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par Mordechay B. LEVIN

Résumé. Soit $\mathbf{x}_0, \mathbf{x}_1, \dots$ une suite de points dans $[0,1)^s$. Un sous-ensemble S de $[0,1)^s$ est appelé un ensemble à restes bornés s'il existe un nombre réel C tel que, pour tout entier positif N,

$$|\mathrm{card}\{n < N \,:\, \mathbf{x}_n \in S\} - \mathrm{mes}(S)N| < C.$$

Soient $(\mathbf{x}_n)_{n\geq 0}$ une suite de Kronecker de dimension s en base $b \geq 2$ et $\gamma = (\gamma_1, \dots, \gamma_s)$, où, pour $i = 1, \dots, s$, le développement en base b de $\gamma_i \in [0,1)$, $\gamma_i = \gamma_{i,1}b^{-1} + \gamma_{i,2}b^{-2} + \cdots$, vérifie $\gamma_{i,j} \neq b-1$ pour une infinité de j. Dans cet article, nous prouvons que $[0,\gamma_1) \times \cdots \times [0,\gamma_s)$ est un ensemble à restes bornés relativement à la suite $(\mathbf{x}_n)_{n\geq 0}$ si et seulement si

$$\max_{1\leq i\leq s} \sup\{j \geq 1 \,:\, \gamma_{i,j} \neq 0\} < \infty.$$

Nous obtenons ce résultat en conséquence d'un énoncé plus général donné dans la Proposition.

Abstract. Let $\mathbf{x}_0, \mathbf{x}_1, \dots$ be a sequence of points in $[0,1)^s$. A subset S of $[0,1)^s$ is called a bounded remainder set if there exists a real number C such that, for every positive integer N,

$$|\mathrm{card}\{n < N \,:\, \mathbf{x}_n \in S\} - \mathrm{meas}(S)N| < C.$$

Let $(\mathbf{x}_n)_{n\geq 0}$ be an s-dimensional digital Kronecker sequence in base $b \geq 2$, $\gamma = (\gamma_1, \dots, \gamma_s)$, $\gamma_i \in [0,1)$ with base-b expansion
$\gamma_i = \gamma_{i,1}b^{-1} + \gamma_{i,2}b^{-2} + \cdots$ for infinitely many $\gamma_{i,j} \neq b-1$, $i = 1, \dots, s$. In this paper, we prove that $[0,\gamma_1) \times \cdots \times [0,\gamma_s)$ is a bounded remainder set with respect to the sequence $(\mathbf{x}_n)_{n\geq 0}$ if and only if

$$\max_{1\leq i\leq s} \sup\{j \geq 1 \,:\, \gamma_{i,j} \neq 0\} < \infty.$$

We get this result as a consequence of a more general statement given in the Proposition.

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1. Introduction

1.1. Discrepancy. Let $\mathbf{x}_0, \mathbf{x}_1, \dots$ be a sequence of points in $[0,1)^s$, and let $S \subseteq [0,1)^s$ be Lebesgue measurable,

$$\Delta\Big(S, (\mathbf{x}_n)_{n=0}^{N-1}\Big) = \sum_{n=0}^{N-1} (\mathbb{1}_S(\mathbf{x}_n) - \lambda(S)), \tag{1.1}$$

where $\mathbb{1}_S(\mathbf{x}) = 1,$ if $\mathbf{x} \in S$, and $\mathbb{1}_S(\mathbf{x}) = 0$, if $\mathbf{x} \notin S$. Here $\lambda(S)$ denotes the s-dimensional Lebesgue-measure of S. We define the star *discrepancy* of an N-point set $(\mathbf{x}_n)_{n=0}^{N-1}$ as

$$D^*\Big((\mathbf{x}_n)_{n=0}^{N-1}\Big) = \sup\nolimits_{0<y_1,\dots,y_s\le 1}\Big|\Delta\Big([\mathbf{0},\mathbf{y}), (\mathbf{x}_n)_{n=0}^{N-1}\Big)/N\Big|, \tag{1.2}$$

where $[\mathbf{0},\mathbf{y}) = [0,y_1) \times \dots \times [0,y_s)$. The sequence $(\mathbf{x}_n)_{n\ge 0}$ is said to be *uniformly distributed* in $[0,1)^s$ if $D^*((\mathbf{x}_n)_{n=0}^{N-1}) \to 0$ for $N \to \infty$.

An s-dimensional sequence $(\mathbf{x}_n)_{n\ge 0}$ is of *low-discrepancy* if $D^*((\mathbf{x}_n)_{n=0}^{N-1}) = O(N^{-1}(\log N)^s)$ for $N \to \infty$.

So far, only three classes of multidimensional low-discrepancy sequences in $[0,1)^s$ have been known: (t,s)-sequences, Halton's sequences and sequences obtained from a module of totally real algebraic number field (see, e.g., [4, 6, 17, 18, 19, 22]).

In 1954, Roth proved that $\limsup_{N\to\infty} N(\log N)^{-\frac{s}{2}} D^*((\mathbf{x}_n)_{n=0}^{N-1}) > 0$. According to the well-known conjecture (see, e.g., [4, p. 283]), this estimate can be improved to

$$\limsup\nolimits_{N\to\infty} N(\log N)^{-s} D^*\Big((\mathbf{x}_n)_{n=0}^{N-1}\Big) > 0. \tag{1.3}$$

In [17, 18, 19], we proved this conjecture for known multidimensional low-discrepancy sequences. For the general case, the best lower bound of the discrepancy were obtained in [5] : $ND^*((\mathbf{x}_n)_{n=0}^{N-1}) > c(s)(\log N)^{\frac{s-1}{2}+\eta(s)}$ for some $c(s), \eta(s) > 0$.

1.2. Digital Kronecker sequence. For an arbitrary prime power b, let $\mathbb{F}_b$ be the finite field of order b, $\mathbb{F}_b^* = \mathbb{F}_b \setminus \{0\}$, $\mathbb{Z}_b = \{0, 1, \dots, b-1\}$. Let $\mathbb{F}_b[z]$ be the ring of polynomials over $\mathbb{F}_b$, and let $\mathbb{F}_b((z^{-1}))$ be the field of formal Laurent series. Every element L of $\mathbb{F}_b((z^{-1}))$ has a unique expansion into a formal Laurent series

$$L = \sum_{k=w}^{\infty} u_k z^{-k} \quad \text{with} \quad u_k \in \mathbb{F}_b, \ \text{and} \ w \in \mathbb{Z} \ \text{where} \ u_w \neq 0. \tag{1.4}$$

The discrete degree valuation ν of L is defined by

$$\nu(L) := -w \ \text{ for } \ L \neq 0, \quad \text{and} \ \nu(0) := -\infty.$$

Furthermore, we define the *fractional part* of L by

$$\{L\} = \sum_{k=\max(1,w)}^{\infty} u_k z^{-k}. \tag{1.5}$$

The elements of $\mathbb{F}_b$ are denoted by $\bar{0}, \bar{1}, \ldots, \overline{b-1}$, respectively, with $\bar{0}$, the neutral element of addition in $\mathbb{F}_b$. We use a bijection $\psi : \mathbb{Z}_b \to \mathbb{F}_b$ with $\psi(j) := \bar{j}$ for $j \in \mathbb{Z}_b$. For $n = 0, 1, \ldots$, let

$$n = \sum_{r=0}^{\infty} a_r(n) b^r \tag{1.6}$$

be the digit expansion of n in base b, satisfying $a_r(n) \in \mathbb{Z}_b$ for $r \geq 0$ and $a_r(n) = 0$ for all sufficiently large r.

With every $n = 0, 1, \ldots$, we associate the polynomial

$$n(z) = \sum_{r=0}^{\infty} \overline{a_r(n)} z^r \in \mathbb{F}_b[z] \tag{1.7}$$

and if $L \in \mathbb{F}_b((z^{-1}))$ is as in (1.4), then we define

$$\{L\}_{|z=b} = \sum_{k=\max(1,w)}^{\infty} \psi^{-1}(u_k) b^{-k}. \tag{1.8}$$

In [22], Niederreiter introduced a non-Archimedean analogue of the classical Kronecker sequences. Here we use a slightly less general construction proposed by Larcher [11, p. 199], see also [16, p. 3]. For every s-tuple $\mathbf{L} = (L_1, \ldots, L_s)$ of elements of $\mathbb{F}_b((z^{-1}))$, we define the sequence $S(\mathbf{L}) = (\mathbf{l}_n)_{n \geq 0}$ by

$$\mathbf{l}_n = (l_n^{(1)}, \ldots, l_n^{(s)}), \quad \text{with } l_n^{(i)} = \{n(z) L_i(z)\}_{|z=b}, \text{ for } 1 \leq i \leq s,\ n \geq 0. \tag{1.9}$$

This sequence is sometimes called a digital Kronecker sequence (or Kronecker-type sequence) (see [16, p. 4]). The similarity to the classical Kronecker sequence is obvious. In analogy to the classical Kronecker sequences, in [14, Theorem 1], the following theorem has been proven.

Theorem A. *A digital Kronecker sequence $S(L)$ is uniformly distributed in $[0,1)^s$ if and only if $1, L_1, \ldots, L_s$ are linearly independent over $\mathbb{Z}_b[x]$.*

Let us consider the famous Littlewood's conjecture:

$$\varliminf_{n \to \infty} n \, \|n\alpha_1\| \cdots \|n\alpha_s\| = 0 \tag{1.10}$$

for all reals $\alpha_1, \ldots, \alpha_s$, where $\|x\| = \min(\{x\}, 1 - \{x\})$. This problem is also known in the function fields case (see, e.g., [1]).

By [23, Theorem 1.1], if (1.10) is false, then the corresponding Kronecker sequence is of low-discrepancy. The same result is true for the digital Kronecker sequence [14, Theorem 2]. In [2], a counterexample for a p-adic variant of the Littlewood conjecture in a positive characteristic was constructed for $s = 2$. From this it could be derived that there exist two-dimensional low-discrepancy digital Kronecker–van der Corput sequences $(l_n^{(1)}, \sum_{r\geq 0} a_r(n)/b^{r+1})_{n\geq 0}$. The digital Kronecker–van der Corput sequences are special cases of the digital Kronecker–Halton sequences (see [10]). For the ordinary Kronecker–Halton sequences see, e.g., [13]. But the problem whether two-dimensional low-discrepancy digital Kronecker sequences (1.9) exist or not is still unresolved, as it is the case for the ordinary Kronecker sequences.

By μ_1 we denote the normalized Haar-measure on $\mathbb{F}_b((z^{-1}))$ and by μ_s the s-fold product measure on $\mathbb{F}_b((z^{-1}))^s$. Below we will talk about almost all the elements $L \in \mathbb{F}_b((z^{-1}))^s$. In this case, we will keep in mind exactly the measure μ_s.

By [24], $\underline{\lim}_{n\to\infty} n \log^{s+\epsilon} n \, \|n\alpha_1\| \cdots \|n\alpha_s\| = \infty$ for all $\epsilon > 0$ and for almost all reals $\alpha_1, \dots, \alpha_s$. The same result is true for function field cases [15, Theorem 9].

In [3, Theorem 1], the following metrical upper bound for the star discrepancy of Kronecker's sequence was proved:

$$D^*\Big(\{\boldsymbol{\alpha} n\}_{n=0}^{N-1}\Big) = O\Big(N^{-1}(\log N)^s(\log\log N)^2\Big)$$

with $\{\boldsymbol{\alpha} n\} = (\{\alpha_1 n\}, \dots, \{\alpha_s n\})$ for almost all reals $\alpha_1, \dots, \alpha_s$. Hence, Kronecker's sequence is of almost low-discrepancy for almost all reals $\alpha_1, \dots, \alpha_s$. The same result is true for function field cases [11, Theorem].

In [3, Theorem 1], the following metrical lower bound for the star discrepancy of Kronecker's sequence was proved:

$$D^*\Big(\{\boldsymbol{\alpha} n\}_{n=0}^{N-1}\Big) \geq c(b,s) N^{-1}(\log N)^s \log\log N$$

for infinitely many $N \geq 1$ for almost all reals $\alpha_1, \dots, \alpha_s$. Therefore, for almost all real numbers $\alpha_1, \dots, \alpha_s$, Kronecker's sequence is not of low-discrepancy. The same result is true for digital Kronecker sequences [16, Theorem 2].

1.3. Bounded remainder set.

Definition 1. *Let* $\mathbf{x}_0, \mathbf{x}_1, \dots$ *be a sequence of points in* $[0,1)^s$*. A Lebesgue measurable subset* S *of* $[0,1)^s$ *is called a* bounded remainder set *for* $(\mathbf{x}_n)_{n\geq 0}$ *if the discrepancy function* $\Delta(S, (\mathbf{x}_n)_{n=0}^{N-1})$ *is bounded in* N*.*

Let α be an irrational number, let I be an interval in $[0,1)$ with length $|I|$, let $\{n\alpha\}$ be the fractional part of $n\alpha$, $n = 1, 2, \dots$. Hecke, Ostrowski and

Kesten proved that $\Delta(S, (\{n\alpha\})_{n=1}^N)$ is bounded if and only if $|I| = \{k\alpha\}$ for some integer k (see references in [7]).

The sets of bounded remainder for the classical s-dimensional Kronecker sequence were studied by Lev and Grepstad [7]. The case of Halton's sequence was studied by Hellekalek [8]. For references to others investigations on bounded remainder set see [7].

Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_s)$, $\gamma_i \in (0,1)$ with the unique b-adic representation $\gamma_i = \gamma_{i,1}b^{-1} + \gamma_{i,2}b^{-2} + \cdots$, $i = 1, \dots, s$ with infinitely many digits not equal to $b-1$ is used. In this paper, we prove the following main

Theorem. *Let $(\mathbf{l}_n)_{n\geq 0}$ be a uniformly distributed digital Kronecker sequence. The set $[0,\gamma_1) \times \cdots \times [0,\gamma_s)$ with $\gamma_i \in (0,1)$ is of bounded remainder with respect to $(\mathbf{l}_n)_{n\geq 0}$ if and only if*

$$\max_{1\leq i\leq s} \sup\{j \geq 1 : \gamma_{i,j} \neq 0\} < \infty. \tag{1.11}$$

In [20], we proved similar results for digital (t,s)-sequences described in [6, Chapter 8]. Note that according to Larcher's conjecture [12, p. 215], the assertion of the Theorem is true for all digital (t,s)-sequences in base b. By Lemma A (see below), a digital Kronecker sequence in base b can be expressed as some digital $(\mathbf{T}, s)$-sequence in base b. Therefore, in the Theorem we consider the generalised conjecture of Larcher.

Let

$$\Gamma = \left\{ \dot{\boldsymbol{\gamma}} = (\dot{\gamma}_1, \dots, \dot{\gamma}_s) \in [0,1)^s \, : \, \dot{\gamma}_i = \sum_{j=1}^{l} \dot{\gamma}_{i,j}/b^j, \; l \geq 1 \right\},$$

$\ddot{\gamma}_i = \dot{\gamma}_i + \gamma_i \in [0,1)$ $(i = 1, \dots, s, \, l \geq 1)$. Applying the Theorem, we get that the interval $[\dot{\gamma}_1, \ddot{\gamma}_1) \times \cdots \times [\dot{\gamma}_s, \ddot{\gamma}_s)$ with $\dot{\boldsymbol{\gamma}} \in \Gamma$ is of bounded remainder with respect to $(\mathbf{l}_n)_{n\geq 0}$ if and only if (1.11) is true. But for $\dot{\boldsymbol{\gamma}} \notin \Gamma$, the problem is still open.

Now we describe the structure of the paper. In Section 2 we recall some notation and results about digital sequences. Section 3 proves the main Theorem based on several lemmas. In Lemmas 1–3, we obtain an estimate of generalized Walsh's series of truncated discrepancy function of digital sequences. In Lemma 4, we use the notation from the *duality theory* (see, e.g., [6, Chapter 7]). The main result of this article is the Proposition. Lemma 5–Lemma 7 are auxiliary.

2. Notations

Let b be an integer greater or equal to 2, and let $s \geq 1$ be a dimension. A subinterval E of $[0,1)^s$ of the form

$$E = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1)b^{-d_i}),$$

with $a_i, d_i \in \mathbb{Z},\ d_i \geq 0,\ 0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$ is called an *elementary interval in base b.*

Definition 2. *Let $0 \leq t \leq m$ be integers. A (t,m,s)-net in base b is a point set $\mathbf{x}_0, \dots, \mathbf{x}_{b^m-1}$ in $[0,1)^s$ such that $\#\{n \in \{0,1,\dots,b^m-1\} : \mathbf{x}_n \in E\} = b^t$ for every elementary interval E in base b with $\operatorname{vol}(E) = b^{t-m}$.*

Definition 3 ([6, Definition 4.30])**.** *For a given function $\mathbf{T} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\mathbf{T}(m) \leq m$ for all $m \in \mathbb{N}_0$, a sequence $(\mathbf{x}_n)_{n \geq 0}$ of points in $[0,1)^s$ is called a $(\mathbf{T}, s)$-sequence in base b if for all integers $m \geq 1$ and $k \geq 0$, the point set consisting of the points $\mathbf{x}_{kb^m}, \dots, \mathbf{x}_{kb^m + b^m - 1}$ forms a $(\mathbf{T}(m), m, s)$-net in base b.*

A $(\mathbf{T}, s)$-sequence in base b is called a *strict* $(\mathbf{T}, s)$-sequence in base b if for all functions $\mathbf{U} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\mathbf{U}(m) \leq m$ for all $m \in \mathbb{N}_0$ and with $\mathbf{U}(m) < \mathbf{T}(m)$ for at least one $m \in \mathbb{N}_0$, it is not a $(\mathbf{U}, s)$-sequence in base b.

Definition 4 ([6, Definition 4.47])**.** *Let $m, s \geq 1$ be integers. Let $C^{(1,m)}, \dots, C^{(s,m)}$ be $m \times m$ matrices over $\mathbb{F}_b$. Now we construct b^m points in $[0,1)^s$. For $n = 0, 1, \dots, b^m - 1$, let $n = \sum_{j=0}^{m-1} a_j(n) b^j$ be the b-adic expansion of n. We map the vectors*

$$y_n^{(i,m)} = (y_{n,1}^{(i,m)}, \dots, y_{n,m}^{(i,m)}), \quad \textit{with} \quad y_{n,j}^{(i,m)} = \sum_{r=0}^{m-1} \overline{a_r(n)} c_{j,r}^{(i,m)} \in \mathbb{F}_b \tag{2.1}$$

to the real numbers

$$x_n^{(i)} = \sum_{j=1}^{m} x_{n,j}^{(i,m)} / b^j, \quad \textit{with} \quad \overline{x_{n,j}^{(i,m)}} = y_{n,j}^{(i,m)}, \tag{2.2}$$

to obtain the point

$$\mathbf{x}_n = (x_n^{(1)}, \dots, x_n^{(s)}) \in [0,1)^s.$$

The point set $\{\mathbf{x}_0, \dots, \mathbf{x}_{b^m-1}\}$ is called a digital net *(over $\mathbb{F}_b$) (with* generating matrices *$(C^{(1,m)}, \dots, C^{(s,m)})$).*

For $m = \infty$, we obtain a sequence $\mathbf{x}_0, \mathbf{x}_1, \dots$ of points in $[0,1)^s$ which is called a digital sequence (*over* $\mathbb{F}_b$) (*with* generating matrices $(C^{(1,\infty)}, \dots, C^{(s,\infty)})$).

We abbreviate $C^{(i,m)}$ as $C^{(i)}$ for $m \in \mathbb{N}$ and for $m = \infty$.

Lemma A ([15, Theorem 3])**.** *A digital Kronecker sequence in base b can be expressed as some digital $(\mathbf{T}, s)$-sequence in base b.*

Details on the generating matrices $(C^{(1)}, \dots, C^{(s)})$ here are as follows. For given s-tuple $(L_1, \dots, L_s)$ of elements of $\mathbb{F}_b((z^{-1}))$ with $L_i = \sum_{k \geq w_i} u_k^{(i)} z^{-k}$, $1 \leq i \leq s$, we define

$$C^{(i)} = (c_{j,r}^{(i)})_{j \geq 1, r \geq 0} \quad \text{with} \quad c_{j,r}^{(i)} = u_{r+j}^{(i)} \quad \text{for} \quad 1 \leq i \leq s,\ j \geq 1,\ r \geq 0.$$

Lemma B ([6, Theorem 4.86]). *Let b be a prime power. A strict digital $(\mathbf{T}, s)$-sequence over $\mathbb{F}_b$ is uniformly distributed in $[0,1)^s$, if and only if $\liminf_{m\to\infty}(m - \mathbf{T}(m)) = \infty$.*

For $k > l$, we put $\sum_{j=k}^{l} c_j = 0$ and $\prod_{j=k}^{l} c_j = 1$. For $x = \sum_{j\geq 1} x_j b^{-j}$, where $x_i \in \mathbb{Z}_b = \{0, \dots, b-1\}$, we define the truncation

$$[x]_m = \sum_{1\leq j\leq m} x_j b^{-j} \quad \text{with} \quad m \geq 1.$$

If $\mathbf{x} = (x^{(1)}, \dots, x^{(s)}) \in [0,1)^s$, then the truncation $[\mathbf{x}]_m$ is defined coordinatewise, that is, $[\mathbf{x}]_m = ([x^{(1)}]_m, \dots, [x^{(s)}]_m)$.

For $x = \sum_{j\geq 1} x_j b^{-j}$ and $y = \sum_{j\geq 1} y_j b^{-j}$ where $x_j, y_j \in \mathbb{Z}_b$, we define the ($b$-adic) digital shifted point v by $v = x \oplus y := \sum_{j\geq 1} v_j b^{-j}$, where $v_j \equiv x_j + y_j \pmod{b}$ and $v_j \in \mathbb{Z}_b$. Let $x \ominus y := \sum_{j\geq 1} v_j b^{-j}$, where $v_j \equiv x_j - y_j \pmod{b}$ and $v_j \in \mathbb{Z}_b$.

For $\mathbf{x} = (x^{(1)}, \dots, x^{(s)}) \in [0,1)^s$ and $\mathbf{y} = (y^{(1)}, \dots, y^{(s)}) \in [0,1)^s$, we define the ($b$-adic) digital shifted point $\mathbf{v}$ by $\mathbf{x} \oplus \mathbf{y} = (x^{(1)} \oplus y^{(1)}, \dots, x^{(s)} \oplus y^{(s)})$. Let $\mathbf{x} \ominus \mathbf{y} := (x^{(1)} \ominus y^{(1)}, \dots, x^{(s)} \ominus y^{(s)})$. For $n_1, n_2 \in \{0, 1, \dots, b^l - 1\}$, we define $n_1 \oplus n_2 := (n_1/b^l \oplus n_2/b^l)b^l$. Let $n_1 \ominus n_2 := (n_1/b^l \ominus n_2/b^l)b^l$.

For $x = \sum_{j\geq 1} x_j b^{-j}$, where $x_j \in \mathbb{Z}_b$, $x_j = 0$ for $j = 1, \dots, k$ and $x_{k+1} \neq 0$, we define the absolute b-adic valuation $\|\cdot\|_b$ of x by $\|x\|_b = b^{-k-1}$. Let $\|n\|_b = b^k$ with $k \in \mathbb{N}_0$ such that $n \in \{b^k, \dots, b^{k+1} - 1\}$.

Definition 5. *A sequence $(\mathbf{x}_n)_{n\geq 0}$ in $[0,1)^s$ is* weakly admissible *in base b if*

$$\varkappa_m := \min_{0\leq k<n<b^m} \|\mathbf{x}_n \ominus \mathbf{x}_k\|_b > 0 \tag{2.3}$$

for all $m \geq 1$ where $\|\mathbf{x}\|_b := \prod_{i=1}^{s} \left\|x^{(i)}\right\|_b$.

In the following, we will use truncations $[\gamma_i]_{\tau_m}$ $(i = 1 \dots, s)$, with $\tau_m = [|\log_b(\varkappa_m)|] + m$. By Lemma 1 and the Proposition, which will be stated and proved in the subsequent Section 3, the weakly admissible property is important for the proof of the Theorem.

3. Proof of the main Theorem

Lemma 1. *Let $(\mathbf{x}_n)_{n\geq 0}$ be an s-dimensional weakly admissible digital sequence in base b, $m \geq 1$, $\tau_m = [|\log_b(\varkappa_m)|] + m$. Then we have for all integers $A \geq 0$*

$$\left|\Delta\Big([\mathbf{0}, \boldsymbol{\gamma}), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1}\Big) - \Delta\Big([\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1}\Big)\right| \leq s, \quad \forall\, N \in \{1, \dots, b^m\}.$$

Proof. Let

$$B=[\mathbf{0},\boldsymbol{\gamma}), \quad B_i=\prod_{1\le j<i}\prod_{i<j\le s}[0,\gamma^{(j)})\times[0,[\gamma^{(i)}]_{\tau_m}) \text{ and } B_0=\bigcup_{i=1}^{s}(B\setminus B_i).$$

It is easy to see that $B=[\mathbf{0},[\boldsymbol{\gamma}]_{\tau_m})\cup B_0$. By (1.1), we get

$$\Delta\Big([\mathbf{0},\boldsymbol{\gamma}),(\mathbf{x}_n)_{n=b^mA}^{b^mA+N-1}\Big) = \Delta\Big([\mathbf{0},[\boldsymbol{\gamma}]_{\tau_m}),(\mathbf{x}_n)_{n=b^mA}^{b^mA+N-1}\Big)+\Delta\Big(B_0,(\mathbf{x}_n)_{n=b^mA}^{b^mA+N-1}\Big).$$

Hence

$$\Big|\Delta\Big([\mathbf{0},\boldsymbol{\gamma}),(\mathbf{x}_n)_{n=b^mA}^{b^mA+N-1}\Big)-\Delta\Big([\mathbf{0},[\boldsymbol{\gamma}]_{\tau_m}),(\mathbf{x}_n)_{n=b^mA}^{b^mA+N-1}\Big)\Big| \le \sum_{i=1}^{s}\Big|\Delta\Big(B\setminus B_i,(\mathbf{x}_n)_{n=b^mA}^{b^mA+N-1}\Big)\Big|. \tag{3.1}$$

Suppose that there exist $i\in\{1,\dots,s\}$, $k,n\in\{0,1,\dots,b^m-1\}$, $k\neq n$ and $A\ge 0$ such that $x_{n+b^mA},x_{k+b^mA}\in B\setminus B_i$. Therefore

$$x^{(i)}_{n+b^mA,j}=x^{(i)}_{k+b^mA,j} \quad \text{for} \quad j=1,\dots,\tau_m.$$

From (1.6), (2.1) and (2.2), we have

$$y^{(i)}_{n+b^mA,j}=y^{(i)}_{k+b^mA,j} \quad \text{for} \quad j=1,\dots,\tau_m,$$

$$y^{(i)}_{n+b^mA,j}=y^{(i)}_{n,j}+y^{(i)}_{b^mA,j}, \text{ and } y^{(i)}_{k+b^mA,j}=y^{(i)}_{k,j}+y^{(i)}_{b^mA,j} \text{ for } j=1,\dots,\tau_m.$$

Hence

$$y^{(i)}_{n,j}=y^{(i)}_{k,j},\ \ j=1,\dots,\tau_m \quad \text{and} \quad x^{(i)}_{n,j}=x^{(i)}_{k,j},\ \ j=1,\dots,\tau_m.$$

Therefore

$$\Big\|x^{(i)}_n\ominus x^{(i)}_k\Big\|_b<b^{-\tau_m}\le\varkappa_m \quad \text{and} \quad \|\mathbf{x}_n\ominus\mathbf{x}_k\|_b\ge\varkappa_m.$$

By (2.3), we have a contradiction. Thus

$$\operatorname{card}\{n\in\{0,1,\dots,b^m-1\}:\mathbf{x}_{n+b^mA}\in B\setminus B_i\}\le 1, \quad \text{and } \Big|\Delta\Big(B\setminus B_i,(\mathbf{x}_n)_{n=b^mA}^{b^mA+N-1}\Big)\Big|\le 1.$$

Using (3.1), we get the assertion of Lemma 1. □

Let p be a prime, $b=p^\kappa$,

$$E(\alpha):=\exp(2\pi i\,\mathrm{Tr}(\alpha)/p), \qquad \alpha\in\mathbb{F}_b,$$

where $\mathrm{Tr}: \mathbb{F}_b \to \mathbb{F}_p$ denotes the usual trace of an element of $\mathbb{F}_b$ in $\mathbb{F}_p$. We identify $\mathbb{F}_p$ with $\mathbb{Z}_p$. Let

$$\delta(\mathfrak{T}) = \begin{cases} 1, & \text{if } \mathfrak{T} \text{ is true}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.2}$$

By [21, Ref. 5.6 and Ref. 5.8], we get

$$\frac{1}{b} \sum_{\beta \in \mathbb{F}_b} E(\alpha\beta) = \delta(\alpha = 0), \quad \text{where} \quad \alpha \in \mathbb{F}_b. \tag{3.3}$$

Let $\beta_1, \dots, \beta_\kappa$ be a $\mathbb{F}_p$ base of $\mathbb{F}_b$, and let Tr be a standard trace function. We need the following special bijection $\omega : \mathbb{F}_b \to \mathbb{Z}_b$:

$$\omega(\alpha) = \sum_{j=1}^{\kappa} p^{j-1} \,\mathrm{Tr}(\alpha\beta_j), \qquad b = p^\kappa. \tag{3.4}$$

We use notations (1.6), (2.1) and (2.2). Let $n = \sum_{r \geq 0} a_r(n) b^r$ be the b-adic expansion of n, and let

$$\widetilde{n} = \sum_{r \geq 0} \omega(\overline{a_r(n)}) b^r. \tag{3.5}$$

Therefore

$$\{\widetilde{n} \,:\, 0 \leq n < b^m\} = \{0, 1, \dots, b^m - 1\}. \tag{3.6}$$

Hence

$$\overline{a_r(n)} = \omega^{-1}(a_r(\widetilde{n})) \tag{3.7}$$

and

$$u^{(i)}_{\widetilde{n},j} := \sum_{r \geq 0} \omega^{-1}(a_r(\widetilde{n})) c^{(i)}_{j,r} = \sum_{r \geq 0} \overline{a_r(n)} c^{(i)}_{j,r} = y^{(i)}_{n,j}, \quad 1 \leq i \leq s. \tag{3.8}$$

Let

$$x^{(s+1)}_n := \{n/b^m\}, \quad x^{(s+1)}_{n,j} := a_{m-j}(n), \quad y^{(s+1)}_{n,j} := \overline{x^{(s+1)}_{n,j}}. \tag{3.9}$$

Bearing in mind that $\overline{a_{m-j}(n)} = \omega^{-1}(a_{m-j}(\widetilde{n}))$, we put

$$u^{(s+1)}_{\widetilde{n},j} := \omega^{-1}(a_{m-j}(\widetilde{n})) = \overline{a_{m-j}(n)} = y^{(s+1)}_{n,j}, \quad j \in \{1, \dots, m\}. \tag{3.10}$$

Let

$$u^{(i)}_n = (u^{(i)}_{n,1}, \dots, u^{(i)}_{n,\tau_m}) \in \mathbb{F}_b^{\tau_m} \quad \text{and} \quad u^{(s+1)}_n = (u^{(s+1)}_{n,1}, \dots, u^{(s+1)}_{n,m}).$$

We abbreviate $s+1$-dimensional vectors $(u^{(1)}_n, \dots, u^{(s+1)}_n)$, $(x^{(1)}_n, \dots, x^{(s+1)}_n)$, $(k^{(1)}, \dots, k^{(s+1)})$ by symbols $\vec{\mathbf{u}}_n$, $\vec{\mathbf{x}}_n$, $\vec{\mathbf{k}}$, and the s-dimensional vector $(x^{(1)}_n, \dots, x^{(s)}_n)$ by the symbol $\mathbf{x}_n$, where $x^{(s+1)}_n = \{n/b^m\}$.

By (3.4)–(3.10), we get $u_n^{(s+1)} = u_{n+b^m A}^{(s+1)}$, $A = 1, 2, \dots,$

$$u_{n_1 \oplus n_2, j}^{(i)} = u_{n_1, j}^{(i)} + u_{n_2, j}^{(i)}, \quad j \geq 1, i \in \{1, \dots, s+1\}, \qquad \vec{\mathbf{u}}_{n_1 \oplus n_2} = \vec{\mathbf{u}}_{n_1} + \vec{\mathbf{u}}_{n_2}. \tag{3.11}$$

In what follows, we will need this linear property. To get this property, we used $u_{\tilde{n}}^{(i)}$ instead of $y_n^{(i)}$ $(i = 1, \dots, s+1)$.

Let $N \in \{1, \dots, b^m\}$, $\gamma^{(s+1)} = N/b^m$, $\Bbbk = (\Bbbk_1, \dots, \Bbbk_l) \neq \mathbf{0}$, with $\Bbbk_j \in \mathbb{F}_b$,

$$v(\Bbbk) := \max\{j \in \{1, \dots, l\} \,:\, \Bbbk_j \neq 0\}, \quad v(\mathbf{0}) = 0, \;\; l \geq 1. \tag{3.12}$$

We introduce inner products as follows

$$\begin{aligned} \vec{\mathbf{k}} \cdot \vec{\mathbf{u}}_n &= \sum_{i=1}^{s} \sum_{j=1}^{\tau_m} k_j^{(i)} u_{n,j}^{(i)} + \sum_{j=1}^{m} k_j^{(s+1)} u_{n,j}^{(s+1)}, \\ \Bbbk \cdot \mathbb{u} &= \sum_{j=1}^{l} \Bbbk_j \mathbb{u}_j, \quad \Bbbk, \mathbb{u} \in \mathbb{F}_b^l, \; l \geq 1. \end{aligned} \tag{3.13}$$

In Lemma 2, we derive the generalized Walsh series decomposition of the discrepancy function.

Although the argument is fairly standard (see, e.g., [6, Lemma 14.8], [9, Lemma 1], [22, Theorem 3.10]), I have not found it stated explicitly in the literature in a form that is easily applicable to our case. So, I give the details in full.

Lemma 2. *Let $A \geq 1$ be an integer, $N \in \{1, \dots, b^m\}$, $\gamma^{(s+1)} = N/b^m$, and let $(\mathbf{x}_n)_{n \geq 0}$ be a digital sequence in base b. Then*

$$\begin{aligned} &\Delta([\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}), (\mathbf{x}_n)_{n=b^m A}^{b^m A + N - 1}) \\ &= \sum_{n=0}^{b^m - 1} \sum_{(k^{(1)}, \dots, k^{(s)}) \in (\mathbb{F}_b^{\tau_m})^s} \sum_{k^{(s+1)} \in \mathbb{F}_b^m} E(\vec{\mathbf{k}} \cdot \vec{\mathbf{u}}_{\widetilde{n + b^m A}}) \widehat{\mathbb{1}}(\vec{\mathbf{k}}) - b^m \prod_{i=1}^{s+1} [\gamma^{(i)}]_{\tau_m}, \end{aligned}$$

where

$$\widehat{\mathbb{1}}(\vec{\mathbf{k}}) = \prod_{i=0}^{s+1} \widehat{\mathbb{1}}^{(i)}(k^{(i)}), \qquad \widehat{\mathbb{1}}^{(i)}(0) = [\gamma^{(i)}]_{\tau_m} \; (1 \leq i \leq s+1), \;\; [\gamma^{(s+1)}]_{\tau_m} = \gamma^{(s+1)}, \tag{3.14}$$

and

$$(3.15)\quad \widehat{\mathbb{1}}^{(i)}(\mathbb{k}) = b^{-v(\mathbb{k})} E\left(-\sum_{j=1}^{v(\mathbb{k})-1} \mathbb{k}_j \overline{\gamma_j^{(i)}}\right) \times \left(\sum_{\mathbb{b}=0}^{\gamma^{(i)}_{v(\mathbb{k})}-1} E(-\mathbb{k}_{v(\mathbb{k})}\overline{\mathbb{b}}) + E(-\mathbb{k}_{v(\mathbb{k})}\overline{\gamma^{(i)}_{v(\mathbb{k})}})\{b^{v(\mathbb{k})}[\gamma^{(i)}]_{\tau_m}\}\right)$$

with $\mathbb{k} = (\mathbb{k}_1, \dots, \mathbb{k}_l)$, $l \in \{\tau_m, m\}$, $i = 1, \dots, s+1$.

Proof. Let $\gamma = \sum_{j=1}^{l} \gamma_j b^{-j} > 0$, $z = \sum_{j=1}^{\infty} z_j b^{-j}$, with $\gamma_j, z_j \in \mathbb{Z}_b$, $l \geq 1$ be integer. It is easy to verify (see also [22, p. 37–38]) that

$$\mathbb{1}_{[0,\gamma)}(z) = \sum_{r=1}^{l} \sum_{\mathbb{b}=0}^{\gamma_r-1} \prod_{j=1}^{r-1} \delta(z_j = \gamma_j)\delta(z_r = \mathbb{b}).$$

By (2.2) and (3.8), we have that

$$x_{j,n}^{(i)} = \mathbb{b} \iff y_{j,n}^{(i)} = \overline{\mathbb{b}} \iff u_{j,\tilde{n}}^{(i)} = \overline{\mathbb{b}},$$

and

$$\begin{aligned}
\mathbb{1}_{[0,[\gamma^{(i)}]_{\tau_m})}(x_n^{(i)}) &= \sum_{r=1}^{\tau_m} \sum_{\mathbb{b}=0}^{\gamma_r^{(i)}-1} \prod_{j=1}^{r-1} \delta(x_{j,n}^{(i)} = \gamma_j^{(i)})\delta(x_{r,n}^{(i)} = \mathbb{b}) \\
&= \sum_{r=1}^{\tau_m} \sum_{\mathbb{b}=0}^{\gamma_r^{(i)}-1} \prod_{j=1}^{r-1} \delta(y_{j,n}^{(i)} = \overline{\gamma_j^{(i)}})\delta(y_{r,n}^{(i)} = \overline{\mathbb{b}}) \\
&= \sum_{r=1}^{\tau_m} \sum_{\mathbb{b}=0}^{\gamma_r^{(i)}-1} \prod_{j=1}^{r-1} \delta(u_{j,\tilde{n}}^{(i)} = \overline{\gamma_j^{(i)}})\delta(u_{r,\tilde{n}}^{(i)} = \overline{\mathbb{b}}), \quad i = 1, \dots, s.
\end{aligned}$$

Similarly, we derive

$$(3.16)\quad \mathbb{1}_{[0,\gamma^{(s+1)})}(x_n^{(s+1)}) = \sum_{r=1}^{m} \sum_{\mathbb{b}=0}^{\gamma_r^{(s+1)}-1} \prod_{j=1}^{r-1} \delta(u_{j,\tilde{n}}^{(s+1)} = \gamma_j^{(s+1)})\delta(u_{r,\tilde{n}}^{(s+1)} = \overline{\mathbb{b}}).$$

By (3.3), we have

$$(3.17)\quad \mathbb{1}_{[0,[\gamma^{(i)}]_{\tau_m})}(x_n^{(i)}) = \sum_{r=1}^{\tau_m} \sum_{\mathbb{b}=0}^{\gamma_r^{(i)}-1} b^{-r} \sum_{\mathbb{k}_1,\dots,\mathbb{k}_r \in \mathbb{F}_b} \dot{\mathbb{1}}^{(i)}(\mathbb{k}),$$

where

$$\dot{\mathbb{1}}^{(i)}(\mathbb{k}) = E\left(\sum_{j=1}^{r-1} \mathbb{k}_j (u_{j,\tilde{n}}^{(i)} - \overline{\gamma_j^{(i)}}) + \mathbb{k}_r (u_{r_i,\tilde{n}}^{(i)} - \overline{\mathbb{b}})\right) = E(\mathbb{k} \cdot u_{\tilde{n}}^{(i)})\widetilde{\mathbb{1}}^{(i)}(\mathbb{k}),$$

with

$$\widetilde{\mathbb{1}}^{(i)}(\Bbbk) = E\left(-\sum_{j=1}^{r-1} \Bbbk_j \overline{\gamma_j^{(i)}} - \Bbbk_r \overline{\mathbb{b}}\right).$$

Hence

$$\begin{aligned}
\mathbb{1}_{[0,[\gamma^{(i)}]_{\tau_m})}(x_n^{(i)}) &= \sum_{r=1}^{\tau_m} \sum_{\mathbb{b}=0}^{\gamma_r^{(i)}-1} b^{-r} \sum_{\Bbbk_1,\dots,\Bbbk_{\tau_m} \in \mathbb{F}_b} \delta(v(\Bbbk) \le r) E(\Bbbk \cdot u_{\tilde{n}}^{(i)}) \widetilde{\mathbb{1}}^{(i)}(\Bbbk) \\
&= \sum_{\Bbbk_1,\dots,\Bbbk_{\tau_m} \in \mathbb{F}_b} \sum_{r=1}^{\tau_m} \sum_{\mathbb{b}=0}^{\gamma_r^{(i)}-1} b^{-r} \delta(v(\Bbbk) \le r) E(\Bbbk \cdot u_{\tilde{n}}^{(i)}) \widetilde{\mathbb{1}}^{(i)}(\Bbbk) \\
&= \sum_{\Bbbk_1,\dots,\Bbbk_{\tau_m} \in \mathbb{F}_b} E(\Bbbk \cdot u_{\tilde{n}}^{(i)}) \ddot{\mathbb{1}}^{(i)}(\Bbbk),
\end{aligned}$$

where

$$\ddot{\mathbb{1}}^{(i)}(\Bbbk) = \sum_{r=v(\Bbbk)}^{\tau_m} \sum_{\mathbb{b}=0}^{\gamma_r^{(i)}-1} b^{-r} \widetilde{\mathbb{1}}^{(i)}(\Bbbk).$$

Applying (3.15) and (3.17), we derive

$$\begin{aligned}
\ddot{\mathbb{1}}^{(i)}(\Bbbk) &= \sum_{r=v(\Bbbk)}^{\tau_m} \sum_{\mathbb{b}=0}^{\gamma_r^{(i)}-1} b^{-r} E\bigg(-\sum_{j=1}^{r-1} \Bbbk_j \overline{\gamma_j^{(i)}} - \Bbbk_r \overline{\mathbb{b}}\bigg) \\
&= \sum_{\mathbb{b}=0}^{\gamma_{v(\Bbbk)}^{(i)}-1} b^{-v(\Bbbk)} E\bigg(-\sum_{j=1}^{v(\Bbbk)-1} \Bbbk_j \overline{\gamma_j^{(i)}} - \Bbbk_{v(\Bbbk)} \overline{\mathbb{b}}\bigg) \\
&\quad + E\bigg(-\sum_{j=1}^{v(\Bbbk)-1} \Bbbk_j \overline{\gamma_j^{(i)}}\bigg) \sum_{r=v(\Bbbk)+1}^{\tau_m} \sum_{\mathbb{b}=0}^{\gamma_r^{(i)}-1} b^{-r} \\
&= b^{-v(\Bbbk)} E\bigg(-\sum_{j=1}^{v(\Bbbk)-1} \Bbbk_j \overline{\gamma_j^{(i)}}\bigg) \\
&\qquad \times \left(\sum_{\mathbb{b}=0}^{\gamma_{v(\Bbbk)}^{(i)}-1} E\big(-\Bbbk_{v(\Bbbk)} \overline{\mathbb{b}}\big) + E\big(-\Bbbk_{v(\Bbbk)} (\gamma_{v(\Bbbk)}^{(i)})\big) \{b^{v(\Bbbk)} [\gamma]_{\tau_m}^{(i)}\}\right) \\
&= \widehat{\mathbb{1}}^{(i)}(\Bbbk).
\end{aligned}$$

Hence

$$\mathbb{1}_{[0,[\gamma^{(i)}]_{\tau_m})}(x_n^{(i)}) = \sum_{\Bbbk_1,\dots,\Bbbk_{\tau_m} \in \mathbb{F}_b} E(\Bbbk \cdot u_{\tilde{n}}^{(i)}) \widehat{\mathbb{1}}^{(i)}(\Bbbk).$$

Similarly, we obtain from (3.15) and (3.16) that

$$\mathbb{1}_{[0,\gamma^{(s+1)})}(x_n^{(s+1)}) = \sum_{\Bbbk_1,\dots,\Bbbk_m \in \mathbb{F}_b} E(\Bbbk \cdot u_{\tilde{n}}^{(s+1)}) \widehat{\mathbb{1}}^{(s+1)}(\Bbbk).$$

Using (3.13), we obtain

$$\prod_{i=1}^{s+1} \mathbb{1}_{[0,[\gamma^{(i)}]_{\tau_m})}(x_n^{(i)}) = \sum_{(k^{(1)},\dots,k^{(s)}) \in (\mathbb{F}_b^{\tau_m})} \sum_{k^{(s+1)} \in \mathbb{F}_b^m} E(\vec{\mathbf{k}} \cdot \mathbf{u}_{\tilde{n}}) \widehat{\mathbb{1}}(\vec{\mathbf{k}}). \tag{3.18}$$

Bearing in mind that $x_{n+b^mA}^{(s+1)} = \{(n + b^m A)/b^m\} = \{n/b^m\}$ and $\gamma^{(s+1)} = N/b^m$, we have

$$\begin{aligned} &\{n \in \{0,1,\dots,N-1\} : \mathbf{x}_{n+b^mA} \in [\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m})\} \\ &\qquad = \left\{ n \in \{0,1,\dots,b^m - 1\} : \begin{array}{r} \vec{\mathbf{x}}_{n+b^mA} = (\mathbf{x}_{n+b^mA}, x_{n+b^mA}^{(s+1)}) \\ \in [\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}) \times [0, \gamma^{(s+1)}) \end{array} \right\}. \end{aligned}$$

From (3.18) and (1.1), we derive

$$\begin{aligned} &\Delta([\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}), (\mathbf{x}_n)_{n=b^mA}^{b^mA+N-1}) \\ &= \sum_{n=0}^{b^m-1} \prod_{i=1}^{s+1} \mathbb{1}_{[\mathbf{0},[\gamma^{(i)}]_{\tau_m})}(x_{n+b^mA}^{(i)}) - b^m \prod_{i=1}^{s+1} [\gamma^{(i)}]_{\tau_m} \\ &= \sum_{n=0}^{b^m-1} \sum_{(k^{(1)},\dots,k^{(s)}) \in (\mathbb{F}_b^{\tau_m})^s} \sum_{k^{(s+1)} \in \mathbb{F}_b^m} E(\vec{\mathbf{k}} \cdot \vec{\mathbf{u}}_{\widetilde{n+b^mA}}) \widehat{\mathbb{1}}(\vec{\mathbf{k}}) - b^m \prod_{i=1}^{s+1} [\gamma^{(i)}]_{\tau_m}. \end{aligned}$$

Hence Lemma 2 is proved. □

Let

$$\vec{\mathbf{k}} = (k^{(1)}, \dots, k^{(s+1)}), \quad k^{(i)} = (k_1^{(i)}, \dots, k_{\tau_m}^{(i)}), \quad i \in \{1, \dots, s\},$$
$$k^{(s+1)} = (k_1^{(s+1)}, \dots, k_m^{(s+1)}),$$
$$G_m = \left\{ \vec{\mathbf{k}} : \begin{array}{r} k_j^{(i)} \in \mathbb{F}_b \text{ with } j \in \{1,\dots,\tau_m\},\ i \in \{1,\dots,s\}, \\ \text{and } j \in \{1,\dots,m\} \text{ for } i = s+1 \end{array} \right\} = (\mathbb{F}_b^{\tau_m})^s \times \mathbb{F}_b^m,$$
$$G_m^* = G_m \setminus \{\mathbf{0}\},$$

and let

$$D_m = \{\vec{\mathbf{k}} \in G_m : \vec{\mathbf{k}} \cdot \vec{\mathbf{u}}_n = 0 \quad \text{for all} \quad n \in \{0,1,\dots,b^m-1\}\}, \tag{3.19}$$

$D_m^* = D_m \setminus \{\mathbf{0}\}$.

It is easy to see that

$$\mu\vec{\mathbf{k}} \in D_m^* \quad \text{for all} \quad \mu \in \mathbb{F}_b^*,\ \vec{\mathbf{k}} \in D_m^*. \tag{3.20}$$

These definitions, Lemma 3 and Lemma 4 are slight modifications of the results of the *duality theory* (see, e.g., [6, Chapter 7]).

Lemma 3. *Let* $(\mathbf{x}_n)_{n\geq 0}$ *be a digital sequence in base* b. *Then*

$$\Delta([\mathbf{0},[\boldsymbol{\gamma}]_{\tau_m}),(\mathbf{x}_n)_{n=b^mA}^{b^mA+N-1}) = \sum_{\vec{\mathbf{k}}\in G_m^*} \widehat{\mathbb{1}}(\vec{\mathbf{k}}) \sum_{n=0}^{b^m-1} E(\vec{\mathbf{k}}\cdot\vec{\mathbf{u}}_n + \vec{\mathbf{k}}\cdot\vec{\mathbf{u}}_{\widetilde{b^mA}}). \tag{3.21}$$

Proof. By (3.14) we have $\widehat{\mathbb{1}}(\mathbf{0}) = \prod_{i=1}^{s+1}[\gamma^{(i)}]_{\tau_m}$. Applying Lemma 2, we get

$$\Delta([\mathbf{0},[\boldsymbol{\gamma}]_{\tau_m}),(\mathbf{x}_n)_{n=b^mA}^{b^mA+N-1}) = \sum_{\vec{\mathbf{k}}\in G_m^*} \widehat{\mathbb{1}}(\vec{\mathbf{k}}) \sum_{n=0}^{b^m-1} E(\vec{\mathbf{k}}\cdot\vec{\mathbf{u}}_{\widetilde{n+b^mA}}).$$

Using (3.5), (3.8) and (3.11), we obtain

$$\widetilde{n+b^mA} = \tilde{n} + \widetilde{b^mA} = \tilde{n}\oplus\widetilde{b^mA} \quad \text{and} \quad \vec{\mathbf{u}}_{\widetilde{n+b^mA}} = \vec{\mathbf{u}}_{\tilde{n}} + \vec{\mathbf{u}}_{\widetilde{b^mA}}.$$

Now from (3.6), we get (3.21). Hence Lemma 3 is proved. □

Lemma 4. *Let* $(\mathbf{x}_n)_{n\geq 0}$ *be a digital sequence in base* b. *Then*

$$\sigma := \sum_{n=0}^{b^m-1} E(\vec{\mathbf{k}}\cdot\vec{\mathbf{u}}_n) = b^m\delta(\vec{\mathbf{k}}\in D_m). \tag{3.22}$$

Proof. Using (3.8), (3.10) and (3.13), we have

$$\begin{aligned}\vec{\mathbf{k}}\cdot\vec{\mathbf{u}}_{\tilde{n}} &= \sum_{i=1}^{s}\sum_{j=1}^{\tau_m}\sum_{r=0}^{m-1} k_j^{(i)}\overline{a_r(n)}c_{j,r}^{(i)} + \sum_{j=1}^{m} k_j^{(s+1)}\overline{a_{m-j}(n)} \\ &= \sum_{r=0}^{m-1}\overline{a_r(n)}\left(\sum_{i=1}^{s}\sum_{j=1}^{\tau_m} k_j^{(i)}c_{j,r}^{(i)} + k_{m-r}^{(s+1)}\right) = \sum_{r=0}^{m-1} f_r\xi_r,\end{aligned}$$

where

$$f_r = \overline{a_r(n)}\in\mathbb{F}_b \quad \text{and} \quad \xi_r = \sum_{i=1}^{s}\sum_{j=1}^{\tau_m} k_j^{(i)}c_{j,r}^{(i)} + k_{m-r}^{(s+1)}. \tag{3.23}$$

By (3.6), (1.6) and (3.3), we obtain

$$\begin{aligned}\sigma = \sum_{n=0}^{b^m-1} E(\vec{\mathbf{k}}\cdot\vec{\mathbf{u}}_n) &= \sum_{n=0}^{b^m-1} E(\vec{\mathbf{k}}\cdot\vec{\mathbf{u}}_{\tilde{n}}) \\ &= \sum_{f_0,\dots,f_{m-1}\in\mathbb{F}_b} E\left(\sum_{r=0}^{m-1} f_r\xi_r\right) = b^m\prod_{r=0}^{m-1}\delta(\xi_r=0).\end{aligned}$$

Now from (3.19) and (3.23), we get that $\vec{\mathbf{k}}\in D_m$ and Lemma 4 follows. □

Let

$$\Lambda_m = \Big\{ \vec{\mathbf{k}} = (k^{(1)}, \dots, k^{(s+1)}) \in G_m : k^{(s+1)} = \mathbf{0} \Big\} = (\mathbb{F}_b^{\tau_m})^s \times \{\mathbf{0}\},$$

$$g_{\vec{\mathbf{w}}} = \Big\{ A \geq 1 : y^{(i)}_{b^m A, j} = w^{(i)}_j, \ i \in \{1, \dots, s\}, \ j \in \{1, \dots, \tau_m\} \Big\},$$

$$(3.24) \qquad \rho_{\vec{\mathbf{w}}} := \begin{cases} 0 & \text{if } g_{\vec{\mathbf{w}}} = \emptyset, \\ \min g_{\vec{\mathbf{w}}} & \text{otherwise,} \end{cases} \qquad M_m = \{\rho_{\vec{\mathbf{w}}} \ : \ \vec{\mathbf{w}} \in \Lambda_m\},$$

$$\vec{\mathbf{w}} = (w^{(1)}, \dots, w^{(s+1)}), \ w^{(i)} = (w^{(i)}_1, \dots, w^{(i)}_{\tau_m}) \in \mathbb{F}_b^{\tau_m}, \ i = 1, \dots, s,$$
$$w^{(s+1)} = \mathbf{0} \in \mathbb{F}_b^m.$$

Bearing in mind (3.8), we get

$$(3.25) \qquad g_{\vec{\mathbf{w}}} = \Big\{ A \geq 1 \ : \ u^{(i)}_{\widetilde{b^m A}} = w^{(i)}, \ \ i \in \{1, \dots, s\} \Big\}.$$

We consider the following condition :

$$(3.26) \qquad g_{\vec{\mathbf{w}}} \neq \emptyset \quad \text{for all} \quad \vec{\mathbf{w}} \in \Lambda_m.$$

Let R_m be a finite set of integers, and let

$$(3.27) \qquad \sigma_1(R_m) := \frac{1}{\operatorname{card}(R_m)} \sum_{A \in R_m} \Big| \Delta\Big([\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}), (\mathbf{x}_n)_{n=b^m A}^{b^m A + N - 1} \Big) \Big|^2.$$

Lemma 5. *Let $(\mathbf{x}_n)_{n\geq 0}$ be a weakly admissible uniformly distributed digital $(\mathbf{T}, s)$-sequence in base b, satisfying to (3.26) for all $m \geq m_0$ with some $m_0 \geq 1$. Then*

$$(3.28) \qquad \sigma_1(M_m) = \sum_{\vec{\mathbf{k}} \in D_m^*} b^{2m} |\widehat{\mathbb{1}}(\vec{\mathbf{k}})|^2,$$

where M_m is defined in (3.24).

Proof. By (3.24), (3.13) and (3.3), we obtain

$$(3.29) \quad \frac{1}{b^{s\tau_m}} \sum_{\vec{\mathbf{w}} \in \Lambda_m} E(\vec{\mathbf{k}} \cdot \vec{\mathbf{w}})$$
$$= \frac{1}{b^{s\tau_m}} \sum_{w_j^{(i)} \in \mathbb{F}_b, \, i \in \{1, \dots, s\}, \, j \in \{1, \dots, \tau_m\}} E\left(\sum_{i=1}^{s} \sum_{j=1}^{\tau_m} k_j^{(i)} w_j^{(i)} \right)$$
$$= \prod_{i=1}^{s} \prod_{j=1}^{\tau_m} \delta(k_j^{(i)} = \bar{0}) = \prod_{i=1}^{s} \delta(k^{(i)} = \mathbf{0}), \quad \text{where} \quad \vec{\mathbf{k}} \in G_m.$$

Using (3.21), we derive

$$(3.30)\quad |\Delta([\mathbf{0},[\boldsymbol{\gamma}]_{\tau_m}),(\mathbf{x}_n)_{n=b^mA}^{b^mA+N-1})|^2$$
$$= \sum_{\vec{\mathbf{k}}',\vec{\mathbf{k}}''\in G_m^*} \widehat{\mathbb{1}}(\vec{\mathbf{k}}')\overline{\widehat{\mathbb{1}}(\vec{\mathbf{k}}'')} \sum_{n_1,n_2=0}^{b^m-1} E(\vec{\mathbf{k}}'\cdot\vec{\mathbf{u}}_{n_1}+\vec{\mathbf{k}}'\cdot\vec{\mathbf{u}}_{\widetilde{b^mA}}-\vec{\mathbf{k}}''\cdot\vec{\mathbf{u}}_{n_2}-\vec{\mathbf{k}}''\cdot\vec{\mathbf{u}}_{\widetilde{b^mA}}).$$

Here $\overline{z}$ is the complex conjugation of z. According to conditions of Lemma 5, (3.26) is true, i.e. $g_{\vec{\mathbf{w}}}\neq\emptyset$ for all $\vec{\mathbf{w}}\in\Lambda_m$. Therefore for all $\vec{\mathbf{w}}\in\Lambda_m$ there exists $\rho_{\vec{\mathbf{w}}}\geq 1$. By (3.24), if $\vec{\mathbf{w}}_1\neq\vec{\mathbf{w}}_2$, then $\rho_{\vec{\mathbf{w}}_1}\neq\rho_{\vec{\mathbf{w}}_2}$. So

$$\mathrm{card}(M_m)=\mathrm{card}(\Lambda_m)=b^{s\tau_m}.$$

From (3.10), we get that $u_{\widetilde{b^mA}}^{(s+1)}=\mathbf{0}$. In view of (3.24), (3.25), we have that if $A_1,A_2\in M_m,\ A_1\neq A_2$, then $\vec{\mathbf{u}}_{\widetilde{b^mA_1}}\neq\vec{\mathbf{u}}_{\widetilde{b^mA_2}}$. Hence

$$\{\vec{\mathbf{u}}_{\widetilde{b^mA}}\ :\ A\in M_m\}=\Lambda_m.$$

Applying (3.24), (3.29), (3.30) and (3.27) with $R_m=M_m$, we have

$$\sigma_1(M_m)$$
$$= \sum_{\vec{\mathbf{k}}',\vec{\mathbf{k}}''\in G_m^*} \widehat{\mathbb{1}}(\vec{\mathbf{k}}')\overline{\widehat{\mathbb{1}}(\vec{\mathbf{k}}'')}$$
$$\times\sum_{n_1,n_2=0}^{b^m-1} b^{-s\tau_m}\sum_{A\in M_m} E(\vec{\mathbf{k}}'\cdot\vec{\mathbf{u}}_{n_1}-\vec{\mathbf{k}}''\cdot\vec{\mathbf{u}}_{n_2}+(\vec{\mathbf{k}}'-\vec{\mathbf{k}}'')\cdot\vec{\mathbf{u}}_{\widetilde{b^mA}})$$
$$= \sum_{\vec{\mathbf{k}}',\vec{\mathbf{k}}''\in G_m^*} \widehat{\mathbb{1}}(\vec{\mathbf{k}}')\overline{\widehat{\mathbb{1}}(\vec{\mathbf{k}}'')}$$
$$\times\sum_{n_1,n_2=0}^{b^m-1} E(\vec{\mathbf{k}}'\cdot\vec{\mathbf{u}}_{n_1}-\vec{\mathbf{k}}''\cdot\vec{\mathbf{u}}_{n_2})b^{-s\tau_m}\sum_{\vec{\mathbf{w}}\in\Lambda_m} E(\vec{\mathbf{k}}'-\vec{\mathbf{k}}'')\cdot\vec{\mathbf{w}})$$
$$= \sum_{\vec{\mathbf{k}}',\vec{\mathbf{k}}''\in G_m^*} \widehat{\mathbb{1}}(\vec{\mathbf{k}}')\overline{\widehat{\mathbb{1}}(\vec{\mathbf{k}}'')}\sum_{n_1,n_2=0}^{b^m-1} E(\vec{\mathbf{k}}'\cdot\vec{\mathbf{u}}_{n_1}-\vec{\mathbf{k}}''\cdot\vec{\mathbf{u}}_{n_2})\prod_{i=1}^{s}\delta({k'}^{(i)}={k''}^{(i)}).$$

Let $n_3=n_2\ominus n_1$. From (1.6), we obtain $\{n_3\ :\ 0\leq n_2<b^m\}=\{0,1,\dots,b^m-1\}$. By (3.5)–(3.11), we get $\vec{\mathbf{u}}_{n_3}=\vec{\mathbf{u}}_{n_2}-\vec{\mathbf{u}}_{n_1}$. Hence

$$\sigma_1(M_m)=\sum_{\vec{\mathbf{k}}',\vec{\mathbf{k}}''\in G_m^*} \widehat{\mathbb{1}}(\vec{\mathbf{k}}')\overline{\widehat{\mathbb{1}}(\vec{\mathbf{k}}'')}\sum_{n_1,n_3=0}^{b^m-1} E((\vec{\mathbf{k}}'-\vec{\mathbf{k}}'')\cdot\vec{\mathbf{u}}_{n_1}-\vec{\mathbf{k}}''\cdot\vec{\mathbf{u}}_{n_3})$$
$$\times\prod_{i=1}^{s}\delta({k'}^{(i)}-{k''}^{(i)}).$$

We get $\vec{\mathbf{k}}'-\vec{\mathbf{k}}''=(\mathbf{0},\dots,\mathbf{0},{k'}^{(s+1)}-{k''}^{(s+1)})$.

From (3.10), we have $u^{(s+1)}_{\tilde{n},j} = \overline{a_{m-j}(n)}$ and

$$(\vec{\mathbf{k}}' - \vec{\mathbf{k}}'') \cdot \vec{\mathbf{u}}_{\tilde{n}} = (k'^{(s+1)} - k''^{(s+1)}) \cdot u^{(s+1)}_{\tilde{n}} = \sum_{j=1}^{m} (k'^{(s+1)}_j - k''^{(s+1)}_j)\overline{a_{m-j}(n)}.$$

Taking into account (3.3), we get

$$\begin{aligned}\sum_{n_1=0}^{b^m-1} E((\vec{\mathbf{k}}' - \vec{\mathbf{k}}'') \cdot \vec{\mathbf{u}}_{n_1}) &= \sum_{n_1=0}^{b^m-1} E((\vec{\mathbf{k}}' - \vec{\mathbf{k}}'') \cdot \vec{\mathbf{u}}_{\widetilde{n_1}}) \\ &= \sum_{n=0}^{b^m-1} E\left(\sum_{j=1}^{m} (k'^{(s+1)}_j - k''^{(s+1)}_j)\overline{a_{m-j}(n)}\right) \\ &= b^m \prod_{j=1}^{m} \delta(k'^{(s+1)}_j = k''^{(s+1)}_j).\end{aligned}$$

Hence $\vec{\mathbf{k}}' = \vec{\mathbf{k}}''$. Now using Lemma 4, we obtain

$$\sigma_1(M_m) = b^m \sum_{\vec{\mathbf{k}}' \in G^*_m} |\widehat{\mathbb{1}}(\vec{\mathbf{k}}')|^2 \sum_{n_3=0}^{b^m-1} E(-\vec{\mathbf{k}}' \cdot \vec{\mathbf{u}}_{n_3}) = \sum_{\vec{\mathbf{k}} \in D^*_m} b^{2m} |\widehat{\mathbb{1}}(\vec{\mathbf{k}})|^2.$$

Therefore Lemma 5 is proved. □

Lemma 6. *Let* $A_{k,\mathbb{c}} := E(-k\bar{\mathbb{c}}) \sum_{\mathbb{b}=0}^{\mathbb{c}-1} E(k\bar{\mathbb{b}})$, $k \in \mathbb{F}_b$, $\mathbb{c} \in \mathbb{Z}_b$, *and let*

$$B_{k,\mathbb{c}}(x) := \sum_{\mathbb{b}=0}^{\mathbb{c}-1} E(k\bar{\mathbb{b}}) + E(k\bar{\mathbb{c}})x = E(k\bar{\mathbb{c}}/b)(A_{k,\mathbb{c}} + x), \quad x \in [0,1]. \tag{3.31}$$

Then there exist $\mathsf{a}_1, \dots, \mathsf{a}_5 \in \mathbb{Z}_b$, $\mathsf{a}_1^2 + \mathsf{a}_2^2 + \mathsf{a}_3^2 > 0$, $\mathsf{a}_4 = \mathsf{a}_5 = 0$ *such that*

$$\left| B_{k,\mathbb{c}}\left(\sum_{j=1}^{5} \frac{\mathsf{a}_j}{b^j} + \frac{y}{b^5}\right)\right| \geq b^{-5}, \quad \forall\, k \in \mathbb{F}_b,\ \mathbb{c} \in \mathbb{Z}_b,\ y \in [0,1] \tag{3.32}$$

and

$$\sum_{k \in \mathbb{F}^*_b} |B_{k,\mathbb{c}}(x)|^2 \geq b^{-2r_0} \quad \forall\, \mathbb{c} \in \mathbb{Z}_b, \quad \textit{where} \quad \|x\| \geq b^{-r_0},\ r_0 \geq 1. \tag{3.33}$$

Proof. Let

$$\mathbb{A}_1 = \{\theta_{k,\mathbb{c}} := \mathrm{Re}(A_{k,\mathbb{c}}) \,:\, k \in \mathbb{F}_b,\ \mathbb{c} \in \mathbb{Z}_b\}.$$

We get $\mathrm{card}(\mathbb{A}_1) \leq b^2$. Let

$$\mathbb{A}_2 = \{\mathbf{a} = (\mathsf{a}_1, \dots, \mathsf{a}_5) \in \mathbb{Z}_b^5 \,:\, \mathsf{a}_1^2 + \mathsf{a}_2^2 + \mathsf{a}_3^2 > 0, \quad \mathsf{a}_4 = \mathsf{a}_5 = 0\}$$

and let $z_{\mathbf{a}} = \mathsf{a}_1/b + \cdots + \mathsf{a}_5/b^5$. By (3.31), we derive

$$|B_{k,\mathbb{c}}(x)| = |A_{k,\mathbb{c}} + x| \geq |\mathrm{Re}(A_{k,\mathbb{c}}) + x|.$$

Suppose that (3.32) is not true. Then for all $\mathbf{a} \in \mathbb{A}_2$ there exist $k(\mathbf{a}), \mathbb{c}(\mathbf{a})$ and $y(\mathbf{a})$ such that

$$b^{-5} > \left| B_{k(\mathbf{a}),\mathbb{c}(\mathbf{a})}\left(\sum_{j=1}^{5} \frac{\mathsf{a}_j}{b^j} + \frac{y(\mathbf{a})}{b^5} \right) \right| \geq \left| \theta_{k(\mathbf{a}),\mathbb{c}(\mathbf{a})} + z_{\mathbf{a}} + \frac{y(\mathbf{a})}{b^5} \right|.$$

Hence $|\theta_{k(\mathbf{a}),\mathbb{c}(\mathbf{a})} + z_{\mathbf{a}}| < b^{-4}$. Suppose that $\theta_{k(\mathbf{a}_1),\mathbb{c}(\mathbf{a}_1)} = \theta_{k(\mathbf{a}_2),\mathbb{c}(\mathbf{a}_2)}$ for some $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{A}_2$, $\mathbf{a}_1 \neq \mathbf{a}_2$. Hence $|z_{\mathbf{a}_1} - z_{\mathbf{a}_2}| < b^{-3}$. Bearing in mind that $|z_{\mathbf{a}_1} - z_{\mathbf{a}_2}| \geq b^{-3}$ for all $\mathbf{a}_1 \neq \mathbf{a}_2$, we get a contradiction. Therefore $\theta_{k(\mathbf{a}_1),\mathbb{c}(\mathbf{a}_1)} \neq \theta_{k(\mathbf{a}_2),\mathbb{c}(\mathbf{a}_2)}$ for all $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{A}_2$, $\mathbf{a}_1 \neq \mathbf{a}_2$. Thus $\operatorname{card}(\mathbb{A}_1) \geq \operatorname{card}(\mathbb{A}_2)$. Hence

$$b^2 > \operatorname{card}(\mathbb{A}_1) \geq \operatorname{card}(\mathbb{A}_2) = b^3 - 1 > b^2.$$

We have a contradiction. Therefore (3.32) is true.

Now we consider assertion (3.33). If $\mathbb{c} = 0$, then $|B_{k,\mathbb{c}}(x)| = |x|$ and (3.33) follows.

Now let $\mathbb{c} \in \{1, \dots, b-1\}$. By (3.31), we have

$$|B_{k,\mathbb{c}}(x)|^2 = |A_{k,\mathbb{c}}|^2 + 2x \operatorname{Re}(A_{k,\mathbb{c}}) + x^2. \tag{3.34}$$

Using (3.3), we get

$$\begin{aligned} \sum_{k \in \mathbb{F}_b^*} |A_{k,\mathbb{c}}|^2 &= \sum_{k \in \mathbb{F}_b^*} \left| \sum_{\mathbb{b}=0}^{\mathbb{c}-1} E(k\bar{\mathbb{b}}) \right|^2 \\ &= -\mathbb{c}^2 + \sum_{\mathbb{b}_1,\mathbb{b}_2=0}^{\mathbb{c}-1} \sum_{k \in \mathbb{F}_b} E(k(\bar{\mathbb{b}}_1 - \bar{\mathbb{b}}_2)) \\ &= -\mathbb{c}^2 + b \sum_{\mathbb{b}_1,\mathbb{b}_2=0}^{\mathbb{c}-1} \delta(\mathbb{b}_1 = \mathbb{b}_2) = -\mathbb{c}^2 + b\mathbb{c}. \end{aligned}$$

We obtain

$$\sum_{k \in \mathbb{F}_b^*} A_{k,\mathbb{c}} = -\mathbb{c} + \sum_{\mathbb{b}=0}^{\mathbb{c}-1} \sum_{k \in \mathbb{F}_b} E(k(\bar{\mathbb{b}} - \bar{\mathbb{c}})) = -\mathbb{c} + b \sum_{\mathbb{b}=0}^{\mathbb{c}-1} \delta(\bar{\mathbb{b}} = \bar{\mathbb{c}}) = -\mathbb{c}.$$

Now from (3.34), we derive

$$\sum_{k \in \mathbb{F}_b^*} |B_{k,\mathbb{c}}(x)|^2 = \mathbb{c}(b-\mathbb{c}) - 2x\mathbb{c} + x^2(b-1) \geq \mathbb{c}(1-2x+x^2) \geq (1-x)^2 \geq \|x\|^2$$

and (3.33) follows. Thus Lemma 6 is proved. □

Corollary. *Let* $\mathsf{a}_1, \dots, \mathsf{a}_5$ *be integers chosen in Lemma 6 and let* $\gamma^{(s+1)}_{v(k^{(s+1)})+j} = \mathsf{a}_j$, $j = 1, \dots, 5$, *with some* $\vec{\mathbf{k}} = (k^{(1)}, \dots, k^{(s+1)}) \in D_m^*$. *Then*

$$|\widehat{\mathbb{1}}^{(s+1)}(\mu k^{(s+1)})| \geq b^{-v(k^{(s+1)})-5} \quad \textit{for all} \quad \mu \in \mathbb{F}_b^* \tag{3.35}$$

and

$$(3.36)\quad \sum_{\mu\in\mathbb{F}_b^*} |\widehat{\mathbb{1}}^{(i)}(\mu k^{(s)})|^2 \geq b^{-2v(k^{(s)})-2r_0}, \quad \textit{where } \left\| b^{v(k^{(s)})}[\gamma^{(s)}]_{\tau_m} \right\| \geq b^{-r_0}.$$

Proof. By (3.15) and (3.31), we get

$$|\widehat{\mathbb{1}}^{(i)}(\mu k^{(i)})|$$
$$= b^{-v(k^{(i)})} \left| \sum_{\mathbb{b}=0}^{\gamma^{(i)}_{v(k^{(i)})}-1} E\Big(-\mu k^{(i)}_{v(k^{(i)})}\overline{\mathbb{b}}\Big) + E\Big(-\mu k^{(i)}_{v(k^{(i)})}\overline{\gamma^{(i)}_{v(k^{(i)})}}\Big)\{b^{v(k^{(i)})}[\gamma^{(i)}]_{\tau_m}\} \right|$$
$$= b^{-v(k^{(i)})}|B_{\mathbb{k}^{(i)},\mathbb{c}^{(i)}}(\mathbb{x}^{(i)})|,$$

where

$$\mathbb{k}^{(i)} = -\mu k^{(i)}_{v(k^{(i)})} \neq \bar{0}, \quad \mathbb{c}^{(i)} = \gamma^{(i)}_{v(k^{(i)})}, \quad \mathbb{x}^{(i)} = \Big\{b^{v(k^{(i)})}[\gamma^{(i)}]_{\tau_m}\Big\}, \quad i = s, s+1.$$

Bearing in mind the conditions of the Corollary, we get that $\mathbb{x}^{(s+1)}$ have the following b-adic expansion :

$$\mathbb{x}^{(s+1)} = \gamma^{(s+1)}_{v(k^{(s+1)})+1}/b + \gamma^{(s+1)}_{v(k^{(s+1)})+2}/b^2 + \cdots = \mathsf{a}_1/b + \cdots + \mathsf{a}_5/b^5 + \cdots .$$

Now applying Lemma 6, we get the assertion of the Corollary. □

Lemma 7. *Let* $(\mathbf{x}_n)_{n\geq 0}$ *be a digital sequence in base* b *and let* $\rho \in \{2, 3, \ldots, m-2\}$ *be an integer. Then there exists* $\vec{\mathbf{k}} \in D_m^*$ *such that*

$$k^{(1)} = \cdots = k^{(s-1)} = \mathbf{0}, \quad k^{(s)}_{v(k^{(s)})} = \bar{1}, \quad 1 \leq v(k^{(s)}) \leq \rho - 1$$

and $v(k^{(s+1)}) \leq m - \rho + 2$*.*

Proof. From (3.5)–(3.10), (3.19), (3.22) and (3.23), we get that $\vec{\mathbf{k}} \in D_m^*$ if and only if

$$(3.37)\qquad \sum_{i=1}^{s}\sum_{j=1}^{\tau_m} k_j^{(i)} c_{j,r}^{(i)} + k_{m-r}^{(s+1)} = \bar{0}, \quad \text{for all } r = 0, 1, \ldots, m-1.$$

We put $k^{(1)} = \cdots = k^{(s-1)} = \mathbf{0}$, $k_j^{(s)} = \bar{0}$, for $j \geq \rho$ and $k_j^{(s+1)} = \bar{0}$, for $j > m - \rho + 2$. Hence (3.37) is true if and only if

$$(3.38)\quad k_{m-r}^{(s+1)} = -\sum_{j=1}^{\rho-1} k_j^{(s)} c_{j,r}^{(s)} \ \text{ for } r = 0, 1, \ldots, m-1,$$
$$k_{m-r}^{(s+1)} = 0 \ \text{ for } m - r > m - \rho + 2.$$

Therefore, in order to obtain the statement of the lemma, it is sufficient to show that there exists a nontrivial solution of the following system of linear equations

$$(3.39)\quad \sum_{j=1}^{\rho-1} k_j^{(s)} c_{j,r}^{(i)} + k_{m-r}^{(s+1)} \delta(m-r \leq m-\rho+2) = \bar{0}, \quad r = 0,\dots,m-1.$$

In this system, we have $m+1$ unknowns $k_1^{(s)},\dots,k_{\rho-1}^{(s)}$, $k_1^{(s+1)},\dots,k_{m-\rho+2}^{(s+1)}$ and m linear equations. Hence there exists a nontrivial solution of (3.39). By (3.39), we get that if $k^{(s)} = \mathbf{0}$, then $k^{(s+1)} = \mathbf{0}$. Hence $k^{(s)} \neq \mathbf{0}$ and $1 \leq v(k^{(s)}) \leq \rho - 1$. Taking into account that if $\vec{\mathbf{k}} \in D_m$ then $\mu\vec{\mathbf{k}} \in D_m$ for all $\mu \in \mathbb{F}_b^*$. Therefore there exists $\vec{\mathbf{k}} \in D_m^*$ such that $k_{v(k^{(s)})}^{(s)} = \bar{1}$ and $1 \leq v(k^{(s)}) \leq \rho - 1$. Thus Lemma 7 is proved. □

Proposition. *Let* $(\mathbf{x}_n)_{n\geq 0}$ *be a weakly admissible uniformly distributed digital* $(\mathbf{T}, s)$*-sequence in base* b*, satisfying to* (3.26) *for all* $m \geq m_0$ *with some* $m_0 \geq 1$*. Then* $[0,\gamma_1)\times\cdots\times[0,\gamma_s)$ *is of bounded remainder with respect to* $(\mathbf{x}_n)_{n\geq 0}$ *if and only if* (1.11) *is true.*

Proof. The sufficient part of the Proposition follows directly from the definition of $(\mathbf{T}, s)$ sequence, Lemma A and Lemma B. We will consider only the necessary part of the Proposition.

Suppose that (1.11) does not hold true. Then

$$\max_{1\leq i\leq s} \operatorname{card}\Big\{ j \geq 1 \,:\, \gamma_j^{(i)} \neq 0 \Big\} = \infty.$$

Suppose, w.l.o.g,

$$\operatorname{card}\Big\{ j \geq 1 \,:\, \gamma_j^{(s)} \neq 0 \Big\} = \infty.$$

Let

$$(3.40)\quad \mathbb{W} = \Big\{ j \geq 1 \,:\, \gamma_j^{(s)} \in \{1,\dots,b-2\} \text{ or } \gamma_j^{(s)} = b-1, \text{ and} \gamma_{j+1}^{(s)} = 0 \Big\}.$$

Bearing in mind that $\{j \geq 1 \,:\, \gamma_l^{(s)} = b-1 \ \ \forall\, l > j\} = \emptyset$, we obtain either $\gamma_j^{(s)} \in \{1,\dots,b-2\}$ for infinitely many $j > 1$ or $\gamma_j^{(s)} = b-1$, and$\gamma_{j+1}^{(s)} = 0$ for infinitely many $j > 1$. In both cases we obtain that $\operatorname{card}(\mathbb{W}) = \infty$.

Suppose that there exists $H > 0$ such that $b^{2H} c_1 > 4H^2$,

$$(3.41)\qquad \Big|\Delta\Big([\mathbf{0},\boldsymbol{\gamma}), (\mathbf{x}_n)_{n=M}^{M+N-1}\Big)\Big| \leq H - s \quad \text{for all} \quad M \geq 0,\ N \geq 1,$$

with $c_1 = \gamma_0^2 b^{-24}$, $[\mathbf{0},\boldsymbol{\gamma}) = [0,\gamma_1)\times\cdots\times[0,\gamma_s)$, $\gamma_0 = \gamma_1\gamma_2\cdots\gamma_{s-1}$.

We arrange the elements $\mathbb{W}$ in an ascending order, $W = \{\mathrm{w}_j \,:\, \mathrm{w}_i < \mathrm{w}_j \text{ for } i < j,\ j = 1,2,\dots\}$. Furthermore we choose a proper subset of $\mathbb{W}$,

$\{r(1), r(2), r(3), \ldots\}$ again arranged in ascending order, with

$$(3.42) \quad r(j+1) = \min\{\mathbb{w}_k \in W : \mathbb{w}_k \geq r(j) + H^2\} \qquad \text{for } j = 0, 1, \ldots,\ r(0) = 1.$$

We choose m and J from the following conditions

$$(3.43) \quad m = r(J), \quad 2\prod_{i=1}^{s-1}[\gamma_i]_{\tau_m} \geq \prod_{i=1}^{s-1}\gamma_i = \gamma_0, \quad J \geq H^2 b^{18}\gamma_0^{-2},\ m \geq m_0,$$

where m_0 is chosen in Lemma 5.

Applying Lemma 1 and (3.41), we have

$$(3.44) \quad \left|\Delta\Big([\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1}\Big)\right| \leq H \quad \forall\ A \geq 0,\ N \in \{1, \ldots, b^m\}.$$

By Lemma 7, we get that there exists a sequence $(\vec{\mathbf{k}}(j))_{j=1}^J$ such that

$$(3.45) \quad \begin{gathered} \vec{\mathbf{k}}(j) \in D_m^*, \quad k^{(1)}(j) = \cdots = k^{(s-1)}(j) = \mathbf{0}, \quad k^{(s)}_{v(k^{(s)}(j))}(j) = \bar{1}, \\ v(k^{(s)}(j)) \leq r(j) - 1,\ \ v(k^{(s+1)}(j)) \leq m - r(j) + 2, \quad j \in \{1, \ldots, J\}. \end{gathered}$$

We see that the sequence $(\vec{\mathbf{k}}(j))_{j=1}^J$ does not depend on $\gamma^{(s+1)}$. We will construct $\gamma^{(s+1)}$ as follows (see (3.48)). We have

$$\{b^{v(k^{(s)}(j))}[\gamma^{(s)}]_{\tau_m}\} = .\gamma^{(s)}_{v(k^{(s)}(j))+1} \cdots \gamma^{(s)}_{r(j)} \cdots .$$

In view of (3.40) and (3.42), we get $\gamma^{(s)}_{r(j)} \in \{1, \ldots, b-2\}$ or $\gamma^{(s)}_{r(j)} = b-1$, and $\gamma^{(s)}_{j+1} = 0$. In both cases, we obtain from (3.45) that

$$(3.46) \quad \left\|b^{v(k^{(s)}(j))}[\gamma^{(s)}]_{\tau_m}\right\| \ \geq\ b^{v(k^{(s)}(j)) - r(j) - 2}, \quad j = 1, \ldots, J.$$

Let $H_1 = \{1, 2, \ldots, J\}$ if

$$(3.47) \quad |v(k^{(s+1)}(j_0)) - v(k^{(s+1)}(j_1))| \geq 6$$

for all $1 \leq j_0 < j_1 \leq J$, and let $H_1 = \{j_0\}$ if there exist $1 \leq j_0 < j_1 \leq J$ such that (3.47) is false.

Hence $\operatorname{card}(H_1) \in \{1, J\}$. Note that the choice of H_1 is not unique for $\operatorname{card}(H_1) = 1$.

Let $\mathsf{a}_1, \ldots, \mathsf{a}_5$ be integers chosen in Lemma 6 and let

$$(3.48) \quad N = b^m\gamma^{(s+1)} \quad \text{with} \quad \gamma^{(s+1)} = \sum_{j \in H_1}\sum_{\nu=1}^{5} \mathsf{a}_\nu b^{-\nu - v(k^{(s+1)}(j))}.$$

From Lemma 5, (3.24), (3.27), (3.44) and conditions of the Proposition, we have

$$(3.49) \quad H^2 \geq \sigma_1(M_m) = \sum_{\vec{\mathbf{k}} \in D_m^*} b^{2m}|\widehat{\mathbb{1}}(\vec{\mathbf{k}})|^2.$$

By (3.20), we obtain that if $\vec{\mathbf{k}}(j) \in D_m$ then $\mu\vec{\mathbf{k}}(j) \in D_m$ for $\mu \in \mathbb{F}_b^*$. Suppose that $\mathrm{card}(H_1) = J$. Taking into account (3.47), we get that $\mu_1\vec{\mathbf{k}}(j_1) \neq \mu_2\vec{\mathbf{k}}(j_2)$ for all $\mu_1, \mu_2 \in \mathbb{F}_b^*$ and $j_1, j_2 \in H_1$, $j_1 \neq j_2$.

For both cases $\mathrm{card}(H_1) = J$ and $\mathrm{card}(H_1) = 1$, we have from (3.14), (3.45) and (3.49) that

$$
\begin{aligned}
(3.50)\quad \sigma_1(M_m) &\geq \sum_{\mu\in\mathbb{F}_b^*}\sum_{j\in H_1} b^{2m}|\widehat{\mathbb{1}}(\mu\vec{\mathbf{k}}(j))|^2 \\
&= ([\gamma_1]_{\tau_m}\cdots[\gamma_{s-1}]_{\tau_m})^2 \sum_{\mu\in\mathbb{F}_b^*}\sum_{j\in H_1} b^{2m}|\widehat{\mathbb{1}}^{(s)}(\mu k^{(s)}(j))|^2|\widehat{\mathbb{1}}^{(s+1)}(\mu k^{(s+1)}(j))|^2.
\end{aligned}
$$

In the proof of the Proposition, we will use the following heuristic. Applying the Corollary, we get that summands in (3.50) are not small (see (3.52) and (3.53)). Hence the case $\mathrm{card}(H_1) = J$ is simple (see (3.53)). The case $\mathrm{card}(H_1) = 1$ is more complicated. We will prove that in (3.50) there is a big summand (see also (3.52)).

Let us consider (3.52). According to (3.45), we have $r(j_1)+v(k^{(s+1)}(j_1)) \leq m+2$ for all $j_1 \in \{1,\dots,J\}$. Taking into account that $r(j_1) - r(j_0) \geq H^2$ and that $|v(k^{(s+1)}(j_1)) - v(k^{(s+1)}(j_0))| \leq 5$ (see (3.47)), we get that $m-(r(j_0)+v(k^{(s+1)}(j_0))) \geq H^2-7$. So, we will get a contradiction in (3.52) (see also (3.54)).

Let us return to (3.50). In view of (3.48), we have that $\gamma^{(s+1)}_{v(k^{(s+1)}(j))+\nu} = \mathsf{a}_\nu$, $\nu = 1,\dots,5$. Therefore, we can use (3.35) :

$$
|\widehat{\mathbb{1}}^{(s+1)}(\mu k^{(s+1)}(j))|^2 \geq b^{-2v(k^{(s+1)}(j))-10} \qquad \text{for all } \mu \in \mathbb{F}_b^*,\ j \in H_1.
$$

Bearing in mind (3.46) and applying (3.36) with $r_0 = r(j)-v(k^{(s)}(j))+2$, we get

$$
\begin{aligned}
(3.51)\quad &\sum_{\mu\in\mathbb{F}_b^*} |\widehat{\mathbb{1}}^{(s)}(\mu k^{(s)}(j))|^2 \\
&\qquad \geq b^{-2v(k^{(s)}(j))-2(r(j)-v(k^{(s)}(j))+2)} = b^{-2r(j)-4}, \quad j \in H_1.
\end{aligned}
$$

By (3.43), we obtain $4 \geq \gamma_0^2([\gamma_1]_{\tau_m}\cdots[\gamma_{s-1}]_{\tau_m})^{-2}$. In view of (3.49)–(3.51), we have

$$
\begin{aligned}
(3.52)\quad 4H^2 \geq 4\sigma_1(M_m) &\geq \sigma_1(M_m)\gamma_0^2([\gamma_1]_{\tau_m}\cdots[\gamma_{s-1}]_{\tau_m})^{-2} \\
&\geq \gamma_0^2 \sum_{j\in H_1}\sum_{\mu\in\mathbb{F}_b^*} 1|\widehat{\mathbb{1}}^{(s+1)}(\mu k^{(s)}(j))|^2 b^{2(m-v(k^{(s+1)}(j))-5)} \\
&\geq \gamma_0^2 \sum_{j\in H_1} b^{2(m-r(j)-v(k^{(s+1)}(j))-7)}.
\end{aligned}
$$

Suppose that $\mathrm{card}(H_1) = J$. From (3.43) and (3.45), we get

$$(3.53)\quad 4H^2 \geq 4\sigma_1(M_m) \geq \gamma_0^2 \sum_{j=1}^{J} b^{2(m-r(j)-v(k^{(s+1)}(j))-7)} \geq \gamma_0^2 J b^{-18} > 4H^2.$$

We have a contradiction. Now let $\mathrm{card}(H_1) = 1$.

By (3.47), we obtain that there exist $j_0, j_1 \in \{1, \dots, J\}$ such that $j_0 \in H_1$, $j_0 < j_1$ and $|v(k^{(s+1)}(j_0)) - v(k^{(s+1)}(j_1))| \leq 5$.

According to (3.42) and (3.45), we have

$$r(j_0) + v(k^{(s+1)}(j_0)) \leq r(j_1) - H^2 + v(k^{(s+1)}(j_1)) + 5$$

and

$$m - r(j_0) - v(k^{(s+1)}(j_0)) \geq m - r(j_1) - v(k^{(s+1)}(j_1)) + H^2 - 5 \geq H^2 - 7.$$

Applying (3.52) and (3.41), we get

$$(3.54)\quad 4H^2 \geq 4\sigma_1(M_m) \geq \gamma_0^2 b^{2(m-r(j_0)-v(k^{(s+1)}(j_0))-5)} \geq \gamma_0^2 b^{2H^2-24} = b^{2H^2} c_1 > 4H^2,$$

with $c_1 = \gamma_0^2 b^{-24}$. We have a contradiction. By (3.53) and (3.54), the Proposition is proved. □

Note that by Definition 3 and Lemma B, the sufficient part of the statement in the Proposition holds true for any uniformly distributed digital sequence.

Completion of the proof of the Theorem. By Theorem A, we get that $1, L_1, \dots, L_s$ are linearly independent over $\mathbb{F}_b[z]$. Hence $1, z^m L_1, \dots, z^m L_s$ are linearly independent over $\mathbb{F}_b[z]$. Let $\mathbf{L}^{(m)} = (z^m L_1, \dots, z^m L_s)$, and let $S(\mathbf{L}^{(m)}) = (\mathbf{l}_n^{(m)})_{n\geq 0}$ (see (1.9)) with

$$\mathbf{l}_n^{(m)} = (l_n^{(m,1)}, \dots, l_n^{(m,s)}), \quad l_n^{(m,i)} = \{n(z) z^m L_i(z)\}_{|z=b}, \quad \text{for } 1 \leq i \leq s,\ n \geq 0.$$

Using Theorem A, we obtain that $S(\mathbf{L}^{(m)})$ is a uniformly distributed sequence in $[0,1)^s$. Therefore, for all $\vec{\mathbf{w}} \in \Lambda_m$ there exists an integer $A \geq 1$ with

$$\overline{l_{b^m A, j}^{(m,i)}} = w_j^{(i)} \quad \text{for} \quad 1 \leq i \leq s,\ 1 \leq j \leq \tau_m.$$

Thus $S(\mathbf{L})$ satisfies the condition (3.26).

Bearing in mind that $1, L_1, \dots, L_s$ are linearly independent over $\mathbb{F}_b[z]$, we get that $\{n(z)L_i\} \neq 0$ for all $n \geq 1$. Hence $\{l^{(i)}(n)\} \neq 0$ for all $n \geq 1$ $(i = 1, \dots, s)$. Therefore the sequence $S(\mathbf{L})$ is weakly admissible.

Applying the Proposition, we get the assertion of the Theorem. □

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