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Résumé. Pour toute unité d'un corps de nombres réel K de degré $n+1$, il n'existe qu'un nombre fini de n-uples dans K^n qui ont un développement purement périodique par l'algorithme de Jacobi–Perron associé à cette unité. Nous donnons un algorithme explicite pour calculer tous ces n-uples pour tous les degrés $n+1$ et toutes les unités de K.

Abstract. For any unit ε in a real number field K of degree $n+1$, there exist only a finite number of n-tuples in K^n which have a purely periodic expansion by the Jacobi–Perron algorithm associated with this unit. We give an explicit algorithm to compute all these n-tuples for any degree $n+1$ and any unit of K.

1. Introduction

One of the generalizations of the continued fraction algorithm to higher dimensions is the Jacobi–Perron Algorithm (JPA). Its main interest lies in the great simplicity of its definition. For $n = 1$, we get the continued fraction algorithm.

The continued fraction expansion of a real number α is ultimately periodic if and only if α is a real quadratic number. Moreover, a quadratic number α has a purely periodic expansion if and only if $\alpha > 1$ and its conjugate $\overline{\alpha}$ satisfies $-1 < \overline{\alpha} < 0$ (we say that α is reduced) [3, p. 50]. In this case, if l denotes a period length and $\left(\frac{p_i}{q_i}\right)$ $(i \geq 0)$ the sequence of convergents of α then we have:

$$\begin{pmatrix} p_l & p_{l-1} \\ q_l & q_{l-1} \end{pmatrix} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \varepsilon \begin{pmatrix} \alpha \\ 1 \end{pmatrix}, \tag{1.1}$$

where $\varepsilon = q_l\alpha + q_{l-1}$ is a unit of $\mathbb{Q}(\alpha)$.

Proof. The characteristic polynomial of the 2×2 matrix on the left-hand side of (1.1) is equal to $x^2 - x\,(p_l + q_{l-1}) + p_l q_{l-1} - q_l p_{l-1}$. P has a root $\varepsilon > 1$ which is a unit since $P(0) = \pm 1$. Then, since the expansion of α is

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purely periodic with a period of length l, we have $\alpha = \frac{p_l\alpha+p_{l-1}}{q_l\alpha+q_{l-1}}$, it is easy to verify that $\varepsilon = q_l\alpha + q_{l-1}$. □

This property is also valid for purely periodic JPA expansions with $n > 1$. It is called the Hasse–Bernstein Theorem [5] and ε is called a Hasse–Bernstein unit. E. Dubois, A. Fahrane and R. Paysant-Le Roux [6] proved that, for $n = 2$, $\deg(K) = 3$, the Hasse–Bernstein units are always Pisot numbers and that this is not always true for $n \geq 3$.

Several authors have presented a few classes of periodic JPA expansions: see [7], [8]. In [9] C. Levesque and G. Rhin presented new periodic expansions in cubic number fields depending on parameters. The length of the period tends to infinity with the parameters. B. Adam [1] generalized [9] and P. Voutier [11] gave new periodic expansions in number fields depending on parameters.

For $n = 1$, it is well-known that if ε is a unit of a real quadratic field K, then there exists only a finite number of reduced elements in K whose continued fraction expansion is associated with ε by (1.1).

In [2] B. Adam and G. Rhin generalised this result as follows. Let K be a real number field of degree $n+1$ and ε a unit of K, there exist only a finite number of elements in K^n whose JPA expansion is purely periodic and associated with ε (as defined in Definition 4).

Moreover in the same paper, for $n = 2$, B. Adam and G. Rhin [2] gave an explicit algorithm which gives, for any unit, all the purely periodic expansions associated with this unit.

Here we give an explicit algorithm to compute all these expansions for any real number field K and any unit in K.

In Section 2, we recall the basic facts that are needed in the other parts of the paper. In Section 3, we give our algorithm and in Section 4, we provide some information about the application of this algorithm to some specific examples.

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2. Preliminaries: JPA and Hasse–Bernstein theorem

We assume now that the $n+1$ real numbers $1, \alpha_1, \alpha_2, \dots, \alpha_n$ are $\mathbb{Q}$-linearly independent.

Definition 1. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a vector in $\mathbb{R}^n$ ($n \geq 1$). The *Jacobi–Perron Algorithm* (*JPA*) expansion [10] of α is given by the two sequences:

- $(a^{(\nu)})_{\nu\geq 0}$ in $\mathbb{Z}^n$ where $a^{(\nu)} = (a_1^{(\nu)}, a_2^{(\nu)}, \dots, a_n^{(\nu)})$;
- $(\alpha^{(\nu)})_{\nu\geq 0}$ in $\mathbb{R}^n$ where $\alpha^{(\nu)} = \left(\alpha_1^{(\nu)}, \alpha_2^{(\nu)}, \dots, \alpha_n^{(\nu)}\right)$

defined by:

$$\begin{cases} \alpha^{(0)} = \alpha; \\ a_i^{(\nu)} = \left[\alpha_i^{(\nu)}\right] \quad \text{for } 1 \le i \le n \text{ and } \nu \ge 0; \\ \text{if } \alpha_1^{(\nu)} \ne a_1^{(\nu)},\ \alpha_n^{(\nu+1)} = \frac{1}{\alpha_1^{(\nu)} - a_1^{(\nu)}};\ \alpha_i^{(\nu+1)} = \frac{\alpha_{i+1}^{(\nu)} - a_{i+1}^{(\nu)}}{\alpha_1^{(\nu)} - a_1^{(\nu)}} \ \text{ for } 1 \le i < n. \end{cases}$$

where $[x]$ is the greatest integer part of x. We define $a_0^{(\nu)} = 1$ and $\alpha_0^{(\nu)} = 1$ for all ν.

Definition 2. The JPA expansion of α is periodic if there exist two integers $k \ge 0$ and $l > 0$ such that $a_i^{(k+\nu)} = a_i^{(k+\nu+l)}$ for all $\nu \ge 0$ and $0 < i \le n$. l is called a period length.
If k and l are the smallest integers which satisfy this equality, then k is called the preperiod length and l is called the primitive period length. If $k = 0$, then the expansion is said to be purely periodic.

Remark. If the Jacobi–Perron Algorithm (JPA) expansion [10, pp. 4–5] of α is purely periodic, then the vectors $a^{(\nu)} = \left(a_1^{(\nu)}, a_2^{(\nu)}, \dots, a_n^{(\nu)}\right)$ in $\mathbb{Z}^n$ satisfy the following Perron Conditions $\mathbf{P}_1$:

$$\begin{cases} \text{for } \nu \ge 0 \text{ and } 0 \le i \le n, \\ \left(a_n^{(\nu)}, a_{n-1}^{(\nu+1)}, \dots, a_{n-i}^{(\nu+i)}\right) \ge \left(a_i^{(\nu)}, a_{i-1}^{(\nu+1)}, \dots, a_1^{(\nu+i-1)}, a_0^{(\nu+i)}\right) \\ \text{in lexicographical order.} \end{cases}$$

Definition 3. We define the sequence $A^{(\nu)} = \left(A_0^{(\nu)}, A_1^{(\nu)}, A_2^{(\nu)}, \dots, A_n^{(\nu)}\right)$ of vectors in $\mathbb{Z}^n$ by:

$$\begin{cases} \text{for } 0 \le i \le n \text{ and } 0 \le j \le n,\ A_i^{(j)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \\ \text{for } \nu \ge 0 \text{ and } 0 \le i \le n, \\ A_i^{(\nu+n+1)} = A_i^{(\nu)} + a_1^{(\nu)} A_i^{(\nu+1)} + \dots + a_n^{(\nu)} A_i^{(\nu+n)}. \end{cases}$$

We have the following formulas, which were shown by Perron [10]:

- $\alpha_i = \dfrac{\sum_{j=0}^n A_i^{(\nu+j)} \alpha_j^{(\nu)}}{\sum_{j=0}^n A_0^{(\nu+j)} \alpha_j^{(\nu)}}$ for all $\nu \ge 0$ and $1 \le i \le n$,
- by writing

$$\mathcal{A}_\nu = \begin{pmatrix} a_n^{(\nu)} & 1 & 0 & \dots & 0 \\ a_{n-1}^{(\nu)} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_1^{(\nu)} & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}, \tag{2.1}$$

we have

$$\mathcal{A}^{(\nu)} = \mathcal{A}_0\mathcal{A}_1 \cdots \mathcal{A}_{\nu-1} = \begin{pmatrix} A_n^{(\nu+n)} & \dots & \dots & A_n^{(\nu)} \\ \vdots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \vdots \\ A_0^{(\nu+n)} & \dots & \dots & A_0^{(\nu)} \end{pmatrix}.$$

Definition 4. If the JPA expansion of $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is purely periodic with a period length l, then

$$\varepsilon = A_0^{(l)} + \alpha_1 A_0^{(l+1)} + \cdots + \alpha_n A_0^{(l+n)} = \prod_{\nu=0}^{l-1} \alpha_n^{(\nu)}.$$

is a unit of $K = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$. If l is the primitive period length, this is the *Hasse–Bernstein unit* [4, Theorem 17, p. 109] and we say that this JPA expansion is *associated with the unit* ε.

We have

$$\mathcal{A}^{(l)} \begin{pmatrix} \alpha_n \\ \vdots \\ \alpha_1 \\ 1 \end{pmatrix} = \epsilon \begin{pmatrix} \alpha_n \\ \vdots \\ \alpha_1 \\ 1 \end{pmatrix}.$$

Notice that this is a higher dimensional version of the 2-dimensional relationship in (1.1).

Definition 5. We say that a matrix $\mathcal{A}$ is a *JPA matrix* of length $l > 0$ if there is a finite sequence of integers $(a_i^{(\nu)})$, $0 \le \nu \le l-1$, $0 < i \le n$ such that $\mathcal{A} = \mathcal{A}_0\mathcal{A}_1 \cdots \mathcal{A}_{l-1}$ where the matrices $\mathcal{A}_i$ are defined by (2.1) and the integers $a_i^{(\nu)}$ satisfy the Perron conditions $\mathbf{P}_1$. We say that each $\mathcal{A}_i$ is an *elementary JPA matrix.*

Definition 6. We say that a matrix $\mathcal{A}$ is a *JPA period matrix* if $\mathcal{A}$ is a *JPA matrix* of length $l > 0$ and if the infinite sequence of integers $(a_i^{(j)})$, $j \ge 0$, $0 < i \le n$ defined by $a_i^{(kl+\nu)} = a_i^{(\nu)}$, for $0 \le \nu \le l-1$, $0 < i \le n$ and $k \ge 1$ satisfies the Perron conditions $\mathbf{P}_1$.

Observe that for any elementary JPA matrix $\mathcal{A}$, the first column of $\mathcal{A}$, say $(a_n, \dots, a_1, 1)^T$, satisfies the Perron conditions $\mathbf{P}_1$, so we have $a_n \ge a_i \ge 0$ for all $1 \le i \le n-1$ and $a_n \ge 1$.

The following theorem has been proved in [2] as Theorem 3.1.

Theorem 7. *Let K be a real number field of degree $n+1$ and $\varepsilon > 1$ a unit of K. There exist only a finite number of elements in K^n whose JPA expansion is purely periodic associated with ε.*

We search for a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in K^n such that the JPA expansion of α is purely periodic and associated with ε, that is to say we search for the JPA period matrices $\mathcal{A}$ such that

$$\mathcal{A} \begin{pmatrix} \alpha_n \\ \vdots \\ \alpha_1 \\ 1 \end{pmatrix} = \varepsilon \begin{pmatrix} \alpha_n \\ \vdots \\ \alpha_1 \\ 1 \end{pmatrix}.$$

With the previous notations we have the following lemma.

Lemma 8. *Suppose that $\mathcal{A}$ is a JPA matrix of length k, where k is an integer satisfying $1 \leq k \leq n+1$. The first k columns of $\mathcal{A}$, denoted here as $\mathcal{A}_1, \ldots, \mathcal{A}_k$, each satisfy the property $\mathbf{P}_2$: $\mathcal{A}_j^1 \geq \mathcal{A}_j^{n+1} \geq 1$, and $\mathcal{A}_j^1 \geq \mathcal{A}_j^i \geq 0$ for $2 \leq i \leq n$. Denote by $\mathcal{E}_j$ the j-th column of the $(n+1) \times (n+1)$ identity matrix. Then, the last $n+1-k$ columns of $\mathcal{A}$ are equal to $\mathcal{E}_{j-k}$ for $k+1 \leq j \leq n+1$.*

Proof. The proof follows by induction on k.

The lemma is true for $k = 1$ because, in this case, $\mathcal{A}$ is an elementary JPA matrix which satisfies $\mathbf{P}_1$.

For a fixed $k \geq 1$ denote by $\mathcal{B}$ the next elementary JPA matrix and $\mathcal{C} = \mathcal{A}\mathcal{B}$. The first column of $\mathcal{B}$ is $(b_n, \ldots, b_1, 1)^T$ and it satisfies the property $\mathbf{P}_2$. So the first column of $\mathcal{C}$ is equal to

$$\mathcal{C}_1 = b_n\mathcal{A}_1 + \cdots + b_{n+1-k}\mathcal{A}_k + b_{n-k}\mathcal{E}_1 + \cdots + b_1\mathcal{E}_{n-k} + \mathcal{E}_{n+1-k}$$

and the next n columns are $\mathcal{A}_1, \ldots, \mathcal{A}_k, \mathcal{E}_1, \ldots, \mathcal{E}_{n-k}$. Since $b_n \geq 1$ and all other b_j are non-negative, it is clear that for each row the i-th term of $\mathcal{C}_1$ is greater or equal to the i-th term of $\mathcal{A}_1$ and, since the columns $\mathcal{A}_1, \ldots, \mathcal{A}_k$ satisfy the property $\mathbf{P}_2$. This proves the lemma. □

3. The Algorithm

Let K be a real field of degree $n+1 \geq 2$ and $\varepsilon > 1$ a unit such that $K = \mathbb{Q}(\varepsilon)$ with Q as its minimal polynomials over $\mathbb{Z}$. Let $\mathcal{A}$ be any JPA period matrix associated with ε, then $Q(\mathcal{A}) = 0$.

3.1. Construction of JPA period matrices. We can write any periodic JPA matrix as $\mathcal{A} = \mathcal{A}_0\mathcal{A}_1 \cdots \mathcal{A}_{l-1}$ for some $l \geq 1$ and build $\mathcal{A}$ by constructing each of the $\mathcal{A}_k$'s inductively for $k = 0, \ldots, l-1$.

Initial step. By Lemma 8, the first column $V = (v_{n+1}, \dots, v_1)^T$ of any periodic JPA matrix $\mathcal{A}$ satisfies the property $\mathbf{P}_2$. So $v_{n+1} \geq v_i \geq 0$ for all $2 \leq i \leq n+1$ and $v_{n+1} \geq v_1 \geq 1$. Moreover, as a consequence of $\mathbf{P}_2$, in particular that all the elements of $\mathcal{A}$ are non-negative, we find that $v_{n+1} \leq \mathrm{trace}(\mathcal{A}) = \mathrm{trace}(Q)$, the equality holding because $Q(\mathcal{A}) = 0$ implies $\mathrm{trace}(\mathcal{A}) = \mathrm{trace}(Q)$.

For all such column vectors, V, we proceed to the inductive step with $k = 0$.

Inductive step. For a given $k \geq 0$, we suppose we have the first column of a JPA matrix $\mathcal{A}_k \cdots \mathcal{A}_{l-1}$ and we will determine both $\mathcal{A}_k$ and the first column of $\mathcal{A}_{k+1} \cdots \mathcal{A}_{l-1}$.

We use $V = (v_{n+1}, \dots, v_1)^T$ to denote the first column of $\mathcal{A}_k \cdots \mathcal{A}_{l-1}$ and W to denote the first column of $\mathcal{A}_{k+1} \cdots \mathcal{A}_{l-1}$. From Lemma 8, there are two possibilities.

(a) We may have $W = \mathcal{E}_1$. In this case, the first column of $\mathcal{A}_k$ is V and $\mathcal{A}_{k+1} \cdots \mathcal{A}_{l-1}$ is the $(n+1) \times (n+1)$ identity matrix. So $\mathcal{A} = \mathcal{A}_0 \cdots \mathcal{A}_k$.

We now check whether $Q(\mathcal{A}) = 0$. If so, then we report $\mathcal{A}$ as a periodic JPA matrix associated with ε.

Our algorithm considers no further cases arising from such $\mathcal{A}_0 \cdots \mathcal{A}_k$.

(b) If $W \neq \mathcal{E}_1$, then we let $(a_n, \dots, a_1, 1)^T$ be the first column of $\mathcal{A}_k$. So $W = \mathcal{A}_k^{-1} V$ and since

$$\mathcal{A}_k^{-1} = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & \dots & \dots & 0 & -a_n \\ 0 & 1 & \dots & \dots & 0 & -a_{n-1} \\ 0 & \dots & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & 0 & 1 & -a_1 \end{pmatrix},$$

we have $W = (v_1, v_{n+1} - a_n v_1, v_n - a_{n-1} v_1, \dots, v_2 - a_1 v_1)^T$. We let the a_i's range over all possibilities such that the Perron conditions $\mathbf{P}_1$ are satisfied for $\nu = 0, \dots, k$ and so that the elements of W are all non-negative. To reduce the number of possibilities even further, we also check that the elements of W satisfy Lemma 8 (to make the program run in a feasible amount of time, such constraints on the a_i's should be applied as early as possible). In this way, we get a collection of all possible $\mathcal{A}_k$'s and their associated W's.

We now apply the inductive step for each such W.

Note that this algorithm does terminate as at every inductive step, we find that the sum of the elements of $\mathcal{A}_{k+1} \cdots \mathcal{A}_{l-1}$ is smaller than the sum of the elements of $\mathcal{A}_k \cdots \mathcal{A}_{l-1}$ (due to the nature of $\mathcal{A}_k^{-1}$) and these elements are all non-negative integers. Hence after finitely many steps, it must stop.

3.2. Computation of $\alpha_1, \ldots, \alpha_n$ in $\mathbb{Q}(\varepsilon)$. From the steps in Subsection 3.1, we have now determined all the JPA period matrices, $\mathcal{A}$, associated to ε. We now show how to use our knowledge of $\mathcal{A}$ and ε to calculate the α_i's exactly. Note that this is the problem of finding the eigenvectors of $\mathcal{A}$ that are associated with ε.

For $1 \le i \le n$, we write the α_i's as

$$\alpha_i = \sum_{j=(n+1)(i-1)+1}^{(n+1)i} c_j \varepsilon^{j-((n+1)(i-1)+1)},$$

where the c_j's are rational numbers.

We have the equation:

$$\mathcal{A} \begin{pmatrix} \alpha_n \\ \vdots \\ \alpha_1 \\ 1 \end{pmatrix} = \varepsilon \begin{pmatrix} \alpha_n \\ \vdots \\ \alpha_1 \\ 1 \end{pmatrix}.$$

We subtract the right-hand term side from the left-hand side and reduce each component of this vector modulo $Q(\varepsilon)$. Then, each component, which is a polynomial in ε of degree at most n, has all its coefficients equal to 0. This gives a linear system $Ec = S$ of $(n+1)^2$ equations and n^2+n unknowns $c = (c_1, \ldots, c_{n^2+n})$. Keeping the first n^2+n equations of this linear system, we get $Fc = T$. Thus $c = F^{-1}T$.

4. Numerical computations

4.1. The units are Pisot numbers. Here we give some examples for the degrees 4, 5 and 6 when the units are Pisot numbers. The periods will not be always primitive. The programs have been written in Pascal. A compiled language like Pascal was used, rather than interpreted languages like Maple or Pari, due to speed, as the computations for each example do take a substantial amount of time.

$[K : \mathbb{Q}] = 4, n = 3$.
Let $Q_1 = x^4 - 430x^3 + 291x^2 - 46x + 1$. All its roots are real and only one is greater than 1, $\varepsilon = 429.322437\ldots$. Note that $\varepsilon = \omega^4$ where ω is the unique root of $R_1 = x^4 - 4x^3 - 3x^2 + 2x + 1$ that is real and greater than 1. There are 666 matrices giving purely periodic JPA developments associated to ε, 596 with period of length 4, 42 of length 6, 8 of length 8 and 20 of length 10.

$[K : \mathbb{Q}] = 5, n = 4$.
Let $Q_2 = x^5 - 57x^4 - 42x^3 - 22x^2 - 7x - 1$. It has $\varepsilon = 57.734109\ldots$ as a root. Note that $\varepsilon = \omega^5$ where $\omega = 1.965948\ldots$ is a root of the polynomial $R_2 = x^5 - x^4 - x^3 - x^2 - x - 1$. There are 576 matrices giving purely periodic JPA developments associated to ε, 1 with period of length 1, 19 with period

of length 2, 43 with period of length 3, 498 with period of length 4, 1 with period of length 6 (its primitive period is of length 1) and 14 with period of length 7.

$[K:\mathbb{Q}]=6, n=5$.
Let $Q_3 = x^6-63x^5+129x^4-111x^3+49x^2-11x+1$. It has $\varepsilon = 60.911887\ldots$ as a root. Here $\varepsilon=\omega^6$, where $\omega = 1.983582\ldots$ is a root of the polynomial $R_3 = x^6-x^5-x^4-x^3-x^2-x-1$. There are 2858 matrices associated to ε, all with period of length 6.

4.2. The units are not Pisot numbers. Here we use examples obtained from the example given on p. 275 of [6], with $a_3=a_4=a_1=3$ and $a_2=0$.

$[K:\mathbb{Q}]=5, n=4$.
Let $R_4 = x^5-3x^4-3x^3-3x-1$. $\omega = 3.8390646\ldots$ is a root of R_4. It is not a Pisot number since it has a real conjugate less than -1. For each $k=1,\ldots,4$, there is only one matrix which is associated to ω^k. Note that the maximum of trace $\left(\omega^k\right)$ occurs for $k=4$ and is 219.

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