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Résumé. Un point d'une conique définie sur $\mathbb{Q}$ est dit singulier s'il est transcendant et admet de très bonnes approximations rationnelles, uniformément en termes de la hauteur. Les nombres extrémaux et les fractions continues sturmiennes sont les abscisses de tels points sur la parabole $y = x^2$. Nous établissons ici une mesure de transcendance de points singuliers sur les coniques définies sur $\mathbb{Q}$ qui, dans ces deux cas, améliore la mesure obtenue précédemment par Adamczewski et Bugeaud. L'outil principal est une version quantitative du théorème du sous-espace de Schmidt due à Evertse.

Abstract. A singular point on a plane conic defined over $\mathbb{Q}$ is a transcendental point of the curve which admits very good rational approximations, uniformly in terms of the height. Extremal numbers and Sturmian continued fractions are abscissa of such points on the parabola $y = x^2$. In this paper we provide a measure of transcendence for singular points on conics defined over $\mathbb{Q}$ which, in these two cases, improves on the measure obtained by Adamczewski and Bugeaud. The main tool is a quantitative version of Schmidt subspace theorem due to Evertse.

1. Introduction

In [1, §5.2], Adamczewski and Bugeaud established a measure of transcendence for the extremal numbers from [9] as well as for the Sturmian continued fractions from [3] (see also [2, §4]). The goal of this paper is to prove the following sharper measure which applies to a larger class of numbers.

Theorem 1. *Let $(\xi, \eta) \in \mathbb{R}^2$. Suppose that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$, and that $f(\xi, \eta) = 0$ for some irreducible polynomial $f(x, y)$ of $\mathbb{Q}[x, y]$ of degree 2, not in $\mathbb{Q}[x]$. Suppose furthermore that there exists a real number $\lambda > 1/2$ such that the inequalities*

$$|x_0| \le X, \quad |x_0\xi - x_1| \le X^{-\lambda}, \quad |x_0\eta - x_2| \le X^{-\lambda} \tag{1.1}$$

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have a non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$ *for each large enough real number* $X \geq 1$*. Then* ξ *is transcendental over* $\mathbb{Q}$ *and there exists a computable constant* $c > 0$ *such that, for each pair of integers* $d \geq 3$ *and* $H \geq 2$ *and each algebraic number* α *of degree* $d(\alpha) \leq d$ *and naive height* $H_0(\alpha) \leq H$*, we have*

$$(1.2) \qquad |\xi - \alpha| \geq H^{-w(d)} \quad \textit{where} \quad w(d) = \exp\big(c(\log d)(\log\log d)\big).$$

By the naive height $H_0(\alpha)$ of an algebraic number α, we mean the largest absolute value of the coefficients of its irreducible polynomial P_α in $\mathbb{Z}[x]$, while its degree $d(\alpha)$ is the degree of P_α.

Fix $(\xi, \eta) \in \mathbb{R}^2$ and, for each $X \geq 1$, define $\Delta(X)$ to be the minimum of the quantities

$$\delta(\mathbf{x}) := \max\{|x_0\xi - x_1|, |x_0\eta - x_2|\}$$

as $\mathbf{x} = (x_0, x_1, x_2)$ runs through the points of $\mathbb{Z}^3$ with $1 \leq x_0 \leq X$. Then the condition that (1.1) has a non-zero integer solution for a given $X \geq 1$ is equivalent to asking that $\Delta(X) \leq X^{-\lambda}$. By Minkowski convex body theorem, we have $\Delta(X) \leq X^{-1/2}$ for each $X \geq 1$. We even have $\Delta(X) \leq cX^{-1}$ with a constant $c > 0$ that is independent of X if $1, \xi, \eta$ are linearly dependent over $\mathbb{Q}$. However, if ξ and η are algebraic over $\mathbb{Q}$ and if $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$, then Schmidt subspace theorem [11, Ch. VI, Theorem 1B] implies that, for any given $\lambda > 1/2$, the inequality

$$|x_0|^{2\lambda}\, |x_0\xi - x_1|\, |x_0\eta - x_2| \leq 1$$

has only finitely many solutions $(x_0, x_1, x_2) \in \mathbb{Z}^3$ with $x_0 \neq 0$. This in turn implies that $\Delta(X) \geq c_\lambda X^{-\lambda}$ for each $X \geq 1$ with a constant $c_\lambda > 0$. Thus any point (ξ, η) satisfying the hypotheses of the theorem has at least one transcendental coordinate. However, if ξ is algebraic over $\mathbb{Q}$, then $f(\xi, y)$ is a non-zero polynomial of degree at most two in y (because $f(x, y)$ is irreducible and depends on y), and therefore η is also algebraic, a contradiction. So, ξ must be transcendental. This proves the first part in the conclusion of the theorem. To establish the measure of transcendence (1.2), we follow Adamczewski and Bugeaud in [1] by using a quantitative version of Schmidt subspace theorem, namely that of Evertse from [6]. We recall the latter result in Section 2 and postpone the proof of the theorem to Sections 3 and 4.

Let $\gamma = (1 + \sqrt{5})/2 \simeq 1.618$ denote the golden ratio. In [1], Adamczewski and Bugeaud consider the case where $(\xi, \eta) = (\xi, \xi^2)$ is a point on the parabola $y = x^2$. For such a point, the condition that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$ amounts to asking that ξ is not rational nor quadratic over $\mathbb{Q}$. In that case, Davenport and Schmidt [5] showed the existence of a constant $c > 0$ such that $\Delta(X) \geq cX^{-1/\gamma}$ for arbitrarily large values of X. In [9], we proved that, conversely, there exist transcendental real numbers ξ, called *extremal numbers*, for which the pair $(\xi, \eta) = (\xi, \xi^2)$

satisfies $\Delta(X) \leq c' X^{-1/\gamma}$ for each $X \geq 1$, with another constant $c' > 0$. Such pairs thus satisfy the hypotheses of the theorem for any choice of λ in $(1/2, 1/\gamma)$. Examples of extremal numbers include all real numbers whose continued fraction expansion is the Fibonacci word on two distinct positive integers [8]. In [3], Bugeaud and Laurent consider more generally the real numbers ξ whose continued fraction expansion is a characteristic Sturmian word on two distinct positive integers. When the slope of such a word has itself bounded partial quotients (like the slope $1/\gamma$ of the Fibonacci word), they determine an explicit and best possible value $\hat{\lambda} > 1/2$ such that the pair (ξ, ξ^2) satisfies the hypotheses of the theorem for each $\lambda \in (1/2, \hat{\lambda})$. For all extremal numbers ξ and all of the above characteristic Sturmian continued fractions ξ, Adamczewski and Bugeaud [1, §5.2] establish a measure of transcendence of the form

$$|\xi - \alpha| \geq H^{-w(d)} \quad \text{where} \quad w(d) = \exp\left(c(\log d)^2 \cdot (\log\log d)^2\right).$$

In [2, §4], they refine this measure for characteristic Sturmian continued fractions ξ to $w(d) = \exp(c(\log d)^2 \cdot \log\log d)$. Here we further remove the square on the term $\log d$. However this is still not enough to conclude that those numbers ξ are S-numbers in the sense of Mahler, as one could expect, since this requires a measure of the form $|\xi - \alpha| \geq H^{-cd}$.

The numbers of Sturmian type introduced by A. Poëls in [7] include all of the above-mentioned numbers, and provide further examples of real numbers ξ for which the point (ξ, ξ^2) satisfies the hypotheses of our theorem. So, our measure (1.2) applies to these numbers as well.

In general, if a polynomial $f \in \mathbb{Q}[x, y]$ of degree 2 admits at least one zero (ξ, η) with $1, \xi, \eta$ linearly independent over $\mathbb{Q}$ then f is irreducible over $\mathbb{Q}$ and its gradient does not vanish at the point (ξ, η). Thus the equation $f(x, y) = 0$ defines a conic in $\mathbb{R}^2$ with infinitely many points. In [10, Theorem 1.2], we show that there are points (ξ, η) on that curve which satisfy the hypotheses of the theorem for any choice of λ in $(1/2, 1/\gamma)$ (and none for any $\lambda > 1/\gamma$). So, if $f \notin \mathbb{Q}[x]$, then ξ is transcendental and satisfies (1.2).

We do not know if the theorem applies to irreducible polynomials f of $\mathbb{Q}[x, y]$ of degree $\deg(f) > 2$. We do not even know if such a polynomial could have a zero (ξ, η) which fulfills the hypotheses of the theorem. In particular, we wonder if there are such "singular" points on the plane cubic $y = x^3$ and if so, what is the supremum of the corresponding values of λ.

2. The quantitative subspace theorem

To state the notion of height used by Evertse in [6], let $\bar{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, let $K \subset \bar{\mathbb{Q}}$ be a subfield of finite degree d over $\mathbb{Q}$, and let $n \geq 2$ be an integer. For each place v of K, we denote by K_v the completion of K at v, by $d_v = [K_v : \mathbb{Q}_v]$ the local degree of K at v, and by $|\cdot|_v$ the absolute value on K_v which extends the usual absolute value on $\mathbb{Q}$ if v is archimedean or the usual p-adic absolute value on $\mathbb{Q}$ (with $|p|_v = p^{-1}$) if v lies above a prime number p. Then the absolute Weil height of a non-zero point $\mathbf{a} = (a_1, \dots, a_n) \in K^n$ is

$$H(\mathbf{a}) = \prod_{v|\infty} (|a_1|_v^2 + \cdots + |a_n|_v^2)^{d_v/(2d)} \prod_{v\nmid\infty} \max\{|a_1|_v, \dots, |a_n|_v\}^{d_v/d}$$

where the first product runs over the archimedean places of K and the second one over all remaining places of K. This height is called absolute because, for a given non-zero $\mathbf{a} \in \bar{\mathbb{Q}}^n$, it is independent of the choice of a number field $K \subset \bar{\mathbb{Q}}$ such that $\mathbf{a} \in K^n$. Moreover it is projective in the sense that $H(\mathbf{a}) = H(c\mathbf{a})$ for any $c \in \bar{\mathbb{Q}} \setminus \{0\}$.

For any non-zero linear form $\ell(\mathbf{x}) = a_1x_1 + \cdots + a_nx_n \in \bar{\mathbb{Q}}x_1 + \cdots + \bar{\mathbb{Q}}x_n$, we define the degree of ℓ to be the degree of the extension of $\mathbb{Q}$ generated by all quotients a_i/a_j with $a_j \neq 0$, and its height to be $H(a_1, \dots, a_n)$. Then [6, Corollary] reads as follows.

Theorem 2 (Evertse, 1996)**.** *Let $n \geq 2$ be an integer, let $\ell_1, \dots, \ell_n$ be n linearly independent linear forms in n variables with coefficients in $\bar{\mathbb{Q}}$, let D be an upper bound for their degrees, and let H be an upper bound for their heights. Then, for every δ with $0 < \delta < 1$, there are proper linear subspaces $T_1, \dots, T_t$ of $\mathbb{Q}^n$ with*

$$t \leq 2^{60n^2} \delta^{-7n} \log(4D) \cdot \log\log(4D)$$

such that every non-zero point $\mathbf{x} \in \mathbb{Z}^n$ with $H(\mathbf{x}) \geq H$ satisfying

$$|\ell_1(\mathbf{x}) \cdots \ell_n(\mathbf{x})| \leq |\det(\ell_1, \dots, \ell_n)| H(\mathbf{x})^{-\delta} \tag{2.1}$$

lies in $T_1 \cup \cdots \cup T_t$.

Note that the precise statement of [6, Corollary] deals only with primitive points $\mathbf{x} \in \mathbb{Z}^n$, namely non-zero integer points whose coordinates are relatively prime as a set. However if a non-zero point $\mathbf{x} \in \mathbb{Z}^n$ satisfies (2.1), then the primitive points $\mathbf{y}$ of which it is an integer multiple are also solutions of (2.1), and so that restriction is not necessary.

3. A sequence of minimal points

Let the notation and the hypotheses be as in the statement of the theorem. The function $\Delta\colon [1, \infty) \to \mathbb{R}$ attached to the pair (ξ, η) is monotone decreasing to zero, and constant in each interval between two consecutive integers. Let $X_1 = 1 < X_2 < X_3 < \cdots$ be its points of discontinuity listed in increasing order, together with 1. For each index $i \geq 1$, we set $\Delta_i = \Delta(X_i)$ and choose a non-zero point $\mathbf{x}_i = (x_{i,0}, x_{i,1}, x_{i,2}) \in \mathbb{Z}^3$ such that

$$x_{i,0} = X_i \quad \text{and} \quad \delta(\mathbf{x}_i) = \Delta_i.$$

Following Davenport and Schmidt in [4, 5], we say that $(\mathbf{x}_i)_{i\geq 1}$ is a sequence of *minimal points* for (ξ,η). In this section, we establish some of its properties starting with the most fundamental one.

Since Δ is constant on $[X_i, X_{i+1})$ with $\Delta(X) \leq X^{-\lambda}$ for each large enough value of X, there exists $i_0 \geq 2$ such that

$$\Delta_i \leq X_{i+1}^{-\lambda} \quad \text{for each} \quad i \geq i_0. \tag{3.1}$$

Lemma 3. *For each $i \geq 1$, the subspace $W_i = \langle \mathbf{x}_i, \mathbf{x}_{i+1}\rangle_{\mathbb{Q}}$ of $\mathbb{Q}^3$ spanned by $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ has dimension 2 and $\{\mathbf{x}_i, \mathbf{x}_{i+1}\}$ forms a basis of $W_i \cap \mathbb{Z}^3$.*

Proof. The points $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ are primitive with $\mathbf{x}_{i+1} \neq \pm\mathbf{x}_i$. So they span a subspace of $\mathbb{Q}^3$ of dimension 2. For the second assertion, it suffices to adapt the argument in the proof of [4, Lemma 2]. □

For any basis $\{\mathbf{x},\mathbf{y}\}$ of $W_i \cap \mathbb{Z}^3$, the cross product $\mathbf{x}\wedge\mathbf{y}$ is a primitive element of $\mathbb{Z}^3$ which, by the lemma, is equal to $\pm\mathbf{x}_i \wedge \mathbf{x}_{i+1}$. Upon defining the height $H(W_i)$ of W_i as the Euclidean norm of that vector, we obtain

$$H(W_i) = \|\mathbf{x}_i \wedge \mathbf{x}_{i+1}\|_2 \ll X_{i+1}\Delta_i \ll X_{i+1}^{1-\lambda} \tag{3.2}$$

with implied constants that do not depend on i.

Lemma 4. *Let I denote the set of indices $i \geq 2$ such that $\mathbf{x}_{i-1}$, $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ are linearly independent over $\mathbb{Q}$. Then I is an infinite set. For any pair of consecutive elements $i < j$ of I (in the natural ordering inherited from $\mathbb{N}$), we have $W_i \neq W_j$ and $X_j \leq H(W_i)H(W_j)$.*

Proof. The argument of [5, Lemma 5] shows that I is infinite. Let $i<j$ be consecutive elements of I. We have $W_{i-1} \neq W_i = \cdots = W_{j-1} \neq W_j$, thus $W_i \neq W_j$. Moreover the points $\mathbf{x}_i \wedge \mathbf{x}_{i+1} = \pm\mathbf{x}_{j-1}\wedge\mathbf{x}_j$ and $\mathbf{x}_j \wedge \mathbf{x}_{j+1}$ being orthogonal to $\mathbf{x}_j$ and not parallel, their cross product is a non-zero integer multiple of $\mathbf{x}_j$, and thus

$$X_j \leq \|\mathbf{x}_j\|_2 \leq \|\mathbf{x}_i \wedge \mathbf{x}_{i+1}\|_2\, \|\mathbf{x}_j \wedge \mathbf{x}_{j+1}\|_2 = H(W_i)H(W_j). \qquad \square$$

Lemma 5. *For each $i \geq 1$, we have $X_{i+1}^{\lambda} \ll X_i$.*

Proof. Let $\varphi \in \mathbb{Q}[x_0,x_1,x_2]$ be the homogeneous quadratic form for which $f(x,y) = \varphi(1,x,y)$. We claim that $\varphi(\mathbf{x}_i) \neq 0$ for each sufficiently large i. If we take it for granted then, for each of those i, we have $1/c \leq |\varphi(\mathbf{x}_i)|$ where c is a common denominator of the coefficients of φ. As $\varphi(1,\xi,\eta)=0$, we also have $|\varphi(\mathbf{x}_i)| \ll \|\mathbf{x}_i\|_2\Delta_i \ll X_i X_{i+1}^{-\lambda}$. Combining the two estimates yields $X_{i+1}^{\lambda} \ll X_i$.

The claim is clear if f has at most one zero in $\mathbb{Q}^2$. Otherwise, [10, Lemma 2.4] shows that there exist $\mu \in \mathbb{Q}^{\times}$ and $T \in \mathrm{GL}_3(\mathbb{Q})$ such that $\mu(\varphi\circ T)(x_0,x_1,x_2) = x_0x_2 - x_1^2$. Then $T^{-1}(1,\xi,\eta)$ is proportional to $\Theta = (1,\theta,\theta^2)$ for some $\theta \in \mathbb{R}$ and, for each $i \geq 1$, the point $T^{-1}(\mathbf{x}_i)$ is proportional to a primitive integral point $\mathbf{y}_i = (y_{i,0}, y_{i,1}, y_{i,2})$ with $\|\mathbf{y}_i\|_2 \asymp X_i$

and $\|\mathbf{y}_i \wedge \Theta\|_2 \asymp \Delta_i$. We now argue as Davenport and Schmidt in the proof of [5, Lemma 2], omitting details. If $\varphi(\mathbf{x}_i) = 0$ for some i, then $\mathbf{y}_i = \pm(m^2, mn, n^2)$ for some coprime integers m, n with $|m| \asymp X_i^{1/2}$ and $|m\theta - n| \ll X_{i+1}^{-\lambda} X_i^{-1/2}$. However, if $i \geq 2$, then $\mathbf{y}_{i-1}$ is not proportional to $\mathbf{y}_i$ and so we have $my_{i-1,j+1} \neq ny_{i-1,j}$ for some $j \in \{0, 1\}$. For that j, we find that $1 \leq |my_{i-1,j+1} - ny_{i-1,j}| \ll X_i^{1/2-\lambda}$ and so i is bounded from above. □

Lemma 6. *Let $\theta > (1-\lambda)/(2\lambda - 1)$. Then, there exists an element i_1 of I with $i_1 \geq i_0$ such that, for any pair of consecutive elements $i < j$ of I with $i \geq i_1$, we have $H(W_i) < H(W_j)$ and $X_{j+1} < X_{i+1}^{\theta}$.*

Proof. Let $i < j$ be consecutive elements of I with $i \geq i_0$. In the situation where $H(W_j) \leq H(W_i)$, Lemma 4 together with (3.2) yields

$$X_{i+1} \leq X_j \leq H(W_i)H(W_j) \leq H(W_i)^2 \ll X_{i+1}^{2(1-\lambda)}.$$

Since $2(1-\lambda) < 1$, this cannot hold when i is large enough. For such i, we thus have $H(W_i) < H(W_j)$. Combining Lemmas 4 and 5 with (3.2), we also find

$$X_{j+1}^{\lambda} \ll X_j \leq H(W_i)H(W_j) \ll X_{i+1}^{1-\lambda} X_{j+1}^{1-\lambda},$$

thus $X_{j+1}^{2\lambda-1} \ll X_{i+1}^{1-\lambda}$ and so $X_{j+1} < X_{i+1}^{\theta}$ if i is large enough. □

4. Proof of the main theorem

In continuation with the preceding section, we suppose that ξ, η, $f(x, y)$ and λ are as in the statement of the theorem. We choose θ and i_1 as in Lemma 4, and list in increasing order $i_1 < i_2 < i_3 < \cdots$ the elements of I that follow i_1. Then, $W_{i_1}, W_{i_2}, W_{i_3}, \ldots$ are subspaces of $\mathbb{Q}^3$ of dimension 2 with strictly increasing heights and so they are pairwise distinct. This will be important in what follows. We also choose $\delta \in (0, 1/2)$ such that

$$6\delta < 2\lambda - 1. \tag{4.1}$$

All constants $C_1, C_2, \ldots$ that appear below depend only, in a simple way, on these data.

Since $f(x, y)$ has degree 2 and $\partial f/\partial y \neq 0$, we deduce from the linear independence of $1, \xi, \eta$ over $\mathbb{Q}$ that $|\partial f/\partial y(\xi, \eta)| \neq 0$. Thus, by the implicit function theorem, there exist $C_1 > 0$ and $C_2 \geq 1$ such that, for any $\alpha \in \mathbb{C}$ with $|\xi - \alpha| \leq C_1$, we can find $\beta \in \mathbb{C}$ satisfying

$$f(\alpha, \beta) = 0 \quad \text{and} \quad |\eta - \beta| \leq C_2|\xi - \alpha|. \tag{4.2}$$

Fix integers $d \geq 3$ and $H \geq 2$ and an algebraic number α with degree $d(\alpha) \leq d$ and naive height $H_0(\alpha) \leq H$. We need to provide a lower bound for

$$\epsilon := |\xi - \alpha|. \tag{4.3}$$

Suppose first that $\epsilon \le C_1$ and choose $\beta \in \mathbb{C}$ as in (4.2). Since $f(x,y)$ is irreducible over $\mathbb{Q}$ and depends on y, it is relatively prime to the irreducible polynomial $P_\alpha(x)$ of α and so β is a root of their resultant in x. Thus β is an algebraic number with

$$d(\beta) \le 2d \quad \text{and} \quad H_0(\beta) \le C_3^d H^2. \tag{4.4}$$

Consider the linear forms with algebraic coefficients

$$\ell_1 = x_0, \quad \ell_2 = x_0\alpha - x_1 \quad \text{and} \quad \ell_3 = x_0\beta - x_2. \tag{4.5}$$

By the above they have degree at most $2d$. To estimate their heights, we note that, for any algebraic number γ, we have

$$H(1,\gamma) \le \sqrt{2} M(\gamma)^{1/d(\gamma)} \le C_4 H_0(\gamma)^{1/d(\gamma)} \le C_4 H_0(\gamma),$$

where $M(\gamma)$ denotes the Mahler measure of γ. Using this crude estimate together with (4.4), we find that the linear forms (4.5) have heights at most

$$\max\{H(1,\alpha), H(1,\beta)\} \le C_4 \max\{H, C_3^d H^2\} \le H^{C_5 d},$$

where the last inequality uses $H \ge 2$. By Theorem 2 of Evertse, there exist proper linear subspaces $T_1, \dots, T_t$ of $\mathbb{Q}^3$ with

$$t \le 2^{540}\delta^{-21} \log(8d) \cdot \log\log(8d) \le C_6 (\log d)(\log\log d) \tag{4.6}$$

such that every non-zero point $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ with $H(\mathbf{x}) \ge H^{C_5 d}$ satisfying

$$|x_0|\,|x_0\alpha - x_1|\,|x_0\beta - x_2| \le H(\mathbf{x})^{-\delta}$$

lies in $T_1 \cup \cdots \cup T_t$.

Let $\ell \ge 1$ be the smallest integer such that

$$X_{i_\ell+1} \ge \max\{\, H^{C_5 d},\, (t+1)^{1/\delta},\, 4^{1/\delta} C_7\} \quad \text{where} \quad C_7 = 3 + |\xi| + |\eta|. \tag{4.7}$$

The subspaces $W_{i_\ell}, \dots, W_{i_{\ell+t}}$ of $\mathbb{Q}^3$ being all distinct, there is at least one index j among $\{i_\ell, \dots, i_{\ell+t}\}$ for which $W_j \notin \{T_1, \dots, T_t\}$. Fix such a choice of j. Since W_j has dimension 2, it is not contained in any T_i. Now, consider the points $\mathbf{x} = a\mathbf{x}_j + \mathbf{x}_{j+1}$ with $a \in \mathbb{Z}$. Since any two of them span W_j, each T_i contains at most of one these points. Thus there is at least one choice of a with $0 \le a \le t$ for which $\mathbf{x} \notin T_1 \cup \cdots \cup T_t$. Fix such a choice of a and denote by (x_0, x_1, x_2) the coordinates of the corresponding point $\mathbf{x} = a\mathbf{x}_j + \mathbf{x}_{j+1}$. Since $\{\mathbf{x}_j, \mathbf{x}_{j+1}\}$ is a basis of $W_j \cap \mathbb{Z}^3$ (see Lemma 3), this point $\mathbf{x}$ is primitive and so $H(\mathbf{x}) = \|\mathbf{x}\|_2$ is its Euclidean norm. This yields $H(\mathbf{x}) \ge x_0 \ge X_{j+1} \ge X_{i_\ell+1} \ge H^{C_5 d}$ and thus, by the result of Evertse, we must have

$$|x_0|\,|x_0\alpha - x_1|\,|x_0\beta - x_2| > \|\mathbf{x}\|_2^{-\delta}. \tag{4.8}$$

Using (3.1), we find

$$|x_0| = x_0 \le X_j + tX_{j+1} \le (t+1)X_{j+1}$$
$$\max\{|x_0\xi - x_1|, |x_0\eta - x_2|\} \le \Delta_j + t\Delta_{j+1} \le (t+1)\Delta_j \le (t+1)X_{j+1}^{-\lambda}.$$

By (4.7), we also have $t+1 \le X_{i_\ell+1}^{\delta} \le X_{j+1}^{\delta}$, thus these inequalities imply

$$|x_0| \le X_{j+1}^{1+\delta} \quad \text{and} \quad \max\{|x_0\xi - x_1|, |x_0\eta - x_2|\} \le X_{j+1}^{-\lambda+\delta}.$$

Since $\delta < 1/2 < \lambda$, this yields

$$\|\mathbf{x}\|_2 \le |x_0| + |x_1| + |x_2| \le C_7|x_0| \le C_7 X_{j+1}^2,$$

where C_7 is as in (4.7). Using (4.2) and (4.3), we also deduce that

$$\begin{aligned} \max\{|x_0\alpha - x_1|, |x_0\beta - x_2|\} &\le |x_0| \max\{|\xi - \alpha|, |\eta - \beta|\} + X_{j+1}^{-\lambda+\delta} \\ &\le C_2 X_{j+1}^{1+\delta}\epsilon + X_{j+1}^{-\lambda+\delta} \\ &\le 2\max\left\{C_2 X_{j+1}^{1+\delta}\epsilon,\ X_{j+1}^{-\lambda+\delta}\right\}. \end{aligned}$$

Substituting these estimates into (4.8), we obtain

$$4X_{j+1}^{1+\delta} \max\left\{C_2 X_{j+1}^{1+\delta}\epsilon,\ X_{j+1}^{-\lambda+\delta}\right\}^2 \ge C_7^{-\delta} X_{j+1}^{-2\delta}. \tag{4.9}$$

Suppose first that $C_2 X_{j+1}^{1+\delta}\epsilon < X_{j+1}^{-\lambda+\delta}$. Then, after simplifications, we obtain, by virtue of the choice of δ in (4.1),

$$4C_7^{\delta} \ge X_{j+1}^{2\lambda-1-5\delta} > X_{j+1}^{\delta} \ge X_{i_\ell+1}^{\delta},$$

in contradiction with (4.7). So, the inequality (4.9) implies that

$$\epsilon \ge 2^{-1} C_2^{-1} C_7^{-\delta/2} X_{j+1}^{-(3+5\delta)/2} \ge X_{j+1}^{-C_8}. \tag{4.10}$$

We now use Lemma 6 to estimate X_{j+1} from above. Since $j \in \{i_\ell, \dots, i_{\ell+t}\}$, we obtain

$$X_{j+1} \le X_{i_{\ell+t}+1} \le X_{i_\ell+1}^{\theta^t}.$$

If $\ell = 1$, then (4.10) yields

$$\epsilon \ge X_{i_1+1}^{-C_8\theta^t} \ge 2^{-C_9\theta^t} \ge H^{-C_9\theta^t}.$$

Otherwise, by the choice of ℓ in (4.7), we have

$$X_{i_{\ell-1}+1} < \max\{\, H^{C_5 d},\ (t+1)^{1/\delta},\ 4^{1/\delta}C_7\} \le H^{C_{10}d}.$$

Since $X_{i_\ell+1} \le X_{i_{\ell-1}+1}^{\theta}$, a similar computation then gives

$$\epsilon \ge X_{i_{\ell-1}+1}^{-C_8\theta^{t+1}} \ge H^{-C_8 C_{10} d\theta^{t+1}}.$$

So, in both cases, we obtain $|\xi - \alpha| = \epsilon \ge H^{-w(d)}$ where

$$w(d) = C_{11} d\,\theta^t \le \exp\big(C_{12}(\log d)(\log\log d)\big),$$

using the upper bound for t in (4.6). Finally, in the case where $\epsilon \geq C_1$, this remains true at the expense of replacing C_{12} by a larger constant if necessary.

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