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Proppant-Induced Opening of Hydraulically Created Fractures

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Abstract. The paper examines the problem of the open configuration created when a hydraulic fracture fluid containing a granular proppant is introduced into the fracture. The mathematical modelling examines the problem of an extended cracked region that is wedged open by a granular material present over a finite region of the crack. The combination of the geostatic stress state and the contact stress created between the granular proppant and the elastic rock mass is used to develop a consistency relationship for estimating the dimension of the region of the fracture that will remain open when the pressures applied to create the fracture are released. The interactive mechanics of the fracture and the proppant has an influence on the geometry of the open region that provides the pathway for extraction of the resource.

Keywords. Hydraulic fracture, proppant-fracture interaction, Griffith fracture models, zones of separation

1. Introduction

The use of hydraulic fracturing to enhance the extraction of energy resources such as oil, natural gas and geothermal fluids is not new, having been studied extensively as far back as the early 20th century. Over the past six decades such research has gained prominence because of the continued interest in extracting deeply located fossil fuel resources that were originally considered largely inaccessible and economically not viable. The classical studies by Zheltov and Khristianovitch [1955], Khristianovitch and Zheltov [1955], Barenblatt [1956, 1957, 1959], Perkins and Kern [1961], Le Tirant and Dupuy [1967], Geertsma and de Klerk [1969], Howard and Fast [1970], Williams [1970], Nordgren [1972], Daneshy [1973, 1976], Geertsma and Haafkens [1979], Cleary et al. [1979, 1980] and Clifton and Abou-Sayed [1979] are recognized for their noteworthy contributions to the development of the subject. Other earlier work summarizing the advances in the application of hydraulic fracturing to resources exploration are given by Nemat-Nasser et al. [1983], Mendelsohn [1984a,b], Mader [1989], Desroches et al. [1994], Valko and Economides [1996], Adachi and Detournay [2008], Smith [2015], Cherepanov [2015], Speight [2016], Ma and Holditch [2016] and many others. Over the past decade, the use of hydraulic fracturing techniques has acquired a great deal of attention in relation to unconventional oil and gas recovery. The literature in this area is vast (several thousands of studies) and no attempt can be made provide an all-encompassing record of the advances and applications.

The objective of hydraulic fracturing is to initiate cracks in the resource-bearing formation to enhance the effective permeability of the domain, leading to improved recovery. The process of hydraulic fracturing, and the commonly-used vulgar term *'fracking'*, refers to the injection of fluids under high pressure into wells installed in the resource-bearing formation to create cracks and fissures, thereby enhancing the production of the resource. The objective of hydraulic fracturing is also to maintain the created crack in an open configuration when the pressures required to create the fracture are released. The geostatic stresses will, in general, initiate closure of the created cracks. To maintain the created crack in an open configuration, the fracturing fluid used in hydraulic fracturing operations usually contains a mixture of a proppant agent, water and chemicals that can enhance the fluidity of the mixture. Some of the fluids used in the hydraulic fracturing operation will flow back to the surface through the well bore and some unknown fraction, along with the chemical additives, will remain underground. This unrecovered fluid fraction is perhaps the major source of concern to environmentally conscious communities that oppose the use of hydraulic fracturing for resource extraction. The potential for the unrecovered chemicals to flow and diffuse through the rock formation and fractures to contaminate the groundwater is real and this is a major drawback [Cooley et al., 2012, Holloway and Rudd, 2013].The proppant agents are typically sand, ceramic beads or other non-deformable particulates (see e.g. the texts cited previously, Liang et al. [2016] and Pangilinan et al. [2016]) The injection of the particulate material into the hydraulically created fracture can ensure that the fracture will remain open. The fissures created during hydraulic fracturing will depend on the geostatic stress state in the vicinity of the pressurization region and the accumulation of the proppant will be decided by the orientation of the fracture, gravity effects and the deposition of the particulates within the fracture through a sedimentation process. The development of an approach to study the interaction between a granular proppant zone and a hydraulically created system of fractures is an extremely complex exercise that invariably requires computational approaches, which can: (i) model the mechanics of fluids with suspended particulates, (ii) generate fractures in fluid-saturated porous media representing the resource-bearing formation during fluid injection, including the extent of fracturing that can exist in a particular geologic medium, and (iii) the interaction between the deposited particulate proppant medium and the deformable resource-bearing formation, where there can be gross particulate movement during closure of a fracture. To the author's knowledge, such an all- encompassing modelling exercise has not yet been realized, even though the use of hydraulic fracturing techniques is ubiquitous.

The objective of this paper is to develop an understanding of the mechanics of the interaction between a proppant region located in a hydraulically created fracture and the surrounding geological medium by appeal to simplified analytical approaches. The paper considers the problem of an intact resource-bearing geological formation that is under geostatic stresses. The hydraulic fracturing approach involves the installation of a directionally drilled well that is oriented in a way that enhances the initiation of either an array of cracks in arbitrary orientations or a single planar crack that could be oriented orthogonal to the direction of the minor principal stress. The fracturing fluids dosed with the particulate proppant are injected into the created crack and the fracturing fluid pressures are released, which causes the closure of regions of the crack that are not propped open by the injected particulates. A schematic view of a typical hydraulic fracturing operation is shown in Figure 1.

Fracture generation in terms of initiation and controlled extension is through the use of sequential fluid injection techniques. At significant depths, this process is largely controlled by the geostatic stress state and the innate fracture toughness of the rock is expected to have only a minor influence. In this article, attention is restricted to a situation where a two-dimensional Griffith-type planar crack is created by hydraulic fracturing. Further, we assume that the injected proppant enters the fracture uniformly and its deposition is controlled by the injection rate of the proppant-dosed fracturing fluid and the process of sedimentation within the pressurized fluid. The exact dimensions of the proppant zone, even in the case of the two-dimensional setting shown in Figure 1, will be difficult to assess precisely; some plausible approaches will be discussed in a subsequent section. When the hydraulic fracturing process is terminated by the release of the injection pressure, the

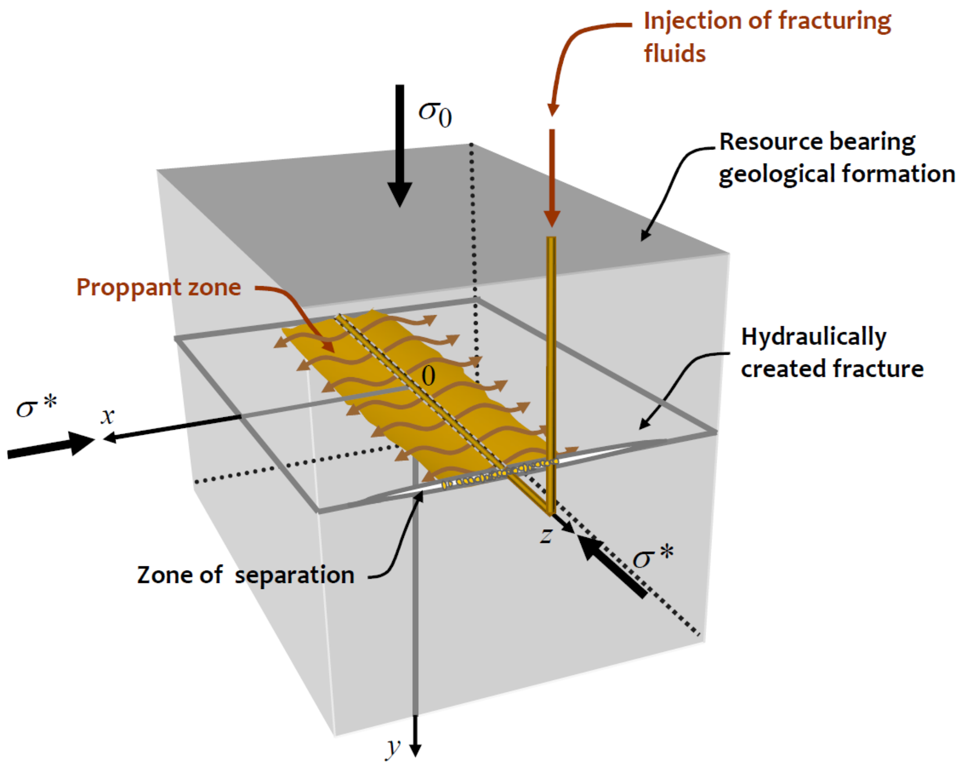

Figure 1. A schematic view of a hydraulic fracturing exercise resulting in a planar separation zone created by the proppant action

fracture surfaces are expected to experience partial closure, which is determined by the restraining action of the proppant region. In terms of the energy resource extraction activity, the important parameter is the extent to which the fracture remains open in the presence of the geostatic stress state and the reactive stresses that are generated during contact between the proppant region and the faces of the fracture. On a related and important theme, the topic of fluid transmissivity in a hydraulically created fracture has also been discussed by Neto and Kotousov [2013] and Khanna et al. [2014]. If it is assumed that the dimensions of the created fracture are large in comparison to the dimensions of the proppant zone, the problem involves the analysis of a unilateral contact problem for the analytical evaluation of the zone of separation. The article by Gladwell and Hara [1981] deals with the compression of an obstacle with spherical boundaries by two elastic halfspace regions. The studies by Selvadurai [1985, 1994] deal with the axisymmetric unilateral contact arising from the compression of a rigid disc-shaped inclusion of finite thickness and finite radius between two halfspace regions. Here the rigid circular disc inclusion can be visualized as an analogue for the proppant region contained within a hydraulically created fracture. In order to solve the problem of the unilateral contact required to estimate the radius of separation, two auxiliary solutions are utilized: the first relates to the indentation of a penny-shaped crack by a smooth rigid disc inclusion Selvadurai and Singh [1984a,b] and the second relates to the internal pressurization of an annular crack [Selvadurai and Singh, 1985]. For both problems the Mode I stress intensity factor at the outer boundary of the indented penny-shaped crack and the outer boundary of the pressurized annular crack are evaluated. The vanishing of the combined Mode I stress intensity factor can be used to determine the radius of the zone of separation. The approach has been used to examine other types of unilateral contact problems encountered in the engineering sciences [Selvadurai, 2003, Selvadurai et al., 2018]. Selvadurai's problem [1985, 1994] related to the compression of a rigid disc by two elastic halfspace regions was re-examined and extended by Gladwell [1995]. The approach presented in these developments has also been applied by Kotousov et al. [2014] to examine the plane strain problem of the compression of a inclusion contained between two halfplane regions.

The methodology used in this paper is to examine the two-dimensional problem of the interaction between the injected proppant region and the hydraulically created crack follows a similar approach: The objective is to determine the dimension of the zone of separation $a < |x| < b$ under the action of the geostatic stress normal to the plane of fracture (σ_0) and the restraining action of the proppant region (Figure 2).

We implicitly assume that the resource-bearing geological medium can be modelled as an isotropic elastic domain even though the medium is fluid-saturated and, strictly speaking, the time-dependent influences of the application and release of fracturing fluids can lead to time-dependent moving boundaries with respect to the zone of

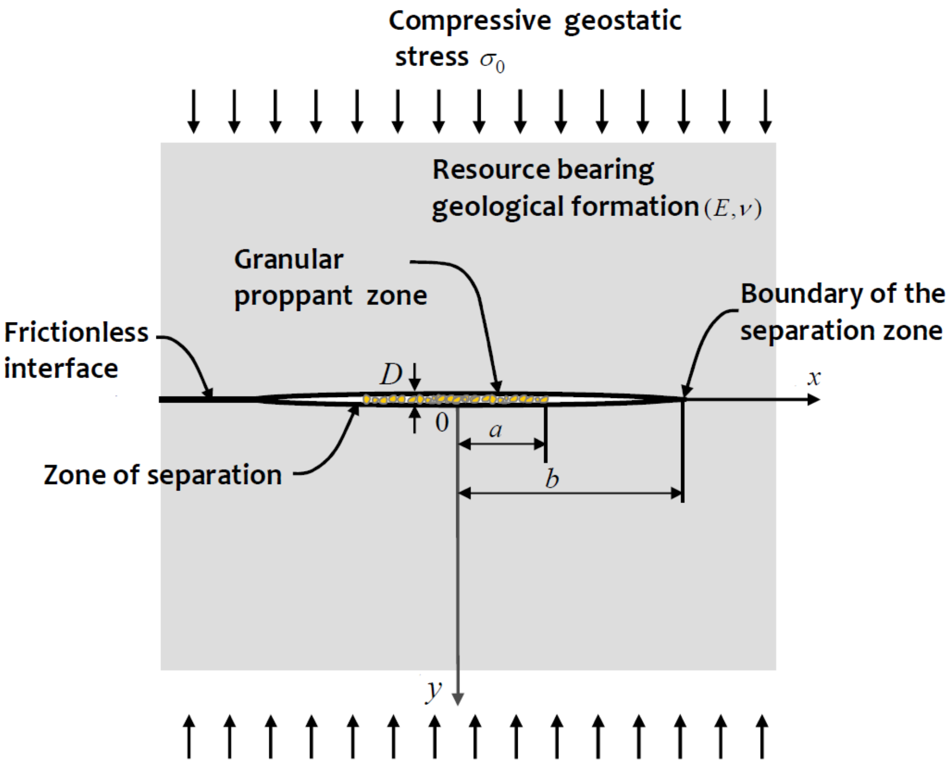

Figure 2. The zone of separation induced by the constraining action of the proppant.

separation/contact [Selvadurai and Mahyari, 1997, 1998]. The first auxiliary problem relates to the estimation of the Mode I stress intensity factor associated with the internal pressurization of the Griffith crack (Figure 3), which is a standard result. The second auxiliary problem relates to the compression of the granular proppant region during release of the fracturing pressures and the resulting reduction in the fracture aperture. First, the granular proppant pile can be of an arbitrary shape and is most likely to have a three-dimensional configuration, unlike the idealized two-dimensional version assumed in this study. Second, the granular proppant pile is an assembly of particles that has no innate strength and will acquire this property through some confinement of the particulate assembly. Third, the particulate assembly is likely to undergo an extrusion-type process due to advance of the fracture surfaces; the limiting compression can be governed by frictional contact between the proppant particles themselves and between the particles and the surfaces of the fracture. Fourth, a "jamming" type process can occur when the approach of the faces of the created fracture reach the largest particle dimension of the proppant, in which case either the movement of the faces will be curtailed, the proppant particle will experience fragmentation or the indentation of the particle will result in the generation of a contact fracture [Selvadurai, 2000a]. To the author's knowledge, a rational solution to this problem is not available in the literature. The approach adopted in this paper is to estimate the distribution of normal stresses that can be developed during the two-dimensional compression of a thin layer (i.e. thickness of the layer, D, is much smaller than its width $2a$) of a granular material by rigid surfaces. These dimensions themselves are also difficult to estimate and can be deduced only from the compression of the injected proppant pile volume, the particle size distribution and its compression to a thickness that can be observed in experiments. As a first approximation, the thickness of the proppant region (D) is assumed to correspond to the maximum particle size of the injected proppant and the width ($2a$) is gauged from the mass of the particulates and a specified void ratio. A schematic view of the second auxiliary problem is shown in Figure 4.

2. The Modeling

We consider the auxiliary problems related to the internal loading of the Griffith crack by (i) uniform stress corresponding to the far field compressive stress state σ_0, and (ii) the proppant region-induced loading over a segment of the Griffith crack. To examine these crack problems we consider a generalized formulation of the mixed boundary value problem for a Griffith crack where the crack surfaces are subjected to a symmetric normal stress $p(x)$ that is applied to the faces of the crack. Excellent expositions of the developments are given by Sneddon and Elliott [1946] and Sneddon and Lowengrub [1969] and in this section the salient results are summarized for completeness. When the Griffith crack is subjected to stresses that are symmetric about the plane $y = 0$, the mixed boundary value problems in elasticity associated with the auxiliary problems can be written in the general forms

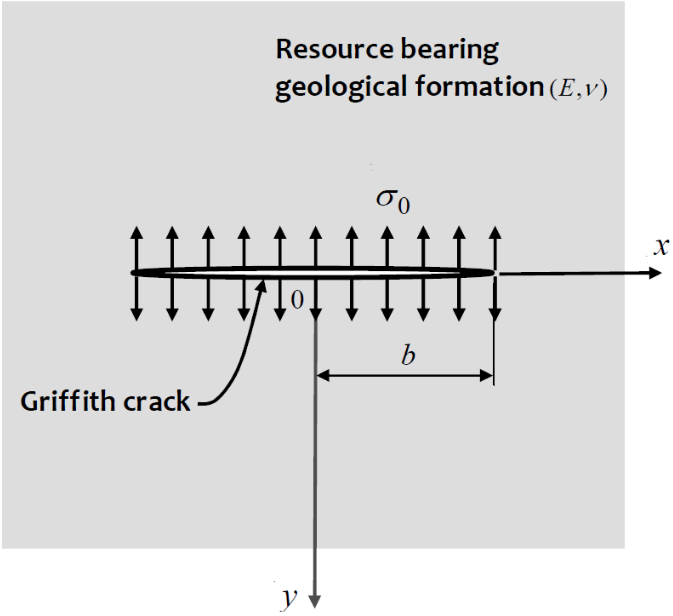

Figure 3. The Griffith crack problem.

$$\sigma_{yy}(x,0) = -p(x); \qquad 0 \leqslant x \leqslant b \tag{1}$$
$$u_y(x,0) = 0; \qquad x > b \tag{2}$$
$$\sigma_{xy}(x,0) = 0; \qquad x \geqslant 0 \tag{3}$$

The two-dimensional problems can be formulated by appeal to the theory of Fourier sine or cosine transforms defined by

$$\mathscr{F}_c[f(\xi,y);\xi \to x] = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty f(\xi,y)\cos(\xi x)d\xi \tag{4}$$
$$\mathscr{F}_s[f(\xi,y);\xi \to x] = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty f(\xi,y)\sin(\xi x)d\xi \tag{5}$$

Considering developments in the classical theory of elasticity for two dimensional problems [Little, 1975, Selvadurai, 2000b], the expressions for the displacements $u_x(x,y)$ and $u_y(x,y)$ can be written as

$$u_x(x,y) = \sqrt{\frac{2}{\pi}}\frac{(1+\nu)}{E}\mathscr{F}_s\left[(1-2\nu-\xi y)\psi(\xi)e^{-\xi y};\xi \to x\right] \tag{6}$$
$$u_y(x,y) = \sqrt{\frac{2}{\pi}}\frac{(1+\nu)}{E}\mathscr{F}_s\left[(2-2\nu+\xi y)\psi(\xi)e^{-\xi y};\xi \to x\right] \tag{7}$$

where E and ν are Young's modulus and Poisson's ratio, respectively, and $\psi(\xi)$ is an arbitrary function. The expressions are applicable to situations involving generalized plane strain and the elastic constants need to be changed to recover the solution applicable to a state of plane stress. The expressions for the stress components indicate that $\sigma_{xy}(x,0)$ vanishes for all real values of x and that

$$\sigma_{yy}(x,0) = -\sqrt{\frac{2}{\pi}}\frac{d}{dx}\mathscr{F}_s[\psi(\xi);x] \tag{8}$$

and the mixed boundary conditions (1) and (2) give the following system of dual integral equations for the unknown function $\psi(\xi)$:

$$\sqrt{\frac{2}{\pi}}\frac{d}{dx}\mathscr{F}_s[\psi(\xi);x] = p(x); \qquad 0 \leqslant x \leqslant b \tag{9}$$
$$\mathscr{F}_c[\psi(\xi);x] = 0; \qquad x > b \tag{10}$$

Considering a representation of the unknown function $\psi(\xi)$ in terms of another function $g(t)$ in the form

$$\psi(\xi) = \int_0^b t g(t) J_0(\xi t)dt \tag{11}$$

where J_0 is the zero-order Bessel function of the first kind, it can be shown that the representation (11) automatically satisfies equation (10) and equation (9) gives rise to an Abel integral equation

$$\frac{2}{\pi}\frac{d}{dx}\int_0^x \frac{t g(t)dt}{\sqrt{x^2-t^2}} = p(x); \qquad 0 \leqslant x \leqslant b \tag{12}$$

The solution of (12) is [Gorenflo and Vessella, 1991]

$$g(t) = \int_0^t \frac{p(x)dx}{\sqrt{t^2-x^2}}; \qquad 0 < t < b \tag{13}$$

and the stress component $\sigma_{yy}(x,0)$ can be expressed in terms of the function $g(t)$ as

$$\sigma_{yy}(x,0) = -\frac{2}{\pi}\frac{d}{dx}\int_0^b \frac{t g(t)dt}{\sqrt{x^2-t^2}}; \qquad x > b \tag{14}$$

The result (14) can be used to determine the Mode I stress intensity factor at the tip of the Griffith crack, in the form

$$K_I = \lim_{x \to b^+}\sqrt{x-b}\,\sigma_{yy}(x,0) = \frac{2\sqrt{b}}{\pi}\int_0^b \frac{p(x)dx}{\sqrt{b^2-x^2}}. \tag{15}$$

The result (15) is in a generalized form that can be used to calculate the Mode I stress intensity factor due to any arbitrary distribution of stress $p(x)$ that is symmetrically applied on both faces of the Griffith crack.

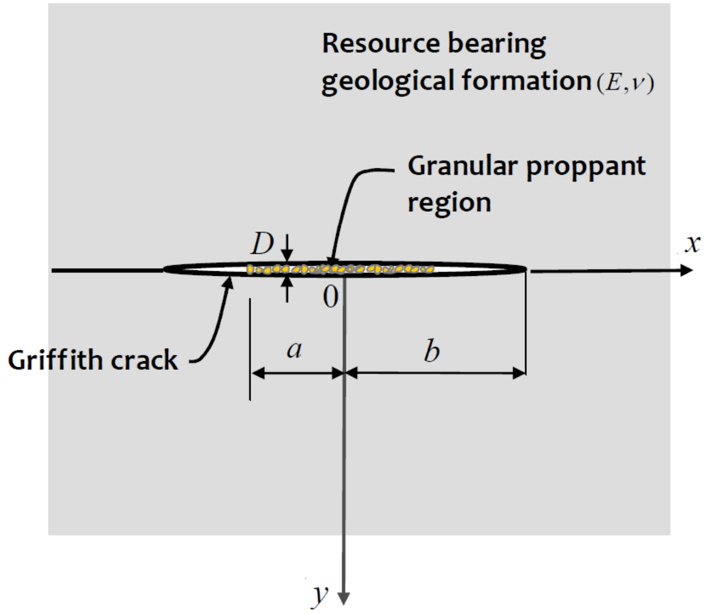

Figure 4. Stressed proppant region within a Griffith crack.

3. The Auxiliary Problems

The objective of the exercise is to determine the extent of the open region of the Griffith crack that will be created when the fracturing pressures are released and the geostatic stresses are allowed to act on the fracture that now contains a proppant region.

3.1. Auxiliary Problem I

We consider the internal pressurization of the Griffith crack by a uniform compressive stress σ_0 applied over both surfaces of the crack. The Mode I stress intensity factor can be obtained from the result (15) by setting $p(x) = \sigma_0$ and this stress intensity factor at the crack tip is.

$$K_I^{(\sigma_0)} = \sigma_0 \sqrt{b} \tag{16}$$

3.2. Auxiliary Problem II

We now focus attention on the evaluation of the Mode I stress intensity factor that results from the contact stresses generated due to the interaction between the granular proppant region and the faces of the crack (Figure 4). As indicated previously, there is no convenient solution to this problem available in the literature. Recourse could be made to the use of theoretical developments that focus on the compression of a thin geomaterial layer between two rigid surfaces. This classical problem in the theory of plasticity was examined by number of investigators including Prandtl [1923], Hartmann [1925], Geiringer [1930], Mandel [1947] and Hill [1950]. The applications of limit analysis techniques for estimating the load carrying capacity of thin layers of non-dilatant geomaterials that possess both cohesion and friction are also given by Chen [1975] and Davis and Selvadurai [2003]. Of particular interest is the problem of the symmetric compression of a thin granular layer by rigid plates, which was examined by Marshall [1967]. This study also provides solutions for the stress field and proceeds to develop the velocity field using the velocity equations for granular materials proposed by Spencer [1964]. The problem of the combined compression and shear of a thin layer of granular material was also examined in an elegant analytical study by Spencer [2005]. Unfortunately, when the results are applied to the study of the compression of a granular layer, a necessary requirement is knowledge of the stresses that are applied at the boundary edges of the thin layer to maintain the integrity of the granular layer. If these stresses are absent, the solution degenerates. The specification of the stress in relation to a passive pressure that can be generated by the region of the proppant extending beyond the compression zone is unreliable. For this reason, attention needs to be focused on developing an alternative model for the interaction between the granular proppant and the resource-bearing geological medium. A plausible model is to assume that the interaction between the Griffith crack and the proppant region can be modelled by the indentation of the crack by a rigid planar punch of width $2a$, which exerts a total force P_0 per unit length. The contact stress at the proppant region and the geologic medium can be approximated by the classical solution for the two-dimensional indentation of a halfplane region developed by Sadowsky [1928] (see also Selvadurai [2000b]). The contact stress distribution can be expressed in the form.

$$\sigma_{yy}(x,0) = \frac{P_0}{\pi\sqrt{a^2 - x^2}}; \qquad -a < x < a \tag{17}$$

The total load per unit length P_0 of the contact region is still unspecified. One procedure for determining the total load is to re-formulate the Griffith crack problem assuming uniform indentation over the region $|x| \leqslant a$, which will result in a three-part mixed boundary value problem; this is non-routine and entails the development of only an approximate result for the Mode I stress intensity factor at the boundary of the indented crack. Here, we develop an alternative procedure that appeals to the granular nature of the proppant region. It is assumed that the fracture zone is propped open by an assembly of the largest group of particles contained in the proppant mixture. Furthermore, the maximum force generated by a wedged particle will correspond to the failure load per unit length for a cylindrical particle of equivalent diameter D. Ideally, the model should examine the failure load for a spherical particle of equivalent diameter D. In keeping with the two-dimensional approach adopted here, we assume that the grain failure load P_G for the cylindrical particle of diameter D and axial length, also equal to D, can be estimated using results applicable to the conventional Brazilian tensile test. i.e.

$$P_G = \pi D^2 \sigma_T \tag{18}$$

If concomitant failure of n large grains occurs in the proppant region, then the peak load is nP_G and the peak stress intensity factor can be estimated from the result

$$K_I^{(G)} = 2nD\sigma_T\sqrt{b}\int_0^a \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} \tag{19}$$

Integrating (19) we obtain

$$K_I^{(G)} = \frac{2nD\sigma_T}{\sqrt{b}}\mathrm{K}(a^2/b^2) \tag{20}$$

where $\mathrm{K}(k)$ is the complete elliptic integral of the first kind defined by

$$\mathrm{K}(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{21}$$

4. Estimation of the Separation Zone

The auxiliary problems, through the evaluation of the Mode I stress intensity factors, allow the application of the constraint proposed by Barenblatt [1962] to estimate the extent of the zone of separation. Since the shear tractions are zero on the plane of symmetry containing the Griffith crack, the vanishing of the Mode I stress intensity factors corresponding to the auxiliary problems will model a separation point where the normal stress vanishes. Adjusting the sign of the stress intensity factor $K_I^{(\sigma_0)}$ to take into consideration the crack opening action, the characteristic equation required to estimate the length of the separation point $x = \pm b$ can be expressed in the form

$$\frac{2nD\sigma_T}{\sqrt{b}}\mathrm{K}(a^2/b^2) - \sigma_0\sqrt{b} = 0 \tag{22}$$

This is a non-linear equation for (b/a) and the smallest root with $(b/a) > 1$ determines the location of the separation region. We can, however, focus on the development of an approximate result by considering a power series approximation for $\mathrm{K}(a^2/b^2)$ in the form

$$\mathrm{K}(a^2/b^2) = \frac{\pi}{2} + \frac{\pi}{8}\left(\frac{a}{b}\right)^2 + \frac{9\pi}{128}\left(\frac{a}{b}\right)^4 + \frac{25\pi}{512}\left(\frac{a}{b}\right)^6 + \ldots \tag{23}$$

If $(a/b) < 1$, we can approximate the complete elliptic integral of the first kind by

$$\mathrm{K}(a^2/b^2) \approx \frac{\pi}{2} \tag{24}$$

and the resulting expression for (22) gives the following result for the normalized approximate estimate for the separation distance:

$$\frac{b}{a} = n\pi\left(\frac{D}{a}\right)\left(\frac{\sigma_T}{\sigma_0}\right) \tag{25}$$

Also, if the larger particles of the proppant zone are densely packed such that $nD \approx 2a$, then we obtain a very rough estimate for the extent of the separation zone in terms of the dimension of the proppant zone as

$$\frac{b}{a} = 2\pi\left(\frac{\sigma_T}{\sigma_0}\right) > 1 \tag{26}$$

Clearly, for a separation region to materialize, $2\pi\sigma_T > \sigma_0$, which is a useful constraint for examining the effectiveness of proppant zones. Usually, proppants consist of quartzitic rocks that have high tensile strength and the above constraint is likely to be realized at depths of interest to hydraulic fracturing operations.

5. Concluding Remarks

The effectiveness of a hydraulically created fracture in terms of resource extraction potential depends on the extent to which proppants can maintain the fracture in an open state when the fracturing pressures are released. The fracture will not remain completely open even with the presence of the proppants and the open fracture regions will contribute significantly to the resource extraction process. The fracture with either a cracked or intact proppant zone will influence oil and gas transport in the proppant filled zones. The fluid transport aspects of particulate-filled fractures that experience closure merits further study, using approaches such as the discrete element method. While this aspect is well appreciated, the methods for estimating the zones that remain open is rarely examined since the problem of interaction between the proppant and the resource-bearing rock is a non trivial contact problem that involves multi-physics of coupled hydro-mechanical processes and moving boundaries. This paper presents an elasticity modelling approach that can be used to provide preliminary estimates for the open area of a fractured region that is under geostatic stresses and kept open by the proppants. The two-dimensional modelling provides an estimate for the separation zones that relates the proppant particle tensile strength to the geostatic normal stress that can induce closure of the fractured zone. Also, in the present treatment, attention is restricted to the situation where the limiting force exerted by the proppant region is estimated by focusing attention on the tensile strength of the proppant grain. In this sense, the methodology presented in the paper provides an elementary result for estimating the separation zone, which takes into account the governing in situ stress and the failure strength of a proppant particle. An alternative approach could involve the case where the limiting force at a proppant grain is estimated by the failure of the geologic medium due to an indentation fracture initiated by proppant grain penetration. A more sophisticated approach will require the consideration of both the poroelasticity of the resource-bearing formation and the mixed boundary value problems that can result from the interaction of the poroelastic rock with the non-deformable granular proppant zones possibly leading to indentational brittle fracture [Selvadurai, 2000a] and indentation damage [Selvadurai, 2004, Selvadurai and Shirazi, 2004] at contact zones. It should be emphasized that Eq. (26) describes only one of many mechanisms that can contribute to the closing of a hydraulically created fracture and is only valid when there is a proppant monolayer and the strength of the rock is far greater than the strength of a proppant particle in relation to the geostatic stress state.

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Review History

This paper was sent for two rounds of reviews. The first round was sent to three Reviewers, two of whom have remained anonymous. Two recommended publication and one was against publication. After a second round of Review by two of the Reviewers, a decision was made to accept the manuscript for publication. Please see the Review History online for the complete reviews: ReviewHistory.pdf.

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