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Boundary conditions and Schwarz waveform relaxation method for linear viscous Shallow Water equations in hydrodynamics

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Abstract. We propose in the present work an extension of the Schwarz waveform relaxation method to the case of viscous shallow water system with advection term. We first show the difficulties that arise when approximating the Dirichlet to Neumann operators if we consider an asymptotic analysis based on large Reynolds number regime and a small domain aspect ratio. Therefore we focus on the design of a Schwarz algorithm with Robin like boundary conditions. We prove the well-posedness and the convergence of the algorithm.

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1. Introduction

As for many other disciplines, numerical modeling studies in the field of hydrodynamics continuously address more and more complex situations. Rather than working with one single numerical model, numerous present studies require the use of a *numerical modeling system*, i.e. several numerical models, possibly solving different systems of equations (e.g. shallow water, Euler, Navier-Stokes, hydrostatic or non hydrostatic, mono-phasic or multi-phasic) and/or working in different dimensions (1D, 2D, 3D). These models must work together, i.e. must regularly exchange relevant information. Coupling algorithms are thus needed, that ensure a correct exchange of information in order for the whole system to actually solve the target problem. However the models are complex and are generally developed independently from each other. Therefore a desirable feature for coupling algorithms is to be non intrusive. In this regard, the Schwarz algorithms, initially designed in the context of domain decomposition [4, 11, 17, 22] are good candidates, since they do not require any change in the models but only exchange information through boundary conditions. So called Schwarz *global-in-time* or *waveform relaxation* algorithms [5, 6, 7] even allow for the different models to use different time steps, thus preventing too many communications between the models by gathering the exchange of information at the end of time windows corresponding typically to several tens or hundreds of time steps. Their main drawback however is their iterative nature, that can potentially lead to huge computation costs. These methods must thus be optimized in order to minimize the number of iterations that are required to make the system converge. This convergence speed is directly linked to the boundary conditions that are used at the interfaces between the models, which means that one must actually optimize these conditions. It can be shown that so-called *perfectly transparent*, or equivalently *perfectly* *absorbing*, boundary conditions [3] lead to an exact convergence in only two iterations [10]. However those boundary conditions are generally non local neither in time nor in space and cannot be applied directly, but must be approximated by local tractable operators [4]. Moreover things are even more complex in the context of coupled models with heterogeneous dimensions (e.g. 2D-3D) since extension and reduction operators must be added. An example of such a study is given in [21] for a simple toy model.

Our long term applicative objective is to design efficient Schwarz algorithms for coupling 1D-2D shallow water (SW) models with 3D Navier-Stokes (NS) models. As an example in river hydraulics, we could consider the coupling between 1D SW in straight parts of the river, with 2D SW in more curly regions and/or with 3D NS equations in other specific regions where accurate non-hydrostatic models should be used (*e.g.* near a hydroelectric power plant). Since efficient interface conditions are required, a necessary preliminary step is to study the derivation of exact and approximate absorbing boundary conditions for those systems of equations. A number of previous works deal with Schwarz-type algorithms for Stokes, Navier-Stokes and Oseen (i.e. linearized Navier-Stokes) systems. They study either Dirichlet-Dirichlet [18], Dirichlet-Neumann [23], Neumann-Neumann [16], Robin-Robin [15, 14] or optimized [2] algorithms. But some work still remains to be done to provide efficient conditions for 3D fully non linear Navier-Stokes equations. Regarding shallow water equations, the question of perfectly absorbing conditions has been studied in [9] in the general case of incompletely parabolic equations. However, the approximate conditions proposed in this work rely on a (strong) approximation neglecting the y-direction. More recently, the optimized Schwarz waveform relaxation method was applied in [13] to the linearized shallow water system but without advection term, which is limiting for realistic applications where one generally needs to linearize around a nonzero velocity. In this work, we propose to extend this approach to the case of viscous shallow water equations linearized around a nonzero velocity, hence considering the advection term. Beyond this generalization, this work is also a first step to set up a Schwarz algorithm for nonlinear shallow water system. Besides, and as mentioned above, it has been proved in [20] that under some assumptions we can use the algorithm developed here to set up efficient multi-dimensional and multi-model Schwarz coupling algorithms.
This paper is organized as follows: in Section 2 we write the equations and study the well-posedness of the system. Due to the similar mathematical nature of viscous shallow water equations and of primitive equations of the ocean (the barotropic, *i*.e. vertically integrated, part of the primitive equations corresponds to the shallow water system with advection), we reuse in our work developments presented in [1] (see also [19]) where the optimized Schwarz waveform relaxation method was applied to the primitive equations. Let us mention however that the work in [1] largely uses the smallness of the Rossby parameter, which is not the case here where we consider non-rotating equations (*i.e.* neglecting the Coriolis force). In Section 3 we define the Schwarz waveform relaxation algorithm and we write the perfectly absorbing boundary conditions. Then we show the difficulties that arise when deriving approximate Dirichlet-to-Neumann operators from an asymptotic analysis. Finally we propose in Section 4 to approximate the Dirichlet-to-Neumann operators by constant values, which leads to Robin-like boundary conditions. We study the well-posedness of the corresponding algorithm and prove its convergence, which is also a novelty of this work in comparison with [1].

2. **Well-posedness of linearized viscous shallow water equations**

In order to derive efficient interface conditions for 2-D viscous shallow water equations (sections 3 and 4), we have first to write their linearized approximation and to prove the well-posedness of this system.

2.1. **Linearized system**

Let us consider the 2-D viscous shallow water equations:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}.\nabla)\,\mathbf{u} + g\nabla\zeta - \mu\Delta\mathbf{u} &= 0, \\ \partial_t \zeta + \text{ div }((H+\zeta)\mathbf{u}) + \mathbf{u}.\nabla\zeta &= 0, \end{cases} \qquad (x,y,t) \text{ in } \omega \times \mathbb{R}^+. \tag{2.1}$$

where ω is an open domain of $\mathbb{R}^2$, $\mathbf{u} = (u,v)^T$ is the velocity, and ζ is the free surface anomaly w.r.t. H, the surface height at rest. The total depth of the water column $H + \zeta$ is supposed to be small w.r.t. the horizontal length scale (*shallow water approximation*), see Figure 2.1. We denote by g the gravity acceleration and by μ the viscosity. This system of equations must of course be complemented with initial and boundary conditions. Note that, since we are interested in river dynamics, we do not consider here the Coriolis force, which must be taken into account in the case of ocean dynamics.

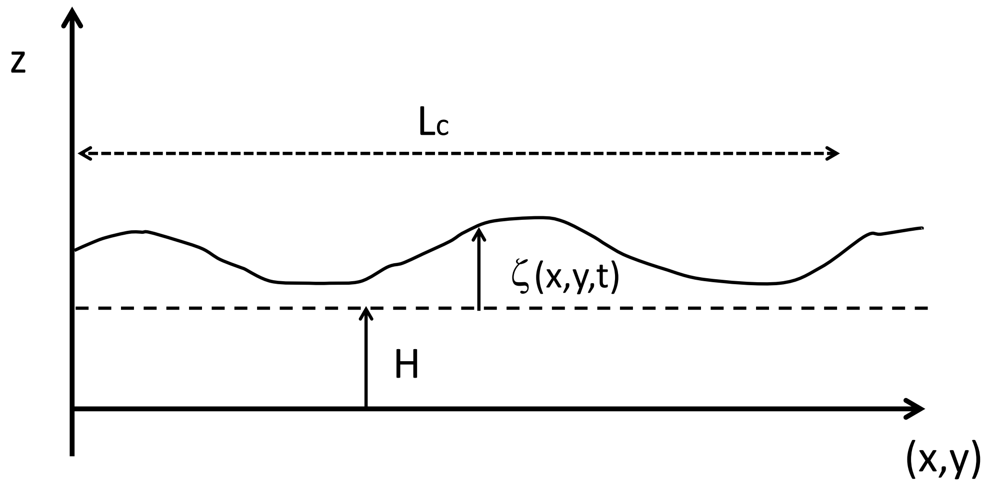

Figure 2.1. Computational 3-D domain where the shallow water approximation is performed.

Linearizing this system around $\mathbf{U}_0 = (u_0, v_0)^t$ and $\zeta_0 = 0$, and adding initial conditions, leads to:

$$\partial_t \mathbf{u} + (\mathbf{U}_0.\nabla)\,\mathbf{u} + g\nabla\zeta - \mu\Delta\mathbf{u} = 0 \ \text{ in } \omega \times \mathbb{R}^+, \tag{2.2a}$$

$$\partial_t \zeta + H \text{ div } \mathbf{u} + \mathbf{U}_0.\nabla\zeta = 0 \ \text{ in } \omega \times \mathbb{R}^+, \tag{2.2b}$$

$$\mathbf{u}(.,0) = \mathbf{u}^{ini} \ \text{ in } \omega, \tag{2.2c}$$

$$\zeta(.,0) = \zeta^{ini} \ \text{ in } \omega. \tag{2.2d}$$

Note that this set of equations can be derived from the linearized hydrostatic Navier-Stokes equations by assuming a shallow 3-D domain and considering a null bottom friction [8].
In the following, we will consider homogeneous boundary conditions for (2.2) on $\partial\omega$ (or when $\|(x,y)\| \to \infty$ if ω is unbounded). This is actually not restrictive, since we will work with error fields, which do satisfy this boundary condition.

2.2. **Well-posedness**

Let us now define a weak formulation of system (2.2) and prove its well-posedness. Since the proof is quite similar to the one for the linearized primitive equations given in [1], we only give here the outline and refer to [1] for more details.

Note first that $\mathbf{u}$ is solution of a linear parabolic problem (2.2a)-(2.2c) with a source term depending on ζ, and that ζ is solution of a linear transport equation (2.2b)-(2.2d) with a source term depending on $\mathbf{u}$. To prove the well-posedness of (2.2) we will thus first study the parabolic system and the transport equation separately, and will then conclude by using fixed-point argument.
Let us first introduce the notion of weak solution for (2.2). In the sequel T denotes the length of the time interval ($0 < T \le \infty$).

Definition 2.1. Let $X^{ini} = (\mathbf{u}^{ini}, \zeta^{ini}) \in L^2(\omega, \mathbb{R}^2) \times L^2(\omega)$. We say that

$$X = (\mathbf{u}, \zeta) \in L^2(0,T; H^1(\omega, \mathbb{R}^2)) \times L^2(\omega \times (0,T))$$

is a weak solution of (2.2) if

$$\begin{cases} \dfrac{d}{dt}(\mathbf{u}, \mathbf{v})_\omega + ((\mathbf{U}_0.\nabla)\mathbf{u}, \mathbf{v})_\omega + \mu(\nabla\mathbf{u}, \nabla\mathbf{v})_\omega = -g\,(\nabla\zeta, \mathbf{v})_\omega \quad \forall\, \mathbf{v} \in H^1(\omega, \mathbb{R}^2,) \\ \mathbf{u}(.,0) = \mathbf{u}^{ini} \quad \text{in } \omega, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \dfrac{d}{dt}(\zeta, \chi)_\omega + (\mathbf{U}_0.\nabla\zeta, \chi)_\omega = -H\,(\text{div}\,\mathbf{u}, \chi)_\omega \quad \forall\, \chi \in L^2(\omega), \\ \zeta(.,0) = \zeta^{ini} \quad \text{in } \omega, \end{cases} \tag{2.4}$$

where $(.,.)_\omega$ is the scalar product in ω.

The following well-posedness result is proven in Appendix A.

Proposition 2.2. *Let $X^{ini} = (\mathbf{u}^{ini}, \zeta^{ini}) \in L^2(\omega, \mathbb{R}^2) \times L^2(\omega)$. There exists a unique weak solution $X = (\mathbf{u}, \zeta)$ of (2.2) in $\big(C(0,T; L^2(\omega, \mathbb{R}^2) \cap L^2(0,T; H^1(\omega, \mathbb{R}^2))\big) \times \big(L^2(\omega \times (0,T)) \cap C(0,T; L^2(\omega))\big)$.*

3. Schwarz waveform relaxation algorithm with absorbing boundary conditions

3.1. Schwarz waveform relaxation algorithm

As explained in Section 1, our goal is to derive efficient boundary conditions for solving shallow water equations with a Schwarz waveform relaxation algorithm. Let us split the computational domain ω in two subdomains ω^- and ω^+. Since our ultimate goal is to couple different systems of equations (corresponding to diverse regimes), these subdomains must not overlap (contrary to usual domain decomposition problems, where the same system of equations is solved on several subdomains). We thus define ω^- and ω^+ by $\omega = \omega^- \cup \omega^+ = (\mathbb{R}^- \times \mathbb{R}) \cup (\mathbb{R}^+ \times \mathbb{R})$. Their interface is $\Gamma = \{0\} \times \mathbb{R}$.
Let $\mathcal{L}_{LSW}$ be the set of operators corresponding to (2.2a)-(2.2b), $X = (\mathbf{u}, \zeta)$ and $X^{ini} = (\mathbf{u}^{ini}, \zeta^{ini})$. The $(k+1)^{\text{th}}$ iteration of the Schwarz waveform relaxation algorithm reads:

$$\left\{\begin{array}{rcl} \mathcal{L}_{LSW}\left(X_-^{k+1}\right) & = & 0 \;\text{ in }\; \omega^- \times (0,T), \\ \mathcal{B}_-\left(X_-^{k+1}\right) & = & \mathcal{B}_-\left(X_+^{k}\right) \;\text{ on }\; \Gamma_T, \\ X_-^{k+1}(.,0) & = & X_-^{ini} \;\text{ in }\; \omega^-, \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcl} \mathcal{L}_{LSW}\left(X_+^{k+1}\right) & = & 0 \;\text{ in }\; \omega^+ \times (0,T), \\ \mathcal{B}_+\left(X_+^{k+1}\right) & = & \mathcal{B}_+\left(X_-^{k+1}\right) \;\text{ on }\; \Gamma_T, \\ X_+^{k+1}(.,0) & = & X_+^{ini} \;\text{ in }\; \omega^+, \end{array}\right. \tag{3.1}$$

where $\Gamma_T = \Gamma \times (0,T)$ and $\mathcal{B}_-$ and $\mathcal{B}_+$ are interface boundary operators to be defined later. These operators must be chosen in order to ensure that the Schwarz algorithm converges, and that this convergence is fast. In this section, our strategy to derive such efficient interface conditions is to rely on so-called *perfectly transparent* (or *perfectly absorbing*) boundary conditions [3].

3.2. Natural transmission conditions

In order to derive perfectly transparent boundary conditions in the next subsection, let us first write the quantities that are naturally preserved through the interface Γ, which are also called *natural transmission conditions.*

Proposition 3.1. *The physical constraint through the interface* $\Gamma \times \mathbb{R}^+$ *is the continuity of*

$$\left(\mu\partial_x \mathbf{u} - u_0\mathbf{u} - g\begin{pmatrix}\zeta\\0\end{pmatrix}, u_0\zeta + H\,u\right). \tag{3.2}$$

Proof. This result is obtained from the variational formulation of (2.2). Let us consider $\mathbf{v} \in \mathcal{D}(\overline{\omega}, \mathbb{R}^2)$. Multiplying (2.2a) by $\mathbf{v}$ and integrating over ω leads to

$$\int_\omega \partial_t \mathbf{u}.\mathbf{v} + \int_\omega (\mathbf{U}_0.\nabla)\,\mathbf{u}.\mathbf{v} - \mu\int_\omega \Delta\mathbf{u}.\mathbf{v} + g\int_\omega \nabla\zeta.\mathbf{v} = 0. \tag{3.3}$$

Integrating by parts and using the fact that $\mathbf{v}$ has a compact support, (3.3) becomes:

$$\int_\omega \partial_t \mathbf{u}.\mathbf{v} - \int_\omega (\mathbf{U}_0.\nabla)\,\mathbf{v}.\mathbf{u} + \mu\int_\omega \nabla\mathbf{u} : \nabla\mathbf{v} - g\int_\omega \zeta \text{ div } \mathbf{v} = 0. \tag{3.4}$$

Since $\omega = \omega^- \cup \omega^+$, (3.3) also reads:

$$\begin{aligned}&\int_{\omega^+} \partial_t \mathbf{u}.\mathbf{v} + \int_{\omega^+} (\mathbf{U}_0.\nabla)\,\mathbf{u}.\mathbf{v} - \mu\int_{\omega^+} \Delta\mathbf{u}.\mathbf{v} + g\int_{\omega^+} \nabla\zeta.\mathbf{v}\\ +&\int_{\omega^-} \partial_t \mathbf{u}.\mathbf{v} + \int_{\omega^-} (\mathbf{U}_0.\nabla)\,\mathbf{u}.\mathbf{v} - \mu\int_{\omega^-} \Delta\mathbf{u}.\mathbf{v} + g\int_{\omega^-} \nabla\zeta.\mathbf{v} \quad = \quad 0.\end{aligned}$$

Integrating by parts in each subdomain, this expression becomes:

$$\begin{aligned}&\int_{\omega^+} \partial_t \mathbf{u}^+.\mathbf{v} - \int_{\omega^+} (\mathbf{U}_0.\nabla)\,\mathbf{v}.\mathbf{u}^+ + \int_\Gamma u_0(\mathbf{u}^+.\mathbf{v})n_1^+ + \int_\Gamma v_0(\mathbf{u}^+.\mathbf{v})n_2^+ + \mu\int_{\omega^+} \nabla\mathbf{u}^+ : \nabla\mathbf{v}\\ &\qquad -\mu\int_\Gamma \partial_{n^+}\mathbf{u}^+.\mathbf{v} - g\int_{\omega^+} \zeta \text{ div } \mathbf{v} + g\int_\Gamma \zeta^+\begin{pmatrix}n_1^+\\n_2^+\end{pmatrix}.\mathbf{v}\\ +&\int_{\omega^-} \partial_t \mathbf{u}^-.\mathbf{v} - \int_{\omega^-} (\mathbf{U}_0.\nabla)\,\mathbf{v}.\mathbf{u}^- + \int_\Gamma u_0(\mathbf{u}^-.\mathbf{v})n_1^- + \int_\Gamma v_0(\mathbf{u}^-.\mathbf{v})n_2^- + \mu\int_{\omega^-} \nabla\mathbf{u}^- : \nabla\mathbf{v}\\ &\qquad -\mu\int_\Gamma \partial_{n^-}\mathbf{u}^-.\mathbf{v} - g\int_{\omega^-} \zeta \text{ div } \mathbf{v} + g\int_\Gamma \zeta^-\begin{pmatrix}n_1^-\\n_2^-\end{pmatrix}.\mathbf{v}\\ &= 0.\end{aligned}$$

where $\mathbf{u}^+$, $\mathbf{u}^-$ and ζ^+, ζ^- denote the value of $\mathbf{u}$ and ζ on both sides of Γ, $n^+ = (n_1^+, n_2^+)^T$ and $n^- = (n_1^-, n_2^-)^T$ are the unit outward vectors normal to ω^+ and ω^- ($n^+ = -n^-$ on Γ). Gathering the terms on ω_+ and ω_- and subtracting (3.4) leads to:

$$\begin{aligned}\forall\,\mathbf{v} \in \mathcal{D}(\overline{\omega}, \mathbb{R}^2) \qquad &\int_\Gamma (u_0\mathbf{u}^+ n_1^+ + v_0\mathbf{u}^+ n_2^+).\mathbf{v} - \mu\int_\Gamma \partial_{n^+}\mathbf{u}^+.\mathbf{v} + g\int_\Gamma \zeta^+\begin{pmatrix}n_1^+\\n_2^+\end{pmatrix}.\mathbf{v}\\ +&\int_\Gamma (u_0\mathbf{u}^- n_1^- + v_0\mathbf{u}^- n_2^-).\mathbf{v} - \mu\int_\Gamma \partial_{n^-}\mathbf{u}^-.\mathbf{v} + g\int_\Gamma \zeta^-\begin{pmatrix}n_1^-\\n_2^-\end{pmatrix}.\mathbf{v} \quad = \quad 0.\end{aligned}$$

Therefore the following equality on Γ holds:

$$-\mu\partial_{n^+}\mathbf{u}^+ + (u_0\mathbf{u}^+ n_1^+ + v_0\mathbf{u}^+ n_2^+) + g\zeta^+\begin{pmatrix}n_1^+\\n_2^+\end{pmatrix} = \mu\partial_{n^-}\mathbf{u}^- - (u_0\mathbf{u}^- n_1^- + v_0\mathbf{u}^- n_2^-) - g\zeta^-\begin{pmatrix}n_1^-\\n_2^-\end{pmatrix}.$$

The interface Γ being $\{x=0\}$, then $n^+ = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $n^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore the preceding relationship corresponds to the continuity through the interface Γ of the quantity $\mu\partial_x \mathbf{u} - u_0\mathbf{u} - g\begin{pmatrix} \zeta \\ 0 \end{pmatrix}$. Working similarly with (2.2b) leads to the continuity of $u_0\zeta + H\,u$ through Γ. ■

3.3. **Perfectly transparent boundary conditions**

Based on these naturally transmitted quantities, let us now define the so-called Dirichlet-to-Neumann operators. We consider in the following the case $u_0 > 0$. The case $u_0 < 0$ can be tackled similarly.

Definition 3.2. Let $X_b = (\mathbf{u}_b, \zeta_b)$ be a Dirichlet data. The operator $\mathcal{S}_-^{\mathbf{u}}$ is defined by:

$$\mathcal{S}_-^{\mathbf{u}} : (\Gamma \times \mathbb{R}^+)^3 \longrightarrow \mathbb{R}^2 \tag{3.5}$$

$$(\mathbf{u}^b, \zeta^b) \longmapsto \left(\mu\partial_x \mathbf{u} - u_0\mathbf{u} - g\begin{pmatrix} \zeta \\ 0 \end{pmatrix}\right)|_{\Gamma\times\mathbb{R}^+} \tag{3.6}$$

where $(\mathbf{u}, \zeta)$ is a solution of the homogeneous system $\mathcal{L}_{LSW} = 0$ on ω^+ with a zero initial condition and Dirichlet boundary condition $(\mathbf{u}_b, \zeta_b)$ on $\Gamma \times (0, T)$.
Similarly the operators $\mathcal{S}_+^{\mathbf{u}}$ and $\mathcal{S}_+^{\zeta}$ are defined by:

$$\left(\mathcal{S}_+^{\mathbf{u}}, \mathcal{S}_+^{\zeta}\right) : (\Gamma \times \mathbb{R}^+)^3 \longrightarrow \mathbb{R}^2 \times \mathbb{R} \tag{3.7}$$

$$(\mathbf{u}^b, \zeta^b) \longmapsto \left(-\mu\partial_x \mathbf{u} + u_0\mathbf{u} + g\begin{pmatrix} \zeta \\ 0 \end{pmatrix}, u_0\zeta + H\,u\right)|_{\Gamma\times\mathbb{R}^+} \tag{3.8}$$

where $(\mathbf{u}, \zeta)$ is a solution of the homogeneous system $\mathcal{L}_{LSW} = 0$ on ω^- with a zero initial condition and Dirichlet boundary condition $\mathbf{u}_b$ on $\Gamma \times (0, T)$. Note that we do not need to prescribe a boundary condition for ζ as we supposed $u_0 > 0$, see [1].

Thanks to this definition of Dirichlet-to-Neumann operators, we can depict perfectly transparent boundary conditions and implement them in the Schwarz waveform relaxation algorithm.

Proposition 3.3. *The Schwarz waveform relaxation algorithm using the optimal boundary conditions:*

$$\mathcal{B}_-(\mathbf{u}, \zeta) = \mathcal{B}_-^{transp}(\mathbf{u}, \zeta) = \mu\partial_x \mathbf{u} - u_0\mathbf{u} - g\begin{pmatrix} \zeta \\ 0 \end{pmatrix} - \mathcal{S}_-^{\mathbf{u}}(\mathbf{u}, \zeta)$$

and

$$\mathcal{B}_+(\mathbf{u}, \zeta) = \mathcal{B}_+^{transp}(\mathbf{u}, \zeta) = \left(-\mu\partial_x \mathbf{u} + u_0\mathbf{u} + g\begin{pmatrix} \zeta \\ 0 \end{pmatrix} - \mathcal{S}_+^{\mathbf{u}}(\mathbf{u}, \zeta)\ ,\ u_0\zeta + Hu - \mathcal{S}_+^{\zeta}(\mathbf{u}, \zeta)\right)$$

converges exactly after two iterations.

Proof. Let us define the errors $\tilde{X}_-^k = X_{|\omega^-} - X_-^k$ and $\tilde{X}_+^k = X_{|\omega^+} - X_+^k$ at iteration k. They satisfy:

$$\left\{\begin{array}{rcl} \mathcal{L}_{LSW}\left(\tilde{X}_-^{k+1}\right) &=& 0 \text{ in } \omega_t^-, \\ \mathcal{B}_-^{\text{transp}}\left(\tilde{X}_-^{k+1}\right) &=& \mathcal{B}_-^{\text{transp}}\left(\tilde{X}_+^{k}\right) \text{ on } \Gamma_t, \\ \tilde{X}_-^{k+1}(.,0) &=& 0 \text{ in } \omega^-, \end{array}\right. \text{ and } \left\{\begin{array}{rcl} \mathcal{L}_{LSW}\left(\tilde{X}_+^{k+1}\right) &=& 0 \text{ in } \omega_t^+, \\ \mathcal{B}_+^{\text{transp}}\left(\tilde{X}_+^{k+1}\right) &=& \mathcal{B}_+^{\text{transp}}\left(\tilde{X}_-^{k+1}\right) \text{ on } \Gamma_t, \\ \tilde{X}_+^{k+1}(.,0) &=& 0 \text{ in } \omega^+, \end{array}\right. \tag{3.9}$$

where $\Gamma_t = \Gamma \times (0, T)$.
At iteration 1, the choice of $\tilde{X}_+^0$ is random and nothing special happens. At iteration 2, the boundary conditions vanish due to the definition of $\mathcal{S}_-^{\mathbf{u}}$, $\mathcal{S}_+^{\mathbf{u}}$ and $\mathcal{S}_+^{\zeta}$. The errors $\tilde{X}_-^2$ and $\tilde{X}_+^2$ are thus equal to zero, since they are solutions of (3.9) with zero right hand sides. ■

Let us now exhibit an analytical expression for these optimal boundary conditions. The expression for $\mathcal{S}_+^\zeta$ is obvious: $\mathcal{S}_+^\zeta(\mathbf{u}^b, \zeta^b) = u_0\zeta + Hu$. However, as we will see later, $\mathcal{S}_+^{\mathbf{u}}$ and $\mathcal{S}_-^{\mathbf{u}}$ are non local operators, both in time and space, and thus are not tractable for actual computations. Therefore we will have to derive local approximations (subsection 3.4), for instance by performing Taylor expansion w.r.t. small parameters. That is why we have first to write the dimensionless form of (2.2), to make such small parameters appear clearly. Let us introduce the dimensionless variables and quantities:

$$(x, y) = L_c(\tilde{x}, \tilde{y}), \quad t = \frac{L_c}{U_c}\tilde{t}, \quad \mathbf{u} = U_c\tilde{\mathbf{u}}, \quad \zeta = H\tilde{\zeta}$$

and

$$\nu = \frac{1}{\mathrm{Re}} = \frac{\mu}{L_c U_c}, \quad \mathrm{Fr} = \frac{U_c}{\sqrt{gH}}$$

where L_c is a characteristic horizontal length, U_c is a characteristic velocity, and Re and Fr denote respectively the Reynolds number and the Froude number. The dimensionless system corresponding to (2.2a)-(2.2b) reads:

$$\begin{aligned} \partial_{\tilde{t}}\tilde{\mathbf{u}} + \left(\tilde{\mathbf{U}}_0.\tilde{\nabla}\right)\tilde{\mathbf{u}} + \frac{1}{\mathrm{Fr}^2}\tilde{\nabla}\tilde{\zeta} - \nu\tilde{\Delta}\tilde{\mathbf{u}} &= 0 \ \text{ in } \tilde{\omega}\times\mathbb{R}^+, && (3.10a)\\ \partial_{\tilde{t}}\tilde{\zeta} + \tilde{\mathrm{div}}\,\tilde{\mathbf{u}} + \tilde{\mathbf{U}}_0.\tilde{\nabla}\tilde{\zeta} &= 0 \ \text{ in } \tilde{\omega}\times\mathbb{R}^+. && (3.10b)\end{aligned}$$

For the sake of simplicity, we will drop the tilde symbols in the following. Computing the Laplace-Fourier transform (Laplace in time, Fourier in the direction normal to Γ, i.e. the y direction) of (3.10a)-(3.10b) with zero initial conditions leads to:

$$\left\{\begin{aligned} -\nu\partial_x^2\hat{\mathbf{u}} + u_0\partial_x\hat{\mathbf{u}} + \left\{s + i\eta v_0 + \nu\eta^2\right\}\hat{\mathbf{u}} + \frac{1}{\mathrm{Fr}^2}\begin{pmatrix}\partial_x\hat{\zeta}\\ \\ i\eta\hat{\zeta}\end{pmatrix} &= 0,\\ u_0\partial_x\hat{\zeta} + (s + i\eta v_0)\hat{\zeta} + \partial_x\hat{u} + i\eta\hat{v} &= 0,\end{aligned}\right. \tag{3.11}$$

where s is the Laplace symbol, η is the Fourier symbol, and $\hat{\cdot}$ denotes the Laplace-Fourier transform. Therefore, as in [9], [13] or [2], one can look for the solution of this system under the form $\hat{X}(x) = (\hat{\mathbf{u}}(x), \hat{\zeta}(x))^T = \mathbf{\Phi}e^{\lambda x}$. System (3.11) becomes

$$\mathbf{M}(\lambda)\mathbf{\Phi} = 0 \tag{3.12}$$

where

$$\mathbf{M}(\lambda) = \begin{pmatrix} P(\lambda) & 0 & \dfrac{\lambda}{\mathrm{Fr}^2}\\ \\ 0 & P(\lambda) & \dfrac{i\eta}{\mathrm{Fr}^2}\\ \\ \lambda & i\eta & u_0\lambda + s + i\eta v_0\end{pmatrix} \quad \text{and} \quad P(\lambda) = -\nu\lambda^2 + u_0\lambda + s + \nu\eta^2 + i\eta v_0.$$

In order to find the nonzero solutions, one thus has to compute the roots of the determinant of this linear system. The determinant of $M(\lambda)$ is a polynomial of degree 5 and can be factorized as $\det(\mathbf{M}(\lambda)) = P(\lambda)Q(\lambda)$. It can be shown [9, 20] that the two roots of $P(\lambda)$ satisfy $\mathfrak{Re}(\lambda_1) > 0$ and $\mathfrak{Re}(\lambda_2) < 0$ (remind that $u_0 > 0$). In the same way the three roots of $Q(\lambda)$ satisfy $\mathfrak{Re}(\lambda_3) < 0$, $\mathfrak{Re}(\lambda_4) < 0$ and $\mathfrak{Re}(\lambda_5) > 0$. Note that in the particular case where $u_0 = v_0 = 0$, these roots coincide as expected with the ones computed in [13].

Since $\hat{X}$ must tend to zero when x tends to infinity, the roots to be considered in ω^+ are λ_2, λ_3 and

λ_4, and the roots to be considered in ω^- are λ_1 and λ_5. Therefore one can compute the exact formulas for $\widehat{\mathcal{S}}^{\mathbf{u}}_-$ and $\widehat{\mathcal{S}}^{\mathbf{u}}_+$:

$$\widehat{\mathcal{S}}^{\mathbf{u}}_- = \begin{pmatrix} \dfrac{(iu_0\eta^2 + is\lambda_2 - \eta\lambda_2 v_0)(-u_0 + \nu\lambda_3 + \nu\lambda_4)}{D} & \dfrac{N_2}{D} & \dfrac{iu_0\eta^2 + is\lambda_2 - \eta\lambda_2 v_0}{D\,\mathrm{Fr}^2} \\ \dfrac{\nu(-\eta^2\nu + \nu\lambda_2\lambda_3 + \nu\lambda_2\lambda_4 - \nu\lambda_3\lambda_4 - u_0\lambda_2 - s - i\eta v_0)\eta\lambda_2}{D} & \dfrac{N_5}{D} & \dfrac{\nu\eta(\eta^2 - \lambda_2)}{D\,\mathrm{Fr}^2} \end{pmatrix} \tag{3.13}$$

where

$$\begin{cases} D = & i\nu\eta^2(\lambda_2 - \lambda_3 - \lambda_4) + i\nu\lambda_2\lambda_3\lambda_4 + iu_0\eta^2 + is\lambda_2 - \eta\lambda_2 v_0 \\ N_2 = & -\nu\eta\,[-\nu\eta^2(\lambda_2 - \lambda_3 - \lambda_4) - \nu\lambda_2\lambda_3\lambda_4 + u_0\lambda_3\lambda_4 - s\lambda_2 + s\lambda_3 + s\lambda_4 \\ & -i\eta\lambda_2 v_0 + i\eta v_0\lambda_3 + i\eta v_0\lambda_4] \\ N_5 = & -[i\nu^2\eta^4 - i\nu^2\eta^2\lambda_2^2 + i\nu^2\eta^2\lambda_3\lambda_4 - i\nu^2\lambda_2^2\lambda_3\lambda_4 + i\nu\eta^2 u_0(\lambda_2 - \lambda_3 - \lambda_4) + i\nu u_0\lambda_2\lambda_3\lambda_4 \\ & + i\nu\eta^2 s + i\eta^2 u_0^2 - i\nu s\lambda_2^2 + isu_0\lambda_2 - \nu\eta^3 v_0 + \nu\eta v_0\lambda_2^2 - \eta\lambda_2 u_0 v_0] \end{cases}$$

and

$$\widehat{\mathcal{S}}^{\mathbf{u}}_+ = \begin{pmatrix} \dfrac{\nu\lambda_1(\lambda_5^2 - \eta^2)}{\eta^2 - \lambda_1\lambda_5} + u_0 & \dfrac{i\nu\eta\lambda_5(\lambda_5 - \lambda_1)}{\eta^2 - \lambda_1\lambda_5} & \dfrac{1}{\mathrm{Fr}^2} \\ \dfrac{i\nu\eta\lambda_1(\lambda_5 - \lambda_1)}{\eta^2 - \lambda_1\lambda_5} & \dfrac{\nu\lambda_5(\lambda_1^2 - \eta^2)}{\eta^2 - \lambda_1\lambda_5} + u_0 & 0 \end{pmatrix}. \tag{3.14}$$

The analytical expressions for $\mathcal{S}^{\mathbf{u}}_-$ and $\mathcal{S}^{\mathbf{u}}_+$ are the inverse Laplace-Fourier transforms of $\widehat{\mathcal{S}}^{\mathbf{u}}_-$ and $\widehat{\mathcal{S}}^{\mathbf{u}}_+$. They are however obviously very complex and involve global integrals in time and space. Therefore the perfectly transparent operators $\mathcal{B}^{\mathrm{transp}}_-$ and $\mathcal{B}^{\mathrm{transp}}_+$ are non local operators, both in time and space, hence not tractable for actual applications. The remaining step is thus now to derive approximations of these operators leading to boundary conditions that are both local and efficient in terms of convergence speed.

3.4. **Approximation of the perfectly transparent operators**

As mentioned previously, a way to derive such approximations consists in computing Taylor expansions of the Laplace-Fourier transform of the transparent operators w.r.t. small parameters. This is the case for instance for 3D primitive equations in the context of oceanic circulation ([1], expansion w.r.t. the Rossby number), for 2D Navier-Stokes equations ([9], expansion w.r.t. ν), or for shallow water equations linearized around a zero velocity ([3], expansion w.r.t. s/η in the inviscid case; [13], expansion w.r.t. ν in the viscous case). In the present context of river dynamics, the aspect ratio is small: $\varepsilon = H/L_c \ll 1$. Moreover the viscosity coefficient is weak, and we can assume that $\nu = \nu_0\varepsilon$, with $\nu_0 \leq \mathcal{O}(1)$. As can be seen in other references in the literature (see *e.g.* [8]), this scaling is necessary to recover viscous shallow water equations ; we will also use this assumption in the approximation of operators $\widehat{\mathcal{S}}^{\mathbf{u}}_-$ and $\widehat{\mathcal{S}}^{\mathbf{u}}_+$ below.

Proposition 3.4. *If $u_0 < 1/Fr$, then the operator $\widehat{\mathcal{S}}^{\mathbf{u}}_+$ reads*

$$\widehat{\mathcal{S}}^{\mathbf{u}}_+ = \begin{pmatrix} u_0 & 0 & \dfrac{1}{Fr^2} \\ 0 & u_0 & 0 \end{pmatrix} + O(\varepsilon). \tag{3.15}$$

Indeed, in the case where $u_0 < 1/\mathrm{Fr}$, we can show (see Appendix B) that λ_1 and λ_5 are $O(1)$ when ε is small. Hence (3.15) directly follows from (3.14).

Proposition 3.5. *If $u_0 > 1/Fr$, then the operator $\widehat{\mathcal{S}}_-^{\mathbf{u}}$ reads*

$$\widehat{\mathcal{S}}_-^{\mathbf{u}} = \begin{pmatrix} -u_0 & 0 & \dfrac{1}{Fr^2} \\ 0 & -u_0 & 0 \end{pmatrix} + O(\varepsilon). \tag{3.16}$$

The reasoning is similar to the previous one but in this case, λ_2, λ_3 and λ_4 are $O(1)$, hence (3.16) directly follows from (3.13) and the expression of D for small ε.

Remark 3.6. In the case where $u_0 < 1/\mathrm{Fr}$ (resp. $u_0 > 1/\mathrm{Fr}$) the first order expansion w.r.t. ε of $\widehat{\mathcal{S}}_-^{\mathbf{u}}$ (resp. $\widehat{\mathcal{S}}_+^{\mathbf{u}}$) is not as simple as for $\widehat{\mathcal{S}}_+^{\mathbf{u}}$ (resp. $\widehat{\mathcal{S}}_-^{\mathbf{u}}$). We provide in Appendix B the computations for the case $u_0 < 1/\mathrm{Fr}$ (the other case is similar). An additional hypothesis is thus required to get a local operator. For instance, a first order expansion w.r.t. η (*i.e.* for small incidence, as proposed for example in [9]) leads to the simplified operator

$$\widehat{\mathcal{S}}_-^{\mathbf{u}} = \begin{pmatrix} -\dfrac{1}{Fr} & 0 & \dfrac{u_0}{Fr} \\ 0 & -u_0 & 0 \end{pmatrix} + O(\varepsilon, \eta). \tag{3.17}$$

Remark 3.7. For the sake of asymptotic analysis, we had to use dimensionless equations in the above computations. Going back to original equations, the local approximations of $\widehat{\mathcal{S}}_+^{\mathbf{u}}$ and $\widehat{\mathcal{S}}_-^{\mathbf{u}}$ corresponding to (3.15) and (3.17) (case where $\tilde{u}_0 < 1/\mathrm{Fr}$, *i.e.* $u_0 < \sqrt{gH}$) thus read:

$$\widehat{\mathcal{S}}_+^{app} = \begin{pmatrix} u_0 & 0 & g \\ 0 & u_0 & 0 \end{pmatrix} \quad i.e. \quad \widehat{\mathcal{S}}_+^{app}(u, v, \zeta) = \begin{pmatrix} u_0\, u + g\,\zeta \\ u_0\, v \end{pmatrix} \tag{3.18}$$

and

$$\widehat{\mathcal{S}}_-^{app} = \begin{pmatrix} -\sqrt{g\,H} & 0 & \sqrt{\dfrac{g}{H}} \\ 0 & -u_0 & 0 \end{pmatrix} \quad i.e. \quad \widehat{\mathcal{S}}_-^{app}(u, v, \zeta) = \begin{pmatrix} -\sqrt{g\,H}\, u + u_0\sqrt{\dfrac{g}{H}}\,\zeta \\ -u_0\, v \end{pmatrix} \tag{3.19}$$

while (3.16) (case where $\tilde{u}_0 > 1/\mathrm{Fr}$, *i.e.* $u_0 > \sqrt{gH}$) becomes

$$\widehat{\mathcal{S}}_-^{app} = \begin{pmatrix} -u_0 & 0 & g \\ 0 & -u_0 & 0 \end{pmatrix} \quad i.e. \quad \widehat{\mathcal{S}}_-^{app}(u, v, \zeta) = \begin{pmatrix} -u_0\, u + g\,\zeta \\ -u_0\, v \end{pmatrix}. \tag{3.20}$$

4. Schwarz waveform relaxation algorithm with Robin boundary conditions

An alternative approach for deriving interface conditions, leading to much simpler calculations, consists in approximating the Dirichlet-to-Neumann operators $\mathcal{S}_-^{\mathbf{u}}$ and $\mathcal{S}_+^{\mathbf{u}}$ by linear functions, thus leading to Robin-like boundary conditions, see [5, 12]. Such conditions make use of free parameters, that can be tuned in order to optimize the convergence rate of the Schwarz algorithm. In the most general case, $\mathcal{S}_-^{\mathbf{u}}$ and $\mathcal{S}_+^{\mathbf{u}}$ would thus be approximated by two 3×2 matrices $\mathcal{S}_-^{app}$ and $\mathcal{S}_+^{app}$, with constant coefficients. However optimizing the convergence rate w.r.t. 12 free parameters is of course out of reach, and the number of degrees of freedom must be significantly reduced. One classical approach in such a case consists in mimicking some properties of the perfectly transparent operators. However, due to the complexity of the expressions (3.13) and (3.14), it seems quite difficult to suppress more than three degrees of freedom without additional hypotheses. There is indeed no obvious relationships between the coefficients of $\mathcal{S}_-^{\mathbf{u}}$, while the expression of $\mathcal{S}_+^{\mathbf{u}}$ leads to

$$\mathcal{S}_+^{app} = \begin{pmatrix} \alpha_1 + u_0 & \alpha_2\alpha_3 & g \\ \alpha_2 & \alpha_1\alpha_3 + u_0 & 0 \end{pmatrix}$$

where α_1, α_2, α_3 are still to be fixed.
A much more drastic approach consists in fixing the number of free parameters to a low value, and to propose corresponding expressions for $\mathcal{S}_-^{app}$ and $\mathcal{S}_+^{app}$. For instance, we present in the following the most drastic choice, which consists in keeping only one free parameter.

Due to the study of the well-posedness of the system (that will be detailed later in this section), the following Robin boundary conditions are proposed:

$$\mathcal{B}_-(\mathbf{u},\zeta) = \begin{pmatrix} \mu\dfrac{\partial u}{\partial x} - g\zeta + \dfrac{(\lambda - u_0)}{2}u \\ \\ \mu\dfrac{\partial v}{\partial x} + \dfrac{(\lambda - u_0)}{2}v \end{pmatrix}, \quad \mathcal{B}_+(\mathbf{u},\zeta) = \begin{pmatrix} -\mu\dfrac{\partial u}{\partial x} + g\zeta + \dfrac{(\lambda + u_0)}{2}u \\ \\ -\mu\dfrac{\partial v}{\partial x} + \dfrac{(\lambda + u_0)}{2}v \\ \\ u_0\zeta \end{pmatrix} \tag{4.1}$$

where λ is a positive constant to be determined.

This corresponds to

$$\mathcal{S}_-^{app} = \frac{u_0 + \lambda}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{S}_+^{app} = \frac{u_0 - \lambda}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that choosing $\lambda = u_0$ in the case $u_0 > \sqrt{gH}$ makes $\mathcal{S}_-^{app}$ correspond to (3.20) for the velocity variable (asymptotic case $\mu \to 0$). Such a value could be an initial choice for the optimization process. Similarly, in the case $u_0 < \sqrt{gH}$, it would be natural to start with $\lambda = -u_0$ but we cannot ensure the well-posedness and/or convergence of the iterative process (see proofs below). Several values for λ could be tested, such as $\lambda = u_0$ or a small (but positive) value.

In the following, $\mathcal{B}_+$ is splitted as $\mathcal{B}_+ = (\mathcal{B}_+^{\mathbf{u}}, \mathcal{B}_+^{\zeta})^T$ with

$$\mathcal{B}_+^{\mathbf{u}}(\mathbf{u},\zeta) = -\mu\frac{\partial \mathbf{u}}{\partial x} + g\begin{pmatrix} \zeta \\ 0 \end{pmatrix} + \frac{(\lambda + u_0)}{2}\mathbf{u} \qquad \text{and} \qquad \mathcal{B}_+^{\zeta}(\mathbf{u},\zeta) = u_0\zeta \tag{4.2}$$

and the following relations hold:

$$\mathcal{B}_+^{\mathbf{u}}(\mathbf{u},\zeta) + \mathcal{B}_-(\mathbf{u},\zeta) = \lambda\mathbf{u}, \tag{4.3}$$

$$\mathcal{B}_+^{\mathbf{u}}(\mathbf{u},\zeta) - \mathcal{B}_-(\mathbf{u},\zeta) = 2\left(-\mu\frac{\partial \mathbf{u}}{\partial x} + \frac{1}{2}u_0\mathbf{u} + g\begin{pmatrix} \zeta \\ 0 \end{pmatrix}\right). \tag{4.4}$$

Note that the expression of $\mathcal{B}_+^{\zeta}$ that is chosen here is a consequence of the simple expression for $\mathcal{S}_+^{\zeta}$: if we define the errors $\tilde{X}_-^k = X_{|\omega^-} - X_-^k$ and $\tilde{X}_+^k = X_{|\omega^+} - X_+^k$ at iteration k, we have:

$$\mathcal{B}_+^{\zeta,trans}(\tilde{X}_+^k) = u_0\tilde{\zeta}_+^k + H\tilde{u}_+^k - \mathcal{S}_+^{\zeta}(\tilde{\mathbf{u}}_+^k, \tilde{\zeta}_+^k).$$

Then due to the definition of $\mathcal{S}_+^{\zeta}$ and (3.9) we obtain:

$$\begin{aligned} \mathcal{B}_+^{\zeta,trans}(\tilde{X}_+^k) &= u_0\tilde{\zeta}_+^k + H\tilde{u}_+^k - u_0\tilde{\zeta}_-^k - H\tilde{u}_-^k, \\ &= u_0\tilde{\zeta}_+^k + H\tilde{u}_+^k - u_0\tilde{\zeta}_-^k - H\tilde{u}_+^k, \\ &= u_0\tilde{\zeta}_+^k - u_0\tilde{\zeta}_-^k, \\ &= 0. \end{aligned}$$

which implies that $u_0\tilde{\zeta}_+^k = u_0\tilde{\zeta}_-^k$ at each iteration k.

Let us now prove the well-posedness of the Schwarz waveform relaxation algorithm with these Robin boundary conditions, and then prove its convergence.

4.1. **Well-posedness of the Schwarz waveform relaxation algorithm**

The Schwarz waveform relaxation algorithm reads:

For $\mathbf{u}^0$ and ζ_+^0 given and for all $k \geq 0$:

- Solve the parabolic system in ω^-:

$$\begin{aligned} \partial_t \mathbf{u}_-^{k+1} + (\mathbf{U}_0.\nabla)\,\mathbf{u}_-^{k+1} - \mu\Delta\mathbf{u}_-^{k+1} &= -g\nabla\zeta_-^{k+1} \quad \text{in } \omega^- \times [0,T], & (4.5a)\\ \mathcal{B}_-(\mathbf{u}_-^{k+1},\zeta_-^{k+1}) &= \mathcal{B}_-(\mathbf{u}_+^{k},\zeta_+^{k}) \quad \text{on } \Gamma\times[0,T], & (4.5b)\\ \mathbf{u}_-^{k+1}(.,0) &= \mathbf{u}_-^{ini} \quad \text{in } \omega^-, & (4.5c) \end{aligned}$$

  and the transport equation in ω^-:

$$\begin{aligned} \partial_t\zeta_-^{k+1} + \mathbf{U}_0.\nabla\zeta_-^{k+1} &= -H\ \mathrm{div}(\mathbf{u}_-^{k+1}) \quad \text{in } \omega^-\times[0,T], & (4.6a)\\ \zeta_-^{k+1}(.,0) &= \zeta_-^{ini} \quad \text{in } \omega^-. & (4.6b) \end{aligned}$$

  Note that, due to the assumption $u_0 > 0$, we do not consider a boundary condition for the transport equation on Γ.

- Solve the parabolic system in ω^+:

$$\begin{aligned} \partial_t \mathbf{u}_+^{k+1} + (\mathbf{U}_0.\nabla_h)\,\mathbf{u}_+^{k+1} - \mu\Delta\mathbf{u}_+^{k+1} &= -g\nabla\zeta_+^{k+1} \quad \text{in } \omega^+ \times [0,T], & (4.7a)\\ \mathcal{B}_+^{\mathbf{u}}(\mathbf{u}_+^{k+1},\zeta_+^{k+1}) &= \mathcal{B}_+^{\mathbf{u}}(\mathbf{u}_-^{k+1},\zeta_-^{k+1}) \quad \text{on } \Gamma\times[0,T], & (4.7b)\\ \mathbf{u}_+^{k+1}(.,0) &= \mathbf{u}_+^{ini} \quad \text{in } \omega^+, & (4.7c) \end{aligned}$$

  and the transport equation in ω^+:

$$\begin{aligned} \partial_t\zeta_+^{k+1} + \mathbf{U}_0.\nabla\zeta_+^{k+1} &= -H\ \mathrm{div}(\mathbf{u}_+^{k+1}) \quad \text{in } \omega^+\times[0,T], & (4.8a)\\ \mathcal{B}_+^{\zeta}\left(\mathbf{u}_+^{k+1},\zeta_+^{k+1}\right) &= \mathcal{B}_+^{\zeta}\left(\mathbf{u}_-^{k+1},\zeta_-^{k+1}\right) \quad \text{on } \Gamma\times[0,T], & (4.8b)\\ \zeta_+^{k+1}(.,0) &= \zeta_+^{ini} \quad \text{in } \omega^+. & (4.8c) \end{aligned}$$

The proof of the well-posedness of this algorithm is similar to what is done in Section 2 and in Appendix A. Therefore, only the main results and definitions are given here.

4.1.1. *Parabolic systems*

To define the weak formulation of the parabolic systems for each subdomain, we take the scalar product of equations (4.5a) and (4.7a) with a test function $\mathbf{v} \in \mathcal{D}(\overline{\omega}^{\pm},\mathbb{R}^2)$. One then needs to know $\partial_x\mathbf{u}_\pm^k$ on the interface Γ in order to define the terms $\left(\mathcal{B}_\pm^{\mathbf{u}}(\mathbf{u}_\pm^k),\mathbf{v}\right)_\Gamma$. If we consider solutions $\mathbf{u}_\pm^{k+1} \in C(0,T,L^2(\omega^\pm,\mathbb{R}^2)) \cap L^2(0,T,H^1(\omega^\pm,\mathbb{R}^2))$, $\partial_x\mathbf{u}_\pm^k$ are in $L^2(\omega^\pm\times(0,T))$ and we cannot define their traces on Γ. The idea in [1] is to choose a first guess $\mathcal{B}_+^0$ in an adequate functional space and then use relation (4.3) to define the terms $\left(\mathcal{B}_\pm^{\mathbf{u}}(\mathbf{u}_\pm^k),\mathbf{v}\right)_\Gamma$.

Let us define $\mathcal{B}_-^k = \mathcal{B}_-(\mathbf{u}^k, \zeta_-^k)$ and $\mathcal{B}_+^k = \mathcal{B}_+^{\mathbf{u}}(\mathbf{u}^k, \zeta_+^k)$.
In the sequel, we denote by $\mathcal{W}_\Gamma$ the space $H^{\frac{1}{2}}(\Gamma, \mathbb{R}^2)$ and by $\mathcal{W}'_\Gamma$ its topological dual.

Definition 4.1. Let $\zeta_-^{k+1} \in L^2(\omega^- \times (0,T))$ and $\zeta_+^{k+1} \in L^2(\omega^+ \times (0,T))$.

- For $k = 0$, let $\mathcal{B}_+^0 \in L^2(0,T,\mathcal{W}'_\Gamma)$.
- For $k \geq 0$, (4.5b) and (4.3) imply that $\mathcal{B}_-^{k+1} = -\mathcal{B}_+^k + \lambda \mathbf{u}_+^k$. Therefore $\mathbf{u}_-^{k+1}$ is said to be a weak solution of (4.5) if and only if for all $\mathbf{v} \in H^1(\omega^-, \mathbb{R}^2)$

$$\begin{aligned}\int_{\omega^-} \partial_t \mathbf{u}_-^{k+1}.\mathbf{v} + \int_{\omega^-} (\mathbf{U}_0.\nabla)\, \mathbf{u}_-^{k+1}.\mathbf{v} + \mu \int_{\omega^-} \nabla \mathbf{u}_-^{k+1} : \nabla \mathbf{v} \\ + \int_\Gamma \mathcal{B}_+^k.\mathbf{v} - \int_\Gamma \lambda \mathbf{u}_+^k.\mathbf{v} + \int_\Gamma \frac{(\lambda - u_0)}{2} \mathbf{u}_-^{k+1}.\mathbf{v} - g \int_{\omega^-} \zeta_-^{k+1} \operatorname{div}(\mathbf{v}) = 0.\end{aligned} \tag{4.9}$$

- Once $\mathbf{u}_-^{k+1}$ is known, (4.3) implies that $\mathcal{B}_+^{k+1} = -\mathcal{B}_-^{k+1} + \lambda \mathbf{u}_-^{k+1}$. Therefore $\mathbf{u}_+^{k+1}$ is said to be a weak solution of (4.7) if and only if for all $\mathbf{v} \in H^1(\omega^+, \mathbb{R}^2)$

$$\begin{aligned}\int_{\omega^+} \partial_t \mathbf{u}_+^{k+1}.\mathbf{v} + \int_{\omega^+} (\mathbf{U}_0.\nabla)\, \mathbf{u}_+^{k+1}.\mathbf{v} + \mu \int_{\omega^+} \nabla \mathbf{u}_+^{k+1} : \nabla \mathbf{v} + \mu \int_{\omega^+} \partial_z \mathbf{u}_+^{k+1}.\partial_z \mathbf{v} \\ + \int_\Gamma \mathcal{B}_-^{k+1}.\mathbf{v} - \int_\Gamma \lambda \mathbf{u}_-^{k+1}.\mathbf{v} + \int_\Gamma \frac{(\lambda + u_0)}{2} \mathbf{u}_+^{k+1}.\mathbf{v} - g \int_{\omega^-} \zeta_+^{k+1} \operatorname{div}(\mathbf{v}) = 0.\end{aligned} \tag{4.10}$$

Then we have the following result:

Proposition 4.2. *Let $\mathbf{u}_+^{ini} \in L^2(\omega^+, \mathbb{R}^2)$, $\mathbf{u}_-^{ini} \in L^2(\omega^-, \mathbb{R}^2)$, $\mathcal{B}_-^{k+1} \in L^2(0,T,\mathcal{W}'^\Gamma))$ and $\mathcal{B}_+^{k+1} \in L^2(0,T,\mathcal{W}'^\Gamma))$. Assume $\zeta_-^{k+1} \in L^2(\omega^- \times [0,T]) \cap C(0,T;L^2(\omega^-))$ and $\zeta_+^{k+1} \in L^2(\omega^+ \times (0,T)) \cap C(0,T;L^2(\omega))^+$. There exists:*

- *a unique solution $\mathbf{u}_-^{k+1}$ of (4.5) in $C(0,T,L^2(\omega^-,\mathbb{R}^2)) \cap L^2(0,T,H^1(\omega^-,\mathbb{R}^2))$,*
- *a unique solution $\mathbf{u}_+^{k+1}$ of (4.7) in $C(0,T,L^2(\omega^+,\mathbb{R}^2)) \cap L^2(0,T,H^1(\omega^+,\mathbb{R}^2))$.*

Moreover we have the following energy inequalities for all $t \in [0,T]$

$$\begin{aligned}\|\mathbf{u}_-^{k+1}\|_{\omega^-}^2 + \mu \int_0^t \|\nabla \mathbf{u}_-^{k+1}\|_{\omega^-}^2 + \int_0^t \int_\Gamma \frac{\lambda}{2} \|\mathbf{u}_-^{k+1}\|^2 \leq \|\mathbf{u}_-^{ini}\|_{\omega^-}^2 + C_1 \int_0^t \|\zeta_-^{k+1}\|_{\omega^-}^2 \\ + C_2 \int_0^t \|\mathcal{B}_+^k\|_\Gamma^2 + C_3 \int_0^t \|\mathbf{u}_+^k\|_\Gamma^2\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\|\mathbf{u}_+^{k+1}\|_{\omega^+}^2 + \mu \int_0^t \|\nabla \mathbf{u}_+^{k+1}\|_{\omega^+}^2 + \int_0^t \int_\Gamma \frac{\lambda}{2} \|\mathbf{u}_+^{k+1}\|^2 \leq \|\mathbf{u}_+^{ini}\|_{\omega^+}^2 + C'_1 \int_0^t \|\zeta_+^{k+1}\|_{\omega^+}^2 \\ + C'_2 \int_0^t \|\mathcal{B}_-^{k+1}\|_\Gamma^2 + C'_3 \int_0^t \|\mathbf{u}_-^{k+1}\|_\Gamma^2\end{aligned} \tag{4.12}$$

where C_1, C_2, C_3, C'_1, C'_2 et C'_3 are positive constants.

4.1.2. *Transport equations*

Let us now study the transport equations in each subdomain.
Definition 4.3.

- Let $\mathbf{u}_-^{k+1}$ be given in $L^2(0,T;H^1(\omega^-,\mathbb{R}^2))$ and ζ_-^{ini} in $L^2(\omega^-)$. The function $\zeta_-^{k+1} \in L^2(\omega^- \times (0,T))$ is a weak solution of (4.6a) et (4.6b) if and only if

$$\begin{cases} \dfrac{d}{dt}(\zeta_-^{k+1},\chi)_{\omega^-} - (\zeta_-^{k+1},\mathbf{U}_0.\nabla\chi)_{\omega^-} = -H\,(\text{div }\mathbf{u}_-^{k+1},\chi)_{\omega^-} \quad \forall\chi \in \mathcal{D}(\omega^-), \\ \zeta_-^{k+1}(.,0) = \zeta_-^{ini} \quad \text{in } \omega^-. \end{cases} \tag{4.13}$$

- Let $\mathbf{u}_+^{k+1}$ be given in $L^2(0,T;L^2(\omega^+,\mathbb{R}^2))$ and $\zeta_+^{ini} \in L^2(\omega^+)$. Once the solutions of (4.5) and (4.13) are known, the transmission condition (4.8b) at the iteration $k+1$ is:

$$\mathcal{B}_+^{\zeta}\left(\mathbf{U}_h^{k+1},\zeta_+^{k+1}\right) = u_0\zeta_+^{k+1}(0,.) = u_0\zeta_-^{k+1}(0,.). \tag{4.14}$$

We denote in the sequel $\zeta_b^{k+1} = \zeta_+^{k+1}(0,.)$.
Assume that $\zeta_b^{k+1} \in L^2(\Gamma\times(0,T))$, ζ_+^{k+1} is a weak solution of (4.8) if and only if

$$\begin{cases} \dfrac{d}{dt}(\zeta_+^{k+1},\chi)_{\omega^+} - (\zeta_+^{k+1},\mathbf{U}_0.\nabla\chi)_{\omega^+} - (u_0\zeta_b^{k+1},\chi)_\Gamma = -H\,(\text{div }\mathbf{u}_+^{k+1},\chi)_{\omega^+} \quad \forall\chi \in \mathcal{D}(\overline{\omega}^+), \\ \zeta_+^{k+1}(.,0) = \zeta_+^{ini} \quad \text{in } \omega^+. \end{cases} \tag{4.15}$$

Therefore we have the following result:
Proposition 4.4.

- *Let $\mathbf{u}_-^{k+1} \in L^2(0,T;H^1(\omega^-,\mathbb{R}^2))$ and $\zeta_-^{ini} \in L^2(\omega^-)$. There exists a unique solution $\zeta_-^{k+1} \in L^2(\omega^-\times[0,T])$ of* (4.6a), (4.6b)*. This solution is obtained from the characteristic formula:*

$$\zeta_-^{k+1}(x,y,t) = \zeta_-^{ini}(x-u_0t,y-v_0t) - H\int_0^t \left(\text{div }\mathbf{u}_-^{k+1}\right)(x-u_0s,y-v_0s,t-s)\,ds. \tag{4.16}$$

This solution is also in $C(0,T;L^2(\omega^-))\cap C((-\infty,0]_x;L^2(\mathbb{R}_y\times(0,T)))$ and for all $x\le 0$ and for all $t\in[0,T]$, ζ_-^{k+1} satisfies the energy inequalities

$$\|\zeta_-^{k+1}(.,t)\|_{\omega^-} \le \|\zeta_-^{ini}\|_{\omega^-} + H\int_0^t \|\text{div }\mathbf{u}_-^{k+1}\|_{\omega^-}\,ds \tag{4.17}$$

and

$$\|\zeta_-^{k+1}(x,.)\|_{\Gamma_t} \le \frac{1}{u_0}\left(\|\zeta_-^{ini}\|_{\omega^-} + H\int_0^t \|\text{div }\mathbf{u}_-^{k+1}\|_{\omega^-}\,ds\right). \tag{4.18}$$

- *Let $\mathbf{u}_+^{k+1} \in L^2(0,T,H^1(\omega^+,\mathbb{R}^2))$ fixed and $\zeta_+^{ini} \in L^2(\omega^+)$. There exists a unique solution $\zeta_+^{k+1} \in L^2(\omega^+\times(0,T))$ of* (4.8a), (4.8b) *and* (4.8c)*. This solution can be written using the characteristic formula:*

$$\zeta_+^{k+1}(x,y,t) = \begin{cases} \zeta_+^{ini}(x-u_0t,y-v_0t) - H\int_0^t \left(\text{div }\mathbf{u}_+^{k+1}\right)(x-u_0s,y-v_0s,t-s)\,ds & \text{if } x>u_0t, \\ \zeta_b^{k+1}(y-\frac{v_0}{u_0}x,t-\frac{x}{u_0}) - H\int_0^{x/u_0}\left(\text{div }\mathbf{u}_+^{k+1}\right)(x-u_0s,y-v_0s,t-s)\,ds & \text{if } x\le u_0t. \end{cases} \tag{4.19}$$

This solution is also in $C(0,T;L^2(\omega^+))\cap C([0,+\infty)_x;L^2(\mathbb{R}_y\times(0,T)))$ and for all $x\ge 0$ and for all $t\in[0,T]$, ζ_+^{k+1} satisfies the energy inequalities:

$$\|\zeta_+^{k+1}(.,t)\|_{\omega^+} \le \|\zeta_+^{ini}\|_{\omega^+} + u_0\|\zeta_b^{k+1}\|_{\Gamma\times(0,t)} + H\int_0^t \|\text{div }\mathbf{u}_+^{k+1}\|_{\omega^-}\,ds, \tag{4.20}$$

$$\|\zeta_+^{k+1}(x,.)\|_{\Gamma_t} \leq \frac{1}{u_0}\left(\|\zeta_+^{ini}\|_{\omega^+} + u_0\|\zeta_b^{k+1}\|_{\Gamma\times(0,t)} + H\int_0^t \|\operatorname{div} \mathbf{u}_+^{k+1}\|_{\omega^-}\, ds\right). \tag{4.21}$$

Finally to conclude that the Schwarz algorithm is well-posed, one can use again the fixed point theorem, as in Section 2 and Appendix A.

4.2. Convergence of the Schwarz waveform relaxation algorithm

The following result completes the theoretical study of the Schwarz waveform relaxation algorithm with Robin boundary conditions.

Proposition 4.5. *Let λ be a positive number.*
If $X^0 = (\mathbf{u}_+^0, \zeta_+^0) \in \left(C(0,T; L^2(\omega^+,\mathbb{R}^2) \cap L^2(0,T;H^1(\omega^+,\mathbb{R}^2))\right) \times \left(L^2(\omega^+ \times (0,T))\right)$ then the Schwarz algorithm (4.5)-(4.8) *is well-posed and the sequences $X_-^{k+1} = \left(\mathbf{u}_-^{k+1}, \zeta_-^{k+1}\right)$ and $X_+^{k+1} = \left(\mathbf{u}_+^{k+1}, \zeta_+^{k+1}\right)$ converge respectively in*
$\left(C(0,T; L^2(\omega^-,\mathbb{R}^2) \cap L^2(0,T;H^1(\omega^-,\mathbb{R}^2))\right) \times \left(L^2(\omega^- \times (0,T)) \cap C(0,T;L^2(\omega^-))\right)$ *and*
$\left(C(0,T; L^2(\omega^+,\mathbb{R}^2) \cap L^2(0,T;H^1(\omega^+,\mathbb{R}^2))\right) \times \left(L^2(\omega^+ \times (0,T)) \cap C(0,T;L^2(\omega^+))\right)$.

Proof. The well-posedness of the algorithm was proved in the last paragraph. Let us now focus on the convergence, proceeding as in [13]. Nevertheless, due to the fact that the equations are linearized around a nonzero velocity $\mathbf{U}_0$, additional terms appear.
Let us introduce in the sequel the errors $\tilde{X}_-^{k+1} = X_{|\omega^-} - X_-^{k+1}$ and $\tilde{X}_+^{k+1} = X_{|\omega^+} - X_+^{k+1}$ where X is the solution of the shallow water system (2.2) throughout the domain $\omega = \mathbf{R}^2$. Therefore the errors are solutions of the systems:

$$\left\{\begin{array}{lll} \mathcal{L}_{LSW}\left(\tilde{X}_-^{k+1}\right) = 0 & \text{in} & \omega^- \times (0,T), \\ \mathcal{B}_-^{ext}\left(\tilde{X}_-^{k+1}\right) = 0 & \text{on} & \partial\omega_{ext}^- \times (0,T), \\ \mathcal{B}_-\left(\tilde{X}_-^{k+1}\right) = \mathcal{B}_-\left(\tilde{X}_+^{k}\right) & \text{on} & \Gamma \times (0,T), \\ \tilde{X}_-^{k+1}(.,0) = 0 & \text{in} & \omega^-, \end{array}\right. \tag{4.22}$$

and

$$\left\{\begin{array}{lll} \mathcal{L}_{LSW}\left(\tilde{X}_+^{k+1}\right) = 0 & \text{in} & \omega^+ \times (0,T), \\ \mathcal{B}_+^{ext}\left(\tilde{X}_+^{k+1}\right) = 0 & \text{on} & \partial\omega_{ext}^+ \times (0,T), \\ \mathcal{B}_+^{\mathbf{u}}\left(\tilde{X}_+^{k+1}\right) = \mathcal{B}_+^{\mathbf{u}}\left(\tilde{X}_-^{k+1}\right) & \text{on} & \Gamma \times (0,T), \\ \mathcal{B}_+^{\zeta}\left(\tilde{X}_+^{k+1}\right) = \mathcal{B}_+^{\zeta}\left(\tilde{X}_-^{k+1}\right) & \text{on} & \Gamma \times (0,T), \\ \tilde{X}_+^{k+1}(.,0) = 0 & \text{in} & \omega + . \end{array}\right. \tag{4.23}$$

Multiplying the first equation of system (4.22) by $(H\tilde{\mathbf{u}}_-^{k+1}, g\tilde{\zeta}_-^{k+1})^T$ and integrating on ω^-, one gets:

$$\begin{aligned} H\int_{\omega^-} \partial_t \tilde{\mathbf{u}}_-^{k+1}.\tilde{\mathbf{u}}_-^{k+1} + H\int_{\omega^-} (\mathbf{U}_0.\nabla)\tilde{\mathbf{u}}_-^{k+1}.\tilde{\mathbf{u}}_-^{k+1} - \mu H\int_{\omega^-} \Delta\tilde{\mathbf{u}}_-^{k+1}.\tilde{\mathbf{u}}_-^{k+1} + gH\int_{\omega^-} \nabla\tilde{\zeta}_-^{k+1}.\tilde{\mathbf{u}}_-^{k+1} + \\ g\int_{\omega^-} \tilde{\zeta}_-^{k+1}\partial_t\tilde{\zeta}_-^{k+1} + gH\int_{\omega^-} \tilde{\zeta}_-^{k+1} \operatorname{div} \tilde{\mathbf{u}}_-^{k+1} + g\int_{\omega^-} \mathbf{U}_0.\nabla\tilde{\zeta}_-^{k+1}\tilde{\zeta}_-^{k+1} \quad = \quad 0. \end{aligned}$$

Integrating by parts and using the relation $\int_{\omega^-} (\mathbf{U}_0.\nabla)\tilde{\mathbf{u}}_-^{k+1}.\tilde{\mathbf{u}}_-^{k+1} = \frac{u_0}{2}\|\tilde{\mathbf{u}}_-^{k+1}\|_\Gamma^2$ leads to:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(H\|\tilde{\mathbf{u}}_-^{k+1}\|_{\omega^-}^2 + g\|\tilde{\zeta}_-^{k+1}\|_{\omega^-}^2\right) + \mu H\|\nabla\tilde{\mathbf{u}}_-^{k+1}\|_{\omega^-}^2 + g\int_{\omega^-} \mathbf{U}_0.\nabla\tilde{\zeta}_-^{k+1}\tilde{\zeta}_-^{k+1} \\ + \int_\Gamma \left(-\mu H\partial_x\tilde{\mathbf{u}}_-^{k+1} + \frac{H}{2}u_0\tilde{\mathbf{u}}_-^{k+1} + gH\begin{pmatrix} \tilde{\zeta}_-^{k+1} \\ 0 \end{pmatrix}\right).\tilde{\mathbf{u}}_-^{k+1} = 0. \end{aligned}$$

Due to (4.3) and (4.4) one has:

$$\left(-\mu\partial_x\tilde{\mathbf{u}}_-^{k+1}+\frac{1}{2}u_0\tilde{\mathbf{u}}_-^{k+1}+g\begin{pmatrix}\tilde{\zeta}_-^{k+1}\\0\end{pmatrix}\right).\tilde{\mathbf{u}}_-^{k+1}=\frac{1}{2\lambda}\left((\mathcal{B}_+^{\mathbf{u},1})^2+(\mathcal{B}_+^{\mathbf{u},2})^2-(\mathcal{B}_-^1)^2-(\mathcal{B}_-^2)^2\right)(\tilde{\mathbf{u}}_-^{k+1},\tilde{\zeta}_-^{k+1})$$

where $\mathcal{B}_-^i$ $(i=1,2)$ denotes the i^{th} coordinate of the vector $\mathcal{B}_-(\tilde{\mathbf{u}}_-^{k+1},\tilde{\zeta}_-^{k+1})$ and $\mathcal{B}_+^{\mathbf{u},i}$ $(i=1,2)$ denotes the i^{th} coordinate of the vector $\mathcal{B}_+^{\mathbf{u}}(\tilde{\mathbf{u}}_-^{k+1},\tilde{\zeta}_-^{k+1})$.
Therefore:

$$\frac{1}{2}\frac{d}{dt}\left(H\|\tilde{\mathbf{u}}_-^{k+1}\|^2_{\omega^-}+g\|\tilde{\zeta}_-^{k+1}\|^2_{\omega^-}\right)+\mu H\|\nabla\tilde{\mathbf{u}}_-^{k+1}\|^2_{\omega^-}+\frac{H}{2\lambda}\int_\Gamma\left((\mathcal{B}_+^{\mathbf{u},1})^2+(\mathcal{B}_+^{\mathbf{u},2})^2\right)(\tilde{\mathbf{u}}_-^{k+1},\tilde{\zeta}_-^{k+1})$$
$$+g\int_{\omega^-}\mathbf{U}_0.\nabla\tilde{\zeta}_-^{k+1}\tilde{\zeta}_-^{k+1}=\frac{H}{2\lambda}\int_\Gamma\left((\mathcal{B}_-^1)^2+(\mathcal{B}_-^2)^2\right)(\tilde{\mathbf{u}}_-^{k+1},\tilde{\zeta}_-^{k+1}).$$

Using the boundary condition on Γ, one can modify the right hand side:

$$\frac{1}{2}\frac{d}{dt}\left(H\|\tilde{\mathbf{u}}_-^{k+1}\|^2_{\omega^-}+g\|\tilde{\zeta}_-^{k+1}\|^2_{\omega^-}\right)+\mu H\|\nabla\tilde{\mathbf{u}}_-^{k+1}\|^2_{\omega^-}+\frac{H}{2\lambda}\int_\Gamma\left((\mathcal{B}_+^{\mathbf{u},1})^2+(\mathcal{B}_+^{\mathbf{u},2})^2\right)(\tilde{\mathbf{u}}_-^{k+1},\tilde{\zeta}_-^{k+1})$$
$$+g\int_{\omega^-}\mathbf{U}_0.\nabla\tilde{\zeta}_-^{k+1}\tilde{\zeta}_-^{k+1}=\frac{H}{2\lambda}\int_\Gamma\left((\mathcal{B}_-^1)^2+(\mathcal{B}_-^2)^2\right)(\tilde{\mathbf{u}}_+^{k},\tilde{\zeta}_+^{k})$$

and using

$$\int_{\omega^-}\mathbf{U}_0.\nabla\tilde{\zeta}_-^{k+1}\tilde{\zeta}_-^{k+1}=-\int_{\omega^-}\mathbf{U}_0.\nabla\tilde{\zeta}_-^{k+1}\tilde{\zeta}_-^{k+1}+\int_\Gamma u_0(\tilde{\zeta}_-^{k+1})^2$$

the relation reads:

$$\frac{1}{2}\frac{d}{dt}\left(H\|\tilde{\mathbf{u}}_-^{k+1}\|^2_{\omega^-}+g\|\tilde{\zeta}_-^{k+1}\|^2_{\omega^-}\right)+\mu H\|\nabla\tilde{\mathbf{u}}_-^{k+1}\|^2_{\omega^-}+\frac{H}{2\lambda}\int_\Gamma\left((\mathcal{B}_+^{\mathbf{u},1})^2+(\mathcal{B}_+^{\mathbf{u},2})^2\right)(\tilde{\mathbf{u}}_-^{k+1},\tilde{\zeta}_-^{k+1})$$
$$+\frac{g}{2}\int_\Gamma u_0(\tilde{\zeta}_-^{k+1})^2=\frac{H}{2\lambda}\int_\Gamma\left((\mathcal{B}_-^1)^2+(\mathcal{B}_-^2)^2\right)(\tilde{\mathbf{u}}_+^{k},\tilde{\zeta}_+^{k}).$$

Integrating between 0 and t for $t\in[0,T]$ and using the initial conditions finally leads to:

$$\begin{aligned}\frac{H}{2}\|\tilde{\mathbf{u}}_-^{k+1}\|^2_{\omega^-}+\frac{g}{2}\|\tilde{\zeta}_-^{k+1}\|^2_{\omega^-}+\mu H\int_0^t\|\nabla\tilde{\mathbf{u}}_-^{k+1}\|^2_{\omega^-} &+ \frac{H}{2\lambda}\int_0^t\int_\Gamma\left((\mathcal{B}_+^{\mathbf{u},1})^2+(\mathcal{B}_+^{\mathbf{u},2})^2\right)(\tilde{\mathbf{u}}_-^{k+1},\tilde{\zeta}_-^{k+1})\\ +\frac{g}{2}\int_0^t\int_\Gamma u_0(\tilde{\zeta}_-^{k+1})^2 &= \frac{H}{2\lambda}\int_0^t\int_\Gamma\left((\mathcal{B}_-^1)^2+(\mathcal{B}_-^2)^2\right)(\tilde{\mathbf{u}}_+^{k},\tilde{\zeta}_+^{k}).\end{aligned}\tag{4.24}$$

In the same way, the following relation holds in ω^+:

$$\begin{aligned}\frac{H}{2}\|\tilde{\mathbf{u}}_+^{k+1}\|^2_{\omega^+}+\frac{g}{2}\|\tilde{\zeta}_+^{k+1}\|^2_{\omega^+}+\mu H\int_0^t\|\nabla\tilde{\mathbf{u}}_+^{k+1}\|^2_{\omega^+} &+ \frac{H}{2\lambda}\int_0^t\int_\Gamma\left((\mathcal{B}_-^1)^2+(\mathcal{B}_-^2)^2\right)(\tilde{\mathbf{u}}_+^{k+1},\tilde{\zeta}_+^{k+1})\\ -\frac{g}{2}\int_0^t\int_\Gamma u_0(\tilde{\zeta}_+^{k+1})^2 &= \frac{H}{2\lambda}\int_0^t\int_\Gamma\left((\mathcal{B}_+^{\mathbf{u},1})^2+(\mathcal{B}_+^{\mathbf{u},2})^2\right)(\tilde{\mathbf{u}}_-^{k+1},\tilde{\zeta}_-^{k+1}).\end{aligned}\tag{4.25}$$

Summing (4.24) and (4.25) yields:

$$\begin{aligned}&\frac{H}{2}\|\tilde{\mathbf{u}}_-^{k+1}\|^2_{\omega^-}+\frac{g}{2}\|\tilde{\zeta}_-^{k+1}\|^2_{\omega^-}+\mu H\int_0^t\|\nabla\tilde{\mathbf{u}}_-^{k+1}\|^2_{\omega^-}+\frac{H}{2}\|\tilde{\mathbf{u}}_+^{k+1}\|^2_{\omega^+}+\frac{g}{2}\|\tilde{\zeta}_+^{k+1}\|^2_{\omega^+}\\ &+\mu H\int_0^t\|\nabla\tilde{\mathbf{u}}_+^{k+1}\|^2_{\omega^+}+\frac{H}{2\lambda}\int_0^t\int_\Gamma\left((\mathcal{B}_-^1)^2+(\mathcal{B}_-^2)^2\right)(\tilde{\mathbf{u}}_+^{k+1},\tilde{\zeta}_+^{k+1})\\ &=\frac{H}{2\lambda}\int_0^t\int_\Gamma\left((\mathcal{B}_-^1)^2+(\mathcal{B}_-^2)^2\right)(\tilde{\mathbf{u}}_+^{k},\tilde{\zeta}_+^{k}).\end{aligned}\tag{4.26}$$

Let us define the following:

$$E^{k+1} = \frac{H}{2}\|\tilde{\mathbf{u}}_-^{k+1}\|^2_{\omega^-} + \frac{g}{2}\|\tilde{\zeta}_-^{k+1}\|^2_{\omega^-} + \mu\, H \int_0^t \|\nabla \tilde{\mathbf{u}}_-^{k+1}\|^2_{\omega^-}$$
$$+ \frac{H}{2}\|\tilde{\mathbf{u}}_+^{k+1}\|^2_{\omega^+} + \frac{g}{2}\|\tilde{\zeta}_+^{k+1}\|^2_{\omega^+} + \mu\, H \int_0^t \|\nabla_h \tilde{\mathbf{u}}_+^{k+1}\|^2_{\omega^+}$$

and

$$F^{k+1} = \frac{H}{2\lambda}\int_0^t \int_\Gamma \left((\mathcal{B}_-^1)^2 + (\mathcal{B}_-^2)^2\right)(\tilde{\mathbf{u}}_+^{k+1}, \tilde{\zeta}_+^{k+1}).$$

Then, summing relation (4.26) for all $k \in \{0, \dots, N\}$, where $N > 1$, one has:

$$\sum_{k=0}^{N} E^{k+1} + F^{N+1} = F^0.$$

This means that the positive series $\sum_{k=0}^N E^{k+1}$ is convergent. The sequence $(E^k)_k$ thus converges to 0, which implies the convergence of $\tilde{X}_+^{k+1}$ and $\tilde{X}_-^{k+1}$ in the functional spaces of proposition 4.5. ■

Both operators $\mathcal{B}_+^{\mathbf{u}}$ and $\mathcal{B}_-$ being functions of the free parameter λ, one can then optimize the convergence rate with respect to λ, see [5, 13]. However, the resulting optimization problem is complicated and one has use a numerical method, see [1].
We will not provide in the present work a numerical validation of this algorithm. Further studies could consider the numerical optimization of the convergence with respect to the free parameter λ, together with the application of the algorithm to the nonlinear viscous shallow water system.

5. Conclusion

We presented in this article an extension of the Schwarz waveform relaxation method to the viscous shallow water equations linearized around a nonzero velocity. We proved the well-posedness and the convergence of the algorithm with zeroth-order approximate transmission conditions. This work can be extended in several directions: higher order approximation of the transmission conditions, numerical optimization of the convergence rate, and design of a Schwarz algorithm for the nonlinear shallow water system.

Moreover, as indicated in the introduction of this paper, our long term goal is to design efficient methods for coupling 1D-2D shallow water models with 3D Navier-Stokes models. As a next step in this direction, we intend to consider a 3D hydrostatic Navier-Stokes model, and to couple it with a shallow water model using the algorithm developed here. To do so, one has first to choose the location of the interface between the two models. It must be chosen in a region where both models are relevant, which implies that it must be within the shallow water regime area, not too close to the full 3D regime area. One has then to supplement each of the two models with adequate boundary conditions on the interface. The work presented in this paper provides a candidate for the shallow water part, and one should consider the boundary conditions provided in [1] for the 3D hydrostatic equations. However, because of the different dimensions of the two models, expansion and reduction operators have also to be introduced in the interface conditions to be able to pass information back and forth from 2D (or 1D) to 3D. Such operators, which must satisfy some physical constraints such as mass conservation, can be defined in several ways (this illustrates the fact that there is not a unique solution to a coupling problem between models with different dimensions). To choose these operators, we will rely on the methodology implemented in [21]. Finally, the convergence rate of the Schwarz coupling algorithm will have to be optimized, typically by tuning free parameters like those occurring in Robin interface conditions.

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Appendix A.

We provide here the technical details inspired from [1] to prove Proposition 2.2. In the sequel we shall denote by $(2.2)_{\mathbf{u}}$ the equations (2.2a)-(2.2c) for $\mathbf{u}$ (with a given ζ), and by $(2.2)_\zeta$ the equations (2.2b)-(2.2d) for ζ (with a given $\mathbf{u}$).

A.1. Well-posedness of the parabolic system

Let us first study the parabolic system $(2.2)_{\mathbf{u}}$, assuming that ζ is a given data.

Proposition A.1. *Let $\mathbf{u}^{ini} \in L^2(\omega, \mathbb{R}^2)$ and $\zeta \in L^2(\omega \times (0,T))$. Then there exists a unique weak solution $\mathbf{u} \in C(0,T; L^2(\omega)) \cap L^2(0,T; H^1(\omega, \mathbb{R}^2))$ of $(2.2)_{\mathbf{u}}$. Moreover we have the following energy inequality:*

$$\|\mathbf{u}\|_\omega^2 + \mu \int_0^t \|\nabla \mathbf{u}\|_\omega^2 \le C \int_0^t \|\zeta\|_\omega^2 + \|\mathbf{u}^{ini}\|_\omega^2, \; \forall t \in [0,T] \tag{A.1}$$

where $C > 0$.

Proof. Multiplying (2.2a) by $\mathbf{u}$ and integrating over ω leads to:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_\omega^2 + \mu \|\nabla \mathbf{u}\|_\omega^2 = g\,(\zeta, \text{div }\mathbf{u})_\omega.$$

Applying the Cauchy-Schwarz inequality to $(\zeta, \text{div }\mathbf{u})_\omega$ and using the fact that $2ab \le \alpha a^2 + \dfrac{b^2}{\alpha}$ for all $a, b, \alpha > 0$, this equation becomes:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_\omega^2 + \mu \|\nabla \mathbf{u}\|_\omega^2 \;&\le\; g\left(\frac{\alpha}{2}\|\zeta\|_\omega^2 + \frac{1}{2\alpha}\|\text{ div }\mathbf{u}\|_\omega^2\right), \\ &\le\; g\left(\frac{\alpha}{2}\|\zeta\|_\omega^2 + \frac{1}{\alpha}\|\nabla \mathbf{u}\|_\omega^2\right).\end{aligned}$$

Choosing α such that $\dfrac{2g}{\alpha} = \mu$ and integrating in time, one gets:

$$\frac{1}{2}\|\mathbf{u}\|_\omega^2 + \frac{\mu}{2}\int_0^t \|\nabla \mathbf{u}\|_\omega^2 \le \frac{g^2}{\mu}\int_0^t \|\zeta\|_\omega^2 + \frac{1}{2}\|\mathbf{u}^{ini}\|_\omega^2.$$

The proof of uniqueness comes then straightforwardly from this energy inequality, while the proof of existence can be obtained using the Galerkin method (i.e. finding a weak solution of $(2.2)_{\mathbf{u}}$ in a space of finite dimension N and making N tends to infinity) and this inequality. ∎

A.2. **Well-posedness of the transport equation**

Let us now study equations $(2.2)_\zeta$ with a given value for $\mathbf{u}$. Similarly to [1] we have the following result:

Proposition A.2. *Let* $\mathbf{u} \in L^2(0,T;H^1(\omega,\mathbb{R}^2))$ *and* $\zeta^{ini} \in L^2(\omega)$*. Then there exists a unique weak solution* $\zeta \in L^2(\omega \times (0,T))$ *of* $(2.2)_\zeta$*. Moreover this solution is given by the characteristic formula:*

$$\zeta(x,y,t) = \zeta^{ini}(x-u_0t, y-v_0t) - H\int_0^t (\operatorname{div}\mathbf{u})(x-u_0s, y-v_0s, t-s)\,ds,\ \forall t \in [0,T]. \tag{A.2}$$

This means that we have also $\zeta \in C(0,T;L^2(\omega)) \cap C(\mathbb{R}_x; L^2(\mathbb{R}_y \times (0,T)))$ *and for all* $t \in [0,T]$ *this solution satisfies the energy inequalities:*

$$\|\zeta(.,t)\|_\omega \le \|\zeta^{ini}\|_\omega + H\int_0^t \|\operatorname{div}\mathbf{u}\|_\omega\,ds \tag{A.3}$$

and

$$\|\zeta(x,.)\|_{\Gamma_{x,t}} \le \frac{1}{u_0}\left(\|\zeta^{ini}\|_\omega + H\int_0^t \|\operatorname{div}\mathbf{u}\|_\omega\,ds\right) \tag{A.4}$$

where $\mathbb{R}_x$ *and* $\mathbb{R}_y$ *denote the sets of real numbers with respect to the variables* x *and* y *respectively and* $\Gamma_{x,t} = \{x\} \times \mathbf{R}_y \times (0,T)$*.*

Proof. The proof for the existence of a solution ζ to $(2.2)_\zeta$ is classic: if ζ satisfies (A.2) and if $\operatorname{div}(\mathbf{u})$ and ζ^{ini} are smooth enough then ζ is a solution of $(2.2)_\zeta$.
The uniqueness can be obtained from the characteristic formula (A.2). Let us suppose that $\mathbf{u}$ and ζ^{ini} vanish. Then we can deduce from (A.2) that $\zeta = 0$ in $\omega \times [0,T]$.
Let us now prove the energy inequality (A.3).
The characteristic formula (A.2) implies that

$$\|\zeta\|_\omega \le \|\zeta^{ini}\|_\omega + H\left\|\int_0^t \operatorname{div}\mathbf{u}\,ds\right\|_\omega.$$

Then, using the Minkowski's integral inequality, the energy estimation (A.3) follows. ∎

A.3. **Well-posedness of the linearized shallow water system**

Finally, to prove the well-posedness of the weak form of the linearized viscous shallow water equations (2.2), we will use a fixed point argument.

Proposition A.3. *Let* $X^{ini} = (\mathbf{u}^{ini}, \zeta^{ini}) \in L^2(\omega,\mathbb{R}^2) \times L^2(\omega)$*. There exists a unique weak solution* $X = (\mathbf{u},\zeta)$ *of* (2.2) *in* $\left(C(0,T;L^2(\omega,\mathbb{R}^2) \cap L^2(0,T;H^1(\omega,\mathbb{R}^2)))\right) \times \left(L^2(\omega\times(0,T)) \cap C(0,T;L^2(\omega))\right)$*.*

Proof. As in [1], given an initial condition X^{ini}, we introduce the following applications:

$$\begin{aligned} \mathcal{S}_1 : L^2(\omega\times(0,T)) \cap C(0,T;L^2(\omega)) &\longrightarrow C(0,T;L^2(\omega,\mathbb{R}^2) \cap L^2(0,T;H^1(\omega,\mathbb{R}^2)) \\ \zeta &\longmapsto \mathbf{u} \quad \text{solution of (2.2a)} \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_2 : C(0,T;L^2(\omega,\mathbb{R}^2) \cap L^2(0,T;H^1(\omega,\mathbb{R}^2)) &\longrightarrow L^2(\omega\times(0,T)) \cap C(0,T;L^2(\omega)) \\ \mathbf{u} &\longmapsto \zeta \quad \text{solution of (2.2b).} \end{aligned}$$

Let us denote by $\mathcal{E}_T$ the functional space:

$$\mathcal{E}_T = \Big(C(0,T;L^2(\omega,\mathbb{R}^2) \cap L^2(0,T;H^1(\omega,\mathbb{R}^2))\Big) \times \Big(L^2(\omega\times(0,T)) \cap C(0,T;L^2(\omega))\Big)$$

Then X is a weak solution of (2.2) if and only if X is a fixed point of the mapping:

$$\begin{array}{rcl}\mathcal{T}:\mathcal{E}_T & \longrightarrow & \mathcal{E}_T\\ (\mathbf{u},\zeta) & \longmapsto & (\mathcal{S}_1(\zeta),\mathcal{S}_2(\mathbf{u})).\end{array}$$

In order to prove the existence of such a fixed point, let us define $X_1=(\mathbf{u}_1,\zeta_1)$, $X_2=(\mathbf{u}_2,\zeta_2)\in\mathcal{E}_T$. By linearity, $\mathcal{S}_1(\zeta_1)-\mathcal{S}_1(\zeta_2)$ satisfies (2.2a) with a null initial condition. The energy inequality (A.1) implies, for all $t\in(0,T)$:

$$\begin{aligned}\|\mathcal{S}_1(\zeta_1)-\mathcal{S}_1(\zeta_2)\|_\omega^2+\mu\int_0^t\|\nabla(\mathcal{S}_1(\zeta_1)-\mathcal{S}_1(\zeta_2))\|_\omega^2 &\leq C\int_0^t\|\zeta_1-\zeta_2\|_\omega^2,\\ &\leq Ct\sup_{s\in[0,t]}\|\zeta_1-\zeta_2\|_\omega^2(s),\end{aligned}$$

where $C>0$.
In the same manner, we deduce from (A.3) and using the Cauchy Schwarz inequality:

$$\|\mathcal{S}_2(\mathbf{u}_1)-\mathcal{S}_2(\mathbf{u}_2)\|_\omega^2(t)\leq 2tH^2\int_0^t\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|_\omega^2(s)ds.$$

These two inequalities imply that for all $T'\in(0,T]$ (with T' small enough), the application $\mathcal{T}$ is strictly contracting in $\mathcal{E}_{T'}$. One has thus simply to repeat this argument on the intervals $[T',2T']$, $[2T',3T']$, etc. ■

Appendix B.

In Section 3.3 we are interested in the zeros of $\det(M)$ where M is:

$$\mathbf{M}(\lambda)=\begin{pmatrix}P(\lambda) & 0 & \dfrac{\lambda}{\mathrm{Fr}^2}\\ 0 & P(\lambda) & \dfrac{i\eta}{\mathrm{Fr}^2}\\ \lambda & i\eta & u_0\lambda+s+i\eta v_0\end{pmatrix}\quad\text{with}\quad P(\lambda)=-\nu\lambda^2+u_0\lambda+s+\nu\eta^2+i\eta v_0.$$

Computing its determinant, we have $\det(\mathbf{M}(\lambda))=P(\lambda)Q(\lambda)$, where

$$\begin{aligned}Q(\lambda) = & -u_0\nu\lambda^3+\left(u_0^2-(s+i\eta v_0)\nu-\frac{1}{Fr^2}\right)\lambda^2+\left(2(s+i\eta v_0)u_0+\nu\eta^2u_0\right)\lambda+\\ & \frac{\eta^2}{Fr^2}+(s+i\eta v_0)(s+i\eta v_0+\nu\eta^2).\end{aligned}$$

Let us define λ_1 and λ_2 the zeros of P, and λ_3, λ_4 and λ_5 those of Q. Regarding Q, it can be shown (see [20]) that 2 roots have a nonpositive real part (let call them λ_3 and λ_4) and the last one λ_5 has a nonnegative real part. We are now interested in the asymptotics $\varepsilon\ll 1$, and recall that $\nu=\nu_0\,\varepsilon$.

Case where $u_0 < 1/Fr$. In this situation, the five roots read:

$$\begin{aligned}
\lambda_1 &= \frac{u_0}{\nu_0\varepsilon} + \frac{s+i\eta v_0}{u_0} + O(\varepsilon),\\
\lambda_2 &= -\frac{s+i\eta v_0}{u_0} + O(\varepsilon),\\
\lambda_3 &= -\frac{3a_1}{\varepsilon} + \frac{b_1 - 3a_1a_2}{a_1} + O(\varepsilon),\\
\lambda_4 &= -\frac{3b_1 + \sqrt{9b_1^2 - 12a_1c_1}}{6a_1} + O(\varepsilon),\\
\lambda_5 &= \frac{-3b_1 + \sqrt{9b_1^2 - 12a_1c_1}}{6a_1} + O(\varepsilon),
\end{aligned}$$

where $a_1 = \dfrac{\frac{1}{Fr^2} - u_0^2}{3u_0\nu_0}$, $a_2 = \dfrac{(s+i\eta v_0)}{3u_0}$, $b_1 = -\dfrac{2(s+i\eta v_0)}{3\nu_0}$ and $c_1 = -\dfrac{\frac{\eta^2}{Fr^2} + (s+i\eta v_0)^2}{u_0\nu_0}$, see [20].

In this case, we have $\lambda_1 = O(1/\varepsilon)$ and $\lambda_3 = O(1/\varepsilon)$ while the other roots are $O(1)$.

At order 1 in ε, the operator $\hat{\mathcal{S}}_+^{\mathbf{u}}$ has the simple expression (3.15), but the operator $\hat{\mathcal{S}}_-^{\mathbf{u}}$ reads:

$$\hat{\mathcal{S}}_-^{\mathbf{u}} = \begin{pmatrix} \frac{i\eta^2u_0^2 - is^2 + 2\eta v_0 s + i\eta^2 v_0^2}{is\alpha_0 - \eta v_0\alpha_0 - iu_0\eta^2}, & -\frac{\eta s u_0 + i\eta^2 u_0 v_0 - u_0^2\eta\alpha_0}{is\alpha_0 - \eta v_0\alpha_0 - iu_0\eta^2}, & \frac{-(i\eta^2u_0^2 - is^2 + 2\eta v_0 s + i\eta^2 v_0^2)u_0}{is\alpha_0 - \eta v_0\alpha_0 - iu_0\eta^2}, \\ 0 & -u_0 & 0 \end{pmatrix} + O(\varepsilon)$$

where

$$\alpha_0 = \sqrt{-\eta^2u_0^2Fr^2 + \eta^2 + s^2Fr^2 + 2is\eta v_0 Fr^2 - \eta^2 v_0^2 Fr^2}.$$

It is thus still fully non local, and far from tractable in actual applications.

Case where $u_0 > 1/Fr$. Here, the five roots read:

$$\begin{aligned}
\lambda_1 &= \frac{u_0}{\nu_0\varepsilon} + \frac{s+i\eta v_0}{u_0} + O(\varepsilon),\\
\lambda_2 &= -\frac{s+i\eta v_0}{u_0} + O(\varepsilon),\\
\lambda_3 &= \frac{-3b_1 + \sqrt{9b_1^2 - 12a_1c_1}}{6a_1} + O(\varepsilon),\\
\lambda_4 &= -\frac{3b_1 + \sqrt{9b_1^2 - 12a_1c_1}}{6a_1} + O(\varepsilon),\\
\lambda_5 &= -\frac{3a_1}{\varepsilon} + \frac{b_1 - 3a_1a_2}{a_1} + O(\varepsilon).
\end{aligned}$$

In that case, we have $\lambda_1 = O(1/\varepsilon)$ and $\lambda_5 = O(1/\varepsilon)$, while the other roots are $O(1)$.
At order 1 in ε, the operator $\hat{\mathcal{S}}_-^{\mathbf{u}}$ has the simple expression (3.16), but the expression of operator $\hat{\mathcal{S}}_+^{\mathbf{u}}$ remains very complex and non local.