\sigma^2(t)$ given in (6.1).

As proved in Corollary 5.1, when $\alpha < 0$ the transition pdf of $Z(t)$ admits the asymptotic behavior (5.8) with

$$\omega_2(t) = \frac{1}{\nu}\left\{\frac{1}{|\alpha|} - \frac{c\,Q}{4\pi^2 + Q^2\,\alpha^2}\left[2\pi\cos\Big(\frac{2\pi t}{Q}\Big) + \alpha\, Q\sin\Big(\frac{2\pi t}{Q}\Big)\right]\right\}^{-1}, \qquad t \geq 0.$$

In Figure 12, the conditional mean and the conditional variance, obtained from (5.5), and the related asymptotic behaviors are shown as function of t for $r(t) = 0.5\,[1 + 0.9\,\sin(2\pi\, t/5)]$, $t_0 = 0$, $x_0 = 5$, $\alpha = -0.08$ and $\xi = 1.0, 2.0$; the dotted functions indicate the corresponding asymptotic mean $\xi/\omega_2(t)$ and asymptotic variances $\xi/[\omega_2(t)]^2$.

As highlighted in the Figures 10–12, the periodic intensity function (6.1), used in the infinitesimal drifts and in the infinitesimal variances of the reflected Wiener, Ornstein-Uhlenbeck and Feller models, affects the shapes of the conditional averages and of the conditional variances.

7. Concluding remarks

For the Wiener, Ornstein Uhlenbeck and Feller processes, restricted to the interval $[0, +\infty)$, with reflecting or a zero-flux condition in the zero-state, we analyze the transition probability density functions and their asymptotic behaviors, paying particular attention to the time-inhomogeneous proportional cases and to the time-homogeneous cases. Some relationships between the transition probability density functions for the restricted Wiener, Ornstein-Uhlenbeck and Feller processes are proved. Moreover, the FPT moments and their asymptotic behaviors are analyzed for the time-homogeneous cases. Finally, some applications of the obtained results to queueing systems are considered. Various numerical computations are performed with Mathematica to illustrate the role played by parameters.

A. Proof of Proposition 4.2

Let $f_Y(x, t|x_0)$ be the transition pdf of the unrestricted time-homogeneous Ornstein-Uhlenbeck process with infinitesimal moments $B_1(x) = \alpha x + \beta$ and $B_2 = \sigma^2$. For $x, x_0 \in \mathbb{R}$, we denote by

$$j_Y(x, t|x_0) = (\alpha\, x + \beta)\, f_Y(x, t|x_0) - \frac{\sigma^2}{2}\,\frac{\partial}{\partial x} f_Y(x, t|x_0) \tag{A.1}$$

the probability current of the unrestricted Ornstein-Uhlenbeck process. The transition pdf $r_Y(x, t|x_0)$ of a time-homogeneous diffusion process restricted to $[0, +\infty)$, with zero reflecting boundary, satisfies the following integral equations (cf. Giorno et al. [44]):

$$\begin{aligned} r_Y(x, t|0) &= 2\, f_Y(x, t|0) - 2\int_0^t j_Y(0, \tau|0)\, r_Y(x, t-\tau|0)\, d\tau, \quad x \geq 0,\\ r_Y(x, t|x_0) &= f_Y(x, t|x_0) - \int_0^t j_Y(0, \tau|x_0)\, r_Y(x, t-\tau|0)\, d\tau, \quad x \geq 0, x_0 > 0. \end{aligned} \tag{A.2}$$

Taking the LT in (A.2) one has:

$$r_\lambda^{(Y)}(x|0) = \frac{2\, f_\lambda^{(Y)}(x|0)}{1 + 2\, j_\lambda^{(Y)}(0|0)}, \qquad r_\lambda^{(Y)}(x|x_0) = f_\lambda^{(Y)}(x|x_0) - j_\lambda^{(Y)}(0|x_0)\, r_\lambda^{(Y)}(x|0), \tag{A.3}$$

where $f_\lambda^{(Y)}(x|x_0)$ is LT of $f_Y(x, t|x_0)$ and $j_\lambda^{(Y)}(0|x_0)$ is LT of $j_Y(0, t|x_0)$. Taking the LT in the first equation in (4.4), with $\alpha(t) = \alpha, \beta(t) = \beta$ and $\sigma^2(t) = \sigma^2$, we have:

$$f_\lambda^{(Y)}(x|x_0) = \begin{cases} \frac{2^{\frac{\lambda}{|\alpha|}-1}}{\sigma\pi\sqrt{|\alpha|}}\Gamma\Big(\frac{\lambda}{2|\alpha|}\Big)\Gamma\Big(\frac{1}{2}+\frac{\lambda}{2|\alpha|}\Big)\exp\Big\{-\frac{|\alpha|}{2\sigma^2}\Big[\Big(x+\frac{\beta}{\alpha}\Big)^2-\Big(x_0+\frac{\beta}{\alpha}\Big)^2\Big]\Big\} \\ \quad\times D_{-\frac{\lambda}{|\alpha|}}\Big(-\frac{\sqrt{2|\alpha|}}{\sigma}\Big[x_0\wedge x+\frac{\beta}{\alpha}\Big]\Big)D_{-\frac{\lambda}{|\alpha|}}\Big(\frac{\sqrt{2|\alpha|}}{\sigma}\Big[x_0\vee x+\frac{\beta}{\alpha}\Big]\Big), & \alpha<0, \\ \\ \frac{2^{\frac{\lambda}{\alpha}}}{\sigma\pi\sqrt{\alpha}}\Gamma\Big(1+\frac{\lambda}{2\alpha}\Big)\Gamma\Big(\frac{1}{2}+\frac{\lambda}{2\alpha}\Big)\exp\Big\{-\frac{\alpha}{2\sigma^2}\Big[\Big(x_0+\frac{\beta}{\alpha}\Big)^2-\Big(x+\frac{\beta}{\alpha}\Big)^2\Big]\Big\} \\ \quad\times D_{-\frac{\lambda}{\alpha}-1}\Big(-\frac{\sqrt{2\alpha}}{\sigma}\Big[x_0\wedge x+\frac{\beta}{\alpha}\Big]\Big)D_{-\frac{\lambda}{\alpha}-1}\Big(\frac{\sqrt{2\alpha}}{\sigma}\Big[x_0\vee x+\frac{\beta}{\alpha}\Big]\Big), & \alpha>0, \end{cases} \tag{A.4}$$

with $x_0 \wedge x = \min(x_0, x)$ and $x_0 \vee x = \max(x_0, x)$. Moreover, taking the LT in Eq. (A.1) one has:

$$j_\lambda^{(Y)}(0|x_0) = \begin{cases} \frac{1}{2}\,\mathcal{I}(x_0) - \frac{2^{\frac{\lambda}{|\alpha|}-\frac{1}{2}}}{\pi} \exp\Big\{\frac{|\alpha|}{2\sigma^2}\Big[\Big(x_0+\frac{\beta}{\alpha}\Big)^2 - \Big(\frac{\beta}{\alpha}\Big)^2\Big]\Big\}\,\Gamma\Big(\frac{1}{2}+\frac{\lambda}{2|\alpha|}\Big) \\ \qquad \times\Gamma\Big(1+\frac{\lambda}{2|\alpha|}\Big)\,D_{-\frac{\lambda}{|\alpha|}}\Big(\frac{\sqrt{2|\alpha|}}{\sigma}\Big(x_0+\frac{\beta}{\alpha}\Big)\Big) D_{-\frac{\lambda}{|\alpha|}-1}\Big(-\frac{\sqrt{2|\alpha|}}{\sigma}\frac{\beta}{\alpha}\Big), \quad \alpha < 0 \\ \\ \frac{1}{2}\,\mathcal{I}(x_0) - \frac{2^{\frac{\lambda}{\alpha}-\frac{1}{2}}}{\pi} \exp\Big\{-\frac{\alpha}{2\sigma^2}\Big[\Big(x_0+\frac{\beta}{\alpha}\Big)^2 - \Big(\frac{\beta}{\alpha}\Big)^2\Big]\Big\}\,\Gamma\Big(\frac{1}{2}+\frac{\lambda}{2\alpha}\Big) \\ \qquad \times\Gamma\Big(1+\frac{\lambda}{2\alpha}\Big)\,D_{-\frac{\lambda}{\alpha}-1}\Big(\frac{\sqrt{2\alpha}}{\sigma}\Big(x_0+\frac{\beta}{\alpha}\Big)\Big) D_{-\frac{\lambda}{\alpha}}\Big(-\frac{\sqrt{2\alpha}}{\sigma}\frac{\beta}{\alpha}\Big), \qquad \alpha > 0 \end{cases} \tag{A.5}$$

where $\mathcal{I}(x_0) = 1$ for $x_0 = 0$ and $\mathcal{I}(x_0) = 0$ for $x_0 > 0$. Therefore, making use of (A.4) and (A.5) in (A.3) and recalling the following relation

$$D_{\nu-1}(z)\,D_\nu(-z) + D_\nu(z)\,D_{\nu-1}(-z) = \frac{\pi\,2^{\nu+1/2}}{\Gamma\Big(1-\frac{\nu}{2}\Big)\Gamma\Big(\frac{1}{2}-\frac{\nu}{2}\Big)},$$

one obtains (4.13).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. V. Giorno, A. G. Nobile, L. M. Ricciardi, On some time non homogeneous diffusion approximations to queueing systems, *Adv. Appl. Prob.*, **19** (1987), 974–994. https://doi.org/10.2307/1427111
2. A. R. Ward, P. W. Glynn, A diffusion approximation for a Markovian queue with reneging, *Queueing Syst.*, **43** (2003), 103–128. https://doi.org/10.1023/A:1021804515162
3. A. Di Crescenzo, B. Martinucci, A. Rhandi, A multispecies birth-death-immigration process and its diffusion approximation, *J. Math. Anal. Appl.*, **442** (2016), 291–316. https://doi.org/10.1016/j.jmaa.2016.04.059
4. A. Di Crescenzo, V. Giorno, B. K. Kumar, A. G. Nobile, $M/M/1$ queue in two alternating environments and its heavy traffic approximation, *J. Math. Anal. Appl.*, **465** (2018), 973–1001. https://doi.org/10.1016/j.jmaa.2018.05.043
5. V. Linetsky, Computing hitting time densities for CIR and OU diffusions, Applications to mean-reverting models, *J. Comput. Finance*, **7** (2004), 1–22. https://doi.org/10.21314/JCF.2004.120

6. D. Veestraeten, Valuing stock options when prices are subject to a lower boundary, *Journal of Futures Markets*, **28** (2008), 231–247. https://doi.org/10.1002/fut.20299

7. P. Lánský, S. Ditlevsen, A review of the methods for signal estimation in stochastic diffusion leaky integrate-and-fire neuronal models, *Biol. Cybern.*, **99** (2008), 253–262. https://doi.org/10.1007/s00422-008-0237-x

8. A. Buonocore, L. Caputo, A. G. Nobile, E. Pirozzi, Gauss-Markov processes in the presence of a reflecting boundary and applications in neuronal models, *Appl. Math. Comp.*, **232** (2014), 799–809. https://doi.org/10.1016/j.amc.2014.01.143

9. A. Buonocore, L. Caputo, A. G. Nobile, E. Pirozzi, Restricted Ornstein-Uhlenbeck process and applications in neuronal models with periodic input signals, *J. Comp. Appl. Math.*, **285** (2015), 59–71. https://doi.org/10.1016/j.cam.2015.01.042

10. G. D'Onofrio, P. Lánský, E. Pirozzi, On two diffusion neuronal models with multiplicative noise: the mean first-passage time properties, *Chaos*, **28** (2018), Article number 043103. https://doi.org/10.1063/1.5009574

11. A. Di Crescenzo, V. Giorno, A. G. Nobile, Analysis of reflected diffusions via an exponential time-based transformation, *J. Stat. Phys.*, **163** (2016), 1425–1453. https://doi.org/10.1007/s10955-016-1525-9

12. V. Giorno, A. G. Nobile, Restricted Gompertz-type diffusion processes with periodic regulation functions, *Mathematics*, **7** (2019), Article number 555. https://doi.org/10.3390/math7060555

13. V. Giorno, A. G. Nobile, On a time-inhomogeneous diffusion process with discontinuous drift. *Appl. Math. Comp.*, **451** (2023), Article number 128012. https://doi.org/10.1016/j.amc.2023.128012

14. Y. Mishura, A. Yurchenko-Tytarenko, Standard and fractional reflected Ornstein-Uhlenbeck processes as the limits of square roots of Cox-Ingersoll-Ross processes, *Stochastics*, **95** (2023), 99–117. https://doi.org/10.1080/17442508.2022.2047188

15. L. M. Ricciardi, A. Di Crescenzo, V. Giorno, A. G. Nobile, An outline of theoretical and algorithmic approaches to first passage time problems with applications to biological modeling, *Math. Japonica*, **50** (1999), 247–322.

16. J. Masoliver, J. Perelló, First-passage and escape problems in the Feller process, *Phys. Rev. E*, **86** (2012), Article number 041116. https://doi.org/10.1103/PhysRevE.86.041116

17. L. Bo, G. Ren, Y. Wang, X. Yang, First passage times of reflected generalized Ornstein-Uhlenbeck processes, *Stochastics Dyn.*, **13** (2013), 1–16. https://doi.org/10.1142/S0219493712500141

18. M. Abundo, E. Pirozzi, Integrated stationary Ornstein-Uhlenbeck process, and double integral processes, *Phys. A*, **494** (2018), 265–275. https://doi.org/10.1016/j.physa.2017.12.043

19. V. Giorno, A. G. Nobile, On the absorbing problems for Wiener, Ornstein-Uhlenbeck and Feller diffusion processes: Similarities and differences, *Fractal Fract.*, **7** (2023), Article number 11. https://doi.org/10.3390/ fractalfract7010011

20. E. B. Dynkin, Kolmogorov and the theory of Markov processes, *Ann. Probab.*, **17** (1989), 822–832. https://www.jstor.org/stable/2244385

21. S. Karlin, H. W. Taylor, *A Second Course in Stochastic Processes*, Academic Press, New York, 1981.

22. A. J. F. Siegert, On the first passage time probability problem, *Phys. Rev.* , **81** (1951), 617—623. https://doi.org/10.1103/PhysRev.81.617

23. V. Giorno, A. G. Nobile, L. M. Ricciardi, On neuronal firing modeling via specially confined diffusion processes, *Scientiae Mathematicae Japonicae*, **58** (2003), 265–294.

24. J. F. C. Kingman, On queue in heavy traffic, *J. R. Stat. Soc. B*, **24** (1962), 383–392. http://www.jstor.org/stable/2984229

25. J. M. Harrison, The diffusion approximation for tandem queues in heavy traffic, *Adv. Appl. Prob.*, **10(4)** (1978), 886–905. https://doi.org/10.2307/1426665

26. V. Linetsky, On the transition densities for reflected diffusions, *Adv. Appl. Prob.*, **37** (2005), 435–460. http://www.jstor.org/stable/30037335

27. A. Molini, P. Talkner, G. G. Katul, A. Porporato, First passage time statistics of Brownian motion with purely time dependent drift and diffusion, *Phys. A*, **390** (2011), 1841–1852. https://doi.org/10.1016/j.physa.2011.01.024

28. D. R. Cox, H. D. Miller, *The Theory of Stochastic Processes*, Chapman & Hall/CRC, Boca Raton, Florida, 1996.

29. A. Buonocore, A. G. Nobile, E. Pirozzi, Simulation of sample paths for Gauss-Markov processes in the presence of a reflecting boundary, *Cogent Math.*, **4** (2017), Article number 1354469. https://doi.org/10.1080/23311835.2017.1354469

30. V. Giorno, A. G. Nobile, R. di Cesare, On the reflected Ornstein-Uhlenbeck process with catastrophes, *Appl. Math. Comp.*, **218** (2012), 11570–11582. https://doi.org/10.1016/j.amc.2012.04.086

31. A. R. Ward, P. W. Glynn, Properties of the reflected Ornstein-Uhlenbeck process, *Queueing Syst.*, **44** (2003), 109–123. https://doi.org/10.1023/A:1024403704190

32. Y. Nie, V. Linetsky, Sticky reflecting Ornstein-Uhlenbeck diffusions and the Vasicek interest rate model with the sticky zero lower bound, *Stochastic Models*, **36** (2020), 1–19. https://doi.org/10.1080/15326349.2019.1630287

33. I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press Inc., 2014. https://doi.org/10.1016/C2010-0-64839-5

34. V. Giorno, A. G. Nobile, Time-inhomogeneous Feller-type diffusion process in population dynamics, *Mathematics*, **9** (2021), Article number 1879. https://doi.org/10.3390/math9161879

35. A. Di Crescenzo, A. G. Nobile, Diffusion approximation to a queueing system with time-dependent arrival and service rates, *Queueing Syst.*, **19** (1995), 41–62. https://doi.org/10.1007/BF01148939

36. V. Giorno, P. Lánský, A. G. Nobile, L. M. Ricciardi, Diffusion approximation and first-passage-time problem for a model neuron. III. A birth-and-death process approach, *Biol. Cyber.*, **58(6)** (1988), 387–404. https://doi.org/10.1007/BF00361346

37. S. Ditlevsen, P. Lánský, Estimation of the input parameters in the Feller neuronal model, *Phys. Rev. E*, **73** (2006), Article number 061910. https://doi.org/10.1103/PhysRevE.73.061910

38. Y. Tian, H. Zhang, Skew CIR process, conditional characteristic function, moments and bond pricing, *Appl. Math. Comput.*, **329** (2018), 230–238. https://doi.org/10.1016/j.amc.2018.02.013

39. J. C. Cox, J. E. Ingersoll Jr., S. A. Ross, A theory of the term structure of interest rates, *Econometrica*, **53** (1985), 385–407. https://doi.org/10.2307/1911242

40. E. Di Nardo, G. D'Onofrio, A cumulant approach for the first-passage-time problem of the Feller square-root process, *Appl. Math. Comput.*, **391** (2021), Article Number 125707. https://doi.org/10.1016/j.amc.2020.125707

41. J. Masoliver, Nonstationary Feller process with time-varying coefficients, *Phys. Rev. E*, **93** (2016), Article number 012122. https://doi.org/10.1103/PhysRevE.93.012122

42. W. Feller, Two singular diffusion problems, *Ann. Math.*, **5** (1951), 173–182. https://doi.org/10.2307/1969318

43. V. Giorno, A. G. Nobile, On the first-passage time problem for a Feller-type diffusion process, *Mathematics*, **9** (2021), Article number 2470. https://doi.org/10.3390/math9192470

44. V. Giorno, A. G. Nobile, L. M. Ricciardi, On the densities of certain bounded diffusion processes, *Ricerche Mat.*, **60** (2011), 89–124. https://doi.org/10.1007/s11587-010-0097-2