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Lubin–Tate Deformation Spaces and Fields of Norms

par Annie CARTER et Matthias STRAUCH

Résumé. On construit une tour de corps à partir des anneaux R_n qui paramétrisent les couples (X, λ), où X est une déformation d'un groupe formel fixé $\mathbb{X}$ de dimension un et de hauteur h, muni d'une structure de Drinfeld λ de niveau n. On choisit des idéaux principaux premiers $\mathfrak{p}_n \,|\, (p)$ de R_n de manière compatible, et on considère le corps K'_n obtenu en localisant R_n en $\mathfrak{p}_n$ et en passant au corps des fractions de la complétion. En prenant le compositum $K_n = K'_n K_0$ de K'_n et de la complétion K_0 d'une certaine extension non-ramifiée de K'_0, on obtient la tour de corps $(K_n)_n$ pour laquelle on démontre qu'elle est 'strictly deeply ramified' au sens de Scholl. Quand $h = 2$, on étudie la question de savoir s'il s'agit d'une tour kummérienne.

Abstract. We construct a tower of fields from the rings R_n which parametrize pairs (X, λ), where X is a deformation of a fixed one-dimensional formal group $\mathbb{X}$ of finite height h, together with a Drinfeld level-n structure λ. We choose principal prime ideals $\mathfrak{p}_n \,|\, (p)$ in each ring R_n in a compatible way and consider the field K'_n obtained by localizing R_n at $\mathfrak{p}_n$ and passing to the field of fractions of the completion. By taking the compositum $K_n = K'_n K_0$ of K'_n with the completion K_0 of a certain unramified extension of K'_0, we obtain a tower of fields $(K_n)_n$ which we prove to be strictly deeply ramified in the sense of Scholl. When $h = 2$ we also investigate the question of whether this is a Kummer tower.

1. Introduction

In this paper we study a tower $K_\bullet$ of complete discrete valuation fields of characteristic zero and residue field of characteristic $p > 0$:

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \tag{1.1}$$

The fields K_n are defined in terms of torsion points of the universal deformation of a formal group $\mathbb{X}$ of dimension one and height $h > 0$ over $\overline{\mathbb{F}}_p$. The extension K_n/K_0 is Galois with $\mathrm{Gal}(K_n|K_0) \simeq (\mathbb{Z}/p^n)^\times \ltimes (\mathbb{Z}/p^n)^{h-1}$. When $h = 1$ we have $\mathbb{X} = \widehat{\mathbb{G}}_m$, and $K_n = \breve{\mathbb{Q}}_p(\mu_{p^n})$, where $\breve{\mathbb{Q}}_p = W(\overline{\mathbb{F}}_p)[1/p]$ is the

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completion of the maximal unramified extension of $\mathbb{Q}_p$. If $h > 1$ the residue field of K_0 is a separable (infinite) Galois extension of $\overline{\mathbb{F}}_p \,(\!(u_1, \ldots, u_{h-1})\!)$ and thus imperfect.

Our starting point is the paper [8] of A.J. Scholl in which he develops a theory of norm fields for certain towers of fields $(K_n)_n$ which he calls *strictly deeply ramified*,[1] and whose residue fields are not necessarily perfect. Our first result is:

Theorem 1.1.1. *The tower 1.1 is strictly deeply ramified in the sense of A.J. Scholl.*

By the general theory of [8], the tower 1.1 therefore gives rise to a complete discretely valued field of norms E of characteristic p whose ring of integers is

$$\mathcal{O}_E = \varprojlim_n \mathcal{O}_{K_n}/p\mathcal{O}_{K_n} \,,$$

where the transition maps are the p-power maps. The field of norms may then be used to study representations of $\mathrm{Gal}(\overline{K_0}|K_0)$ over fields of characteristic p.

In order to study p-adic representations of the absolute Galois group of K_0, it is important to know whether the field of norms E lifts to characteristic zero in a way that is compatible with the action of $\Gamma = \mathrm{Gal}(K_\infty|K_0)$ and Frobenius (where, as usual, $K_\infty = \bigcup_n K_n$). If that is the case, then one can describe p-adic representations of $\mathrm{Gal}(\overline{K_0}|K_0)$ in terms of (ϕ, Γ)-modules.

While we do not settle here the question whether our norm field E lifts to characteristic zero (in a way that is compatible with the action of Γ and Frobenius), we investigate if $K_\bullet$ is a Kummer tower. By this we mean that there are elements $t_1, \ldots, t_{h-1} \in K_0$ such that for all $n \geq 0$ one has $K_n = K_0(\mu_{p^n}, \sqrt[p^n]{t_1}, \ldots, \sqrt[p^n]{t_{h-1}})$. Scholl shows in [8, §2.3], that the field of norms associated to a Kummer tower lifts to characteristic zero.

In order to explain the result that we have in this direction, we need to briefly sketch the construction of the tower 1.1; cf. Section 2.2 for more details. Let R_n be the ring which represents (isomorphism classes of) triples (X, ι, λ), where (X, ι) is a deformation of $\mathbb{X}$ and $\lambda : p^{-n}\mathbb{Z}/\mathbb{Z} \to X[p^n]$ is a Drinfeld level-n structure. The ring R_0 is non-canonically isomorphic to $W(\overline{\mathbb{F}}_p)[\![u_1, \ldots, u_{h-1}]\!]$, and $R_n[1/p]/R_0[1/p]$ is an unramified Galois extension of rings with Galois group isomorphic to $\mathrm{GL}_n(\mathbb{Z}/p^n\mathbb{Z})$. Set $\mathfrak{p}_0 = pR_0$. For $n > 0$ let $\mathfrak{p}_n \in R_n$ be a prime ideal of height one which has the property that $\mathfrak{p}_{n+1}$ divides $\mathfrak{p}_n$ in R_{n+1}, for all $n \geq 0$. Let R'_n be the completion of the localization $(R_n)_{\mathfrak{p}_n}$ (with respect to the topology defined by the maximal ideal), and $K'_n = \mathrm{Frac}(R'_n)$ the field of fractions of R'_n. Let $K'_{n,u} \subseteq K'_n$ be the largest subfield which is unramified over K'_0, and put $\widetilde{K}_0 = \bigcup_n K'_{n,u}$.

[1]We refer to the body of the paper for the discussion of this concept.

Then define K_0 to be the p-adic completion of $\widetilde{K}_0$ and define $K_n = K_n'K_0$ to be the composition of K_n' and K_0. We have only investigated the question whether $K_\bullet = (K_n)_n$ is a Kummer tower when $h = 2$. In this case, we have

Theorem 1.1.2. *Let $h = 2$ and let $K_\bullet = (K_n)_n$ be the tower as defined above.*

(i) *For every $n \geq 0$ there is an element $t_n \in K_0$ such that $K_n = K_0(\mu_{p^n}, \sqrt[p^n]{t_n})$.*

(ii) *If $p > 2$, there does not exist an element $t \in \widetilde{K}_0$ such that for all sufficiently large n one has $K_n = K_0(\mu_{p^n}, \sqrt[p^n]{t})$.*

Our method of proof leaves open the possibility that there is an element $t \in K_0$ such that $K_n = K_0(\mu_{p^n}, \sqrt[p^n]{t})$ for all $n \geq 0$. However, even if $K_\bullet$ fails to be a Kummer tower, it might still be possible that the norm field E lifts to characteristic zero (together with Galois action and Frobenius), but we do not have positive evidence with regard to this problem.

The motivation to consider the tower of fields $K_\bullet$ stems from the fact that, for $\ell \neq p$, the ℓ-adic étale cohomology of the rigid analytic spaces associated to the formal schemes $\mathrm{Spf}(R_n)$ realizes the ℓ-adic local Langlands correspondence for $\mathrm{GL}_h(\mathbb{Q}_p)$, as was shown by M. Harris and R. Taylor [5]. Furthermore, P. Scholze's work [9] shows that the p-adic cohomology of the Lubin–Tate tower carries information about the conjectural p-adic (local) Langlands correspondence. The tower $K_\bullet$ comes equipped with an action of a maximal parabolic subgroup in $\mathrm{GL}_h(\mathbb{Z}_p)$ and with an action of the group of elements of norm one in the division algebra D of invariant $\frac{1}{h}$, and these actions commute with each other. More generally, it is possible to consider a whole family of towers of fields $K_{v,\bullet}$, indexed by points $v \in \mathbb{P}^{h-1}(\mathbb{Q}_p)$, and this family of fields carries an action of $\mathrm{GL}_h(\mathbb{Z}_p) \times \mathcal{O}_D^\times$. The original motivation behind the present paper is to clarify the meaning of those group actions on this family of towers, and their associated norm fields, and this article provides the first step in this direction.

Remark 1.1.3. The tower of fields $(K_n')_n$ and the field $K_\infty' = \bigcup_{n\geq 0} K_n'$ have also been studied in [6] (where K_n' is denoted L_n), but the objective in loc. cit. is quite different in that it is concerned with the composite field

$$K_\infty'.K_0'\left(\mu_{p^\infty}, u_1^{1/p^\infty}, \ldots, u_{h-1}^{1/p^\infty}\right).$$

In particular, the fields K_n do not appear in loc. cit., and the question whether the tower $K_\bullet$ is strictly deeply ramified is not addressed in there.

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2. Deformations with level structures and the tower of fields $K_\bullet$

2.1. Formal modules, deformations, and level structures. We briefly recall some facts about deformations of formal $\mathcal{O}$-modules and level structures, following [3, §§1 and 4].

We fix a finite extension $F/\mathbb{Q}_p$ with ring of integers $\mathcal{O} = \mathcal{O}_F$, uniformizer π, and residue field $k_F = \mathcal{O}/(\pi)$ of cardinality q. Let $\breve{\mathcal{O}}$ be the completion of the maximal unramified extension of $\mathcal{O}$. The residue field of $\breve{\mathcal{O}}$ is an algebraic closure of k_F which we denote by $\overline{k}_F$. We also fix a formal $\mathcal{O}$-module $\mathbb{X}$ of dimension one and finite F-height $h \geq 1$ over $\overline{k}_F$. Up to isomorphism there is only one formal $\mathcal{O}$-module over $\overline{k}_F$ of given F-height h [3, 1.7]. Given a formal $\mathcal{O}$-module X over some ring R we denote by $[\,\cdot\,]_X : \mathcal{O} \to \mathrm{End}_R(X)$ the corresponding ring homomorphism.

By $\mathcal{C}$ we denote the category of $\breve{\mathcal{O}}$-algebras R with the following properties:

(i) R is a complete, local, noetherian ring, whose maximal ideal we denote by $\mathfrak{m}_R$;
(ii) the structure homomorphism $\breve{\mathcal{O}} \to R$ is local;
(iii) the canonical field homomorphism $\overline{k}_F = \breve{\mathcal{O}}/\pi\breve{\mathcal{O}} \to R/\mathfrak{m}_R$ is an isomorphism.

Morphisms in $\mathcal{C}$ are local homomorphisms of $\breve{\mathcal{O}}$-algebras.

By a *deformation* of $\mathbb{X}$ over $R \in \mathrm{ob}(\mathcal{C})$, we mean a pair (X, ι) consisting of a formal $\mathcal{O}$-module X over R, together with an isomorphism $\iota : \mathbb{X} \xrightarrow{\simeq} X \otimes_R R/\mathfrak{m}_R$. Two deformations (X_1, ι_1) and (X_2, ι_2) are defined to be *isomorphic* if there is an isomorphism $f : X_1 \to X_2$ of formal $\mathcal{O}$-modules over R such that $(f \otimes R/\mathfrak{m}_R) \circ \iota_1 = \iota_2$. In that case we write $f : (X_1, \iota_1) \xrightarrow{\simeq} (X_2, \iota_2)$.

Let (X, ι) be a deformation of $\mathbb{X}$. We fix a coordinate T on X, and using T, we equip the maximal ideal $\mathfrak{m}_R$ with the structure of an $\mathcal{O}$-module. Let n denote a positive integer. A *structure of level* n on X is an $\mathcal{O}$-module homomorphism

$$\lambda : \left(\pi^{-n}\mathcal{O}/\mathcal{O}\right)^h \longrightarrow \mathfrak{m}_R$$

such that the power series $[\pi]_X(T)$ is divisible by

$$\prod_{\alpha \in (\pi^{-1}\mathcal{O}/\mathcal{O})^h} (T - \lambda(\alpha)) \,.$$

Remark 2.1.1. A structure of level zero is, by definition, the unique homomorphism from the trivial group $(\pi^0\mathcal{O}/\mathcal{O})^h$ to $\mathfrak{m}_R$. In the definition of $\mathrm{Def}_{\mathbb{X},n}$ below, the datum of λ can be ignored when $n = 0$.

Let (X_1, ι_1) and (X_2, ι_2) be two deformations of $\mathbb{X}$, and let λ_i be a level-n structure on X_i for $i = 1, 2$. The triples $(X_1, \iota_1, \lambda_1)$ and $(X_2, \iota_2, \lambda_2)$ are defined to be *isomorphic* if there is an isomorphism $f : (X_1, \iota_1) \xrightarrow{\simeq} (X_2, \iota_2)$

of deformations satisfying $f \circ \lambda_1 = \lambda_2$. Define the functor

$$\mathrm{Def}_{\mathbb{X},n} : \mathcal{C} \longrightarrow \mathrm{Sets}$$

by associating to $R \in \mathrm{ob}(\mathcal{C})$ the set of isomorphism classes of triples (X, ι, λ), where (X, ι) is a deformation of $\mathbb{X}$ and λ is a level-n structure on X. For $n' \geq n$, the restriction of any level-n' structure λ' on X to $\left(\pi^{-n}\mathcal{O}/\mathcal{O}\right)^h \subseteq \left(\pi^{-n'}\mathcal{O}/\mathcal{O}\right)^h$ is a level-n structure. We thus get a natural transformation $\mathrm{Def}_{\mathbb{X},n'} \to \mathrm{Def}_{\mathbb{X},n}$. Moreover, we have a right action of $\mathrm{GL}_h(\mathcal{O}/(\pi^n))$ on the functor $\mathrm{Def}_{\mathbb{X},n}$ which is defined by $[X, \iota, \lambda].g = [X, \iota, \lambda \circ g]$, where $[X, \iota, \lambda]$ denotes the isomorphism class of the triple (X, ι, λ).

Parts (i)–(iii) of the following result are due to V.G. Drinfeld [3, 4.2, 4.3], and part (iv) has been shown in [12, 2.1.2].

Theorem 2.1.2.

(i) *For every $n \geq 0$ the functor $\mathrm{Def}_{\mathbb{X},n}$ is representable, i.e., there is an $\breve{\mathcal{O}}$-algebra $R_n \in \mathrm{ob}(\mathcal{C})$ and an isomorphism of functors*

$$\mathrm{Def}_{\mathbb{X},n} \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(R_n, -)\,.$$

(ii) *The ring R_n in (i) is a regular local ring. For all $n' \geq n$ the ring homomorphism $R_n \to R_{n'}$ (induced by the natural transformation $\mathrm{Def}_{\mathbb{X},n'} \to \mathrm{Def}_{\mathbb{X},n}$) is finite and flat.*

(iii) *The ring R_0 is (non-canonically) isomorphic to $\breve{\mathcal{O}}[\![u_1, \ldots, u_{h-1}]\!]$.*

(iv) *The ring extension $R_n[1/\pi]/R_0[1/\pi]$ is Galois with Galois group isomorphic to $\mathrm{GL}_h(\mathcal{O}/(\pi^n))$. (The left action of this group on R_n is induced by its right action on the functor $\mathrm{Def}_{\mathbb{X},n}$.)*

Remarks 2.1.3.

(i) When $F = \mathbb{Q}_p$, hence $\mathcal{O} = \mathbb{Z}_p$, part (iii) is due to Lubin and Tate [7], which is why the formal scheme $\mathrm{Spf}(R_0)$ (or its rigid analytic generic fiber) is called a Lubin–Tate deformation space. More generally, the formal schemes $\mathrm{Spf}(R_n)$ (or their rigid analytic generic fibers) are also called Lubin–Tate deformation spaces.

(ii) Let $[X^{\mathrm{univ}}, \iota^{\mathrm{univ}}] \in \mathrm{Def}_{\mathbb{X},0}(R_0)$ be the element which corresponds to the identity map $\mathrm{id}_{R_0} \in \mathrm{Hom}_{\mathcal{C}}(R_0, R_0)$. Then X^{univ} is called the *universal deformation* of $\mathbb{X}$. Furthermore, consider the isomorphism class of triples $[X^{\mathrm{univ}}, \iota^{\mathrm{univ}}, \lambda_n^{\mathrm{univ}}] \in \mathrm{Def}_{\mathbb{X},n}(R_n)$ corresponding to the identity map $\mathrm{id}_{R_n} \in \mathrm{Hom}_{\mathcal{C}}(R_n, R_n)$. The map $\lambda_n^{\mathrm{univ}}$ is called the *universal level-n structure*. Moreover, for $n' \geq n$, the restriction of $\lambda_{n'}^{\mathrm{univ}}$ to $\left(\pi^{-n}\mathcal{O}/\mathcal{O}\right)^h \subseteq \left(\pi^{-n'}\mathcal{O}/\mathcal{O}\right)^h$ is equal to the composition of $\lambda_n^{\mathrm{univ}}$ with the inclusion $\mathfrak{m}_{R_n} \hookrightarrow \mathfrak{m}_{R_{n'}}$.

(iii) In the following we will often consider the action of $\mathrm{GL}_h(\mathcal{O})$ on R_n which is induced by the canonical map $\mathrm{GL}_h(\mathcal{O}) \to \mathrm{GL}_h(\mathcal{O}/(\pi^n))$, and we write $g.a$ for the image of $a \in R_n$ under the action of

$g \in \mathrm{GL}_h(\mathcal{O})$, and we write $g.A$ for the image of a subset $A \subseteq R_n$ under the action of g.

(iv) When $h = 1$ the universal deformation X^{univ} is, up to isomorphism, the unique lift of $\mathbb{X}$ to $\breve{\mathcal{O}}$. This implies the well-known fact that all Lubin–Tate formal groups for $\mathcal{O}$ (i.e., one-dimensional $\mathcal{O}$-modules of F-height one over $\mathcal{O}$) become isomorphic over $\breve{\mathcal{O}}$, cf. [10, §3.7, Lem. 1]. Let $LT_{\mathcal{O}}$ be any Lubin–Tate formal group for $\mathcal{O}$, and let F_n be the extension of F generated by the π^n-torsion points of $LT_{\mathcal{O}}$. This is a purely ramified extension of degree $(q-1)q^{n-1}$, and the composite field $\breve{F}_n := F_n.\breve{F}$, where $\breve{F} = \breve{\mathcal{O}}[1/\pi]$, does not depend on the choice of $LT_{\mathcal{O}}$. When $h = 1$, the ring R_n is the ring of integers of $\breve{F}_n$.

2.2. Construction of the tower of fields $K_\bullet$.

2.2.1. *Sequences of prime ideals.* The construction which we are going to perform depends on the choice of a sequence $\mathfrak{p}_\bullet = (\mathfrak{p}_n)_{n>0}$ of ideals $\mathfrak{p}_n \subseteq R_n$ with the following properties

(i) For all $n > 0$ the ideal $\mathfrak{p}_n$ is a prime ideal of height one.
(ii) $\mathfrak{p}_1 | (\pi)$ in R_1, and $\mathfrak{p}_{n+1} | \mathfrak{p}_n$ in R_{n+1} for all $n > 0$.

In the following we set $\mathfrak{p}_0 := \pi R_0$. We note that any prime ideal of height one of R_n is a principal ideal, because R_n is a regular local ring, hence a unique factorization domain. Put $R_\infty = \bigcup_{n\geq 0} R_n$.

2.2.2. Note that the group $\mathrm{GL}_h(\mathcal{O})$ acts on the set of all such sequences $\mathfrak{p}_\bullet$: if $g \in \mathrm{GL}_h(\mathcal{O})$, and if $\mathfrak{p}_\bullet = (\mathfrak{p}_n)_n$ is such a sequence, then $g.\mathfrak{p}_\bullet = (g.\mathfrak{p}_n)_n$ is another such sequence. We call $\alpha = (\alpha_1, \ldots, \alpha_h) \in (\pi^{-n}\mathcal{O}/\mathcal{O})^h$ *primitive* if α is not divisible by π, i.e., $\alpha \notin (\pi^{-(n-1)}\mathcal{O}/\mathcal{O})^h$. Denote by $(\pi^{-n}\mathcal{O}/\mathcal{O})^h_{\mathrm{prim}} \subseteq (\pi^{-n}\mathcal{O}/\mathcal{O})^h$ the set of primitive elements. Note that the group of units $(\mathcal{O}/(\pi^n))^\times$ acts on $(\pi^{-n}\mathcal{O}/\mathcal{O})^h_{\mathrm{prim}}$, and let $\mathcal{P}_n = (\pi^{-n}\mathcal{O}/\mathcal{O})^h_{\mathrm{prim}}/(\mathcal{O}/(\pi^n))^\times$ be the set of orbits under this group. We denote by $[\alpha] \in \mathcal{P}_n$ the orbit of $\alpha \in (\pi^{-n}\mathcal{O}/\mathcal{O})^h_{\mathrm{prim}}$. We also call $v = (v_1, \ldots, v_h) \in \mathcal{O}^h$ *primitive* if it is not divisible by π, we let $\mathcal{O}^h_{\mathrm{prim}}$ be the subset of primitive vectors, and denote, as usual, by $\mathbb{P}^{h-1}(\mathcal{O}) = \mathcal{O}^h_{\mathrm{prim}}/\mathcal{O}^\times$ the set of orbits under the action of $\mathcal{O}^\times$, and we denote by $[v]$ its class in $\mathbb{P}^{h-1}(\mathcal{O})$.

Most statements of the following proposition have already been shown in the literature, but as we use them later on, we repeat them here. For elements $x, y \in R_n$ we write $x \sim y$ if x and y are associate, i.e., there is $u \in R_n^\times$ such that $y = ux$.

Proposition 2.2.3. *Let n be a positive integer.*

(i) *Let $\alpha_1, \dots, \alpha_h \in (\pi^{-n}\mathcal{O}/\mathcal{O})^h$ be a basis of $(\pi^{-n}\mathcal{O}/\mathcal{O})^h$ over $\mathcal{O}/(\pi^n)$. Then*
$$\Big(\lambda_n^{\mathrm{univ}}(\alpha_1), \dots, \lambda_n^{\mathrm{univ}}(\alpha_h)\Big)$$
is a regular system of parameters for R_n. Moreover, R_n is generated as an R_0-algebra by $\lambda_n^{\mathrm{univ}}(\alpha_1), \dots, \lambda_n^{\mathrm{univ}}(\alpha_h)$.

(ii) *For every $\alpha \in (\pi^{-n}\mathcal{O}/\mathcal{O})^h_{\mathrm{prim}}$, the element $\lambda_n^{\mathrm{univ}}(\alpha)$ is a prime element of R_n.*

(iii) *For $0 \neq \alpha \in (\pi^{-n}\mathcal{O}/\mathcal{O})^h$ and $a \in (\mathcal{O}/(\pi^n))^\times$, one has $\lambda_n^{\mathrm{univ}}(a\alpha) \sim \lambda_n^{\mathrm{univ}}(\alpha)$. Moreover, for $\alpha, \beta \in (\pi^{-n}\mathcal{O}/\mathcal{O})^h_{\mathrm{prim}}$ one has $\lambda_n^{\mathrm{univ}}(\alpha) \sim \lambda_n^{\mathrm{univ}}(\beta)$ if and only if $[\alpha] = [\beta]$ in $\mathcal{P}_n$.*

(iv) *For $\beta \in (\pi^{-n}\mathcal{O}/\mathcal{O})^h_{\mathrm{prim}}$, and $\alpha \in (\pi^{-(n+1)}\mathcal{O}/\mathcal{O})^h_{\mathrm{prim}}$, the prime element $\lambda_{n+1}^{\mathrm{univ}}(\alpha)$ divides $\lambda_n^{\mathrm{univ}}(\beta)$ in R_{n+1} if and only if $[\pi\alpha] = [\beta]$ in $\mathcal{P}_n$.*

(v) *Let $\breve{F}_n/\breve{F}$ be as in 2.1.3(iv). Then there is an embedding of $\breve{\mathcal{O}}$-algebras $\mathcal{O}_{\breve{F}_n} \hookrightarrow R_n$.*

(vi) *Let $\varpi_n \in \mathcal{O}_{\breve{F}_n}$ be a uniformizer. One has*
$$\varpi_n \sim \prod_{[\alpha] \in \mathcal{P}_n} \lambda_n^{\mathrm{univ}}(\alpha)\,.$$

(vii) *For $\beta \in (\pi^{-n}\mathcal{O}/\mathcal{O})^h_{\mathrm{prim}}$ we have the following prime factorization of $\lambda_n^{\mathrm{univ}}(\beta)$:*
$$\lambda_n^{\mathrm{univ}}(\beta) \sim \prod_{[\alpha] \in \mathcal{P}_{n+1}, [\pi\alpha] = [\beta]} \lambda_{n+1}^{\mathrm{univ}}(\alpha)^q\,.$$

(viii) *For every $[v] \in \mathbb{P}^{h-1}(\mathcal{O})$ the sequence of ideals $\big((\lambda_n^{\mathrm{univ}}(\pi^{-n}v + \mathcal{O}^h))\big)_{n>0}$ satisfies the conditions in 2.2.1.*

(ix) *Conversely, for every sequence of prime ideals $(\mathfrak{p}_n)_{n>0}$ as in 2.2.1 there is a unique $[v] \in \mathbb{P}^{h-1}(\mathcal{O})$ such that $(\lambda_n^{\mathrm{univ}}(\pi^{-n}v + \mathcal{O}^h)) = \mathfrak{p}_n$ for all $n > 0$.*

Proof. (i). The first statement is contained in [3, 4.3]. For the second statement let $S \subseteq R_n$ be the subring generated over R_0 by the elements $\lambda_n^{\mathrm{univ}}(\alpha_1), \dots, \lambda_n^{\mathrm{univ}}(\alpha_h)$.

Claim. $R_n/\mathfrak{m}_{R_0}R_n$ is a finite-dimensional $\overline{k}_F$-vector space generated by finitely many monomials in the $\lambda_n^{\mathrm{univ}}(\alpha_i)$.

Proof of the Claim. To see this, consider for any $t \geq 1$ the exact sequence of vector spaces over $R_0/\mathfrak{m}_{R_0} = \overline{k}_F$:
$$\begin{aligned} 0 \;\longrightarrow\; & (\mathfrak{m}_{R_0}.R_n + (\mathfrak{m}_{R_n})^t)/(\mathfrak{m}_{R_0}.R_n + (\mathfrak{m}_{R_n})^{t+1}) \\ \longrightarrow\; & R_n/(\mathfrak{m}_{R_0}.R_n + (\mathfrak{m}_{R_n})^{t+1}) \;\longrightarrow\; R_n/(\mathfrak{m}_{R_0}.R_n + (\mathfrak{m}_{R_n})^t) \;\longrightarrow\; 0. \end{aligned}$$

Note that the term on the left is a quotient of $\mathfrak{m}_{R_n}^t/\mathfrak{m}_{R_n}^{t+1}$, and is thus generated by finitely many monomials in the $\lambda_n^{\mathrm{univ}}(\alpha_i)$. For $t = 1$ the term on the right is equal to the residue field of R_n, which is equal to the residue field of R_0 (which is $\overline{k}_F$). Using induction on t we see that $R_n/(\mathfrak{m}_{R_0}.R_n + (\mathfrak{m}_{R_n})^t)$ is generated by finitely many monomials in the $\lambda_n^{\mathrm{univ}}(\alpha_i)$. Next, because $\mathfrak{m}_{R_0}R_n$ is $\mathfrak{m}_{R_n}$-primary[2] there is a $t > 0$ such that $\mathfrak{m}_{R_n}^t \subset \mathfrak{m}_{R_0}R_n$, by [1, 7.16], and for such a t we have $R_n/(\mathfrak{m}_{R_0}.R_n + (\mathfrak{m}_{R_n})^t) = R_n/\mathfrak{m}_{R_0}R_n$. ☐

In particular, $R_n = S + \mathfrak{m}_{R_0}R_n$. Hence $R_n = S$ by a corollary of Nakayama's Lemma.

(ii). Follows easily from (i), cf. [13, 4.2 (i)].

(iii). We fix a coordinate T on X^{univ}. Then the multiplication by $a \in \mathcal{O}$ on X^{univ} is given by a power series $[a]_{X^{\mathrm{univ}}}(T) = aT + T^2P(T)$ with a power series $P(T) \in R_0[\![T]\!]$. If a is a unit in $\mathcal{O}$, then we see that $[a]_{X^{\mathrm{univ}}}(x)/x = a + xP(x)$ is a unit in R_n for all non-zero $x \in \mathfrak{m}_{R_n}$. It follows that

$$\lambda_n^{\mathrm{univ}}(a\alpha) = [a]_{X^{\mathrm{univ}}}\Big(\lambda_n^{\mathrm{univ}}(\alpha)\Big) = \lambda_n^{\mathrm{univ}}(\alpha) \cdot \frac{[a]_{X^{\mathrm{univ}}}\Big(\lambda_n^{\mathrm{univ}}(\alpha)\Big)}{\lambda_n^{\mathrm{univ}}(\alpha)}$$

is associate to $\lambda_n^{\mathrm{univ}}(\alpha)$. This shows that $\lambda_n^{\mathrm{univ}}(\alpha) \sim \lambda_n^{\mathrm{univ}}(\beta)$ if $[\alpha] = [\beta]$ in $\mathcal{P}_n$. The converse is in [13, 4.2 (i)].

(iv). Suppose $[\pi\alpha] = [\beta]$. By (iii) we have $\lambda_n^{\mathrm{univ}}(\beta) \sim \lambda_n^{\mathrm{univ}}(\pi\alpha)$, and

$$\lambda_n^{\mathrm{univ}}(\pi\alpha) = \lambda_{n+1}^{\mathrm{univ}}(\pi\alpha) = [\pi]_{X^{\mathrm{univ}}}\Big(\lambda_{n+1}^{\mathrm{univ}}(\alpha)\Big) = \lambda_{n+1}^{\mathrm{univ}}(\alpha) \cdot \frac{[\pi]_{X^{\mathrm{univ}}}\Big(\lambda_{n+1}^{\mathrm{univ}}(\alpha)\Big)}{\lambda_{n+1}^{\mathrm{univ}}(\alpha)},$$

hence $\lambda_{n+1}^{\mathrm{univ}}(\alpha)$ divides $\lambda_n^{\mathrm{univ}}(\pi\alpha)$ in R_{n+1}. Therefore, $(\lambda_{n+1}^{\mathrm{univ}}(\alpha)) \cap R_n \supset (\lambda_n^{\mathrm{univ}}(\pi\alpha))$. Since R_0 is integrally closed, the going-down theorem [1, 5.16] is applicable and implies that the height of the prime ideal $(\lambda_{n+1}^{\mathrm{univ}}(\alpha)) \cap R_n$ must be equal to the height of $(\lambda_{n+1}^{\mathrm{univ}}(\alpha))$ which is one. This implies $(\lambda_{n+1}^{\mathrm{univ}}(\alpha)) \cap R_n = (\lambda_n^{\mathrm{univ}}(\pi\alpha))$.

Now suppose $\lambda_{n+1}^{\mathrm{univ}}(\alpha)$ divides $\lambda_n^{\mathrm{univ}}(\beta)$ in R_{n+1}. Then, using the same argument as above, we have $(\lambda_{n+1}^{\mathrm{univ}}(\alpha)) \cap R_n = (\lambda_n^{\mathrm{univ}}(\beta))$. On the other hand, we have just seen that $(\lambda_{n+1}^{\mathrm{univ}}(\alpha)) \cap R_n = (\lambda_n^{\mathrm{univ}}(\pi\alpha))$. Therefore, $\lambda_n^{\mathrm{univ}}(\beta) \sim \lambda_n^{\mathrm{univ}}(\pi\alpha)$. Now we use (iii) to conclude.

(v) and (vi). These assertions are in [13, 3.4, 4.2 (ii)].

---

[2] Because R_n is noetherian, $\mathfrak{m}_{R_0}R_n$ has a primary decomposition $\bigcap_i \mathfrak{q}_i$ [1, 7.13], where each $\mathfrak{q}_i$ is a primary ideal, which in turn implies that $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ a prime ideal in R_n. Therefore, $R_0 \cap \mathfrak{p}_i$ contains $\mathfrak{m}_{R_0}$, which is maximal in R_0. The prime ideal $\mathfrak{p}_i$ is then maximal in R_n [1, 5.3], i.e., $\mathfrak{p}_i = \mathfrak{m}_{R_n}$. Hence $\mathfrak{m}_{R_0}R_n$ is $\mathfrak{m}_{R_n}$-primary [1, 4.3].

(vii). We apply statement (vi) twice, for n and for $n+1$, and obtain:

$$\prod_{[\beta]\in\mathcal{P}_n} \lambda_n^{\mathrm{univ}}(\beta) \sim \varpi_n \sim \varpi_{n+1}^q \sim \prod_{[\alpha]\in\mathcal{P}_{n+1}} \lambda_{n+1}^{\mathrm{univ}}(\alpha)^q$$
$$= \prod_{[\beta]\in\mathcal{P}_n} \prod_{[\alpha]\in\mathcal{P}_{n+1},[\pi\alpha]=[\beta]} \lambda_{n+1}^{\mathrm{univ}}(\alpha)^q\,.$$

Assertion (vii) follows now from statement (iv).

(viii). In $\breve{F}_1$ we have $\pi \sim \varpi_1^{q-1}$, and thus $\pi \sim \prod_{[\alpha]\in\mathcal{P}_1} \lambda_1^{\mathrm{univ}}(\alpha)^{q-1}$, by (vi). This shows the first condition in 2.2.1. The second condition now follows from statement (iv).

(ix). As we have seen in the proof of (viii), any principal prime ideal $\mathfrak{p}_1$ of R_1 dividing (π) must be generated by one of $\lambda_1(\alpha_1)$ for a unique $[\alpha_1] \in \mathcal{P}_1$. By (vii), any principal prime ideal $\mathfrak{p}_{n+1}$ of R_{n+1} dividing $(\lambda_n(\alpha_n))$, with $\alpha_n \in \mathcal{P}_n$, must be generated by an element $\lambda_{n+1}(\alpha_{n+1})$ with $\alpha_{n+1} \in \mathcal{P}_{n+1}$ and $[\pi\alpha_{n+1}] = [\alpha_n]$. One can choose elements $\widetilde{\alpha}_n \in \pi^{-n}\mathcal{O}^h$ such that $\widetilde{\alpha}_n + \mathcal{O}^h = \alpha_n$ and $\pi\widetilde{\alpha}_{n+1} + \mathcal{O}^h = \alpha_n$. It is easily seen that the limit $v = \lim_{n\to\infty} \pi^n\widetilde{\alpha}_n$ exists and is an element in $\mathcal{O}^h_{\mathrm{prim}}$, and $\pi^{-n}v + \mathcal{O}^h = \alpha_n$ for all $n > 0$. This proves statement (ix). $\square$

Corollary 2.2.4. *The prime ideals of height one of $R_\infty = \bigcup_n R_n$ lying over (π) are naturally parametrized by elements in $\mathbb{P}^{h-1}(\mathcal{O})$, and the action of $\mathrm{GL}_h(\mathcal{O})$ on the set of those prime ideals of R_∞ is transitive.*

Proof. This is an immediate consequence of 2.2.3 (viii) and (ix). $\square$

Convention 2.2.5. In the remainder of this section we will describe certain Galois groups. Their description will involve terms like $1 + \pi^m\mathcal{O}/(\pi^n)$ or $1 + \pi^m M_{h-1}(\mathcal{O}/(\pi^n))$, for $n \geq m \geq 0$. When $m = 0$ we will interpret these terms as meaning $(\mathcal{O}/(\pi^n))^\times$ and $\mathrm{GL}_{h-1}(\mathcal{O}/(\pi^n))$, respectively.

2.2.6. *The fields $\mathcal{K}_n$.* We denote by $\mathcal{K}_n$ and $\mathcal{K}_\infty$ the fields of fractions of R_n and R_∞, respectively. Furthermore, we let $e_1 = (1,0,\ldots,0),\ldots,e_h = (0,\ldots,0,1)$ be the standard generators of $\mathcal{O}^h$.

Corollary 2.2.7.

(i) *For every $\sigma \in \mathrm{Gal}(\mathcal{K}_n \mid \mathcal{K}_0)$ there is a unique matrix $(a_{i,j})_{1\leq i,j,\leq h} \in \mathrm{GL}_h(\mathcal{O}/(\pi^n))$ such that for all $j = 1,\ldots,h$:*

$$\sigma\Big(\lambda_n^{\mathrm{univ}}(\pi^{-n}e_j + \mathcal{O}^h)\Big) = [a_{1,j}]_{X^{\mathrm{univ}}}\Big(\lambda_n^{\mathrm{univ}}(\pi^{-n}e_1 + \mathcal{O}^h)\Big) +_{X^{\mathrm{univ}}} \ldots$$
$$\ldots +_{X^{\mathrm{univ}}} \; [a_{h,j}]_{X^{\mathrm{univ}}}\Big(\lambda_n^{\mathrm{univ}}(\pi^{-n}e_h + \mathcal{O}^h)\Big)$$

The map $\gamma : \mathrm{Gal}(\mathcal{K}_n \mid \mathcal{K}_0) \to \mathrm{GL}_h(\mathcal{O}/(\pi^n))$ defined thus is an isomorphism.

(ii) *For varying $n \geq m \geq 0$, the isomorphism γ in (i) is compatible with the obvious transition maps*

$$\mathrm{Gal}(\mathcal{K}_n \mid \mathcal{K}_0) \longrightarrow \mathrm{Gal}(\mathcal{K}_m \mid \mathcal{K}_0)\,,$$
$$\mathrm{GL}_h(\mathcal{O}/(\pi^n)) \longrightarrow \mathrm{GL}_h(\mathcal{O}/(\pi^m))\,,$$

and thus induces an isomorphism $\mathrm{Gal}(\mathcal{K}_\infty \mid \mathcal{K}_0) \xrightarrow{\simeq} \mathrm{GL}_h(\mathcal{O})$.

(iii) *For $n \geq m \geq 0$, the isomorphism γ in (i) induces an isomorphism*

$$\mathrm{Gal}(\mathcal{K}_n \mid \mathcal{K}_m) \xrightarrow{\simeq} 1 + \pi^m M_h(\mathcal{O}/(\pi^n))\,.$$

Proof. (i). Recall that by 2.1.2 (iv) the action of $\mathrm{GL}_h(\mathcal{O}/(\pi^n))$ on the functor $\mathrm{Def}_{\mathbb{X},n}$ induces an action of this group on R_n, which is trivial on R_0, and such that the resulting map

$$\mathrm{GL}_h(\mathcal{O}/(\pi^n)) \longrightarrow \mathrm{Gal}(R_n[1/\pi] \mid R_0[1/\pi])$$

is an isomorphism of groups. As R_n is a regular local ring, it is integrally closed, and is thus the integral closure of R_0 in $\mathcal{K}_n$. Hence $R_n[1/\pi]$ is the integral closure of $R_0[1/\pi]$ in $\mathcal{K}_n$ [1, 5.12]. A Galois automorphism of $\mathcal{K}_n$ over $\mathcal{K}_0$ is trivial on $R_0[1/\pi]$, and hence maps $R_n[1/\pi]$ to itself. Therefore, the canonical map

$$\mathrm{Gal}(R_n[1/\pi] \mid R_0[1/\pi]) \longrightarrow \mathrm{Gal}(\mathcal{K}_n \mid \mathcal{K}_0)$$

is an isomorphism. This map is given explicitly as follows.

Recall the isomorphism of functors $\mathrm{Def}_{\mathbb{X},n} \to \mathrm{Hom}_{\mathcal{C}}(R_n, -)$ from 2.1.2 (i), which we will here denote by ψ. Given an $\breve{\mathcal{O}}$-algebra R in $\mathcal{C}$ and a triple $[X, \iota, \lambda] \in \mathrm{Def}_{\mathbb{X},n}(R)$, we have $\psi_R([X, \iota, \lambda]) = \alpha : R_n \to R$ if and only if

$$[X, \iota, \lambda] = [X^{\mathrm{univ}} \otimes_{R_0^{\mathrm{univ}},\alpha} R, (X^{\mathrm{univ}} \otimes_{R_0} \overline{k}_F \to X^{\mathrm{univ}} \otimes_{R_0^{\mathrm{univ}},\bar{\alpha}} R/\mathfrak{m}_R) \circ \iota, \alpha \circ \lambda]\,,$$

where $\bar{\alpha} : R_n \to R/\mathfrak{m}_R$ is the morphism induced by α. To $g \in \mathrm{GL}_h(\mathcal{O}/(\pi^n))$ we associate the natural transformation $\phi_g : \mathrm{Def}_{\mathbb{X},n} \to \mathrm{Def}_{\mathbb{X},n}$ given by sending $[X, \iota, \lambda] \in \mathrm{Def}_{\mathbb{X},n}(R)$ to $[X, \iota, \lambda \circ g]$. As ψ is an isomorphism of functors, there is a morphism $\gamma_g : R_n \to R_n$ in the category $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
\mathrm{Def}_{\mathbb{X},n} \arrow[r, "\phi_g"] \arrow[d, "\psi"] & \mathrm{Def}_{\mathbb{X},n} \arrow[d, "\psi"] \\
\mathrm{Hom}_{\mathcal{C}}(R_n, -) \arrow[r, "\gamma_g^*"] & \mathrm{Hom}_{\mathcal{C}}(R_n, -)
\end{tikzcd}$$

is commutative. An analogous diagram exists for any $n' \leq n$ and these commutative diagrams, for n and $n' \leq n$, form a commutative cube. Since the map ϕ_g induced on $\mathrm{Def}_{\mathbb{X},0}$ is the identity, it follows that γ_g is the identity on R_0.

We can evaluate the functors in the commutative diagram above on R_n and use the fact that $\psi_{R_n}([X^{\mathrm{univ}}, \iota^{\mathrm{univ}}, \lambda_n^{\mathrm{univ}}]) = \mathrm{id}_{R_n}$ to obtain $\psi_{R_n}([X^{\mathrm{univ}},$

$\iota^{\mathrm{univ}}, \lambda_n^{\mathrm{univ}} \circ g]) = \gamma_g$, which is equivalent to $[X^{\mathrm{univ}}, \iota^{\mathrm{univ}}, \lambda_n^{\mathrm{univ}} \circ g] = [X^{\mathrm{univ}}, \iota^{\mathrm{univ}}, \gamma_g \circ \lambda_n^{\mathrm{univ}}]$ (because γ_g is the identity on R_0). By the definition of the equivalence relation on triples (cf. the paragraph before 2.1.2), there is an isomorphism $f : (X^{\mathrm{univ}}, \iota^{\mathrm{univ}}) \to (X^{\mathrm{univ}}, \iota^{\mathrm{univ}})$ of (rigidified) formal groups over R_n such that $f \circ \lambda_n^{\mathrm{univ}} \circ g = \gamma_g \circ \lambda_n^{\mathrm{univ}}$. This implies that f induces that identity on $X^{\mathrm{univ}} \otimes_{R_0} \overline{k}_F$, and must hence be the identity. Therefore, we have $\gamma_g \circ \lambda^{\mathrm{univ}} = \lambda^{\mathrm{univ}} \circ g$. The assertion now follows from the fact that $\lambda_n^{\mathrm{univ}}$ is a homomorphism of groups $(\pi^{-n}\mathcal{O}/\mathcal{O})^h \to (\mathfrak{m}_{R_n}, +_{X^{\mathrm{univ}}})$.

(ii), (iii). These statements are an easy consequence of the first. □

2.2.8. *The fields K'_n.* In the following it will be convenient to fix a particular sequence of primes $\mathfrak{p}_n$ satisfying 2.2.1, namely the sequence corresponding to the first standard basis vector $v = e_1 = (1, 0, \dots, 0)$ by 2.2.3 (viii), i.e., $\mathfrak{p}_n = (\pi_n)$ where $\pi_n = \lambda_n^{\mathrm{univ}}(\pi^{-n}e_1 + \mathcal{O}^h)$. Let R'_n be the completion of the localization $(R_n)_{\mathfrak{p}_n}$ (with respect to the topology defined by the maximal ideal) and denote by $K'_n = \mathrm{Frac}(R'_n)$ its field of fractions. The field extension $K'_n \mid K'_0$ is finite and Galois and $\mathrm{Gal}(K'_n \mid K'_0)$ is the decomposition group of the prime ideal $\mathfrak{p}_n$, cf. [11, Ch. II, §3, Cor. 4]. Let k'_n be the residue field of K'_n. Let $k'_{n,\mathrm{sep}} \subseteq k'_n$ be the separable closure of k'_0 in k'_n. Set $K'_\infty = \bigcup_{n \ge 0} K'_n$.

Corollary 2.2.9. *The ramification index of the extension $K'_n \mid K'_0$ is*

$$e(K'_n \mid K'_0) = (q-1)q^{n-1}.$$

Proof. In the proof of 2.2.3 (viii) we have shown $\pi \sim \prod_{[\alpha] \in \mathcal{P}_1} \lambda_1^{\mathrm{univ}}(\alpha)^{q-1}$, which shows that $e(K'_1 \mid K'_0) = q - 1$. For $n > 0$ it follows from 2.2.3 (vii) that $e(K'_{n+1} \mid K'_n) = q$. □

Proposition 2.2.10.

(i) *R'_n is generated over R'_0 by $\lambda_n^{\mathrm{univ}}(\pi^{-n}e_1 + \mathcal{O}^h), \dots, \lambda_n^{\mathrm{univ}}(\pi^{-n}e_h + \mathcal{O}^h)$, and the residue field k'_n of K'_n is generated over k'_0 by the images of $\lambda_n^{\mathrm{univ}}(\pi^{-n}e_2 + \mathcal{O}^h), \dots, \lambda_n^{\mathrm{univ}}(\pi^{-n}e_h + \mathcal{O}^h)$.*

(ii) *The isomorphism $\gamma : \mathrm{Gal}(\mathcal{K}_n \mid \mathcal{K}_0) \xrightarrow{\simeq} \mathrm{GL}_h(\mathcal{O}/(\pi^n))$ in 2.2.7 induces an isomorphism*

$$\mathrm{Gal}(K'_n \mid K'_0) \xrightarrow{\simeq} \left\{ \left(\begin{array}{c|ccc} a_{1,1} & a_{1,2} & \cdots & a_{1,h} \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array} \right) \;\middle|\; \begin{array}{l} a_{1,1} \in (\mathcal{O}/\pi^n\mathcal{O})^\times, \\ a_{1,j} \in \mathcal{O}/(\pi^n) \;\; \text{for } j > 1, \\ A' \in \mathrm{GL}_{h-1}(\mathcal{O}/(\pi^n)) \end{array} \right\},$$

which we again denote by γ.

(iii) *Let $I(K'_n \mid K'_0) \subseteq \mathrm{Gal}(K'_n \mid K'_0)$ be the inertia subgroup. Then the isomorphism γ in (ii) maps $I(K'_n \mid K'_0)$ isomorphically onto the subgroup*

$$\left\{ \left(\begin{array}{c|ccc} a_{1,1} & a_{1,2} & \cdots & a_{1,h} \\ \hline 0 & & & \\ \vdots & & I_{h-1} & \\ 0 & & & \end{array} \right) \;\middle|\; \begin{array}{l} a_{1,1} \in (\mathcal{O}/\pi^n\mathcal{O})^\times, \\ a_{1,j} \in \mathcal{O}/(\pi^n) \ \ \text{for } j > 1 \end{array} \right\},$$

where I_{h-1} is the $(h-1)\times(h-1)$-identity matrix. Furthermore, the residue field extension $k'_n \mid k'_0$ is normal and the subextsion $k'_{n,\mathrm{sep}} \mid k'_0$ is Galois. The isomorphism γ in (ii) induces an isomorphism $\mathrm{Aut}(k'_n \mid k'_0) = \mathrm{Gal}(k'_{n,\mathrm{sep}} \mid k'_0) \xrightarrow{\simeq} \mathrm{GL}_{h-1}(\mathcal{O}/(\pi^n))$.

(iv) *For any $n \geq m \geq 0$, the isomorphism γ in (ii) induces an isomorphism*

$$\mathrm{Gal}(K'_n \mid K'_m) \simeq \left\{ \left(\begin{array}{c|ccc} a_{1,1} & a_{1,2} & \cdots & a_{1,h} \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array} \right) \;\middle|\; \begin{array}{l} a_{1,1} \in (1+\pi^m\mathcal{O})/(1+\pi^n\mathcal{O}), \\ a_{1,j} \in (\pi^m)/(\pi^n) \ \ \text{for } j > 1, \\ A' \in I_{h-1} + \pi^m M_{h-1}(\mathcal{O}/(\pi^n)) \end{array} \right\}.$$

(v) *The isomorphism γ in (ii) induces, for every $m \in \mathbb{N}$, an isomorphism*

$$\mathrm{Gal}(K'_\infty \mid K'_m) \simeq \left\{ \left(\begin{array}{c|ccc} a_{1,1} & a_{1,2} & \cdots & a_{1,h} \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array} \right) \;\middle|\; \begin{array}{l} a_{1,1} \in 1+\pi^m\mathcal{O}, \\ a_{1,j} \in \pi^m\mathcal{O} \ \ \text{for } j > 1, \\ A' \in I_{h-1} + \pi^m M_{h-1}(\mathcal{O}) \end{array} \right\}.$$

Proof. (i). Follows from 2.2.3 (i).

(ii). This is statement [14, 4.1 (ii)]. The integer n (resp. m, resp. h) in this reference corresponds to h (resp. n, resp. 1) here. The universal Drinfeld level structure is denoted by ϕ in loc. cit., and the ideal $\mathfrak{p}_{1,m}$ of loc. cit. corresponds to the ideal (π_n) considered here.

(iii). This is statement [14, 4.1 (iii)]. It is a general fact that the residue field extension $k'_n \mid k'_0$ is normal (and the extension $k'_{n,\mathrm{sep}} \mid k'_0$ is therefore Galois), and that the map from $\mathrm{Gal}(K'_n \mid K'_0)/I(K'_n \mid K'_0)$ to the automorphism group $\mathrm{Aut}(k'_n \mid k'_0)$ is an isomorphism, cf. [2, Ch. 5, §2.2, Thm. 2].

(iv). Follows straightforwardly from the isomorphism in (ii).

(v). It follows from 2.2.7 that, for fixed m, the isomorphism in (iv) is compatible with the natural transition maps on both sides, as n varies. Passing to the projective limit shows the assertion. □

2.2.11. *The fields $\widetilde{K}_n$ and K_n.* For $\ell \in \mathbb{N}$ let $K'_{\ell,u} \subseteq K'_\ell$ be the maximal subextension of $K'_\ell \mid K'_0$ which is unramified over K'_0. The residue field of $K'_{\ell,u}$ is the separable closure $k'_{\ell,\mathrm{sep}}$ of k'_0 in k'_ℓ. Set $\widetilde{K}_0 = \bigcup_{\ell \geq 0} K'_{\ell,u}$, and define K_0 to be the π-adic completion of $\widetilde{K}_0$. For $n \geq 0$, put

$$\widetilde{K}_n = K'_n \widetilde{K}_0 \quad \text{ and } \quad K_n = K'_n K_0\,.$$

The fields $\widetilde{K}_n$ and K_n are discretely valued and K_n is complete. Denote by A_n (resp. $\widetilde{A}_n$) the ring of integers of K_n (resp. $\widetilde{K}_n$) and by k_n (resp. $\widetilde{k}_n$) its residue field. As completion does not affect the residue field, the canonical map $\widetilde{k}_n \to k_n$ is an isomorphism.

Remark 2.2.12. For $n \geq m \geq 0$ the extension $\widetilde{K}_n \mid \widetilde{K}_m$ is Galois of finite degree, and $\widetilde{K}_n$ is also a Galois extension of any of the fields $K'_{\ell,u}$ (this extension is of infinite degree). Similarly, the extension $K_n \mid K_m$ is Galois of finite degree.[3]

Proposition 2.2.13.

(i) *For $\ell' \geq n \geq \ell \geq m \geq 0$, the isomorphism γ in 2.2.10(ii) induces an isomorphism*

$$\operatorname{Gal}(K'_n K'_{\ell',u} \mid K'_m K'_{\ell,u}) \simeq \left\{ \left(\begin{array}{c|ccc} a_{1,1} & a_{1,2} & \cdots & a_{1,h} \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array} \right) \;\middle|\; \begin{array}{l} a_{1,1} \in (1+\pi^m\mathcal{O})/(1+\pi^n\mathcal{O}), \\ a_{1,j} \in (\pi^m)/(\pi^n) \ \ \text{for } j>1, \\ A' \in I_{h-1} + \pi^\ell M_{h-1}(\mathcal{O}/(\pi^{\ell'})) \end{array} \right\},$$

which we again denote by γ. This isomorphism is compatible with the natural transition maps on both sides as $\ell' \geq n \geq \ell \geq m$ vary.

(ii) *For any n and ℓ the isomorphism γ in (i) induces an isomorphism*

$$\operatorname{Gal}(\widetilde{K}_n \mid K'_{\ell,u}) \simeq \left\{ \left(\begin{array}{c|ccc} a_{1,1} & a_{1,2} & \cdots & a_{1,h} \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array} \right) \;\middle|\; \begin{array}{l} a_{1,1} \in (\mathcal{O}/(\pi^n))^\times, \\ a_{1,j} \in \mathcal{O}/(\pi^n) \ \ \text{for } j>1, \\ A' \in I_{h-1} + \pi^\ell M_{h-1}(\mathcal{O})) \end{array} \right\}.$$

(iii) *For any $n \geq m$ the isomorphism γ in (i) induces an isomorphism*

$$\operatorname{Gal}(\widetilde{K}_n \mid \widetilde{K}_m) \simeq 1 + \pi^m\mathcal{O}/(\pi^n) \ltimes ((\pi^m)/(\pi^n))^{h-1}\,,$$

[3] If $h > 1$, the p-adically complete field K_n is presumably not algebraic over any of the fields K'_m $(m \leq n)$, $\widetilde{K}_m$ $(m \leq n)$, or $K'_{\ell,u}$ (any ℓ).

(iv) *For all* $n \geq m \geq 0$*, the extension* $K_n \mid K_m$ *is Galois and the restriction map* $\mathrm{Gal}(K_n \mid K_m) \to \mathrm{Gal}(\widetilde{K}_n \mid \widetilde{K}_m)$ *is an isomorphism. We thus have* $[K_n : K_m] = q^{(n-m)h}$*, if* $m > 0$*, and* $[K_n : K_0] = (q-1)q^{nh-1}$.

(v) *For all* $n \geq 1$ *one has* $[k_n : k_{n-1}] = q^{h-1}$.

Proof. (i). We consider $K'_n K'_{\ell',u}$ as a subfield of $K'_{\ell'}$. Then we have the short exact sequence

$$1 \longrightarrow \mathrm{Gal}(K'_{\ell'} \mid K'_n K'_{\ell',u}) \longrightarrow \mathrm{Gal}(K'_{\ell'} \mid K'_0) \longrightarrow \mathrm{Gal}(K'_n K'_{\ell',u} \mid K'_0) \longrightarrow 1\,.$$

Consider $\sigma \in \mathrm{Gal}(K'_{\ell'} \mid K'_0)$ and write

$$\gamma(\sigma) = \left(\begin{array}{c|ccc} a_{1,1} & a_{1,2} & \cdots & a_{1,h} \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array}\right)$$

with $a_{1,1} \in \mathcal{O}/(\pi^{\ell'})$, $a_{1,j} \in \mathcal{O}/(\pi^{\ell'})$, for $j > 1$, and $A' \in \mathrm{GL}_{h-1}(\mathcal{O}/(\pi^{\ell'}))$. For σ to act trivially on $K'_n K'_{\ell',u}$ it must, in particular, act trivially on $K'_{\ell',u}$. By 2.2.10 (iii), the matrix A' must therefore be the identity matrix. Since K'_n is generated by the elements $\lambda_n^{\mathrm{univ}}(\pi^{-n} e_j + \mathcal{O}^h)$, for $j = 1, \ldots, h$, we must also have $a_{1,1} \in 1 + (\pi^n)/(\pi^{\ell'})$ and $a_{1,j} \in (\pi^n)/(\pi^{\ell'})$ for $j > 1$. This proves the assertion when $\ell = m = 0$. Using similar arguments we see that the subgroup $\mathrm{Gal}(K'_n K'_{\ell',u} \mid K'_m K'_{\ell,u})$ of $\mathrm{Gal}(K'_n K'_{\ell',u} \mid K'_0)$ is mapped by γ to the group as stated.

(ii). This follows from (i) when we take $m = 0$ and when pass to the projective limit as $\ell' \to \infty$.

(iii). This statement follows from (ii) when we take the projective limit (which is just an intersection) as $\ell \to \infty$.

(iv). The extension $K'_n \mid K'_0$ is Galois, and so is the extension $K_n = K'_n K_0 \mid K_m = K'_m K_0$ that we obtain by taking the composite fields with K_0. The restriction map $\mathrm{Gal}(K_n \mid K_m) \to \mathrm{Gal}(\widetilde{K}_n \mid \widetilde{K}_m)$ is injective, because if $\sigma \in \mathrm{Gal}(K_n \mid K_m)$ acts trivially on $\widetilde{K}_n$, which is dense in K_n (for the p-adic topology), then it acts trivially on K_n. But this map is also surjective because any Galois automorphism in $\mathrm{Gal}(\widetilde{K}_n \mid \widetilde{K}_m)$ extends continuously to an automorphism of K_n over K_m, because $\widetilde{K}_n$ is dense in K_n.

(v). By (iii) and (iv) we have $[K_n : K_0] = (q-1)q^{n(h-1)+n-1}$ for all $n \geq 1$. For the ramification index we have $e(K_n \mid K_0) = e(\widetilde{K}_n \mid \widetilde{K}_0) =$

$e(K_n' \mid K_0') = (q-1)q^{n-1}$; cf. 2.2.9, since K_0 is the completion of the unramified extension $\widetilde{K}_0$ of K_0'. It follows that $[k_n : k_0] = q^{n(h-1)}$, from which we conclude that $[k_n : k_{n-1}] = q^{h-1}$. □

3. Strictly deeply ramified towers

In the rest of this paper we will only consider the tower $K_\bullet$ constructed in Section 2 when $\mathcal{O} = \mathbb{Z}_p$. In particular, we have

$$q = p \text{ and } \pi = p\,.$$

In the following we will always write p instead of q, when using formulas from the preceding section (involving cardinalities), but we keep writing π instead of p when referring to the uniformizer of $\mathcal{O} = \mathbb{Z}_p$.

The reason for restricting our attention to the case of $\mathbb{Z}_p$ is because of the way the theory of strictly deeply ramified towers of fields has been developed by Scholl. We note, however, that Scholl's theory can be generalized to a setting which would allow us to work here with the ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_p$, cf. [8, 2.3.1].

For $i = 1, \dots, h$ we put $Y_{n,i} = \lambda_n^{\mathrm{univ}}(p^{-n}e_i + \mathbb{Z}_p^h)$; we consider these as elements of A_n. Recall that, with this definition, $\pi_n = Y_{n,1}$ (cf. 2.2.8).

Recall that we denote by k_n the residue field of $K_n = K_n'K_0$ (cf. 2.2.11), and by k_n' the residue field of K_n' (cf. 2.2.8). By 2.2.3 the elements $Y_{n,1}, \dots,$ $Y_{n,h}$ form a system of parameters of R_n, and generate R_n as R_0-algebra. Therefore, $k_n' = \mathrm{Frac}(R_n/\pi_n R_n)$ is generated by

$$y_{n,i} := Y_{n,i} \bmod \mathfrak{p}_n\,, \quad i = 2, \dots, h\,,$$

over k_0'.

Proposition 3.1.1. *For all $n \geq 1$ and $i = 1, \dots, h$, the minimal polynomial of $Y_{n,i}$ over K_{n-1} is*

$$Q_i(T) = Q_{n,i}(T) = \prod_{a \in \mathbb{F}_p} (T - (Y_{n,i} +_{X^{\mathrm{univ}}} [a]_{X^{\mathrm{univ}}}(Y_{1,1})))\,,$$

and has coefficients in A_{n-1}, except if $n = 1 = i$, in which case

$$Q_{1,1}(T) = \prod_{a \in \mathbb{F}_p,\, a \neq -1} (T - (Y_{1,1} +_{X^{\mathrm{univ}}} [a]_{X^{\mathrm{univ}}}(Y_{1,1})))\,.$$

For $2 \leq i \leq h$, the reduction of $Q_{n,i}$ modulo the maximal ideal (π_{n-1}) of A_{n-1} is $T^p - y_{n,i}^p$.

Proof. We only treat the case $n > 1$; the case $n = 1$ is very similar (with the obvious modifications). By 2.2.13 (iii) and (iv), the Galois group of K_n/K_{n-1} is isomorphic to the group

$$(1 + \pi^{n-1}\mathcal{O})/(1 + \pi^n\mathcal{O}) \ltimes (\pi^{n-1}\mathcal{O}/\pi^n\mathcal{O})^{h-1}\,,$$

and the Galois action is given by the formula in 2.2.7. This means the Galois conjugates of $Y_{n,i}$ are the elements

$$Y_{n,i} +_{X^{\text{univ}}} [a\pi^{n-1}]_{X^{\text{univ}}}(Y_{n,1})$$

for $a \in \mathbb{F}_p$. This shows that the minimal polynomial of $Y_{n,i}$ over K_{n-1} is

$$Q_i(T) = \prod_{a\in\mathbb{F}_p} (T - (Y_{n,i} +_{X^{\text{univ}}} [a\pi^{n-1}]_{X^{\text{univ}}}(Y_{n,1}))) .$$

Since all roots are in A_n, the coefficients of this polynomial are in $A_n \cap K_{n-1} = A_{n-1}$.

The universal formal group law, as any one-dimensional formal group law, has the property that

$$T_1 +_{X^{\text{univ}}} T_2 = T_1 + T_2 + T_1T_2 \cdot (\,\cdot\,)$$

This implies that $Q_{n,i}(T) \equiv \prod_{a\in\mathbb{F}_p}(T - (Y_{n,i} \bmod \pi_n)) = T^p - y_{n,i}^p \bmod \pi_n$. As the coefficients of $Q_{n,i}$ are in A_{n-1}, we also have $Q_{n,i}(T) \bmod \pi_{n-1} = T^p - y_{n,i}^p$. □

Proposition 3.1.2.

(i) *The residue field k_n is generated as a field over k_{n-1} by the elements $y_{n,2}, \ldots, y_{n,h}$, and $\{y_{n,2}^{i_2} \cdots y_{n,h}^{i_h} \mid 0 \le i_j \le p-1\}$ is a basis of k_n over k_{n-1}.*

(ii) *The elements $Y_{n,1}, \ldots, Y_{n,h}$ generate A_n as an algebra over A_{n-1}, and A_n is a free A_{n-1}-module with basis*

$$\{Y_{n,1}^{i_1} \cdots Y_{n,h}^{i_h} \mid \forall\, j \in \{1, \ldots, h\} : 0 \le i_j \le p-1\} ,$$

except if $n = 1$ in which case A_1 is free over A_0 with basis

$$\{Y_{n,1}^{i_1} \cdots Y_{n,h}^{i_h} \mid \forall\, j \in \{1, \ldots, h\} : 0 \le i_1 \le p-2,\ 0 \le i_j \le p-1 \textit{ for } j > 1\}.$$

Proof. (i). Recall that K_0 is the p-adic completion of $\widetilde{K}_0$. Since K_n' is finite over K_0', it follows that K_n is also the p-adic completion of $\widetilde{K}_n = K_n'\widetilde{K}_0$. By definition, $\widetilde{K}_0 = \bigcup_{\ell\ge0} K_{\ell,u}'$, cf. 2.2.11, and we thus have $\widetilde{K}_n = \bigcup_{\ell\ge0} K_n'K_{\ell,u}'$. Because the residue field does not change after passing to the completion, the residue field k_n of K_n is equal to the residue field of $\widetilde{K}_n$, and the residue field of $\widetilde{K}_n$ is the union of the residue fields of the $K_n'K_{\ell,u}'$. By definition, $K_{\ell,u}'$ is unramified over K_0' and its residue field is the separable closure $k_{\ell,\text{sep}}'$ of k_0' in k_ℓ'. By [4, 2.4.8] the residue field of the composite field $K_n'K_{\ell,u}'$ is thus equal to $k_n'k_{\ell,\text{sep}}'$, the composition of the residue fields. The union of these fields is then $k_n'k_0$. By the remark before 3.1.1, the field k_n' is generated over k_{n-1}' by $y_{n,i}$, $2 \le i \le h$, and k_n is thus generated over k_{n-1} by those same elements. By 3.1.1, these elements are of degree $\le p$, and because $[k_n : k_{n-1}] = p^{h-1}$, cf. 2.2.13(v), they are indeed of degree p.

(ii). We observe that A_{n-1} is a local ring with maximal ideal $\pi_{n-1}A_{n-1}$. Since $\pi_{n-1}A_n = \pi_n^p A_n = Y_{n,1}^p A_n$ (except if $n=1$ in which case $\pi_0 A_1 = pA_1 = \pi_1^{p-1}A_1$), we have a filtration

$$0 \subseteq (\pi_n^{p-1})/(\pi_n^p) \subseteq \ldots \subseteq (\pi_n)/(\pi_n^p) \subseteq A_n/(\pi_n^p) = A_n/\pi_{n-1}A_n$$

(and similarly when $n=1$, when we replace π_n^p by π_1^{p-1}).

Claim. $A_n/\pi_{n-1}A_n$ is generated as k_{n-1}-algebra by the set $\{\overline{Y}_{n,i} | 1 \le i \le h\}$, where $\overline{Y}_{n,i}$ denotes the image of $Y_{n,i} \in A_n$ modulo π_{n-1}.

Proof of the Claim. We show by descending induction on i that for $1 \le i \le p$ (resp. $1 \le i \le p-1$ if $n=1$) the map

$$k_{n-1}[X_2, \ldots, X_h] \longrightarrow (\pi_n^i)/(\pi_{n-1}), \tag{3.1}$$

induced by

$$X_j \mapsto \pi_n^i Y_{n,j} \bmod \pi_{n-1}, \quad 2 \le j \le h,$$

is surjective. When $i=p$ (or $i=p-1$ if $n=1$) there is nothing to show. Given $x \in (\pi_n^{i-1})/(\pi_{n-1})$, write $x = \pi_n^{i-1}y \bmod \pi_{n-1}$ with $y \in A_n$. Using part (i) we can write $y = f + \pi_n z$ with $f \in A_{n-1}[Y_{n,2}, \ldots, Y_{n,h}]$ and $z \in A_n$. Therefore, $x = \pi_n^{i-1}f + \pi_n^i z \bmod \pi_{n-1}$. Applying our induction hypothesis to z, we see that $\pi_n^i z$ is in the image of the map 3.1. This proves the surjectivity of 3.1 in the case $i-1$ instead of i. □

By 3.1.1, the degree of $Y_{n,1}$ over A_{n-1} is p (resp. $p-1$, if $n=1$). Thus the k_{n-1}-vector space $A_n/\pi_{n-1}A_n$ is generated by

$$\{\overline{Y}_{n,1}^{i_1} \cdots \overline{Y}_{n,h}^{i_h} \mid \forall\, j \in \{1, \ldots, h\} : 0 \le i_j \le p-1\}$$

(if $n=1$ then it suffices that $i_1 \le p-2$). By Nakayama's Lemma, this generating set can be lifted to a generating set of A_n as an A_{n-1}-module, which is to say that A_n is generated by the set

$$\{Y_{n,1}^{i_1} \cdots Y_{n,h}^{i_h} \mid \forall\, j \in \{1, \ldots, h\} : 0 \le i_j \le p-1\}$$

(if $n=1$ then it suffices that $i_1 \le p-2$) as an A_{n-1}-module. It is a general fact that A_n is free over A_{n-1} of degree $[K_n : K_{n-1}] = p^h$ (resp. $[K_1 : K_0] = (p-1)p^{h-1}$), cf. [11, Ch. II, §2, Prop. 3], and those elements must then be basis of A_n as A_{n-1}-module. □

Proposition 3.1.3. *For every $n \ge 0$ we have $[k_n : k_n^p] = p^{h-1}$. Therefore, each field K_n is a d-big local field in the sense of Scholl* [8, 1.1] *with $d = h-1$.*

Proof. Recall that $k_0' = \operatorname{Frac} \overline{\mathbb{F}}_p[\![u_1, \ldots, u_{h-1}]\!]$. Thus the extension $k_0'/(k_0')^p$ has degree p^{h-1}, because it is generated by the elements $u_1, \ldots, u_{h-1}$, each of which is of degree p over $(k_0')^p$ and thus totally inseparable over $(k_0')^p$.

Because $k'_{n,\mathrm{sep}}/k'_0$ is separable, so is $(k'_{n,\mathrm{sep}})^p/(k'_0)^p$. The fields $(k'_{n,\mathrm{sep}})^p$ and k'_0 are thus linearly disjoint over $(k'_0)^p$. This implies that

$$[k'_0(k'_{n,\mathrm{sep}})^p : (k'_0)^p] = p^{h-1}[(k'_{n,\mathrm{sep}})^p : (k'_0)^p] \\ = p^{h-1}[k'_{n,\mathrm{sep}} : k'_0] = [k'_{n,\mathrm{sep}} : (k'_0)^p]$$

and thus $k'_{n,\mathrm{sep}} = (k'_{n,\mathrm{sep}})^p k'_0$. Recall that $k_0 = \bigcup_n k'_{n,\mathrm{sep}}$. By what we have just shown, $k_0 = \bigcup_n (k'_{n,\mathrm{sep}})^p k'_0 = k_0^p k'_0$. As k_0^p is separable over $(k'_0)^p$ and k'_0 is totally inseparable over $(k'_0)^p$, these fields are linearly disjoint over $(k'_0)^p$, and we thus have

$$[k_0 : k_0^p] = [k_0^p \otimes_{(k'_0)^p} k'_0 : k_0^p] = [k'_0 : (k'_0)^p] = p^{h-1}\,.$$

Suppose towards induction that k_{n-1}/k_{n-1}^p has degree p^{h-1}. By 2.2.13 (v), the extension k_n/k_{n-1} has degree p^{h-1}, and via the isomorphism $x \mapsto x^p$, we conclude that the extension k_n^p/k_{n-1}^p also has degree p^{h-1}. By 3.1.2 we have

$$k_n^p = (k_{n-1}(y_{n,2}, \ldots, y_{n,h}))^p = k_{n-1}^p(y_{n,2}^p, \ldots, y_{n,h}^p)\,.$$

By 3.1.1, the reduction of the minimal polynomial of $Y_{n,i}$ over K_{n-1} to k_{n-1} is equal to $T^p - y_{n,i}^p$, and thus $y_{n,i}^p \in k_{n-1}$ for each i. It follows that $k_n^p = k_{n-1}^p(y_{n,2}^p, \ldots, y_{n,h}^p) \subseteq k_{n-1}$. As both have the same degree over k_{n-1}^p, we must have $k_n^p = k_{n-1}$. It then follows that $[k_n : k_n^p] = [k_n : k_{n-1}] = p^{h-1}$. □

We recall the definition of a strictly deeply ramified tower.

Definition 3.1.4 ([8, 1.3])**.** Let d be a non-negative integer, and let

$$L_\bullet = (L_0 \subseteq L_1 \subseteq L_2 \subseteq \ldots)$$

be a tower of d-big local fields. The tower $L_\bullet$ is called *strictly deeply ramified* if there exists an integer $n_0 \geq 0$ and an ideal $\xi \subseteq \mathcal{O}_{L_{n_0}}$ with $0 < v_p(\xi) \leq 1$ such that the following condition holds: for every $n \geq n_0$ the extension L_n/L_{n-1} has degree p^{d+1}, and there exists a surjection

$$\Omega_{\mathcal{O}_{L_n}/\mathcal{O}_{L_{n-1}}} \twoheadrightarrow (\mathcal{O}_{L_n}/\xi\mathcal{O}_{L_n})^{d+1}\,.$$

We now arrive at our first goal, namely the proof of Result 1.1.1.

Proposition 3.1.5. *The tower $(K_n)_n$ is strictly deeply ramified.*

Proof. By 2.2.13, we have $[K_n : K_{n-1}] = p^h$ for $n \geq 2$. It remains to show that for all $n \geq 2$, there exists a surjection $\Omega_{A_n \mid A_{n-1}} \to (A_n/\pi A_n)^h$. By 3.1.1, the minimal polynomial of $Y_{n,i}$ over K_{n-1} is

$$Q_{n,i}(T) = \prod_{a \in \mathbb{F}_p} \left(T - (Y_{n,i} +_{X^{\mathrm{univ}}} [a\pi^{n-1}]_{X^{\mathrm{univ}}}(Y_{n,1}))\right),$$

which has coefficients in A_{n-1}. It then follows that

$$\begin{aligned}\left[\frac{\mathrm{d}}{\mathrm{d}T}Q_{n,i}\right](Y_{n,i}) &= \sum_{a\in\mathbb{F}_p}\prod_{b\neq a}(Y_{n,i} - (Y_{n,i} +_{X^{\mathrm{univ}}} [b]_{X^{\mathrm{univ}}}(Y_{1,1}))) \\ &= \prod_{b\neq 0}(Y_{n,i} - (Y_{n,i} +_{X^{\mathrm{univ}}} [b]_{X^{\mathrm{univ}}}(Y_{1,1}))) \\ &= \prod_{b\neq 0}(Y_{n,i} - (Y_{n,i} + \widetilde{b}\pi_1 + \widetilde{b}\pi_1 Y_{n,i}\cdot(\,\cdot\,))) \\ &= \prod_{b\neq 0}(-\widetilde{b}\pi_1)(1+Y_{n,i}\cdot(\,\cdot\,))\,,\end{aligned}$$

where $(\,\cdot\,)$ is an element of A_n and $\widetilde{b}$ is a representative of b in $\mathbb{Z}_p$. As $-\widetilde{b}(1+Y_{n,i}(\,\cdot\,))$ is a unit, and K_1/K_0 has ramification index $p-1$, we have $|\frac{\mathrm{d}}{\mathrm{d}T}Q_{n,i}(Y_{n,i})| = |\pi_1^{p-1}| = |\pi|$. In particular,

$$\left[\frac{\mathrm{d}}{\mathrm{d}T}Q_{n,i}\right](Y_{n,i}) \in \pi A_n\,. \tag{3.2}$$

We now show that $\Omega_{A_n/A_{n-1}}$ is a free $A_n/\pi A_n$-module of rank h. As $Q_{n,i}(Y_{n,i}) = 0$, we have

$$0 = \mathrm{d}\Big(Q_i(Y_{n,i})\Big) = \left[\frac{\mathrm{d}}{\mathrm{d}T}Q_{n,i}\right](Y_{n,i})\cdot \mathrm{d}Y_{n,i} = \varepsilon\pi \mathrm{d}Y_{n,i}\,,$$

for some unit $\varepsilon \in A_n$. Therefore $\pi\mathrm{d}Y_{n,i} = 0$. Because the elements $Y_{n,i}$ generate A_n as an algebra over A_{n-1}, the $\mathrm{d}Y_{n,i}$ generate $\Omega_{A_n/A_{n-1}}$ as a module over A_n. Hence we have shown that π annihilates $\Omega_{A_n/A_{n-1}}$, and $\Omega_{A_n/A_{n-1}}$ is thus a module over $A_n/\pi A_n$. By the definition of the polynomials $Q_{n,i}$, the map

$$\theta : A_{n-1}[T_1,\dots,T_h]/(Q_{n,1}(T_1),\dots,Q_{n,h}(T_h)) \longrightarrow A_n\,,\ \ T_i \longmapsto Y_{n,i}\,,$$

is well defined, and by 3.1.2 it is surjective. By 3.1.1 the domain of θ is a free A_{n-1}-module of degree $[K_n : K_{n-1}]$, and so is the target of θ. Therefore, θ is an isomorphism of A_{n-1}-algebras.

Let $\widetilde{\theta}$ be the composition of

$$A_{n-1}[T_1,\dots,T_h] \longrightarrow A_{n-1}[T_1,\dots,T_h]/(Q_{n,1}(T_1),\dots,Q_{n,h}(T_h))$$

and θ. By 3.2 we have $\theta\big(\frac{\mathrm{d}}{\mathrm{d}T_i}Q_{n,i}(T_i)\big) \in \pi A_n$. This implies that the map

$$A_{n-1}[T_1,\dots,T_h] \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}T_i}} A_{n-1}[T_1,\dots,T_h] \xrightarrow{\widetilde{\theta}} A_n \longrightarrow A_n/\pi A_n$$

factors via $A_{n-1}[T_1,\dots,T_h]/(Q_{n,1}(T_1),\dots,Q_{n,h}(T_h)) \cong A_n$ and induces a A_{n-1}-linear derivation $\partial_i : A_n \to A_n/\pi A_n$. By the universal property of $\Omega_{A_n/A_{n-1}}$, there is a unique A_n-linear map $\psi_i : \Omega_{A_n/A_{n-1}} \to A_n/\pi A_n$ such that $\partial_i(a) = \psi_i(\mathrm{d}a)$ for all $a \in A_n$.

Suppose that in $\Omega_{A_n/A_{n-1}}$ we have a relation $\sum a_j \mathrm{d}Y_{n,j} = 0$ for some $a_j \in A_n/\pi A_n$. Applying ψ_i to this equation we find $0 = \sum a_j \psi_i(\mathrm{d}Y_{n,j}) = a_i$. Thus $\Omega_{A_n/A_{n-1}}$ is a free module over $A_n/\pi A_n$, with basis $\mathrm{d}Y_{n,1}, \dots, \mathrm{d}Y_{n,h}$. The tower $(K_n)_n$ is thus strictly deeply ramified. □

4. Is the Lubin–Tate Tower a Kummer tower?

In this section we investigate whether the Lubin–Tate tower of fields introduced above is a Kummer tower (as recalled in the introduction). We will only consider the case when the formal group $\mathbb{X}$ has height 2.

In this section we use the following convention. When we study group cohomology, we let a group G act on a abelian group A from the left: $G \times A \to A$, $(g, a) \mapsto g \cdot a$. Furthermore, we write a 1-cocycle c on G with values in A as $g \mapsto c_g$, i.e., c_g is the value of c on $g \in G$. The cocyle c then satisfies $c_{gh} = g \cdot c_h + c_g$.

4.1. Preliminaries on Galois cohomology. We begin by recalling the following elementary result:

Lemma 4.1.1 ([15, 6.2.2])**.** *Let $U = \langle u \rangle$ be a cyclic group of order d, and let M be an abelian U-module. There exists a canonical isomorphism*

$$\left\{ m \in M \;\middle|\; \sum_{j=1}^{d-1} u^j m = 0 \right\} \Big/ \Big\{ um - m \mid m \in M \Big\} \longrightarrow H^1(U, M)$$

$$\overline{m} \longmapsto \left[c(m)_{u^i} = \sum_{j=0}^{i-1} u^j \cdot m \right].$$

Furthermore, if the action of U on M is trivial and $dM = 0$, then there exists a canonical isomorphism $M \simeq H^1(U, M)$.

Corollary 4.1.2. *Suppose $p > 2$. Let $U = (\mathbb{Z}/p^n\mathbb{Z})^\times$ and $M = \mu_{p^n}$. The group U acts on M by setting $u \cdot \zeta = \zeta^u$. Then $H^1(U, M) = 0$.*

Proof. It is well known (and easy to prove) that the group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ is isomorphic to $\mu_{p-1} \times \mathbb{Z}/p^{n-1}\mathbb{Z}$ (if p is odd), and it is thus itself cyclic of order $d = (p-1)p^{n-1}$. Let $u \in \mathbb{Z} \setminus p\mathbb{Z}$ be such that $u + p^n\mathbb{Z} \in U$ is a generator. Since $u \not\equiv 1 \bmod p$, we have $p \nmid u - 1$. On the other hand, $p^n \mid u^d - 1$, so for all $\zeta \in \mu_{p^n}$, we have

$$\prod_{j=0}^{d-1} \zeta^{u^j} = \zeta^{(u^d-1)/(u-1)} = 1\,,$$

and so

$$\left\{ \zeta \in \mu_{p^n} \;\middle|\; \prod_{j=0}^{d-1} u^j \cdot \zeta = 1 \right\} = M\,.$$

On the other hand, as $p \nmid u-1$, we have $u-1 \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, so

$$\{\zeta^{u-1} \mid \zeta \in \mu_{p^n}\} = M\,.$$

Thus by Lemma 4.1.1, we have $H^1(U,M) = 0$. □

Proposition 4.1.3. *Let μ be a finite cyclic group of order k which we will write multiplicatively. Put $U = (\mathbb{Z}/k\mathbb{Z})^\times$ and $E = \mathbb{Z}/k\mathbb{Z}$, and let $G = U \ltimes E$ be the semi-direct product of U with E, with multiplication given by*

$$(\bar{u}_1, \bar{e}_1)\cdot(\bar{u}_2,\bar{e}_2) = (\bar{u}_1\bar{u}_2, \bar{u}_1\bar{e}_2 + \bar{e}_1)\,,$$

where $\bar{x}$ denotes the class of $x \in \mathbb{Z}$ modulo k. The group G acts on μ via U, with $(\bar{u},\bar{e})\cdot\zeta = \zeta^u$ for $(\bar{u},\bar{e}) \in G = (\mathbb{Z}/k\mathbb{Z})^\times \ltimes \mathbb{Z}/k\mathbb{Z}$, $\zeta \in \mu$. Then there exists a split exact sequence

$$1 \longrightarrow H^1(U,\mu) \longrightarrow H^1(G,\mu) \longrightarrow \mu \to 1,$$

where the splitting is given by mapping an element $\zeta \in \mu$ to the cohomology class of the 1-cocycle $\widetilde{c}(\zeta)_{(\bar{u},\bar{e})} = \zeta^e$. Further, if $k = p^n$ for some prime $p > 2$, then $H^1(G,\mu) \simeq \mu$.

Proof. We begin with the inflation-restriction exact sequence

$$0 \longrightarrow H^1(U,\mu) \longrightarrow H^1(G,\mu) \longrightarrow H^0(U, H^1(E,\mu))\,,$$

cf. [15, 6.8.3]. As E acts trivially on μ, we have, by Lemma 4.1.1, $H^1(E,\mu) \simeq \mu$, where the element $\zeta \in \mu$ corresponds to the cocycle $c(\zeta)_{\bar{e}} = \zeta^e$, where $\bar{e} \in E = \mathbb{Z}/k\mathbb{Z}$. The group U acts on $H^1(E,\mu)$ by

$$(u\cdot c(\zeta))_{\bar{e}} = u \cdot c(\zeta)_{\overline{u^{-1}e}} = (\zeta^{u^{-1}e})^u = \zeta^e = c(\zeta)_{\bar{e}}\,,$$

which is to say that the action of U on $H^1(E,\mu)$ is trivial. Thus

$$H^0(U, H^1(E,\mu)) = H^1(E,\mu) \simeq \mu\,.$$

Define the splitting map as in the statement of the proposition. We check that it satisfies the cocycle condition:

$$\widetilde{c}(\zeta)_{(\bar{u}_1,\bar{e}_1)(\bar{u}_2,\bar{e}_2)} = \widetilde{c}(\zeta)_{(\bar{u}_1\bar{u}_2,\bar{u}_1\bar{e}_2+\bar{e}_1)} = \zeta^{u_1e_2+e_1} = (\widetilde{c}(\zeta)_{(\bar{u}_2,\bar{e}_2)})^{u_1}\cdot\widetilde{c}(\zeta)_{(\bar{u}_1,\bar{e}_1)}.$$

It is straightforward to check that the map $\mu \to H^1(G,\mu)$, $\zeta \mapsto \widetilde{c}(\zeta)$, is a group homomorphism and that it is a right inverse for the map $H^1(G,\mu) \to \mu$. Finally, if $k = p^n$, then by Corollary 4.1.2 we have $H^1(U,\mu) = 0$, and thus $H^1(G,\mu) \simeq \mu$. □

Proposition 4.1.4. *Let k be a positive integer. Suppose L/K is a Galois extension of fields with Galois group $G = U \ltimes E$, where $U = (\mathbb{Z}/k\mathbb{Z})^\times$ and $E = \mathbb{Z}/k\mathbb{Z}$, and the multiplication in G is given by $(\bar{u}_1,\bar{e}_1)\cdot(\bar{u}_2,\bar{e}_2) = (\bar{u}_1\bar{u}_2,\bar{u}_1\bar{e}_2+\bar{e}_1)$, where $\bar{x} = x \bmod k$. Suppose L contains a primitive k-th root of unity (and therefore all k-th roots of unity), and suppose G acts on the group μ_k of k-th roots of unity by $(\bar{u},\bar{e})\cdot\zeta = \zeta^u$. Then there exists a $t \in K^\times$ such that $L = K(\mu_k, t^{1/k})$.*

Proof. Let $M = L^E$. Then $\mathrm{Gal}(L \mid M) = E$ is cyclic of order k. As E acts trivially on μ_k, we have $\mu_k \subseteq M$. By Kummer theory, L/M is a Kummer extension of the form $L = M(t^{1/k})$ for some $t \in M^\times$. We now want to show that one can find such an element t already in K.

It suffices to show that there exists a $t \in K^\times$ which is a k-th power in L, $t = s^k$, on which the Galois group acts by $(\bar{u}, \bar{e}) \cdot s = \zeta^e s$ for some primitive k-th root of unity ζ, since in this case $[M(s) : M] = |E|$, and thus $M(s) = L$.

Consider the exact sequence of G-modules

$$1 \longrightarrow \mu_k \longrightarrow L^\times \longrightarrow (L^\times)^k \to 1.$$

From this we get the sequence of cohomology groups

$$H^0(G, L^\times) \longrightarrow H^0(G, (L^\times)^k) \longrightarrow H^1(G, \mu_k) \longrightarrow H^1(G, L^\times).$$

But the zero-th cohomology group on the left is just $K^\times$, and by Hilbert's Theorem 90 the group $H^1(G, L^\times)$ is trivial, so we obtain the sequence

$$K^\times \longrightarrow (L^\times)^k \cap K^\times \longrightarrow H^1(G, \mu_k) \longrightarrow 1\,.$$

In particular, the map $(L^\times)^k \cap K^\times \to H^1(G, \mu_k)$ is surjective. As in Proposition 4.1.3, the map $(\bar{u}, \bar{e}) \mapsto \zeta^e$ is a 1-cocycle, and we have a group homomorphism

$$\mu_k \longrightarrow H^1(G, \mu_k)\,, \quad \zeta \longmapsto \widetilde{c}(\zeta) = [(\bar{u}, \bar{e}) \mapsto \zeta^e]\,.$$

Suppose that there exists a $\xi \in \mu_k$ such that $\xi^e = \xi^{u-1}$ for all $(\bar{u}, \bar{e}) \in G$, i.e., the map $(\bar{u}, \bar{e}) \mapsto \xi^e$ is a 1-coboundary. Then $\xi^1 = \xi^{1-1} = 1$. Thus the map $\zeta \mapsto \widetilde{c}(\zeta)$ is injective.

Let ζ be a primitive k-th root of unity. As the map $(L^\times)^k \cap K^\times \to H^1(G, \mu_k)$ is surjective, there exists a $t \in (L^\times)^k \cap K^\times$ which maps to the 1-cocycle $\widetilde{c}(\zeta)$. Let $s' \in L^\times$ be such that $(s')^k = t$. By definition, under the map

$$(L^\times)^k \cap K^\times = H^0(G, (L^\times)^k) \to H^1(G, \mu_k)\,,$$

the element t maps to the cohomology class of the 1-cocyle $\left[g \mapsto \frac{g(s')}{s'}\right]$ $(g \in G)$. Thus $\widetilde{c}(\zeta)$ and $\left[g \mapsto \frac{g(s')}{s'}\right]$ must be equal up to a coboundary $\left[g \mapsto \frac{g(\xi)}{\xi}\right]$, for some $\xi \in \mu_k$. That is to say: $\frac{g(s')}{s'} = \frac{g(\xi)}{\xi} \cdot \widetilde{c}(\zeta)_g$ for all $g \in G$. Putting $s = s'\xi^{-1}$ we still have $s^k = t$ and $\frac{g(s)}{s} = \widetilde{c}(\zeta)_g$ for all $g \in G$, i.e., $\frac{(\bar{u},\bar{e})(s)}{s} = \zeta^e$, which is equivalent to $(\bar{u}, \bar{e})(s) = \zeta^e \cdot s$. Hence, as mentioned above, s generates L over $K(\mu_k)$. ☐

4.2. Applications to the Lubin–Tate tower. In this section we consider the fields K_n and $\widetilde{K}_n$ constructed in Section 2.2.11, but we assume throughout that $\mathcal{O} = \mathbb{Z}_p$ and $h = 2$. The following is Result 1.1.2 (i) of the introduction.

Corollary 4.2.1. *The extension K_n/K_0 is a Kummer extension, i.e., there is a $t_n \in K_0$ such that $K_n = K_0(\mu_{p^n}, \sqrt[p^n]{t_n})$. The same is true for $\widetilde{K}_n/\widetilde{K}_0$.*

Proof. By Proposition 2.2.13, K_n/K_0 and $\widetilde{K}_n/\widetilde{K}_0$ are both Galois extensions with Galois group $(\mathbb{Z}/p^n\mathbb{Z})^\times \ltimes \mathbb{Z}/p^n\mathbb{Z}$. Let $\breve{\mathbb{Q}}_p$ be the completion of the maximal unramified extension of $\mathbb{Q}_p$. It has been shown in [13, Cor. 3.4] that the field K'_n contains $\breve{\mathbb{Q}}_p(\mu_{p^n})$, which is a Lubin–Tate extension for the multiplicative formal group over $\breve{\mathbb{Q}}_p$. Since $K'_n \subseteq \widetilde{K}_n \subseteq K_n$, both fields K_n and $\widetilde{K}_n$ contain μ_{p^n}. By loc. cit., the action of

$$\mathrm{Gal}(K'_n \mid K'_0) \simeq \left\{ g_{a,b,d} := \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a, d \in (\mathbb{Z}/p^n\mathbb{Z})^\times \,,\ b \in \mathbb{Z}/p^n\mathbb{Z} \right\},$$

cf. 2.2.10, on μ_{p^n} is given by $g_{a,b,d}.\zeta = \zeta^{ad}$, $\zeta \in \mu_{p^n}$. The subgroup $\mathrm{Gal}(\widetilde{K}_n \mid \widetilde{K}_0) = \mathrm{Gal}(K_n \mid K_0)$ of $\mathrm{Gal}(K'_n \mid K'_0)$ consists precisely of those $g_{a,b,d}$ with $d = 1$. Therefore, elements $g_{a,b,1} \in \mathrm{Gal}(\widetilde{K}_n \mid \widetilde{K}_0) = \mathrm{Gal}(K_n \mid K_0)$ act on $\zeta \in \mu_{p^n}$ as $g_{a,b,1}.\zeta = \zeta^a$. We are thus in the situation of Proposition 4.1.4 from which our assertion follows. $\square$

4.2.2. We now turn to the question whether the tower $K_\bullet$ is a Kummer tower. As mentioned in the introduction, $K_\bullet$ being a Kummer means that there is a $t \in K_0$ such that for every $n \geq 0$, one has $K_n = K_0(\mu_{p^n}, \sqrt[p^n]{t})$. Our methods, however, are such that we can only investigate this question under the restriction that t lies in the field $\widetilde{K}_0$, of which K_0 is the p-adic completion. The point is that the $\widetilde{K}_n/K'_0$ are Galois extensions whereas K_0/K'_0 is not, cf. 2.2.12, and our methods are tied to the fact that $\widetilde{K}_n/K'_0$ is a Galois extension.[4]

Recall the field $K'_{\ell,u} \subseteq K'_\ell$ which is the maximal unramified subextension of K'_ℓ/K'_0, cf. 2.2.11. We have, by definition, $\widetilde{K}_0 = \bigcup_{m \geq 0} K'_{\ell,u}$, and $\widetilde{K}_n = \widetilde{K}_0 K'_n$. We recall from 2.2.13 (ii) that the universal Drinfeld basis induces an isomorphism

$$\begin{aligned} G_{n,\ell} &:= \mathrm{Gal}(\widetilde{K}_n \mid K'_{\ell,u}) \\ &\simeq \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a \in (\mathbb{Z}/p^n\mathbb{Z})^\times \,,\ b \in \mathbb{Z}/p^n\mathbb{Z} \,,\ d \in 1 + p^\ell \mathbb{Z}_p \right\}, \end{aligned}$$

[4] A more sophisticated approach might possibly give the stronger result about the non-existence of such a t in K_0 (and not only in $\widetilde{K}_0$), but we do not know how to do this.

where $1+p^\ell\mathbb{Z}_p$ is to be interpreted as $\mathbb{Z}_p^\times$ when $\ell=0$. In the remainder of this paper we use the notation $g_{a,b,d} = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, as in the proof of 4.2.1.

Proposition 4.2.3. *Suppose $p>2$. The group $G_{n,\ell} = \mathrm{Gal}(\widetilde{K}_n \mid K'_{\ell,u})$ acts on $\mu_{p^n} \subseteq K'_n \subseteq \widetilde{K}_n$ by $g_{a,b,d}\cdot\zeta = \zeta^{ad}$ for $\zeta\in\mu_{p^n}$. Furthermore,*

$$H^1(G_{n,\ell},\mu_{p^n}) \simeq \{\zeta\in\mu_{p^n} \mid \forall\, d\in 1+p^\ell\mathbb{Z}_p : \zeta^{d^2}=\zeta\,\}\,.$$

If $\ell>0$ or $p>3$, then $H^1(G_{n,\ell},\mu_{p^n}) = \mu_{p^{\min\{n,\ell\}}}$. If $p=3$ and $\ell=0$ then $H^1(G_{n,\ell},\mu_{p^n}) = \mu_{p^{\min\{n,1\}}}$.

Proof. As we have already recalled in the proof of Proposition 4.2.1, the first assertion about the action of $G_{n,\ell}$ on μ_{p^n} is [13, Cor. 3.4]. Let

$$G_n := \mathrm{Gal}(\widetilde{K}_n \mid \widetilde{K}_0) = \{g_{a,b,d}\in G_{n,\ell} \mid d=1\} \simeq (\mathbb{Z}/p^n\mathbb{Z})^\times \ltimes \mathbb{Z}/p^n\mathbb{Z}\,;$$

cf. 2.2.13 (iii). Then G_n is a normal subgroup of $G_{n,\ell}$. Put $D=G_{n,\ell}/G_n \simeq 1+p^\ell\mathbb{Z}_p$. Since $\mu_{p^n}^{G_n}=\{1\}$, it follows from the inflation-restriction sequence [15, 6.8.3] that

$$H^1(G_{n,\ell},\mu_{p^n}) \simeq H^0(D, H^1(G_n,\mu_{p^n}))\,.$$

By Proposition 4.1.3, we have $H^1(G_n,\mu_{p^n})\simeq\mu_{p^n}$, where $\zeta\in\mu_{p^n}$ corresponds to the class of $c(\zeta)_{g_{a,b,1}} = \zeta^b$. The projection $G_{n,\ell}\twoheadrightarrow D$ has the section $D\to G_{n,\ell}$, $d\mapsto \widetilde{d} := g_{1,0,d}$. Then $\widetilde{d}^{-1}g_{a,b,1}\widetilde{d} = g_{a,bd,1}$, and so

$$(\widetilde{d}\cdot c(\zeta))_{g_{a,b,1}} = \widetilde{d}\cdot c(\zeta)_{\tilde{d}^{-1}g_{a,b,1}\tilde{d}} = \widetilde{d}\cdot c(\zeta)_{g_{a,bd,1}} = \widetilde{d}\cdot\zeta^{bd} = \zeta^{bd^2}\,.$$

Thus the cocycle is fixed by D if and only if $\zeta^{d^2}=\zeta$ for all $d\in 1+p^\ell\mathbb{Z}_p$. Suppose in the following that this is the case.

Case $\ell>0$. Then $(1+p^\ell\mathbb{Z}_p)^2 = 1+p^\ell\mathbb{Z}_p$ (as we assume $p>2$), and hence $\zeta\in\mu_{p^\ell}\cap\mu_{p^n} = \mu_{p^{\min\{n,\ell\}}}$.

Case $p>3$ and $\ell=0$. Then there is $d\in\mathbb{Z}_p^\times$ such that $p\nmid d^2-1$ (e.g., $d=2$), and hence $\zeta=1$.

Case $p=3$ and $\ell=0$. Then $p\mid d^2-1$ for all $d\in\mathbb{Z}_p^\times$ and $p^2\nmid d^2-1$ for some $d\in\mathbb{Z}_p^\times$ (e.g. $d=2$). Hence $\zeta\in\mu_p\cap\mu_{p^n} = \mu_{p^{\min\{n,1\}}}$. □

Remark 4.2.4. In the proof given above we have used the hypothesis $p>2$ when applying 4.1.3, and 4.1.3 in turn relies on 4.1.2, where this hypothesis is also made. If one strengthened 4.1.3 and 4.1.2 by including results pertaining to the case $p=2$ (which would certainly be possible), then one should also be able to obtain a description of $H^1(G_{n,\ell},\mu_{p^n})$ in the case $p=2$.

Lemma 4.2.5. *The action of $G_{n,\ell} = \mathrm{Gal}(\widetilde{K}_n \mid K'_{\ell,u})$ on $\widetilde{K}_n$ extends by continuity to an action on K_n, and $H^0(G_{n,\ell},K_n) = K'_{\ell,u}$.*

Proof. As the Galois automorphisms in $G_{n,\ell}$ are p-adically continuous, they extend uniquely to the p-adic completion K_n of $\widetilde{K}_n$. Consider the tautological exact sequence

$$1 \longrightarrow G_n = \mathrm{Gal}(\widetilde{K}_n \mid \widetilde{K}_0) \longrightarrow G_{n,\ell} = \mathrm{Gal}(\widetilde{K}_n \mid K'_{\ell,u}) \\ \longrightarrow D := \mathrm{Gal}(\widetilde{K}_0 \mid K'_{\ell,u}) \longrightarrow 1\,.$$

Recall that the canonical map $\mathrm{Gal}(K_n \mid K_0) \to \mathrm{Gal}(\widetilde{K}_n \mid \widetilde{K}_0)$ is an isomorphism, by Proposition 2.2.13 (iv). We therefore have

$$H^0(G_{n,\ell}, K_n) = H^0(D, H^0(G_n, K_n)) = H^0(D, K_0)\,.$$

Recall that K_0 is, by definition, the p-adic completion of $\widetilde{K}_0$, cf 2.2.11. Recall also that we defined $\widetilde{k}_0 = \bigcup_n k'_{n,\mathrm{sep}}$ to be the residue field of $\widetilde{K}_0$ (which is also the residue field of K_0) (cf. 2.2.11). The field $\widetilde{k}_0$ is a Galois extension of $k'_{\ell,\mathrm{sep}}$ whose Galois group is canonically isomorphic to $D \simeq 1 + p^\ell\mathbb{Z}_p$. If we put $K''_0 = H^0(D, K_0)$, then group D acts trivially on K''_0, and it therefore also acts trivially on its residue field, which must then be $k'_{\ell,\mathrm{sep}}$. The field K''_0 is therefore a discretely valued complete subfield of K_0 with residue field k'_0 and must then be equal to K'_0. ☐

Proposition 4.2.6. *Suppose $p > 2$. There is no $t \in \widetilde{K}_0$ such that for all sufficiently large $n \gg 0$ one has $K_n = K_0(\mu_{p^n}, \sqrt[p^n]{t})$.*

Proof. Suppose on the contrary that such a $t \in \widetilde{K}_0$ exists. Then it is contained in some subfield $K_{\ell,u} \subseteq \widetilde{K}_0$. We may increase ℓ and assume henceforth that $\ell > 0$, and we choose $n > \ell$. By 4.2.5, the action of $G_{n,\ell} = \mathrm{Gal}(\widetilde{K}_n \mid K'_{\ell,u})$ on $\widetilde{K}_n$ extends by continuity to an action on K_n. We can thus consider the 1-cocycle

$$G_{n,\ell} \longrightarrow \mu_{p^n}\,, \quad s \longmapsto s\big(t^{1/p^n}\big)/t^{1/p^n}\,.$$

Because $H^1(G_{n,\ell}, \mu_{p^n}) = \mu_{p^\ell}$, cf. 4.2.3, the map $s \mapsto (s(t^{1/p^n})/t^{1/p^n})^{p^\ell}$ is a coboundary, so there exists a $\zeta \in \mu_{p^n}$ such that

$$s\big(t^{1/p^{n-\ell}}\big)/t^{1/p^{n-\ell}} = s(\zeta)/\zeta\,.$$

If $s \in \mathrm{Gal}(\widetilde{K}_n \mid K_{\ell,u}(\mu_{p^n}))$, then we further have $s(t^{1/p^{n-\ell}})/t^{1/p^{n-\ell}} = 1$. Using 4.2.5 again, we conclude that $t^{1/p^{n-\ell}} \in K_{\ell,u}(\mu_{p^n})$. It then follows that $[K'_{\ell,u}(\mu_{p^n}, t^{1/p^n}) : K'_{\ell,u}(\mu_{p^n})] \leq p^\ell$, and thus

$$\Big[K_0\big(\mu_{p^n}, t^{1/p^n}\big) : K_0\Big] \leq \Big[K_{\ell,u}\big(\mu_{p^n}, t^{1/p^n}\big) : K_{\ell,u}\Big] \leq (p-1)p^{n-1}p^\ell\,.$$

But $[K_n : K_0] = (p-1)p^{n-1}p^n$. Thus there cannot exist such an element $t \in \widetilde{K}_0$. ☐

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