Journal de Théorie des Nombres de Bordeaux
anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 31, n° 1 (2019), p. 215-226.
<http://jtnb.cedram.org/item?id=JTNB_2019__31_1_215_0>

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de Bordeaux **31** (2019), 215–226

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Résumé. Nous donnons une condition nécessaire et suffisante pour qu'une racine strictement supérieure à 1 d'un polynôme réciproque unitaire de degré pair $\geqslant 4$ à coefficients entiers soit un nombre de Salem. Cette condition exige que le polynôme minimal d'une certaine puissance de cet entier algébrique ait un coefficient linéaire assez grand. Pour les nombres de Salem de certains petits degrés nous déterminons également la probabilité qu'une puissance d'un tel nombre satisfasse à cette condition.

Abstract. We present a necessary and sufficient condition for a root greater than unity of a monic reciprocal polynomial of an even degree at least four, with integer coefficients, to be a Salem number. This condition requires that the minimal polynomial of some power of the algebraic integer has a linear coefficient that is relatively large. We also determine the probability that an arbitrary power of a Salem number, of certain small degrees, satisfies this condition.

1. Introduction

A Salem number is a real algebraic integer $\tau > 1$ of degree at least four, conjugate to τ^{-1}, all of whose conjugates, excluding τ and τ^{-1}, are unimodal i.e., lie on $|z| = 1$. The corresponding minimal polynomial $P(x)$ of degree d of these numbers, called a Salem polynomial, is (self-)reciprocal, that is $x^d P(1/x) = P(x)$. Since $P(x)$ is self-reciprocal and irreducible it must have even degree. It is well known [7] that τ^n should also be a Salem number of degree d for any natural n. Fractional parts of τ^n are dense in the unit interval $[0,1]$, but are not uniformly distributed [1, 8]. Salem numbers have appeared in quite different areas of mathematics (number theory, harmonic analysis, knot theory, etc.). Throughout, when we speak about a conjugate, the minimal polynomial or the degree of an algebraic number we mean over the field of the rationals $\mathbb{Q}$.

Manuscrit reçu le 21 mai 2018, révisé le 25 juin 2018, accepté le 16 juillet 2018.

2010 *Mathematics Subject Classification.* 11R06.

Mots-clefs. Salem number, $\mathbb{Q}$ -linearly independent numbers, reciprocal polynomial, Galois automorphism.

Partially supported by Serbian Ministry of Education and Science, Project 174032.

In [9] Vieira, extending a result of Lakatos and Losonczi [3], presented a sufficient condition for a self-reciprocal polynomial to have a fixed number of roots on the complex unit circle $U = \{z \in \mathbb{C} : |z| = 1\}$. Let $p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$ be a d-th degree self-reciprocal polynomial. If the inequality

$$|a_{d-l}| > \frac{1}{2}\left(\frac{d}{d-2l}\right) \sum_{k=0,k\neq l,d-l}^{d} |a_k|, \quad l < d/2 \tag{1.1}$$

holds, then $p(z)$ has exactly $d-2l$ roots on U and these roots are simple. Here we present, in a sense, a result which lies in the opposite direction of a special case of this theorem. Namely, we shall prove the following

Theorem 1.1. *A real algebraic integer $\tau > 1$ is a Salem number if and only if its minimal polynomial $P(x)$ is reciprocal of even degree $d \geq 4$, and there is $n \in \mathbb{N}$, $n \geq 2$ such that τ^n has the minimal polynomial $P_n(x) = 1 + a_{1,n}x + a_{2,n}x^2 + \cdots + a_{d-1,n}x^{d-1} + x^d$, which is also reciprocal of degree d, and satisfies the condition*

$$|a_{d-1,n}| > \frac{1}{2}\left(\frac{d}{d-2}\right)\left(2 + \sum_{k=2}^{d-2} |a_{k,n}|\right). \tag{1.2}$$

Notice that the condition (1.2) is the special case when $l = 1$ of the condition (1.1) applied to $P_n(x)$.

We present a method, easy for implementation, for the calculation of the coefficients of $P_n(x)$ starting with $P(x)$ without determination of its roots. We can use the companion matrix C of a monic polynomial $P(x) = x^d + a_{d-1}x^{d-1} + a_{d-2}x^{d-2} + \cdots + a_0$ defined as

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{d-2} & -a_{d-1} \end{bmatrix}_{d\times d}$$

It is well known [4, 5] that $P(x)$ is the characteristic polynomial of C so the root λ of $P(x)$ is an eigenvalue of C. If v is an eigenvector of C associated with λ then $C^n v = C^{n-1}Cv = C^{n-1}\lambda v = \cdots = \lambda^n v$. Thus C^n should have an eigenvalue λ^n and the characteristic polynomial of C^n must be $P_n(x)$, i.e. $P_n(x) = det(xI - C^n)$. It is easy to show that $v = [1\ \lambda\ \lambda^2\ \ldots\ \lambda^{d-1}]^T$.

Using this method we are able, for a Salem number τ, to find at least one n such that the minimal polynomial $P_n(x)$ of τ^n satisfies condition (1.2). In Table 1.1 we present examples of Salem numbers and the smallest n which we have found. The forth example in the table is the root of the Lehmer polynomial which is the smallest known Salem number. We notice that n

becomes large as d increases. It would be interesting to find at least one n for all small Salem numbers in the Mossinghoff's list [6].

The relative frequency of n such that the minimal polynomial $P_n(x)$ of τ^n satisfies (1.2), which we approximated for $d = 4$ (2.12) and for $d = 6$ (2.13), significantly decreases when d increases. One might ask what is the probability of fulfilling the condition (1.2) for an arbitrary power of the root. We determined the exact value of the probability for $d = 4, 6$ and we approximated the probability for $d = 8, 10$.

Theorem 1.2. *Let τ be a Salem number of degree d, $n, n_0 \in \mathbb{N}$ and let $P_n(x)$ be the minimal polynomial of τ^n. Let p_d denotes the limit of the probability that coefficients of $P_n(x)$ satisfy* (1.2) *when $n \geq n_0$ is randomly chosen, as n_0 approaches infinity. Then:*

(1) *p_4 is equal to* $1/3$ *and,*
(2)

$$p_6 = \frac{4}{\pi^2}\left[\int_{\arccos\frac{\sqrt{30}}{6}}^{\arccos\frac{\sqrt{19}-1}{6}} \left(\arccos\frac{-5-6\cos t}{6+6\cos t} - (\pi - t)\right)\mathrm{d}t + \int_{\arccos\frac{\sqrt{19}-1}{6}}^{\arccos\frac{\sqrt{6}}{6}} \left(\arccos\frac{1-6\cos t}{6-6\cos t} - (\pi - t)\right)\mathrm{d}t\right] = 0.0717258\ldots. \tag{1.3}$$

Furthermore, we have approximated the probabilities for $d = 8$ and $d = 10$ using a numerical method and have got $p_8 \approx 0.012173$, $p_{10} \approx 0.0018$. These results suggest that p_d decreases approximately five times when d is increased by two.

If $P(x)$ is monic, reciprocal, with integer coefficients then $P_n(x)$ is a periodic sequence of polynomials if and only if $P(x)$ is the product of cyclotomic polynomials. In fact, if $P_n(x)$ is a periodic sequence, among these polynomials there are only finitely many distinct ones. Then the set of roots of these polynomials is also finite, and all the powers $\alpha, \alpha^2, \alpha^3, \ldots$ of a root α of $P(x)$ are in this set. Therefore for some p, q, $\alpha^p = \alpha^q$, $p \neq q$. Since $\alpha \neq 0$ it follows that $\alpha^{p-q} = 1$. Vice versa, if $P(x)$ is the product of cyclotomic polynomials then all its roots are roots of 1 so the set of its powers is finite and the set of coefficients $a_{k,n}$ for $k = 1, 2, \ldots, d-1$, $n = 1, 2, \ldots$ of $P_n(x)$ is also finite. Thus $P_n(x)$ is a periodic sequence of polynomials.

Acknowledgments. The author would like to thank the unknown referee for valuable comments that resulted in an improvement of this paper, especially for the calculation of the smallest n of the last seven Salem numbers in the Table 1.1, marked with *, which the referee generously permitted to be added in the Table.

TABLE 1.1. Salem number τ and the smallest n such that the minimal polynomial $P_n(x)$ of τ^n satisfies (1.2)

	d	τ	Coefficients	n
1.	4	1.72208381	1 −1 −1	9
2.	6	1.50613568	1 −1 0 −1	14
3.	8	1.28063816	1 0 0 −1 −1	72
4.	10	1.17628082	1 1 0 −1 −1 −1	605
5.	10	1.21639166	1 0 0 0 −1 −1	53
6.	10	1.23039143	1 0 0 −1 0 −1	240
7.	10	1.26123096	1 0 −1 0 0 −1	43
8.*	10	1.29348595	1 0 −1 −1 0 1	1367
9.*	12	1.24072642	1 −1 1 −1 0 0 −1	5894
10.*	14	1.20002652	1 0 0 −1 −1 0 0 1	61739
11.*	14	1.20261674	1 0 −1 0 0 0 0 −1	40389
12.*	14	1.25509351	1 0 −1 −1 0 1 0 −1	10824
13.*	14	1.26729644	1 −1 0 0 0 0 −1 1	48159
14.*	16	1.23631793	1 −1 0 0 0 0 0 0 −1	68667

2. Proofs of Theorems

In order to prove Theorem 1.1 we shall use a theorem of Kronecker [1, Theorem 4.6.4], which is a consequence of Weyl's theorems [2]. Suppose $\alpha = (\alpha_k)_{1\le k\le p} \in \mathbb{R}^p$ has the property that the real numbers $1, \alpha_1, \dots, \alpha_p$ are $\mathbb{Q}$-linearly independent, and let μ denote an arbitrary vector in $\mathbb{R}^p$, N an integer and ε a positive real number. Then Kronecker's theorem states that there exists an integer $n > N$ such that $\|n\alpha_k - \mu_k\| < \varepsilon$, $(k = 1, \dots, p)$ where $\|x\| = \min\{|x - m| : m \in \mathbb{Z}\}$ is the distance from x to the nearest integer.

Proof of Theorem 1.1.

Necessity. Suppose that $\tau > 1$ is a Salem number. The essence of the proof is to show that there is n such that each of $d - 2$ unimodal roots of $P_n(x)$ could be arbitrarily close to exactly one root of $x^{d-2} + 1$ (see [10, Lemma 2]) and to show that then the coefficients of $P_n(x)$ will satisfy the condition (1.2). It is obvious that roots of $x^{d-2} + 1$ are $\exp(\pm\frac{\pi+2j\pi}{d-2}i)$, $j = 0, 1, \dots, d/2 - 2$. We denote conjugates of τ by

$$\tau^{-1},\ \exp(\pm 2i\pi\omega_1), \dots, \exp(\pm 2i\pi\omega_{d/2-1}). \tag{2.1}$$

Numbers 1, $\omega_1, \dots, \omega_{d/2-1}$ are $\mathbb{Q}$-linearly independent [1, proof of Theorem 5.3.2]. According to the Kronecker's theorem consider $(w_j)_{2\le j\le d/2} \in \mathbb{R}^{d/2-1}$ with $\mu = (\frac{1/2+0}{d-2}, \frac{1/2+1}{d-2}, \dots, \frac{1/2+d/2-2}{d-2})$. It is clear that for every

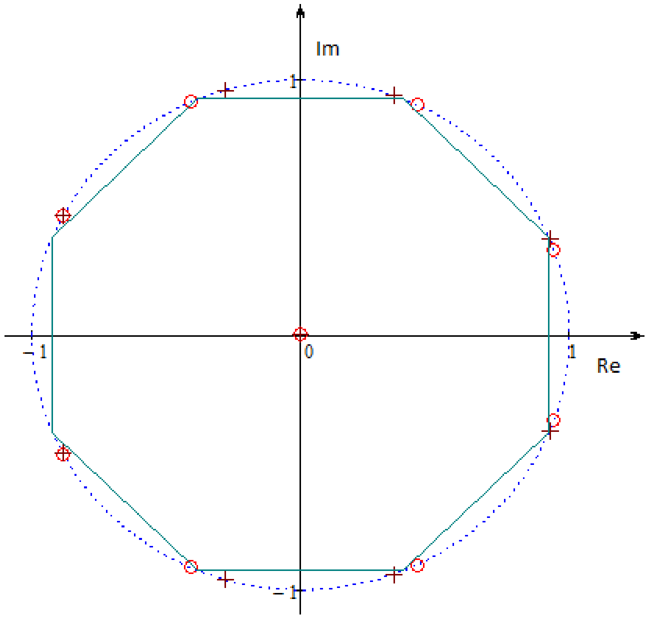

FIGURE 2.1. If τ is the seventh Salem number in Table 1.1 of degree 10 then conjugates of τ^{43} (represented with $\circ$) and of τ^{80} (represented with $+$), whose minimal polynomials $P_{43}(x)$, $P_{80}(x)$ satisfy (2), are close to roots of x^8+1, vertices of the regular octagon.

$\varepsilon > 0$ there exists an arbitrarily large integer n such that

$$\left| n\omega_j - \frac{1/2 + j - 1}{d-2} \right| < \varepsilon \pmod 1 \quad (j = 1, 2, \ldots, d/2 - 1). \tag{2.2}$$

Since a coefficient of a polynomial is a continuous function of its roots, for every $\epsilon > 0$ there exists an arbitrarily large integer n such that the minimal polynomial

$$\begin{aligned} P_n(x) &= (x - \tau^n)(x - \tau^{-n}) \\ &\quad \cdot \left(x^{d-2} + 1 + \sum_{j=1}^{d/2-2} \epsilon_j (x^{d-2-j} + x^j) + \epsilon_{d/2-1} x^{d/2-1} \right), \end{aligned} \tag{2.3}$$

of the Salem number τ^n satisfies $|\epsilon_k| < \epsilon$, $k = 1, \ldots, d/2 - 1$. We denote

$$-\tau^n - \tau^{-n} = T \tag{2.4}$$

$$
\begin{aligned}
P_n(x) &= (x^2 + Tx + 1) \\
&\qquad \cdot \left(x^{d-2} + 1 + \sum_{j=1}^{d/2-2} \epsilon_j (x^{d-2-j} + x^j) + \epsilon_{d/2-1} x^{d/2-1} \right) \\
&= x^d + 1 + (T + \epsilon_1)(x^{d-1} + x) + (\epsilon_2 + T\epsilon_1 + 1)(x^{d-2} + x^2) \\
&\qquad + \sum_{j=3}^{d/2-1} (\epsilon_j + T\epsilon_{j-1} + \epsilon_{j-2})(x^{d-j} + x^j) \\
&\qquad + (2\epsilon_{d/2-2} + T\epsilon_{d/2-1}) x^{d/2}.
\end{aligned}
$$

Now we consider the coefficients of $P_n(x)$ to show they satisfy the condition (1.2). It is obvious that $|a_{d-1,n}| = |T + \epsilon_1| \geq |T| - |\epsilon|$. We need to estimate

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{d}{d-2}\right)\left(2 + \sum_{k=2}^{d-2} |a_{k,n}|\right) \\
&\quad = \frac{1}{2}\left(\frac{d}{d-2}\right)\Bigg(2 + 2|\epsilon_2 + T\epsilon_1 + 1| + 2\sum_{j=3}^{d/2-1} |\epsilon_j + T\epsilon_{j-1} + \epsilon_{j-2}| \\
&\qquad\qquad + |2\epsilon_{d/2-2} + T\epsilon_{d/2-1}|\Bigg) \\
&\quad \leq \frac{1}{2}\left(\frac{d}{d-2}\right)\Bigg(2 + 2\epsilon_2 + 2|T|\epsilon_1 + 2 + 2\sum_{j=3}^{d/2-1} (\epsilon_j + |T|\epsilon_{j-1} + \epsilon_{j-2}) \\
&\qquad\qquad + 2\epsilon_{d/2-2} + |T|\epsilon_{d/2-1}\Bigg) \\
&\quad \leq \frac{1}{2}\left(\frac{d}{d-2}\right)(4 + (2d-8)\epsilon + (d-3)|T|\epsilon).
\end{aligned}
$$

So the condition (1.2) will be satisfied if

$$
|T| - |\epsilon| > \frac{1}{2}\left(\frac{d}{d-2}\right)(4 + (2d-8)\epsilon + (d-3)|T|\epsilon),
$$

which is equivalent to

$$
\frac{(2d-4)|T| - 4d}{(d^2-3d)|T| + 2d^2 - 6d - 4} > \epsilon, \quad d \geq 4. \tag{2.5}
$$

Since $|T| = \tau^n + \tau^{-n}$ tends to ∞ as $n \to \infty$ it is obvious that the left side of (2.5) tends to $D := \frac{2d-4}{d(d-3)}$ as $n \to \infty$. The determination of n such that coefficients of $P_n(x)$ satisfies (1.2) has to be done in following four steps:

(i) we choose ϵ such that $D > \epsilon > 0$;

(ii) we choose an integer N such that (2.5) will be fulfilled for all $n \geq N$;
(iii) we chose an $\varepsilon > 0$ such that if each of $d-2$ unimodal roots of a $P_n(x)$ is at the distance $< \varepsilon$ in modulus of exactly one root of $x^{d-2}+1$ then $|\epsilon_k| < \epsilon$, $k = 1, \dots, d/2-1$ is fulfilled in (2.3);
(iv) we chose $n \geq N$ such that (2.2) is fulfilled.

Sufficiency. Suppose that $\tau > 1$ is a real algebraic integer with conjugates $\tau_1 = \tau$, $\tau_2, \dots, \tau_d$ over $\mathbb{Q}$ such that τ^n has the minimal polynomial $P_n(x)$ which is also reciprocal of degree d, and satisfies the condition (1.2). If τ is a conjugate of τ' then τ^n is a conjugate of τ'^n. Since the minimal polynomial $P_n(x)$ of τ^n is of degree d so τ_1^n, $\tau_2^n, \dots, \tau_d^n$ must be different numbers and their product has to be 1 because $P_n(x)$ is monic and reciprocal. The polynomial $P_n(x)$ satisfies the condition (1.2) so it satisfies the condition (1.1) of Vieira's theorem where $l = 1$. According to the theorem there are $d-2$ roots of $P_n(x)$ on the boundary of the unit disc $|z| = 1$. Since they occur in conjugate complex pairs their product is equal to 1. It follows that τ^{-n} should be a conjugate of τ^n which allow us to conclude that τ^n is a Salem number. If $|\tau'^n| = 1$ then $|\tau'| = 1$ thus it follows that there are $d-2$ conjugates of τ on the boundary of the unit disc. Finally, in the same manner as for τ^n, we conclude that τ is also a Salem number. □

Proof of Theorem 1.2.

(1). If we use (2.1) and denote $D := \tau^n + 1/\tau^n$ $(d = 4)$ we have

$$P_n(x) = (x^2 - Dx + 1)\left(x^2 - 2\cos(2\pi n\omega_1)x + 1\right).$$

We denote $2\pi\{n\omega_1\}$ by θ_1 and $2\cos(\theta_1)$ by s_1 where $\{\cdot\}$ denotes the fractional part. Since $n\omega_1$ is uniformly distributed modulo one θ_1 is uniformly distributed on $[0, 2\pi]$. For $d = 4$ the condition (1.2) is reduced to $|a_{3,n}| > 2 + |a_{2,n}|$. Since $P_n(x) = (x^2 - Dx + 1)(x^2 - s_1x + 1)$ the condition becomes

$$|-D - s_1| > 2 + |Ds_1 + 2|. \tag{2.6}$$

From the definition of D it is obvious that $D \to \infty$ when $n \to \infty$. Since $|s_1| \leq 2$ we have $D + s_1 \to \infty$ so that $|-D - s_1| = |D + s_1|$ is equal, for every sufficiently large n, to $D + s_1$. Finally (2.6) becomes $D + s_1 > 2 + |Ds_1 + 2|$ i.e. $D + s_1 - 2 > Ds_1 + 2 > -D - s_1 + 2$. Solving this double inequality for s_1 we get

$$-\frac{D}{D+1} < s_1 < \frac{D-4}{D-1}.$$

When n_0 tends to infinity we obtain $-1 < s_1 < 1$ i.e. $-1/2 < \cos\theta_1 < 1/2$. It follows that $\pi/3 < \theta_1 < 2\pi/3$ or $4\pi/3 < \theta_1 < 5\pi/3$ so that the limit of the probability has to be $p_4 = \frac{2\pi/3}{2\pi} = \frac{1}{3}$.

(2). Using (2.1) with $d=6$ and the definition of D we have

$$P_n(x) = (x^2 - Dx + 1)(x^2 - 2\cos(2\pi n\omega_1)x + 1)(x^2 - 2\cos(2\pi n\omega_2)x + 1).$$

We denote $\theta_1 := 2\pi\{n\omega_1\}$, $\theta_2 := 2\pi\{n\omega_2\}$. Coefficients of $P_n(x)$ depend only on real parts of unimodal roots so that we can chose the complex conjugates from the upper half (complex) plane. Thus we define

$$t_i = \begin{cases} \theta_i & \text{if } \theta_i \in (0,\pi); \\ 2\pi - \theta_i & \text{if } \theta_i \in (\pi, 2\pi), \end{cases} \qquad i = 1,2. \tag{2.7}$$

Since $n\omega_1$, $n\omega_2$ are uniformly distributed modulo one θ_1, θ_2 are uniformly distributed on $[0,2\pi]$ and t_1, t_2 are uniformly distributed on $[0,\pi]$. We denote

$$s_1 := 2\cos(t_1), \quad s_2 := 2\cos(t_2). \tag{2.8}$$

For $d=6$ the condition (1.2) is reduced to $|a_{5,n}| > \frac{6}{8}(2 + 2|a_{4,n}| + |a_{3,n}|)$. Since

$$P_n(x) = (x^2 - Dx + 1)(x^2 - s_1x + 1)(x^2 - s_2x + 1)$$

the condition becomes

$$\begin{aligned} &|-D - s_1 - s_2| \\ &\quad > \frac{6}{8}(2 + 2|Ds_1 + Ds_2 + s_1s_2 + 3| + |-2D - 2s_1 - 2s_2 - Ds_1s_2|). \end{aligned} \tag{2.9}$$

The main idea of the proof is to determine the region S in s_1Os_2 plane such that every point $(s_1,s_2) \in S$ satisfies (2.9). Since $D \to \infty$ when $n \to \infty$, $|s_1| \le 2$, $|s_2| \le 2$ we conclude that the left side in (2.9) $|-D - s_1 - s_2| = |D + s_1 + s_2|$ is equal, for every sufficiently large n, to $D + s_1 + s_2$. We can find the boundary of S if we replace $>$ in (2.9) with $=$ and if we replace both $|\cdot|$ on the right side with $\pm(\cdot)$. There are four possibilities for replacing so we get four equations which we solve for s_2. We get rational functions $s_2 = f_i(D,s_1)$ which tends to $s_2 = F_i(s_1)$ when $n_0 \to \infty$, $i = 1,2,3,4$:

$$\begin{aligned} f_1(D,s_1) &= \frac{10D + 10s_1 - 6Ds_1 - 24}{6D + 6s_1 - 3Ds_1 - 10}, & F_1(s_1) &= \frac{10 - 6s_1}{6 - 3s_1}, \\ f_2(D,s_1) &= -\frac{2D + 2s_1 + 6Ds_1 + 24}{6D + 6s_1 + 3Ds_1 + 2}, & F_2(s_1) &= -\frac{2 + 6s_1}{6 + 3s_1}, \\ f_3(D,s_1) &= -\frac{10D + 10s_1 + 6Ds_1 + 12}{6D + 6s_1 + 3Ds_1 + 10}, & F_3(s_1) &= -\frac{10 + 6s_1}{6 + 3s_1}, \\ f_4(D,s_1) &= \frac{2D + 2s_1 - 6Ds_1 - 12}{6D + 6s_1 - 3Ds_1 - 2}, & F_4(s_1) &= \frac{2 - 6s_1}{6 - 3s_1}. \end{aligned}$$

The boundary of S consists of parts of graphs of $F_i(s_1)$. We have to find intersection points of these graphs. Therefore we solve four equations:

$$\begin{aligned} &F_1(s_1) = F_2(s_1) \Rightarrow s_1 = 1/3 \pm \sqrt{19}/3, \\ &F_1(s_1) = F_3(s_1) \Rightarrow s_1 = \pm\sqrt{30}/3, \\ &F_4(s_1) = F_2(s_1) \Rightarrow s_1 = \pm\sqrt{6}/3, \\ &F_4(s_1) = F_3(s_1) \Rightarrow s_1 = -1/3 \pm \sqrt{19}/3. \end{aligned} \tag{2.10}$$

We have to determine the area of the region T in t_1Ot_2 plane such that for every point $(s_1, s_2) \in S$ there is unique $(t_1, t_2) \in T$ where

$$t_1 = \arccos(s_1/2),\ t_2 = \arccos(s_2/2), \tag{2.11}$$

using (2.8). The ratio of the area of T to the area of all possible values (t_1, t_2), i.e. π^2, is equal to the probability p_6. Since $s_2 = F_i(s_1)$ it follows that $t_2 = \arccos(F_i(2\cos(t_1))/2) =: G_i(t_1)$ using (2.8). For the determination of the area of T it is convenient to show that T has reflection symmetry across the line $t_2 = \pi - t_1$. Let the graph of $t_2 = G_i(t_1)$ be Γ_i. We claim that Γ_1 can be obtained by reflecting of Γ_3 about the line $t_2 = \pi - t_1$ i.e. if $(t_1, t_2) \in \Gamma_1$ then $(\pi - t_2, \pi - t_1) \in \Gamma_3$ (see Figure 2.2). Indeed, if $t_2 = G_1(t_1)$ then

$$\begin{aligned} G_3(\pi - t_2) &= \arccos(F_3(2\cos(\pi - t_2))/2) \\ &= \arccos(F_3(-2\cos(t_2))/2) \\ &= \arccos(F_3(-2\cos(G_1(t_1)))/2) \\ &= \arccos(F_3(-F_1(s_1))/2) \\ &= \arccos(-s_1/2) \\ &= \arccos(-\cos(t_1)) \\ &= \arccos(\cos(\pi - t_1)) \\ &= \pi - t_1. \end{aligned}$$

In the same manner we can show that Γ_2 is a reflection of Γ_4 in the line $t_2 = \pi - t_1$. Therefore T consists of four congruent curve-triangles, each of them has the same area A (see Figure 2.2). If we bring to mind the intersection points (2.10) and formulas (2.11) we find out the intersection points of graphs Γ_i, $i = 1, 2, 3, 4$ which are the limits of two definite integrals that occur in (1.3). We conclude that A is equal to sum of these integrals (see Figure 2.2) and that $p_6 = 4A/\pi^2$ as it is claimed. □

If we use the same method for the determination of p_8, p_{10} etc. it requires multiple definite integrals applied on the regions with complicated boundaries. Thus it is much more convenient to use a numerical approach. For each pair of conjugate complex roots of a Salem polynomial we define a variable $t_i \in (0, \pi)$, as in (2.7) and s_i as in (2.8) $i = 1, 2, \ldots, H$ where we

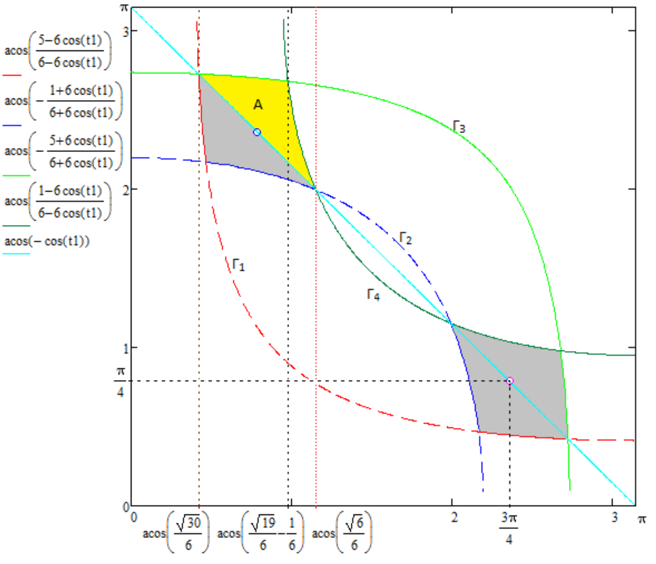

FIGURE 2.2. The event T that a power of a Salem number of degree 6 has the minimal polynomial which satisfies the condition (1.2) is shaded in the figure. It consists of four congruent curve-triangles, each of them has the same area A which is equal to the definite integral. Thus the probability of T is $4A/\pi^2 = 0.0717258\ldots$.

denoted $(d-2)/2$ by H. Let $m \in \mathbb{N}$ and let $0 = t_{i,0}, t_{i,1}, \ldots, t_{i,m} = \pi$, $i = 1, 2, \ldots, H$ be nodes arranged consecutively with equal spacing $h = \pi/m$. Starting from

$$P_n(x) = (x^2 - Dx + 1) \prod_{i=1}^{H} (x^2 - 2\cos(t_i)x + 1)$$

we calculate the coefficients of $P_n(x)$ which obviously depend on D, t_i so that there are the functions $A_{k,n}$ such that

$$a_{k,n} = A_{k,n}(D; t_1, t_2, \ldots, t_H),\ k = 1, 2, \ldots, d-1.$$

For D fixed and for each H-tuple $(t_{1,j_1}, t_{2,j_2}, \ldots, t_{H,j_H})$ we calculate

$$a_{k,n} = A_{k,n}(D; t_{1,j_1}, t_{2,j_2}, \ldots, t_{H,j_H}),\ j_i = 0, 1, \ldots, m,$$

and replace them into the condition (1.2). The number N_c of all H-tuples, i.e. of all points of π^H, which satisfy this condition, divided with $(m+1)^H$, the number of all H-tuples, approximates p_d. If we take a large $D = 10^9$ and a small $h \geq 0.002$ we get $p_8 \approx 0.012173$, $p_{10} \approx 0.0018$. Since there are four nested loops the calculation of p_{10} requires much CPU time. Thus it

TABLE 2.1. Coefficients of $P_{43}(x)$, $P_{80}(x)$ which satisfy (2) and of $P_{100}(x)$ which does not, where $P(x)$ is the minimal polynomial of the seventh Salem number in Table 1.1. The modulus of the linear coefficient of $P_{43}(x)$, $P_{80}(x)$ is relatively large.

$P_{43}(x)$	$P_{80}(x)$	$P_{100}(x)$
1	1	1
−21586	−115763027	−12007769482
3611	23986075	29164508197
688	−39926871	−18134706516
5418	20167702	−25180138718
−6193	4830711	52256753515

was necessary to improve our programm. We use the fact that all H-tuples which satisfy (1.2) are close to the point $P(\pi/H, 3\pi/H, \ldots, (d-3)\pi/H)$ or to $H!$ points obtained by permuting the coordinates of P, because these coordinates are the arguments of the roots of $x^{d-2}+1$. Therefore to get N_c we have to check and count only points in a small region around the P and then to multiply the number of them by $H!$.

We have also verified p_4 and p_6 experimentally. For the first Salem number in the Table 1.1 of degree 4 we have found that if $1 \le n \le 300$ then the coefficients of $P_n(x)$ satisfy (1.2) 98 times: for 9, 13, 16, 17, 20, 24, 27, 31, 35, 38, 42, 45, 46, 49, 53, 56, 57, 60, 64, 67, 68, 71, 75, 78, 79, 82, 86, 89, 93, 97, 100, 104, 107, 108, 111, 115, 118, 122, 126, 129, 130, 133, 137, 140, 141, 144, 148, 151, 155, 159,162, 166, 169, 170, 173, 177, 180, 181, 184, 188, 191, 192, 195, 199, 202, 203, 206, 210, 213, 217, 221, 224, 228, 231, 232, 235, 239, 242, 243, 244, 246, 250, 253, 254, 257, 261, 264, 265, 268, 272, 275, 279, 283, 286, 290, 293, 294, 297, so that the relative frequency is

$$98/300 \approx 0.33. \tag{2.12}$$

For the second Salem number in the Table 1.1 of degree 6 we have found that if $101 \le n \le 300$ then the event that $P_n(x)$ satisfies (1.2) occurs fourteen times: for $n = 116$, 144, 157, 167, 187, 195, 206, 225, 238, 246, 257, 276, 287, 295 so that the relative frequency is $14/200 = 0.07$. If $1001 \le n \le 1200$ then $P_n(x)$ satisfy (1.2) sixteen times: for $n = 1001$, 1029, 1031, 1039, 1050, 1052, 1063, 1080, 1082, 1101, 1103, 1120, 1131, 1133, 1152, 1182 with the relative frequency

$$16/200 = 0.08. \tag{2.13}$$

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