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Résumé. Nous étudions la discrépance L_p ($p \in [1, \infty)$) de réseaux digitaux de dimension 2. En 2001, Larcher et Pillichshammer ont identifié une classe de $(0, n, 2)$-réseaux pour lesquels la version symétrisée au sens de Davenport a une discrépance L_2 d'ordre $\sqrt{\log N}/N$, qui est optimal grâce au résultat célèbre de Roth. Cependant la question de savoir si la même borne s'applique à la discrépance des réseaux originaux est restée ouverte.

Dans cet article, nous identifions les réseaux digitaux de la classe susmentionnée pour lesquels la symétrisation n'est pas nécessaire pour obtenir l'ordre optimal de la discrépance L_p pour $p \in [1, \infty)$.

Ce résultat est dans l'esprit d'un article de Bilyk de 2013, qui a étudié la discrépance L_2 des ensembles des points de la forme $(k/N, \{k\alpha\})$ pour $k = 0, 1, \ldots, N-1$ et a donné des propriétés diophantiennes de α qui garantissent l'ordre optimal de la discrépance L_2.

Abstract. We study the L_p discrepancy of two-dimensional digital nets for finite p. In the year 2001 Larcher and Pillichshammer identified a class of digital nets for which the symmetrized version in the sense of Davenport has L_2 discrepancy of the order $\sqrt{\log N}/N$, which is best possible due to the celebrated result of Roth. However, it remained open whether this discrepancy bound also holds for the original digital nets without any modification.

In the present paper we identify nets from the above mentioned class for which the symmetrization is not necessary in order to achieve the optimal order of L_p discrepancy for all $p \in [1, \infty)$.

Our findings are in the spirit of a paper by Bilyk from 2013, who considered the L_2 discrepancy of lattices consisting of the elements $(k/N, \{k\alpha\})$ for $k = 0, 1, \ldots, N-1$, and who gave Diophantine properties of α which guarantee the optimal order of L_2 discrepancy.

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1. Introduction

Discrepancy is a measure for the irregularities of point distributions in the unit interval (see, e.g., [10]). Here we study point sets $\mathcal{P}$ with N elements in the two-dimensional unit interval $[0,1)^2$. We define the *discrepancy function* of such a point set by

$$\Delta_{\mathcal{P}}(\boldsymbol{t}) = \frac{1}{N}\sum_{\boldsymbol{z}\in\mathcal{P}} \mathbf{1}_{[\mathbf{0},\boldsymbol{t})}(\boldsymbol{z}) - t_1 t_2,$$

where for $\boldsymbol{t} = (t_1,t_2) \in [0,1]^2$ we set $[\mathbf{0},\boldsymbol{t}) = [0,t_1)\times[0,t_2)$ with area t_1t_2 and denote by $\mathbf{1}_{[\mathbf{0},\boldsymbol{t})}$ the indicator function of this interval. The L_p *discrepancy* for $p\in[1,\infty)$ of $\mathcal{P}$ is given by

$$L_p(\mathcal{P}) := \|\Delta_{\mathcal{P}}\|_{L_p([0,1]^2)} = \left(\int_{[0,1]^2} |\Delta_{\mathcal{P}}(\boldsymbol{t})|^p \,\mathrm{d}\boldsymbol{t}\right)^{\frac{1}{p}}$$

and the *star discrepancy* or L_∞ *discrepancy* of $\mathcal{P}$ is defined as

$$L_\infty(\mathcal{P}) := \|\Delta_{\mathcal{P}}\|_{L_\infty([0,1]^2)} = \sup_{\boldsymbol{t}\in[0,1]^2} |\Delta_{\mathcal{P}}(\boldsymbol{t})|.$$

The L_p discrepancy is a quantitative measure for the irregularity of distribution of a point set. Furthermore, it is intimately related to the worst-case integration error of quasi-Monte Carlo rules; see [3, 10, 13, 16].

Throughout this paper we use the following notation: for functions $f,g : \mathbb{N}\to\mathbb{R}^+$, we write $g(N)\lesssim f(N)$, if there exists a $C>0$ such that $g(N)\le Cf(N)$ for all $N\in\mathbb{N}$ with a positive constant C that is independent of N. Likewise, we write $g(N)\gtrsim f(N)$ if $g(N)\ge Cf(N)$ for all $N\in\mathbb{N}$. Further, we write $f(N)\asymp g(N)$ if the relations $g(N)\lesssim f(N)$ and $g(N)\gtrsim f(N)$ hold simultaneously. If the implied constants depend on some parameter, say p, then this is denoted by $\lesssim_p$, $\gtrsim_p$, or $\asymp_p$, respectively.

It is well known that for every $p\in[1,\infty)$ we have

$$L_p(\mathcal{P}) \gtrsim_p \frac{\sqrt{\log N}}{N}, \tag{1.1}$$

for every $N\in\mathbb{N}$ and every N-element point set $\mathcal{P}$ in $[0,1)^2$. Here log denotes the natural logarithm. This was first shown by Roth [18] for $p=2$ and hence for all $p\in[2,\infty]$ and later by Schmidt [20] for all $p\in(1,2)$. The case $p=1$ was added by Halász [5]. For the star discrepancy we have according to Schmidt [19] that

$$L_\infty(\mathcal{P}) \gtrsim \frac{\log N}{N}, \tag{1.2}$$

for every $N\in\mathbb{N}$ and every N-element point set $\mathcal{P}$ in $[0,1)^2$.

Irrational lattices. It is well-known, that the lower bounds in (1.1) and (1.2) are best possible in the order of magnitude in N. For example, when the irrational number $\alpha = [a_0; a_1, a_2, \ldots]$ has bounded partial quotients in it's continued fraction expansion, then the lattice $\mathcal{P}_\alpha$ consisting of the points $(k/N, \{k\alpha\})$ for $k = 0, 1, \ldots, N-1$, where $\{\cdot\}$ denotes reduction modulo one, has optimal order of star discrepancy in the sense of (1.2) (see, e.g., [14] or [16, Corollary 3.5 in combination with Lemma 3.7]). This is, in this generality, not true anymore when, e.g., the L_2 discrepancy is considered. However, in 1956 Davenport [2] showed that the symmetrized version $\mathcal{P}_\alpha^{\mathrm{sym}} := \mathcal{P}_\alpha \cup \mathcal{P}_{-\alpha}$ of $\mathcal{P}_\alpha$ consisting of $2N$ points has L_2 discrepancy of the order $\sqrt{\log N}/N$ which is optimal with respect to (1.1).

Later Bilyk [1] introduced a further condition on α which guarantees the optimal order of L_2 discrepancy without the process of symmetrization. He showed that if α has bounded partial quotients, then

$$L_2(\mathcal{P}_\alpha) \asymp_\alpha \frac{\sqrt{\log N}}{N} \quad \text{if and only if} \quad \left|\sum_{k=0}^{N-1} (-1)^k a_k\right| \lesssim_\alpha \sqrt{n}.$$

Digital nets. In this paper we study analog questions for digital nets over $\mathbb{Z}_2$, which are an important class of point sets with low star discrepancy. Since we only deal with digital nets over $\mathbb{Z}_2$ and in dimension 2 we restrict the necessary definitions to this case. For the general setting we refer to the books of Niederreiter [16] (see also [15]), of Dick and Pillichshammer [3], or of Leobacher and Pillichshammer [13].

Let $n \in \mathbb{N}$ and let $\mathbb{Z}_2$ be the finite field of order 2, which we identify with the set $\{0, 1\}$ equipped with arithmetic operations modulo 2. A two-dimensional digital net over $\mathbb{Z}_2$ is a point set $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{2^n-1}\}$ in $[0, 1)^2$, which is generated by two $n \times n$ matrices over $\mathbb{Z}_2$. The procedure is as follows.

(1) Choose two $n \times n$ matrices C_1 and C_2 with entries from $\mathbb{Z}_2$.
(2) For $r \in \{0, 1, \ldots, 2^n - 1\}$ let $r = r_0 + 2r_1 + \cdots + 2^{n-1} r_{n-1}$ with $r_i \in \{0, 1\}$ for all $i \in \{0, \ldots, n-1\}$ be the dyadic expansion of r, and set $\vec{r} = (r_0, \ldots, r_{n-1})^\top \in \mathbb{Z}_2^n$.
(3) For $j = 1, 2$ compute $C_j \vec{r} =: (y_{r,1}^{(j)}, \ldots, y_{r,n}^{(j)})^\top \in \mathbb{Z}_2^n$, where all arithmetic operations are over $\mathbb{Z}_2$.
(4) For $j = 1, 2$ compute $x_r^{(j)} = \frac{y_{r,1}^{(j)}}{2} + \cdots + \frac{y_{r,n}^{(j)}}{2^n}$ and set $\boldsymbol{x}_r = (x_r^{(1)}, x_r^{(2)}) \in [0, 1)^2$.
(5) Set $\mathcal{P} := \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{2^n-1}\}$. We call $\mathcal{P}$ a *digital net over* $\mathbb{Z}_2$ generated by C_1 and C_2.

One of the most well-known digital nets is the *2-dimensional Hammersley net* $\mathcal{P}^{\mathrm{Ham}}$ *in base 2* which is generated by the matrices

$$C_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Due to the choice of C_1 the first coordinates of the elements of the Hammersley net are $x_r^{(1)} = r/2^n$ for $r = 0, 1, \ldots, 2^n - 1$.

$(0, n, 2)$-nets in base 2. A point set $\mathcal{P}$ consisting of 2^n elements in $[0,1)^2$ is called a $(0, n, 2)$-*net in base 2*, if every dyadic box

$$\left[\frac{m_1}{2^{j_1}}, \frac{m_1+1}{2^{j_1}}\right) \times \left[\frac{m_2}{2^{j_2}}, \frac{m_2+1}{2^{j_2}}\right),$$

where $j_1, j_2 \in \mathbb{N}_0$ and $m_1 \in \{0, 1, \ldots, 2^{j_1} - 1\}$ and $m_2 \in \{0, 1, \ldots, 2^{j_2} - 1\}$ with volume 2^{-n}, i.e. with $j_1 + j_2 = n$, contains exactly one element of $\mathcal{P}$.

It is well known that a digital net over $\mathbb{Z}_2$ is a $(0, n, 2)$-net in base 2 if and only if the following condition holds: For every choice of integers $d_1, d_2 \in \mathbb{N}_0$ with $d_1 + d_2 = n$ the first d_1 rows of C_1 and the first d_2 rows of C_2 are linearly independent.

Every digital $(0, n, 2)$-net achieves the optimal order of star discrepancy in the sense of (1.2), whereas there exist nets which do not have the optimal order of L_p discrepancy for finite p. One example is the Hammersley net as defined above for which we have (see [4, 12, 17])

$$L_p(\mathcal{P}^{\mathrm{Ham}}) = \left(\left(\frac{n}{8 \cdot 2^n}\right)^p + O(n^{p-1})\right)^{1/p} \quad \text{for all } p \in [1, \infty)$$

and

$$L_\infty(\mathcal{P}^{\mathrm{Ham}}) = \frac{1}{2^n}\left(\frac{n}{3} + \frac{13}{9} - (-1)^n \frac{4}{9 \cdot 2^n}\right).$$

Symmetrized nets. Motivated by the results of Davenport for irrational lattices, Larcher and Pillichshammer [11] studied the symmetrization of digital nets. Let $\boldsymbol{x}_r = (x_r, y_r)$ for $r = 0, 1, \ldots, 2^n - 1$ be the elements of a digital net generated by the matrices

$$C_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ 0 & 1 & \cdots & a_{2,n-1} & a_{2,n} \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & \cdots & 1 & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

with entries $a_{j,k} \in \mathbb{Z}_2$ for $1 \leq j < k \leq n$. The matrix C_2 is a so-called "*non-singular upper triangular (NUT) matrix*". Then the *symmetrized net*

$\mathcal{P}^{\mathrm{sym}}$ consisting of (x_r, y_r) and $(x_r, 1 - y_r)$ for $r = 0, 1, \dots, 2^n - 1$ has L_2 discrepancy of optimal order

$$L_2(\mathcal{P}^{\mathrm{sym}}) \asymp \frac{\sqrt{n}}{2^{n+1}} \quad \text{for every } n \in \mathbb{N}.$$

In the present paper we show in the spirit of the paper of Bilyk [1] that there are NUT matrices C_2 such that symmetrization is not required in order to achieve the optimal order of L_2 discrepancy. Our result will be true for the L_p discrepancy for all finite p and not only for the L_2 case.

2. The result

The central aim of this paper is to provide conditions on the generating matrices C_1, C_2 which lead to the optimal order of L_p discrepancy of the corresponding nets. We do so for a class of nets which are generated by $n \times n$ matrices over $\mathbb{Z}_2$ of the following form:

$$C_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{2.1}$$

and a NUT matrix of the special form

$$C_2 = \begin{pmatrix} 1 & a_1 & a_1 & \cdots & a_1 & a_1 & a_1 \\ 0 & 1 & a_2 & \cdots & a_2 & a_2 & a_2 \\ 0 & 0 & 1 & \cdots & a_3 & a_3 & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{n-2} & a_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & 1 & a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \tag{2.2}$$

where $a_i \in \mathbb{Z}_2$ for all $i \in \{1, \dots, n-1\}$. We study the L_p discrepancy of the digital net $\mathcal{P}_{\boldsymbol{a}}$ generated by C_1 and C_2, where $\boldsymbol{a} = (a_1, \dots, a_{n-1}) \in \mathbb{Z}_2^{n-1}$. The set $\mathcal{P}_{\boldsymbol{a}}$ can be written as

$$\mathcal{P}_{\boldsymbol{a}} = \left\{ \left(\frac{t_n}{2} + \cdots + \frac{t_1}{2^n}, \frac{b_1}{2} + \cdots + \frac{b_n}{2^n} \right) : t_1, \dots, t_n \in \{0, 1\} \right\}, \tag{2.3}$$

where $b_k = t_k \oplus a_k(t_{k+1} \oplus \cdots \oplus t_n)$ for $k \in \{1, \dots, n-1\}$ and $b_n = t_n$. The operation $\oplus$ denotes addition modulo 2.

The following result states that the order of the L_p discrepancy of the digital nets $\mathcal{P}_{\boldsymbol{a}}$ is determined by the number of zero elements in $\boldsymbol{a}$.

Theorem 2.1. *Let $h_n = h_n(\boldsymbol{a}) = \sum_{i=1}^{n-1}(1-a_i)$ be the number of zeroes in the tuple $\boldsymbol{a}$. Then we have for all $p \in [1,\infty)$*

$$L_p(\mathcal{P}_{\boldsymbol{a}}) \asymp_p \frac{\max\{\sqrt{n}, h_n(\boldsymbol{a})\}}{2^n}.$$

In particular, the net $\mathcal{P}_{\boldsymbol{a}}$ achieves the optimal order of L_p discrepancy for all $p \in [1,\infty)$ if and only if $h_n(\boldsymbol{a}) \lesssim \sqrt{n}$.

The proof of Theorem 2.1, which will be given in Section 3, is based on Littlewood–Paley theory and tight estimates of the Haar coefficients of the discrepancy function $\Delta_{\mathcal{P}_{\boldsymbol{a}}}$.

For example, if $\boldsymbol{a} = \boldsymbol{0} := (0,0,\dots,0)$ we get the Hammersley net $\mathcal{P}^{\rm Ham}$ in dimension 2. We have $h_n(\boldsymbol{0}) = n-1$ and hence

$$L_p(\mathcal{P}_{\boldsymbol{0}}) \asymp_p \frac{n}{2^n}.$$

If $\boldsymbol{a} = \boldsymbol{1} := (1,1,\dots,1)$, then we have $h_n(\boldsymbol{1}) = 0$ and hence

$$L_p(\mathcal{P}_{\boldsymbol{1}}) \asymp_p \frac{\sqrt{n}}{2^n}.$$

Remark 2.2. The approach via Haar functions allows the precise computation of the L_2 discrepancy of digital nets via Parseval's identity. We did so for a certain class of nets in [9]. It would be possible but tedious to do the same for the class $\mathcal{P}_{\boldsymbol{a}}$ of nets considered in this paper. However, we only executed the massive calculations for the special case where $\boldsymbol{a} = \boldsymbol{1} := (1,1,\dots,1)$, hence where C_2 is a NUT matrix filled with ones in the upper right triangle. The exact value of its L_2 discrepancy is given by

$$L_2(\mathcal{P}_{\boldsymbol{1}}) = \frac{1}{2^n}\left(\frac{5n}{192} + \frac{15}{32} + \frac{1}{4\cdot 2^n} - \frac{1}{72\cdot 2^{2n}}\right)^{1/2}. \tag{2.4}$$

We omit the lengthy proof, but its correctness may be checked with Warnock's formula [22] (see also [3, Proposition 2.15]) for small values of n. Compare (2.4) with the exact L_2 discrepancy of $\mathcal{P}^{\rm Ham} = \mathcal{P}_{\boldsymbol{0}}$ which is given by (see [4, 6, 17, 21])

$$L_2(\mathcal{P}_{\boldsymbol{0}}) = \frac{1}{2^n}\left(\frac{n^2}{64} + \frac{29n}{192} + \frac{3}{8} - \frac{n}{16\cdot 2^n} + \frac{1}{4\cdot 2^n} - \frac{1}{72\cdot 2^{2n}}\right)^{1/2}.$$

3. The proof of Theorem 2.1 via Haar expansion of the discrepancy function

A dyadic interval of length $2^{-j}, j \in \mathbb{N}_0$, in $[0,1)$ is an interval of the form

$$I = I_{j,m} := \left[\frac{m}{2^j}, \frac{m+1}{2^j}\right) \quad \text{for} \quad m \in \{0,1,\dots,2^j-1\}.$$

The left and right half of $I_{j,m}$ are the dyadic intervals $I_{j+1,2m}$ and $I_{j+1,2m+1}$, respectively. The Haar function $h_{j,m}$ is the function on $[0,1)$ which is $+1$

on the left half of $I_{j,m}$, -1 on the right half of $I_{j,m}$ and 0 outside of $I_{j,m}$. The L_∞-normalized Haar system consists of all Haar functions $h_{j,m}$ with $j \in \mathbb{N}_0$ and $m = 0, 1, \ldots, 2^j - 1$ together with the indicator function $h_{-1,0}$ of $[0,1)$. Normalized in $L_2([0,1))$ we obtain the orthonormal Haar basis of $L_2([0,1))$.

Let $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$ and define $\mathbb{D}_j = \{0, 1, \ldots, 2^j - 1\}$ for $j \in \mathbb{N}_0$ and $\mathbb{D}_{-1} = \{0\}$. For $\boldsymbol{j} = (j_1, j_2) \in \mathbb{N}_{-1}^2$ and $\boldsymbol{m} = (m_1, m_2) \in \mathbb{D}_{\boldsymbol{j}} := \mathbb{D}_{j_1} \times \mathbb{D}_{j_2}$, the Haar function $h_{\boldsymbol{j},\boldsymbol{m}}$ is given as the tensor product

$$h_{\boldsymbol{j},\boldsymbol{m}}(\boldsymbol{t}) = h_{j_1,m_1}(t_1) h_{j_2,m_2}(t_2) \quad \text{ for } \boldsymbol{t} = (t_1, t_2) \in [0,1)^2.$$

We speak of $I_{\boldsymbol{j},\boldsymbol{m}} = I_{j_1,m_1} \times I_{j_2,m_2}$ as dyadic boxes with level

$$|\boldsymbol{j}| = \max\{0, j_1\} + \max\{0, j_2\},$$

where we set $I_{-1,0} = \mathbf{1}_{[0,1)}$. The system

$$\left\{ 2^{\frac{|\boldsymbol{j}|}{2}} h_{\boldsymbol{j},\boldsymbol{m}} : \boldsymbol{j} \in \mathbb{N}_{-1}^2, \boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}} \right\}$$

is an orthonormal basis of $L_2([0,1)^2)$ and we have Parseval's identity which states that for every function $f \in L_2([0,1)^2)$ we have

$$\|f\|_{L_2([0,1)^2)}^2 = \sum_{\boldsymbol{j} \in \mathbb{N}_{-1}^2} 2^{|\boldsymbol{j}|} \sum_{\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2, \tag{3.1}$$

where the numbers $\mu_{\boldsymbol{j},\boldsymbol{m}} = \mu_{\boldsymbol{j},\boldsymbol{m}}(f) = \langle f, h_{\boldsymbol{j},\boldsymbol{m}} \rangle = \int_{[0,1)^2} f(\boldsymbol{t}) h_{\boldsymbol{j},\boldsymbol{m}}(\boldsymbol{t}) \, \mathrm{d}\boldsymbol{t}$ are the so-called Haar coefficients of f. There is no such identity for the L_p norm of f for $p \neq 2$; however, for a function $f \in L_p([0,1)^2)$ we have a so-called Littlewood–Paley inequality. It involves the square function $S(f)$ of a function $f \in L_p([0,1)^2)$ which is given as

$$S(f) = \left(\sum_{\boldsymbol{j} \in \mathbb{N}_{-1}^2} \sum_{\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} 2^{2|\boldsymbol{j}|} \, |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 \, \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right)^{1/2},$$

where $\mathbf{1}_I$ is the indicator function of I.

Lemma 3.1 (Littlewood–Paley inequality)**.** *Let $p \in (1, \infty)$ and let $f \in L_p([0,1)^2)$. Then*

$$\|S(f)\|_{L_p} \asymp_p \|f\|_{L_p}.$$

In the following let $\mu_{\boldsymbol{j},\boldsymbol{m}}$ denote the Haar coefficients of the local discrepancy function $\Delta_{\mathcal{P}_{\boldsymbol{a}}}$, i.e.,

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = \int_{[0,1)^2} \Delta_{\mathcal{P}_{\boldsymbol{a}}}(\boldsymbol{t}) h_{\boldsymbol{j},\boldsymbol{m}}(\boldsymbol{t}) \, \mathrm{d}\boldsymbol{t}.$$

In order to estimate the L_p discrepancy of $\mathcal{P}_{\boldsymbol{a}}$ by means of Lemma 3.1 we require good estimates of the Haar coefficients $\mu_{\boldsymbol{j},\boldsymbol{m}}$. This is a very technical and tedious task which we defer to the appendix. In the following we just collect the obtained bounds:

Lemma 3.2. *Let $\boldsymbol{j} = (j_1, j_2) \in \mathbb{N}_0^2$. Then*

(1) *if $j_1 + j_2 \leq n - 3$ and $j_1, j_2 \geq 0$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}| \lesssim 2^{-2n}$.*
(2) *if $j_1 + j_2 \geq n - 2$ and $0 \leq j_1, j_2 \leq n$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}| \lesssim 2^{-n-j_1-j_2}$ and $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = 2^{-2j_1-2j_2-4}$ for all but at most 2^n coefficients $\mu_{\boldsymbol{j},\boldsymbol{m}}$ with $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$.*
(3) *if $j_1 \geq n$ or $j_2 \geq n$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = 2^{-2j_1-2j_2-4}$.*

Now let $\boldsymbol{j} = (-1, k)$ or $\boldsymbol{j} = (k, -1)$ with $k \in \mathbb{N}_0$. Then

(4) *if $k < n$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}| \lesssim 2^{-n-k}$.*
(5) *if $k \geq n$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = 2^{-2k-3}$.*

Finally, if $h_n = \sum_{i=1}^{n-1}(1 - a_i)$, then

(6) $\mu_{(-1,-1),(0,0)} = 2^{-n-3}(h_n + 5) + 2^{-2n-2}$.

Remark 3.3. We remark that Proposition 3.2 shows that the only Haar coefficient that is relevant in our analysis is the coefficient $\mu_{(-1,-1),(0,0)}$. All other coefficients do not affect the order of L_p discrepancy significantly: they are small enough such that their contribution to the overall L_p discrepancy is of the order of Roth's lower bound.

The proof of Proposition 3.2 is split into several cases which take several pages of very technical and tedious computations. We would like to mention that the proof of the formula for the important coefficient $\mu_{(-1,-1),(0,0)}$ is manageable without excessive effort.

Now the proof of Theorem 2.1 can be finished by inserting the upper bounds on the Haar coefficients of $\Delta_{\mathcal{P}_{\boldsymbol{a}}}$ into Lemma 3.1. This shows the upper bound. For details we refer to the paper [8] where the same method was applied (we remark that our Proposition 3.2 is a direct analog of [8, Lemma 1]; hence the proof of Theorem 2.1 runs along the same lines as the proof of [8, Theorem 1] but with [8, Lemma 1] replaced by Proposition 3.2).

The matching lower bound is a consequence of

$$L_p(\mathcal{P}_{\boldsymbol{a}}) \geq L_1(\mathcal{P}_{\boldsymbol{a}}) = \int_{[0,1]^2} |\Delta_{\mathcal{P}_{\boldsymbol{a}}}(\boldsymbol{t})| \,\mathrm{d}\boldsymbol{t} \geq \left| \int_{[0,1]^2} \Delta_{\mathcal{P}_{\boldsymbol{a}}}(\boldsymbol{t}) \,\mathrm{d}\boldsymbol{t} \right| = |\mu_{(-1,-1),(0,0)}|$$

and item (6) of Lemma 3.2.

Appendix. Computation of the Haar coefficients $\mu_{\boldsymbol{j},\boldsymbol{m}}$

Let $\mathcal{P}$ be an arbitrary 2^n-element point set in the unit square. The Haar coefficients of its discrepancy function $\Delta_{\mathcal{P}}$ are given as follows (see [7]). We write $\boldsymbol{z} = (z_1, z_2)$.

- If $\boldsymbol{j} = (-1,-1)$, then

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = \frac{1}{2^n} \sum_{\boldsymbol{z}\in\mathcal{P}} (1-z_1)(1-z_2) - \frac{1}{4}. \tag{A.1}$$

- If $\boldsymbol{j} = (j_1,-1)$ with $j_1 \in \mathbb{N}_0$, then

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = -2^{-n-j_1-1} \sum_{\boldsymbol{z}\in\mathcal{P}\cap I_{\boldsymbol{j},\boldsymbol{m}}} (1-|2m_1+1-2^{j_1+1}z_1|)(1-z_2) + 2^{-2j_1-3}. \tag{A.2}$$

- If $\boldsymbol{j} = (-1,j_2)$ with $j_2 \in \mathbb{N}_0$, then

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = -2^{-n-j_2-1} \sum_{\boldsymbol{z}\in\mathcal{P}\cap I_{\boldsymbol{j},\boldsymbol{m}}} (1-|2m_2+1-2^{j_2+1}z_2|)(1-z_1) + 2^{-2j_2-3}. \tag{A.3}$$

- If $\boldsymbol{j} = (j_1,j_2)$ with $j_1, j_2 \in \mathbb{N}_0$, then

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = 2^{-n-j_2-j_2-2} \sum_{\boldsymbol{z}\in\mathcal{P}\cap I_{\boldsymbol{j},\boldsymbol{m}}} (1-|2m_1+1-2^{j_1+1}z_1|) \times (1-|2m_2+1-2^{j_2+1}z_2|) - 2^{-2j_1-2j_2-4}. \tag{A.4}$$

In all these identities the first summands involving the sum over $\boldsymbol{z} \in \mathcal{P}\cap I_{\boldsymbol{j},\boldsymbol{m}}$ come from the counting part $\frac{1}{N}\sum_{\boldsymbol{z}\in\mathcal{P}} \mathbf{1}_{[\mathbf{0},\boldsymbol{t})}(\boldsymbol{z})$ and the second summands come from the linear part $-t_1t_2$ of the discrepancy function, respectively. Note that we could also write $\boldsymbol{z} \in \mathring{I}_{\boldsymbol{j},\boldsymbol{m}}$, where $\mathring{I}_{\boldsymbol{j},\boldsymbol{m}}$ denotes the interior of $I_{\boldsymbol{j},\boldsymbol{m}}$, since the summands in the formulas (A.2)–(A.4) vanish if $\boldsymbol{z}$ lies on the boundary of the dyadic box. Hence, in order to compute the Haar coefficients of the discrepancy function, we have to deal with the sums over $\boldsymbol{z}$ which appear in the formulas above and to determine which points $\boldsymbol{z} = (z_1,z_2) \in \mathcal{P}$ lie in the dyadic box $I_{\boldsymbol{j},\boldsymbol{m}}$ with $\boldsymbol{j} \in \mathbb{N}_{-1}^2$ and $\boldsymbol{m} = (m_1,m_2) \in \mathbb{D}_{\boldsymbol{j}}$. If m_1 and m_2 are non-negative integers, then they have a dyadic expansion of the form

$$m_1 = 2^{j_1-1}r_1 + \cdots + r_{j_1} \quad\text{and}\quad m_2 = 2^{j_2-1}s_1 + \cdots + s_{j_2} \tag{A.5}$$

with digits $r_{i_1}, s_{i_2} \in \{0,1\}$ for all $i_1 \in \{1,\dots,j_1\}$ and $i_2 \in \{1,\dots,j_2\}$, respectively. Let $\boldsymbol{z} = (z_1,z_2) = \left(\frac{t_n}{2} + \cdots + \frac{t_1}{2^n}, \frac{b_1}{2} + \cdots + \frac{b_n}{2^n}\right)$ be a point of our point set $\mathcal{P}_{\boldsymbol{a}}$. Then $\boldsymbol{z} \in \mathcal{P}_{\boldsymbol{a}} \cap I_{\boldsymbol{j},\boldsymbol{m}}$ if and only if

$$t_{n+1-k} = r_k \text{ for all } k \in \{1,\dots,j_1\} \quad \text{and } b_k = s_k \text{ for all } k \in \{1,\dots,j_2\}. \tag{A.6}$$

Further, for such a point $\boldsymbol{z} = (z_1,z_2) \in I_{\boldsymbol{j},\boldsymbol{m}}$ we have

$$2m_1 + 1 - 2^{j_1+1}z_1 = 1 - t_{n-j_1} - 2^{-1}t_{n-j_1-1} - \cdots - 2^{j_1-n+1}t_1 \tag{A.7}$$

and

$$2m_2 + 1 - 2^{j_2+1}z_2 = 1 - b_{j_2+1} - 2^{-1}b_{j_2+2} - \cdots - 2^{j_2-n+1}b_n. \tag{A.8}$$

There are several parallel tracks between the proofs in this section and the proofs in [9, Section 3], where we computed the Haar coefficients for a simpler class of digital nets.

Let in the following $\mathcal{H}_j := \{i \in \{1,\dots,j\} : a_i = 0\}$ for $j \in \{1,\dots,n-1\}$. Then $h_n = |\mathcal{H}_{n-1}|$ is the parameter as defined in Theorem 2.1.

Case 1: $\boldsymbol{j} \in \mathcal{J}_1 := \{(-1,-1)\}$.

Proposition A.4. *Let $\boldsymbol{j} \in \mathcal{J}_1$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. Then we have*

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = \frac{h_n+5}{2^{n+3}} + \frac{1}{2^{2n+2}}.$$

Proof. By (A.1) we have

$$\begin{aligned}
\mu_{\boldsymbol{j},\boldsymbol{m}} &= \frac{1}{2^n} \sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}} (1-z_1)(1-z_2) - \frac{1}{4} \\
&= 1 - \frac{1}{2^n}\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}} z_1 - \frac{1}{2^n}\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}} z_2 + \frac{1}{2^n}\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}} z_1z_2 - \frac{1}{4} \\
&= -\frac{1}{4} + \frac{1}{2^n} + \frac{1}{2^n}\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}} z_1z_2,
\end{aligned}$$

where we regarded $\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}} z_1 = \sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}} z_2 = \sum_{l=0}^{2^n-1} l/2^n = 2^{n-1} - 2^{-1}$ in the last step. It remains to evaluate $\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}} z_1z_2$. Using the representation of $\mathcal{P}_{\boldsymbol{a}}$ in (2.3), we have

$$\begin{aligned}
\sum_{\boldsymbol{z}\in\mathcal{P}} z_1z_2 &= \sum_{t_1,\dots,t_n=0}^{1} \left(\frac{t_n}{2}+\dots+\frac{t_1}{2^n}\right)\left(\frac{b_1}{2}+\dots+\frac{b_n}{2^n}\right) \\
&= \sum_{k=1}^{n}\sum_{t_1,\dots,t_n=0}^{1} \frac{t_kb_k}{2^{n+1-k}2^k} + \sum_{\substack{k_1,k_2=1\\k_1\neq k_2}}^{n}\sum_{t_1,\dots,t_n=0}^{1} \frac{t_{k_1}b_{k_2}}{2^{n+1-k_1}2^{k_2}} =: S_1 + S_2.
\end{aligned}$$

Note that b_k only depends on $t_k, t_{k+1}, \dots, t_n$ and $b_n = t_n$. We have

$$\begin{aligned}
S_1 &= \frac{1}{2^{n+1}}\sum_{k=1}^{n} 2^{k-1} \sum_{t_k\dots,t_n=0}^{1} t_kb_k \\
&= \frac{1}{2^{n+2}}2^n \sum_{t_n=0}^{1} t_nb_n + \frac{1}{2^{n+2}}\sum_{k=1}^{n-1} 2^k \sum_{t_k\dots,t_n=0}^{1} t_kb_k \\
&= \frac{1}{4} + \frac{1}{2^{n+2}}\sum_{k=1}^{n-1} 2^k \sum_{t_{k+1}\dots,t_n=0}^{1} (1\oplus a_k(t_{k+1}\oplus\dots\oplus t_n))
\end{aligned}$$

$$= \frac{1}{4} + \frac{1}{2^{n+2}} \sum_{k=1}^{n-1} 2^k 2^{n-k-1}(2-a_k) = \frac{1}{4} + \frac{1}{8}\left(n-1+\sum_{k=1}^{n-1}(1-a_k)\right)$$
$$= \frac{1}{8}(n+h_n+1).$$

To compute S_2, assume first that $k_1 < k_2$. Then

$$\sum_{t_1,\ldots,t_n=0}^{1} t_{k_1} b_{k_2} = 2^{k_1-1} \sum_{t_{k_1},\ldots,t_n=0}^{1} t_{k_1} b_{k_2} = 2^{k_1-1} \sum_{t_{k_1+1},\ldots,t_n=0}^{1} b_{k_2}$$
$$= 2^{k_1-1} 2^{k_2-k_1-1} \sum_{t_{k_2},\ldots,t_n=0}^{1} b_{k_2} = 2^{k_1-1}2^{k_2-k_1-1}2^{n-k_2} = 2^{n-2}.$$

Similarly, we observe that we obtain the same result also for $k_1 > k_2$ and hence

$$S_2 = \frac{1}{2^{n+1}} \sum_{\substack{k_1,k_2=0\\ k_1\neq k_2}}^{n} 2^{k_1-k_2}2^{n-2} = \frac{1}{8} \sum_{\substack{k_1,k_2=0\\ k_1\neq k_2}}^{n} 2^{k_1-k_2}$$
$$= \frac{1}{8}\left(-n+2^{n+1}-4+\frac{2}{2^n}\right).$$

Now we put everything together to arrive at the claimed formula. □

Case 2: $\boldsymbol{j} \in \mathcal{J}_2 := \{(-1, j_2) : 0 \le j_2 \le n-2\}$.

Proposition A.5. *Let $\boldsymbol{j} = (-1, j_2) \in \mathcal{J}_2$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. If $\mathcal{H}_{j_2} = \{1,\ldots,j_2\}$, then*

$$\text{(A.9)}\quad mu_{\boldsymbol{j},\boldsymbol{m}} = 2^{-2n-2j_2-4}\biggl(-2^{2j_2+2}(a_{j_2+1}-1) + 2^{n+j_2}(a_{j_2+1}a_{j_2+2}-2) + 2^{2n+2}\sum_{k=1}^{j_2} \frac{s_k}{2^{n+1-k}}\biggr),$$

where the latter sum is zero for $j_2 = 0$. Otherwise, let $w \in \{1,\ldots,j_2\}$ be the greatest index with $a_w = 1$. If $a_{j_2+1} = 0$, then

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = 2^{-2n-2} - 2^{-n-j_2-3} + 2^{-n-2j_2+w-5} + 2^{-2j_2-2}\varepsilon + 2^{-2n-j_2+w-4}a_{j_2+2}(1-2(s_w\oplus\cdots\oplus s_{j_2})).$$

If $a_{j_2+1} = 1$, then

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = -2^{-n-j_2-3} + 2^{-j_2+w-2n-3} + 2^{-2j_2-n+w-4} + 2^{-2j_2-2}\varepsilon - 2^{-2n-j_2+w-2}(s_w\oplus\cdots\oplus s_{j_2}) + 2^{-n-j_2-4}a_{j_2+2}.$$

In the latter two expressions, we put $\varepsilon = \sum_{\substack{k=1\\k\neq w}}^{j_2} \frac{t_k(m_2)}{2^{n+1-k}}$*, where the values* $t_k(m_2)$ *depend only on* m_2 *and are either 0 or 1. Hence, in any case we have* $|\mu_{\boldsymbol{j},\boldsymbol{m}}| \lesssim 2^{-n-j_2}$*.*

Proof. We only show the case where $j_2 \geq 1$ and $\mathcal{H}_{j_2} \neq \{1,\dots,j_2\}$, since the other case is similar but easier. Let $w \in \{1,\dots,j_2\}$ be the greatest index with $a_w = 1$. By (A.3), we need to evaluate the sum

$$\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}\cap I_{\boldsymbol{j},\boldsymbol{m}}} (1-z_1)(1-|2m_2+1-2^{j_2+1}z_2|).$$

By (A.6), the condition $\boldsymbol{z} \in \mathcal{P}_{\boldsymbol{a}} \cap I_{\boldsymbol{j},\boldsymbol{m}}$ yields the identities $b_k = s_k$ for all $k \in \{1,\dots,j_2\}$, which lead to $t_k = s_k$ for all $k \in \{1,\dots,j_2\}$ such that $a_k = 0$. Assume that

$$\{k \in \{1,\dots,j_2\} : a_k = 1\} = \{k_1,\dots,k_l\}$$

for some $l \in \{1,\dots j_2\}$, where $k_1 < k_2 < \dots < k_l$ and $k_l = w$. We have $t_{k_i} = s_{k_i} \oplus s_{k_i+1} \oplus \dots \oplus s_{k_{i+1}}$ for all $i \in \{1,\dots,l-1\}$ and $t_w = s_w \oplus \dots \oplus s_{j_2} \oplus t_{j_2+1} \oplus \dots \oplus t_n$. Hence, we can write

$$1 - z_1 = 1 - u - \frac{t_{j_2+1}}{2^{n-j_2}} - \frac{s_w \oplus \dots \oplus s_{j_2} \oplus t_{j_2+1} \oplus \dots \oplus t_n}{2^{n+1-w}} - \varepsilon,$$

where $u = 2^{-1}t_n + \dots + 2^{-(n-j_2-1)}t_{j_2+2}$ and

$$\varepsilon = \varepsilon(m_2) = \sum_{\substack{k=1\\k\neq w}}^{j_2} \frac{t_k(m_2)}{2^{n+1-k}}.$$

For the expression $1-|2m_2+1-2^{j_2+1}z_2|$ we find by (A.8)

$$1-|2m_2+1-2^{j_2+1}z_2| = 1-|1-t_{j_2+1}\oplus a_{j_2+1}(t_{j_2+2}\oplus\dots\oplus t_n)-v|,$$

where $v = v(t_{j_2+2},\dots,t_n) = 2^{-1}b_{j_2+2} + \dots + 2^{-(n-j_2-1)}b_n$. With these observations, we find (writing $T_j = t_j \oplus \dots \oplus t_n$ for $1 \leq j \leq n-1$ and

$t_w(t_{j_2+1}) = s_w \oplus \cdots \oplus s_{j_2} \oplus t_{j_2+1} \oplus T_{j_2+2})$

$$\begin{aligned}
&\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}\cap I_{\boldsymbol{j},\boldsymbol{m}}} (1-z_1)(1-|2m_2+1-2^{j_2+1}z_2|) \\
&\quad= \sum_{t_{j_2+1},\dots,t_n=0}^{1} \left(1-u-\frac{t_{j_2+1}}{2^{n-j_2}}-\frac{t_w(t_{j_2+1})}{2^{n+1-w}}-\varepsilon\right) \\
&\qquad\qquad \times (1-|1-t_{j_2+1}\oplus a_{j_2+1}T_{j_2+2}-v|) \\
&\quad= \sum_{t_{j_2+2},\dots,t_n=0}^{1} \left\{ \left(1-u-\frac{a_{j_2+1}T_{j_2+2}}{2^{n-j_2}}-\frac{t_w(a_{j_2+1}T_{j_2+1})}{2^{n+1-w}}-\varepsilon\right) v \right. \\
&\qquad\qquad \left. + \left(1-u-\frac{a_{j_2+1}T_{j_2+2}\oplus 1}{2^{n-j_2}}-\frac{t_w(a_{j_2+1}T_{j_2+1}\oplus 1)}{2^{n+1-w}}-\varepsilon\right)(1-v)\right\} \\
&\quad= \sum_{t_{j_2+2},\dots,t_n=0}^{1} 2^{-n-1}\Big(-2^{j_2+1}-2^w+2^{n+1}-2^{n+1}\varepsilon \\
&\qquad\qquad + 2^w t_w(a_{j_2+1}T_{j_2+1}) - 2^{n+1}u + 2^{j_2+1}v + 2^w v \\
&\qquad\qquad - 2^{w+1}t_w(a_{j_2+1}T_{j_2+1})v - 2^{j_2+1}(a_{j_2+1}T_{j_2+2})(2v-1)\Big).
\end{aligned}$$

Let first $a_{j_2+1} = 1$ and hence $t_w(a_{j_2+1}T_{j_2+2}) = t_w(T_{j_2+2}) = s_w \oplus \cdots \oplus s_{j_2}$ does not depend on t_i. Since

$$\sum_{t_{j_2+2},\dots,t_n=0}^{1} u = \sum_{t_{j_2+2},\dots,t_n=0}^{1} v = \sum_{l=0}^{2^{n-j_2-1}-1} \frac{l}{2^{n-j_2+1}} = 2^{n-j_2-2} - \frac{1}{2},$$

we obtain

$$\begin{aligned}
&\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}\cap I_{\boldsymbol{j},\boldsymbol{m}}} (1-z_1)(1-|2m_2+1-2^{j_2+1}z_2|) \\
&\qquad = 2^{-n-1}\Big((-2^{j_2+1}-2^w+2^{n+1}-2^{n+1}\varepsilon+2^w t_w(T_{j_2+2}))2^{n-j_2-1} \\
&\qquad\qquad + (2^w+2^{j_2+1}-2^{n+1}-2^{w+1}t_w(T_{j_2+2}))\left(2^{n-j_2-2}-\frac{1}{2}\right) \\
&\qquad\qquad - 2^{j_2+1}\sum_{t_{j_2+2},\dots,t_n=0}^{1} T_{j_2+2}(2v-1)\Big).
\end{aligned}$$

We analyze the last expression. We find

$$\begin{aligned}\sum_{t_{j_2+2},\dots,t_n=0}^{1} T_{j_2+2}(2v-1) &= 2\sum_{t_{j_2+2},\dots,t_n=0}^{1} T_{j_2+2}v - \sum_{t_{j_2+2},\dots,t_n=0}^{1} T_{j_2+2}\\ &= 2\sum_{t_{j_2+2},\dots,t_n=0}^{1} T_{j_2+1}v - 2^{n-j_2-2},\end{aligned}$$

where

$$\begin{aligned}&\sum_{t_{j_2+2},\dots,t_n=0}^{1} T_{j_2+2}v\\ &\quad= \sum_{t_{j_2+2},\dots,t_n=0}^{1} (t_{j_2+2}\oplus T_{j_2+3})\\ &\qquad\times\left(\frac{t_{j_2+2}\oplus a_{j_2+2}T_{j_2+3}}{2}+\frac{b_{j_2+3}}{4}+\dots+\frac{b_n}{2^{n-j_1-1}}\right)\\ &\quad= \sum_{t_{j_2+3},\dots,t_n=0}^{1} \left(\frac{(T_{j_2+3}\oplus 1)\oplus a_{j_2+2}T_{j_2+3}}{2}+\frac{b_{j_2+3}}{4}+\dots+\frac{b_n}{2^{n-j_2-1}}\right)\\ &\quad= \sum_{t_{j_2+3},\dots,t_n=0}^{1} \frac{1\oplus(1-a_{j_2+2})T_{j_2+3}}{2}+\sum_{l=0}^{2^{n-j_2-2}-1}\frac{l}{2^{n-j_2-1}}\\ &\quad= \frac{1}{2}\sum_{t_{j_2+3},\dots,t_n=0}^{1}(1-(1-a_{j_2+2})T_{j_2+3})+2^{n-j_2-4}-\frac{1}{4}\\ &\quad= \frac{1}{2}\left(2^{n-j_2-2}-(1-a_{j_2+2})2^{n-j_2-3}\right)+2^{n-j_2-4}-\frac{1}{4}\\ &\quad= 2^{n-j_2-4}(1+a_{j_2+2})+2^{n-j_2-4}-\frac{1}{4}.\end{aligned}$$

We put everything together and apply (A.3) to find the result for $a_{j_2+1}=1$.

Now assume that $a_{j_2+1}=0$. Then

$$t_w(a_{j_2+1}T_{j_2+2}) = t_w(0) = s_w\oplus\dots\oplus s_{j_2}\oplus T_{j_2+2}.$$

Hence we have

$$\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}\cap I_{\boldsymbol{j},\boldsymbol{m}}}(1-z_1)(1-|2m_2+1-2^{j_2+1}z_2|)$$
$$= 2^{-n-1}\Bigg((-2^{j_1+1}+2^{n+1}-2^w-2^{n+1}\varepsilon)2^{n-j_2-1} + (2^{j_2+1}-2^{n+1})\left(2^{n-j_2-2}-\frac{1}{2}\right) + 2^w\cdot 2^{n-j_2-2}-2^{w+1}\sum_{t_{j_2+2},\dots,t_n=0}^{1} vt_w(0)\Bigg).$$

We considered $\sum_{t_{j_2+2},\dots,t_n=0}^{1} t_w(0) = 2^{n-j_2-2}$. It remains to evaluate

$$\sum_{t_{j_2+2},\dots,t_n=0}^{1} vt_w(0).$$

We find

$$\sum_{t_{j_2+2},\dots,t_n=0}^{1}(s_w\oplus\cdots\oplus s_{j_2}\oplus t_{j_2+2}\oplus T_{j_2+3}) \times\left(\frac{t_{j_2+2}\oplus a_{j_2+2}T_{j_2+3}}{2}+\frac{b_{j_2+2}}{4}+\cdots+\frac{b_n}{2^{n-j_2-1}}\right)$$
$$=\sum_{t_{j_2+3},\dots,t_n=0}^{1}\Bigg(\frac{(s_w\oplus\cdots\oplus s_{j_2}\oplus T_{j_2+3}\oplus 1)\oplus a_{j_2+2}T_{j_2+3}}{2} + \frac{b_{j_2+2}}{4}+\cdots+\frac{b_n}{2^{n-j_2-1}}\Bigg)$$
$$=\frac{1}{2}\sum_{t_{j_2+3},\dots,t_n=0}^{1}(1-a_{j_2+2})T_{j_2+3}\oplus s_w\oplus\cdots\oplus s_{j_2}\oplus 1 + \sum_{l=0}^{2^{n-j_2-2}-1}\frac{l}{2^{n-j_2-1}}$$
$$=2^{n-j_2-4}(1+a_{j_2+2}(1-2(s_w\oplus\cdots\oplus s_{j_2}))+2^{n-j_2-4}-\frac{1}{4}.$$

Again, we put everything together and apply (A.3) to find the result for $a_{j_2+1}=0$. □

Case 3: $\boldsymbol{j}\in\mathcal{J}_3 := \{(k,-1) : k\geq n\}\cup\{(-1,k) : k\geq n\}$.

Proposition A.6. *Let $\boldsymbol{j}\in\mathcal{J}_3$ and $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$. Then we have*

$$\mu_{\boldsymbol{j},\boldsymbol{m}}=\frac{1}{2^{2k+3}}.$$

Proof. This claim follows from (A.2) and (A.3) together with the fact that no point of $\mathcal{P}_{\boldsymbol{a}}$ is contained in the interior of $I_{\boldsymbol{j},\boldsymbol{m}}$ if $j_1 \geq n$ or $j_2 \geq n$. Hence, only the linear part of $\Delta_{\mathcal{P}_{\boldsymbol{a}}}$ contributes to the Haar coefficients in this case. □

Case 4: $\boldsymbol{j} \in \mathcal{J}_4 := \{(0,-1)\}$.

Proposition A.7. *Let $\boldsymbol{j} \in \mathcal{J}_4$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. Then we have*

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = -\frac{1}{2^{n+3}} + \frac{1}{2^{2n+2}}.$$

Proof. For $\boldsymbol{z} = (z_1, z_2) \in \mathcal{P}_{\boldsymbol{a}} \cap I_{\boldsymbol{j},\boldsymbol{m}} = \mathcal{P}_{\boldsymbol{a}}$ we have

$$1 - z_2 = 1 - \frac{b_1}{2} - \cdots - \frac{b_n}{2^n}$$

and

$$1 - |2m_1 + 1 - 2z_1| = 1 - \left|1 - t_n - \frac{t_{n-1}}{2} - \cdots - \frac{t_1}{2^{n-1}}\right|$$

by (A.7). We therefore find, after summation over t_n,

$$\begin{aligned}
&\sum_{\boldsymbol{z} \in \mathcal{P}_{\boldsymbol{a}} \cap I_{\boldsymbol{j},\boldsymbol{m}}} (1 - |2m_1 + 1 - 2z_1|)(1 - z_2) \\
&\qquad = \sum_{t_1,\ldots,t_n=0}^{1} \left(1 - \left|1 - t_n - \frac{t_{n-1}}{2} - \cdots - \frac{t_1}{2^{n-1}}\right|\right) \\
&\qquad\qquad \times \left(1 - \frac{b_1(t_n)}{2} - \cdots - \frac{b_n(t_n)}{2^n}\right) \\
&\qquad = \sum_{t_1,\ldots,t_{n-1}=0}^{1} \left(u(1 - v(0)) + (1-u)\left(1 - v(1) - \frac{1}{2^n}\right)\right) \\
&\qquad = \sum_{t_1,\ldots,t_{n-1}=0}^{1} \left(1 - \frac{1}{2^n} - v(1) + \frac{1}{2^n}u + uv(1) - uv(0)\right) \\
&\qquad = 2^{n-1}\left(1 - \frac{1}{2^n}\right) + \left(\frac{1}{2^n} - 1\right)(2^{n-2} - 2^{-1}) \\
&\qquad\qquad + \sum_{t_1,\ldots,t_{n-1}=0}^{1} uv(1) - \sum_{t_1,\ldots,t_{n-1}=0}^{1} uv(0).
\end{aligned}$$

Here we use the short-hands $u = 2^{-1}t_{n-1} + \cdots + 2^{-n+1}t_1$ and

$$v(t_n) = 2^{-1}b_1(t_n) + \cdots + 2^{-n+1}b_{n-1}(t_n)$$

and the fact that $\sum_{t_1,\dots,t_{n-1}=0}^{1} u = \sum_{t_1,\dots,t_{n-1}=0}^{1} v(1) = 2^{n-2} - 2^{-1}$. It is not difficult to observe that $\sum_{t_1,\dots,t_{n-1}=0}^{1} uv(0) = \sum_{t_1,\dots,t_{n-1}=0}^{1} uv(1)$; hence

$$\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}} (1-|2m_1+1-2z_1|)(1-z_2) = \frac{1}{4} + 2^{n-2} - \frac{1}{2^{n+1}}.$$

The rest follows with (A.2). □

For the following two propositions, we use the shorthand

$$R = r_1 \oplus \cdots \oplus r_{j_1}.$$

Case 5: $\boldsymbol{j} \in \mathcal{J}_5 := \{(j_1,-1) : 1 \le j_1 \le n-2\}$.

Proposition A.8. *Let $\boldsymbol{j} \in \mathcal{J}_5$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. Then we have*

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = 2^{-2n-2} - 2^{-n-j_1-3} + 2^{-2j_1-2}\varepsilon - 2^{-2n-1}R - 2^{-n-j_1-3}a_{n-j_1-1}(1-2R),$$

where

$$\varepsilon = \varepsilon(m_1) = \frac{r_1}{2^n} + \sum_{k=2}^{j_1} \frac{r_k \oplus a_{n+1-k}(r_{k-1}\oplus\cdots\oplus r_1)}{2^{n+1-k}}. \tag{A.10}$$

Hence, we have $|\mu_{\boldsymbol{j},\boldsymbol{m}}| \lesssim 2^{-n-j_1}$.

Proof. By (A.2), we need to evaluate the sum

$$\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}\cap I_{\boldsymbol{j},\boldsymbol{m}}} (1-|2m_1+1-2^{j_1+1}z_1|)(1-z_2).$$

The condition $\boldsymbol{z} \in \mathcal{P}_{\boldsymbol{a}} \cap I_{\boldsymbol{j},\boldsymbol{m}}$ forces $t_n = r_1, \dots t_{n+1-j_1} = r_{j_1}$ and therefore

$$1 - z_2 = 1 - \frac{b_1}{2} - \cdots - \frac{b_n}{2^n} = 1 - v(t_{n-j_1}) - \frac{t_{n-j_1} \oplus a_{n-j_1}R}{2^{n-j_1-1}} - \varepsilon,$$

where

$$\begin{aligned} v(t_{n-j_1}) &= \frac{b_1}{2} + \cdots + \frac{b_{n-j_1-1}}{2^{n-j_1-1}} \\ &= \frac{t_1 \oplus a_1(t_2 \oplus \cdots \oplus t_{n-j_1} \oplus R)}{2} + \dots \\ &\quad + \frac{t_{n-j_1-1} \oplus a_{n-j_1-1}(t_{n-j_1} \oplus R)}{2^{n-j_1-1}} \end{aligned}$$

and ε as in (A.10). Further, by (A.7) we write $2m_1 + 1 - 2^{j_1+1}z_1 = 1 - t_{n-j_1} - u$, where $u = 2^{-1}t_{n-j_1-1} + \cdots + 2^{j_1-n+1}t_1$. Then

$$\begin{aligned}
&\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}\cap I_{\boldsymbol{j},\boldsymbol{m}}} (1-|2m_1+1-2^{j_1+1}z_1|)(1-z_2)\\
&\quad= \sum_{t_1,\dots,t_{n-j_1}=0}^{1} \left(1 - v(t_{n-j_1}) - \frac{t_{n-j_1}\oplus a_{n-j_1}R}{2^{n-j_1-1}} - \varepsilon\right)\\
&\qquad\qquad\qquad \times (1-|1-t_{n-j_1}-u|)\\
&\quad= \sum_{t_1,\dots,t_{n-j_1-1}=0}^{1} \left\{ \left(1 - v(0) - \frac{a_{n-j_1}R}{2^{n-j_1-1}} - \varepsilon\right) u\right.\\
&\qquad\qquad\qquad \left. + \left(1 - v(1) - \frac{1\oplus a_{n-j_1}R}{2^{n-j_1-1}} - \varepsilon\right)(1-u)\right\}\\
&\quad= \sum_{t_1,\dots,t_{n-j_1-1}=0}^{1} \{1 - 2^{j_1-n} - \varepsilon + 2^{j_1-n}a_{n-j_1}R + 2^{j_1-n}u - v(1)\\
&\qquad\qquad\qquad - 2^{1+j_1-n}a_{n-j_1}Ru + uv(1) - uv(0)\}\\
&\quad= 2^{n-j_1-1}(1 - 2^{j_1-n} - \varepsilon + 2^{j_1-n}a_{n-j_1}R)\\
&\qquad + \left(2^{n-j_1-2} - 2^{-1}\right)(2^{j_1-n} - 1 - 2^{1+j_1-n}a_{n-j_1}R)\\
&\qquad + \sum_{t_1,\dots,t_{n-j_1-1}=0}^{1} (uv(1) - uv(0)).
\end{aligned}$$

We understand $b_1, \dots, b_{n-j_1-1}$ as functions of t_{n-j_1} and have

$$\begin{aligned}
&\sum_{t_1,\dots,t_{n-j_1-1}=0}^{1} uv(0)\\
&\quad= \sum_{t_1,\dots,t_{n-j_1-1}=0}^{1} \left(\frac{t_{n-j_1-1}}{2} + \cdots + \frac{t_1}{2^{n-j_1-1}}\right)\\
&\qquad\qquad \times \left(\frac{b_1(0)}{2} + \cdots + \frac{b_{n-j_1-1}(0)}{2^{n-j_1-1}}\right)\\
&\quad= \sum_{t_1,\dots,t_{n-j_1-1}=0}^{1} \left(\sum_{k=1}^{n-j_1-1} \frac{t_k b_k(0)}{2^{n-j_1-k}2^k} + \sum_{\substack{k_1,k_2=0\\k_1\neq k_2}}^{n-j_1-1} \frac{t_{k_1}b_{k_2}(0)}{2^{n-j_1-k_1}2^{k_2}}\right).
\end{aligned}$$

The first sum simplifies to

$$\begin{aligned}
&\sum_{k=1}^{n-j_1-1} 2^{k-1} \sum_{t_k,\dots,t_{n-j_1-1}=0}^{1} \frac{t_k b_k(0)}{2^{n-j_1-k}2^k} \\
&\quad = \frac{1}{2^{n-j_1}} \sum_{k=1}^{n-j_1-2} 2^{k-1} \sum_{t_k,\dots,t_{n-j_1-1}=0}^{1} t_k(t_k \oplus a_k(t_{k+1} \oplus t_{n-j_1-1} \oplus R)) \\
&\qquad + \frac{1}{2^{n-j_1}} 2^{n-j_1-2} \sum_{t_{n-j_1-1}=0}^{1} t_{n-j_1-1}(t_{n-j_1-1} \oplus a_{n-j_1-1}R) \\
&\quad = \frac{1}{2^{n-j_1}} \sum_{k=1}^{n-j_1-2} 2^{k-1} \sum_{t_{k+1},\dots,t_{n-j_1-1}=0}^{1} (1 \oplus a_k(t_{k+1} \oplus t_{n-j_1-1} \oplus R)) \\
&\qquad + \frac{1}{4}(1 \oplus a_{n-j_1-1}(R \oplus 1)) \\
&\quad = \frac{1}{2^{n-j_1}} \sum_{k=1}^{n-j_1-2} 2^{k-1} 2^{n-j_1-k-2}(2-a_k) + \frac{1}{4}(1 \oplus a_{n-j_1-1}(R \oplus 1)) \\
&\quad = \frac{1}{8} \sum_{k=1}^{n-j_1-2} (2-a_k) + \frac{1}{4}(1 \oplus a_{n-j_1-1}(R \oplus 1)).
\end{aligned}$$

Basically by the same arguments as in the proof of Proposition A.4 we also find

$$\sum_{t_1,\dots,t_{n-j_1-1}=0}^{1} \sum_{\substack{k_1,k_2=0 \\ k_1 \neq k_2}}^{n-j_1-1} \frac{t_{k_1} b_{k_2}}{2^{n-j_1-k_1}2^{k_2}} = \frac{1}{8} \sum_{\substack{k_1,k_2=0 \\ k_1 \neq k_2}}^{1} 2^{k_1-k_2}.$$

Hence, we obtain

$$\begin{aligned}
\sum_{t_1,\dots,t_{n-j_1-1}=0}^{1} uv(0) = \frac{1}{8} \sum_{k=1}^{n-j_1-2} (2-a_k) + \frac{1}{4}(1 \oplus a_{n-j_1-1}(R \oplus 1)) \\
+ \frac{1}{8} \sum_{\substack{k_1,k_2=0 \\ k_1 \neq k_2}}^{1} 2^{k_1-k_2}.
\end{aligned}$$

We can evaluate $\sum_{t_1,\dots,t_{n-j_1-1}=0}^{1} uv(1)$ in almost the same way; the result is

$$\sum_{t_1,\dots,t_{n-j_1-1}=0}^{1} uv(1) = \frac{1}{8} \sum_{k=1}^{n-j_1-2} (2-a_k) + \frac{1}{4}(1 \oplus a_{n-j_1-1}R) + \frac{1}{8} \sum_{\substack{k_1,k_2=0 \\ k_1 \neq k_2}}^{1} 2^{k_1-k_2}.$$

Hence the difference of these two expressions is given by

$$\sum_{t_1,\dots,t_{n-j_1-1}=0}^{1} uv(1) - \sum_{t_1,\dots,t_{n-j_1-1}=0}^{1} uv(0) = \frac{1}{4}a_{n-j_1-1}(2R-1).$$

Now we put everything together and use (A.2) to find the claimed result on the Haar coefficients. □

Case 6: $\boldsymbol{j} \in \mathcal{J}_6 := \{(j_1, j_2) : j_1 + j_2 \leq n-3\}$.

Proposition A.9. *Let $\boldsymbol{j} \in \mathcal{J}_6$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. If $\mathcal{H}_{j_2} = \{1,\dots,j_2\}$ or if $j_2 = 0$, then we have*

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = 2^{-2n-2}(1-2a_{n-j_1}R)(1-a_{j_2+1}).$$

Otherwise, let $w \in \{1,\dots,j_2\}$ be the greatest index with $a_w = 1$. If $a_{j_2+1} = 0$, then

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = 2^{-2n-2}(1-2a_{n-j_1}R).$$

If $a_{j_2+1} = 1$, then

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = -2^{-2n-j_2+w-3}(1-2a_{n-j_1}R)(1-2(s_w \oplus \cdots \oplus s_{j_2})).$$

Note that for $j_1 = 0$ we set $a_{n-j_1}R = 0$ in all these formulas. Hence, in any case we have $|\mu_{\boldsymbol{j},\boldsymbol{m}}| \lesssim 2^{-2n}$

Proof. The proof is similar in all cases; hence we only treat the most complicated case where $j_2 \geq 1$ and $\mathcal{H}_{j_2} \neq \{1,\dots,j_2\}$. By (A.4), we need to study the sum

$$\sum_{\boldsymbol{z} \in \mathcal{P}_{\boldsymbol{a}} \cap I_{\boldsymbol{j},\boldsymbol{m}}} (1-|2m_1+1-2^{j_1+1}z_1|)(1-|2m_2+1-2^{j_2+1}z_2|),$$

where the condition $\boldsymbol{z} \in \mathcal{P}_{\boldsymbol{a}} \cap I_{\boldsymbol{j},\boldsymbol{m}}$ forces $t_{n+1-k} = r_k$ for all $k \in \{1,\dots,j_1\}$ as well as $b_k = s_k$ for all $k \in \{1,\dots,j_2\}$. We have already seen in the proof of Proposition 2 that the latter equalities allow us to express the digits t_k by the digits $s_1,\dots,s_{j_2}$ of m_2 for all $k \in \{1,\dots,j_2\} \setminus \{w\}$. We also have $t_w = s_w \oplus \cdots \oplus s_{j_2} \oplus t_{j_2+1} \oplus \cdots \oplus t_n$. With (A.7), these observations lead to

$$2m_1+1-2^{j_1+1}z_1 = 1 - t_{n-j_1} - u - 2^{j_1+j_2-n+1}t_{j_2+1} - 2^{j_1+w-1}t_w - \varepsilon_2(m_2),$$

where $u = 2^{-1}t_{n-j_1-1} + \cdots + 2^{j_1+j_2-n+2}t_{j_2+2}$ and ε_2 is determined by m_2. Further, we write with (A.8)

$$2m_2+1-2^{j_2+1}z_2 = 1 - b_{j_2+1} - v - 2^{j_1+j_2-n+1}b_{n-j_1} - \varepsilon_1(m_1),$$

where $v = v(t_{n-j_1}) = 2^{-1}b_{j_1+2} + \cdots + 2^{j_1+j_2-n+2}b_{n-j_1-1}$ and ε_1 is obviously determined by m_1. Hence, we have

$$\begin{aligned}
&\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}\cap I_{\boldsymbol{j},\boldsymbol{m}}} (1-|2m_1+1-2^{j_1+1}z_1|)(1-|2m_2+1-2^{j_2+1}z_2|)\\
&\quad= \sum_{\substack{t_{j_2+1},\dots,\\ t_{n-j_1}=0}}^{1} \Big(1-|1-t_{n-j_1}-u-2^{j_1+j_2-n+1}t_{j_2+1}-2^{j_1+w-1}t_w-\varepsilon_2(m_2)|\Big)\\
&\qquad\times\Big(1-|1-t_{j_2+1}\oplus a_{j_2+1}(t_{j_2+2}\oplus\cdots\oplus t_{n-j_1}\oplus R)-v(t_{n-j_1})\\
&\qquad\qquad-2^{j_1+j_2-n+1}(t_{n-j_1}\oplus a_{n-j_1}R)-\varepsilon_1)|\Big).
\end{aligned}$$

Recall we may write $t_w = s_w \oplus \cdots \oplus s_{j_2} \oplus t_{j_2+1} \oplus t_{j_2+2} \oplus \cdots \oplus t_{n-j_1-1} \oplus t_{n-j_1} \oplus R$. We stress the dependence of t_w on $t_{j_2+1} \oplus t_{n-j_1}$ by writing $t_w(t_{j_2+1} \oplus t_{n-j_1})$. If $a_{j_2+1} = 0$, then we obtain after summation over t_{j_2+1} and t_{n-j_1}

$$\begin{aligned}
&\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}\cap I_{\boldsymbol{j},\boldsymbol{m}}} (1-|2m_1+1-2^{j_1+1}z_1|)(1-|2m_2+1-2^{j_2+1}z_2|)\\
&= \sum_{\substack{t_{j_2+2},\dots,\\ t_{n-j_1-1}=0}}^{1} \Big\{(u+2^{j_1+w-1}t_w(0)+\varepsilon_2)(v(0)+2^{j_1+j_2-n+1}a_{n-j_1}R+\varepsilon_1)\\
&\qquad+(u+2^{j_1+j_2-n+1}+2^{j_1+w-1}t_w(1)+\varepsilon_2)\\
&\qquad\quad\times(1-v(0)-2^{j_1+j_2-n+1}a_{n-j_1}R-\varepsilon_1)\\
&\qquad+(1-u-2^{j_1+w-1}t_w(1)-\varepsilon_2)(v(0)+2^{j_1+j_2-n+1}(a_{n-j_1}R\oplus 1)+\varepsilon_1)\\
&\qquad+(1-u-2^{j_1+j_2-n+1}-2^{j_1+w-1}t_w(0)-\varepsilon_2)\\
&\qquad\quad\times(1-v(1)-2^{j_1+j_2-n+1}(a_{n-j_1}R\oplus 1)-\varepsilon_1)\Big\}\\
&= \sum_{\substack{t_{j_2+2},\dots,\\ t_{n-j_1-1}=0}}^{1} \Big\{1+2^{2(n+j_1+j_2+1)}+2^{j_1+w-1}-2^{2j_1+j_2-n+w}\\
&\qquad-2^{2j_1+2j_2-2n+3}a_{n-j_1}R+(2^{1+2j_1+j_2-n+w}-2^{j_1+w})t_w(0)\\
&\qquad+2^{j_1+w}(2t_w(0)-1)+2^{n+j_1+j_2+1}(v(1)-v(0))\\
&\qquad-2^{w+j_1-1}(v(1)+v(0))+2^{j_1+w}(t_w(0)v(0)+t_w(0)v(1))\Big\}.
\end{aligned}$$

We regarded $t_w(1)=1-t_w(0)$. By standard argumentation, we find

$$\sum_{t_{j_2+2},\dots,t_{n-j_1-1}=0}^{1} v(0)=\sum_{t_{j_2+2},\dots,t_{n-j_1-1}=0}^{1} v(1)=2^{n-j_1-j_2-3}-\frac{1}{2}$$

and

$$\sum_{t_{j_2+2},\dots,t_{n-j_1-1}=0}^{1} t_w(0)=\sum_{t_{j_2+2},\dots,t_{n-j_1-2}=0}^{1} 1=2^{n-j_1-j_2-3}.$$

We use the short-hand $T=t_{j_2+3}\oplus\cdots\oplus t_{n-j_1-1}$, which allows us to write

$$\begin{aligned}
&\sum_{t_{j_2+2},\dots,t_{n-j_1-1}=0}^{1} t_w(0)v(0)\\
&\quad=\sum_{t_{j_2+2},\dots,t_{n-j_1-1}=0}^{1}(s_w\oplus\cdots\oplus s_{j_2}\oplus t_{j_2+2}\oplus\cdots\oplus t_{n-j_1-1}\oplus R)\\
&\qquad\quad\times\Big(\frac{t_{j_2+2}\oplus a_{j_2+2}(t_{j_2+3}\oplus\cdots\oplus t_{n-j_1-1}\oplus R)}{2}\\
&\qquad\qquad+\frac{t_{j_2+3}\oplus a_{j_2+3}(t_{j_2+4}\oplus\cdots\oplus t_{n-j_1-1}\oplus R)}{4}\\
&\qquad\qquad+\cdots+\frac{t_{n-j_1-1}\oplus a_{n-j_1-1}R}{2^{n-j_1-j_2-2}}\Big)\\
&\quad=\sum_{t_{j_2+3},\dots,t_{n-j_1-1}=0}^{1}\frac{1}{2}(s_w\oplus\cdots\oplus s_{j_2}\oplus T\oplus R\oplus 1\oplus a_{j_2+2}(T\oplus R))\\
&\qquad\quad+\sum_{l=0}^{2^{n-j_1-j_2-3}-1}\frac{l}{2^{n-j_1-j_2-2}}\\
&\quad=\sum_{t_{j_2+3},\dots,t_{n-j_1-1}=0}^{1}\frac{1}{2}(s_w\oplus\cdots\oplus s_{j_2}\oplus 1\oplus(1-a_{j_2+2})(T\oplus R))\\
&\qquad\quad+2^{n-j_1-j_2-5}-\frac{1}{4}.
\end{aligned}$$

Similarly, we can show

$$\begin{aligned}
&\sum_{t_{j_2+2},\dots,t_{n-j_1-1}=0}^{1} t_w(0)v(1)\\
&\quad=\sum_{t_{j_2+3},\dots,t_{n-j_1-1}=0}^{1}\frac{1}{2}(s_w\oplus\cdots\oplus s_{j_2}\oplus(1-a_{j_2+2})(T\oplus R\oplus 1))\\
&\qquad\qquad\qquad\qquad\qquad\qquad+2^{n-j_1-j_2-5}-\frac{1}{4}
\end{aligned}$$

and therefore

$$\sum_{t_{j_2+2},\dots,t_{n-j_1-1}=0}^{1} t_w(0)(v(0)+v(1)) = 2^{n-j_1-j_2-4} + 2^{n-j_1-j_2-5} - \frac{1}{4},$$

a fact which can be found by distinguishing the cases $a_{j_2+1}=0$ and $a_{j_2+1}=1$. We put everything together and obtain

$$\begin{aligned}\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}\cap I_{\boldsymbol{j},\boldsymbol{m}}} &(1-|2m_1+1-2^{j_1+1}z_1|)(1-|2m_2+1-2^{j_2+1}z_2|)\\ &= 2^{j_1+j_2-n} + 2^{n-j_1-j_2-2} - 2^{-n+j_1+j_2+1}a_{n-j_1}R,\end{aligned}$$

which leads to the claimed result for $a_{j_2+1}=0$ via (A.4).

Now assume that $a_{j_2+1}=1$. In this case, it is more convenient to consider t_w as a function of $t_{j_2+1}\oplus\cdots\oplus t_{n-j_1}\oplus R$. We obtain after summation over t_{j_2+1} and t_{n-j_1}

$$\begin{aligned}\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}\cap I_{\boldsymbol{j},\boldsymbol{m}}} &(1-|2m_1+1-2^{j_1+1}z_1|)(1-|2m_2+1-2^{j_2+1}z_2|)\\
= \sum_{t_{j_2+2},\dots,t_{n-j_1-1}=0}^{1} &\Big\{(u+2^{j_1+j_2-n+1}(T\oplus R)+2^{j_1+w-1}t_w(0)+\varepsilon_2)\\
&\quad\times(v(0)+2^{j_1+j_2-n+1}a_{n-j_1}R+\varepsilon_1)\\
&+(u+2^{j_1+j_2-n+1}(T\oplus R\oplus 1)+2^{j_1+w-1}t_w(1)+\varepsilon_2)\\
&\quad\times(1-v(0)-2^{j_1+j_2-n+1}a_{n-j_1}R-\varepsilon_1)\\
&+(1-u-2^{j_1+j_2-n+1}(T\oplus R\oplus 1)-2^{j_1+w-1}t_w(0)-\varepsilon_2)\\
&\quad\times(v(1)+2^{j_1+j_2-n+1}(a_{n-j_1}R\oplus 1)+\varepsilon_1)\\
&+(1-u-2^{j_1+j_2-n+1}(T\oplus R)-2^{j_1+w-1}t_w(1)-\varepsilon_2)\\
&\quad\times(1-v(1)-2^{j_1+j_2-n+1}(a_{n-j_1}R\oplus 1)-\varepsilon_1)\Big\}\\
= \sum_{t_{j_2+2},\dots,t_{n-j_1-1}=0}^{1} &2^{-2n}\Big\{2^{n+j_1+j_2+1}\\
&+2^{2j_1+j_2+w+1}(1-2t_w(0)+2a_{n-j_1}R(2t_w(0)-1))\\
&-2^{2(j_1+j_2+1)}+2^{2n}+(2^{2j_1+2j_2+3}-2^{n+j_1+j_2+2})(T\oplus R)\\
&+2^{n+j_1+j_2+2}\varepsilon_1(2(T\oplus R)-1)\\
&-2^{n+j_1+j_2+1}(v(0)+v(1))+2^{n+j_1+j_2+1}(2t_w(0)-1)(v(1)-v(0))\\
&+2^{n+j_1+w}(v(0)+v(1))+2^{n+j_1+j_2+2}(T\oplus R)(v(1)+v(0))\Big\}.\end{aligned}$$

Again, we used $t_w(1) = 1 - t_w(0)$. Note that $t_w(0) = s_w \oplus \cdots \oplus s_{j_2}$ is independent of the digits $t_{j_2+2}, \ldots, t_{n-j_1-1}$. We have

$$\sum_{t_{j_2+2},\ldots,t_{n-j_1-1}=0}^{1} T = \sum_{t_{j_2+2},\ldots,t_{n-j_1-2}=0}^{1} 1 = 2^{n-j_1-j_2-3}$$

and we know the sums $\sum_{t_{j_2+2},\ldots,t_{n-j_1-1}=0}^{1} v(0)$ and $\sum_{t_{j_2+2},\ldots,t_{n-j_1-1}=0}^{1} v(1)$ from above. Similarly as above we can show

$$\begin{aligned}&\sum_{t_{j_2+2},\ldots,t_{n-j_1-1}=0}^{1} (T \oplus R)v(0) \\ &\quad = \frac{1}{2} \sum_{t_{j_2+3},\ldots,t_{n-j_1-1}=0}^{1} (1 \oplus (1-a_{j_2+2})(t_{j_2+3} \oplus \cdots \oplus t_{n-j_1-1} \oplus R)) \\ &\qquad\qquad + \sum_{l=0}^{2^{n-j_1-j_2-3}-1} \frac{l}{2^{n-j_1-j_2-2}}\end{aligned}$$

as well as

$$\begin{aligned}&\sum_{t_{j_2+2},\ldots,t_{n-j_1-1}=0}^{1} (T \oplus R)v(1) \\ &\quad = \frac{1}{2} \sum_{t_{j_2+3},\ldots,t_{n-j_1-1}=0}^{1} (1-a_{j_2+2})(t_{j_2+3} \oplus \cdots \oplus t_{n-j_1-1} \oplus R \oplus 1) \\ &\qquad\qquad + \sum_{l=0}^{2^{n-j_1-j_2-3}-1} \frac{l}{2^{n-j_1-j_2-2}},\end{aligned}$$

which yields

$$\begin{aligned}&\sum_{t_{j_2+2},\ldots,t_{n-j_1-1}=0}^{1} (T \oplus R)(v(1)+v(0)) \\ &\qquad = 2^{n-j_1-j_2-4} + 2 \sum_{l=0}^{2^{n-j_1-j_2-3}-1} \frac{l}{2^{n-j_1-j_2-2}}.\end{aligned}$$

Now we can combine our results with (A.4) to obtain the claimed result. $\square$

Case 7: $\boldsymbol{j} \in \mathcal{J}_7 := \{(j_1, j_2) : 0 \leq j_1, j_2 \leq n-1$ and $j_1 + j_2 \geq n-2\}$.

Proposition A.10. *Let $\boldsymbol{j} \in \mathcal{J}_7$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. Then we have $|\mu_{\boldsymbol{j},\boldsymbol{m}}| \lesssim 2^{-n-j_1-j_2}$ for all $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$ and $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = 2^{-2j_1-2j_2-4}$ for all but at most 2^n elements $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$.*

Proof. At most 2^n of the $2^{|\boldsymbol{j}|}$ dyadic boxes $I_{\boldsymbol{j},\boldsymbol{m}}$ for $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$ contain points. For the empty boxes, only the linear part of the discrepancy function contributes to the corresponding Haar coefficients; hence $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = 2^{-2j_1-2j_2-4}$ for all but at most 2^n elements $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. The non-empty boxes contain at most 4 points. Hence we find by (A.4)

$$\begin{aligned}
|\mu_{\boldsymbol{j},\boldsymbol{m}}| \le{}& 2^{-n-j_1-j_2-2} \\
&\times \sum_{\boldsymbol{z} \in \mathcal{P} \cap I_{\boldsymbol{j},\boldsymbol{m}}} |(1-|2m_1+1-2^{j_1+1}z_1|)(1-|2m_2+1-2^{j_2+1}z_2|)| \\
&+ 2^{-2j_1-2j_2-4} \\
&\le 2^{-n-j_1-j_2-2}4 + 2^{-2j_1-2j_2-4} \le 2^{-n-j_1-j_2} + 2^{-j_1-j_2-(n-2)-4} \\
&\lesssim 2^{-n-j_1-j_2}.
\end{aligned}$$

□

Case 8: $\boldsymbol{j} \in \mathcal{J}_8 := \{(n-1,-1),(-1,n-1)\}$.

Proposition A.11. *Let $\boldsymbol{j} \in \mathcal{J}_8$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. Let $\boldsymbol{j} = (n-1,-1)$ or $\boldsymbol{j} = (-1,n-1)$. Then $\mu_{\boldsymbol{j},\boldsymbol{m}} \lesssim 2^{-2n}$.*

Proof. At most 2 points lie in $I_{\boldsymbol{j},\boldsymbol{m}}$. Hence, if $\boldsymbol{j} = (n-1,-1)$, then by (A.2) we have

$$\begin{aligned}
|\mu_{\boldsymbol{j},\boldsymbol{m}}| \le{}& 2^{-n-j_1-1} \sum_{\boldsymbol{z} \in \mathcal{P} \cap I_{\boldsymbol{j},\boldsymbol{m}}} |(1-|2m_1+1-2^{j_1+1}z_1|)(1-z_2)| + 2^{-2j_1-3} \\
&= 2^{-n-j_1-1}2 + 2^{-2j_1-3} = 2^{-2n+1} + 2^{-2n-1} \lesssim 2^{-2n}.
\end{aligned}$$

The case $\boldsymbol{j} = (-1,n-1)$ can be shown the same way. □

Case 9: $\boldsymbol{j} \in \mathcal{J}_9 := \{(j_1,j_2) : j_1 \ge n \text{ or } j_2 \ge n\}$.

Proposition A.12. *Let $\boldsymbol{j} \in \mathcal{J}_9$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. Then $\mu_{\boldsymbol{j},\boldsymbol{m}} = -2^{-2j_1-2j_2-4}$.*

Proof. The reason is that no point is contained in the interior of $I_{\boldsymbol{j},\boldsymbol{m}}$ in this case and hence only the linear part of the discrepancy function contributes to the Haar coefficient in (A.4). □

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