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On the genera of semisimple groups defined over an integral domain of a global function field

par Rony A. BITAN

Résumé. Soit $K = \mathbb{F}_q(C)$ un corps de fonctions global, i.e. le corps des fonctions d'une courbe projective lisse C définie sur un corps fini $\mathbb{F}_q$. L'anneau des fonctions régulières sur $C - S$, où $S \neq \emptyset$ est un ensemble fini de points fermés sur C, est un domaine de Dedekind $\mathcal{O}_S$ de K. Étant donné un $\mathcal{O}_S$-groupe $\underline{G}$ semisimple dont le groupe fondamental $\underline{F}$ est lisse, on aimerait décrire l'ensemble des genres de $\underline{G}$ et encore (dans le cas où le groupe $\underline{G} \otimes_{\mathcal{O}_S} K$ est isotrope à S) son genre principal en termes des groupes abéliens ne dépendant que de $\mathcal{O}_S$ et de $\underline{F}$. Ceci conduit à une condition nécessaire et suffisante pour que le principe local-global de Hasse soit valable pour certains groupes $\underline{G}$. Nous l'utilisons aussi pour exprimer le nombre de Tamagawa $\tau(G)$ d'un K-groupe semisimple $\underline{G}$ par l'invariant d'Euler–Poincaré et faciliter le calcul de $\tau(G)$ pour les K-groupes tordus.

Abstract. Let $K = \mathbb{F}_q(C)$ be the global function field of rational functions over a smooth and projective curve C defined over a finite field $\mathbb{F}_q$. The ring of regular functions on $C - S$ where $S \neq \emptyset$ is any finite set of closed points on C is a Dedekind domain $\mathcal{O}_S$ of K. For a semisimple $\mathcal{O}_S$-group $\underline{G}$ with a smooth fundamental group $\underline{F}$, we aim to describe both the set of genera of $\underline{G}$ and its principal genus (the latter if $\underline{G} \otimes_{\mathcal{O}_S} K$ is isotropic at S) in terms of abelian groups depending on $\mathcal{O}_S$ and $\underline{F}$ only. This leads to a necessary and sufficient condition for the Hasse local-global principle to hold for certain $\underline{G}$. We also use it to express the Tamagawa number $\tau(G)$ of a semisimple K-group G by the Euler–Poincaré invariant. This facilitates the computation of $\tau(G)$ for twisted K-groups.

1. Introduction

Let C be a projective algebraic curve defined over a finite field $\mathbb{F}_q$, assumed to be geometrically connected and smooth. Let $K = \mathbb{F}_q(C)$ be the global field of rational functions over C, and let Ω be the set of all closed points of C. For any point $\mathfrak{p} \in \Omega$, let $v_\mathfrak{p}$ be the induced discrete valuation on K, $\hat{\mathcal{O}}_\mathfrak{p}$ the complete valuation ring with respect to $v_\mathfrak{p}$, and $\hat{K}_\mathfrak{p}, k_\mathfrak{p}$ its fraction field and residue field at $\mathfrak{p}$, respectively. Any *Hasse set* of K, namely,

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a non-empty finite set $S \subset \Omega$, gives rise to an integral domain of K called a *Hasse domain*:

$$\mathcal{O}_S := \{x \in K : v_{\mathfrak{p}}(x) \geq 0 \ \ \forall\, \mathfrak{p} \notin S\}.$$

This is a regular and one dimensional Dedekind domain. Group schemes defined over $\operatorname{Spec} \mathcal{O}_S$ are underlined, being omitted in the notation of their generic fibers.

Let $\underline{G}$ be an affine, smooth and of finite type group scheme defined over $\operatorname{Spec} \mathcal{O}_S$. We define $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ to be the set of isomorphism classes of $\underline{G}$-torsors over $\operatorname{Spec} \mathcal{O}_S$ relative to the étale or the flat topology (the classification for the two topologies coincide when $\underline{G}$ is smooth; cf. [2, VIII, Cor. 2.3]). The sets $H^1(K, G)$ and $H^1_{\text{ét}}(\hat{\mathcal{O}}_{\mathfrak{p}}, \underline{G}_{\mathfrak{p}})$, for every $\mathfrak{p} \notin S$, are defined similarly. All these three sets are naturally pointed: the distinguished point of $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ (resp., $H^1(K, G)$, $H^1_{\text{ét}}(\hat{\mathcal{O}}_{\mathfrak{p}}, \underline{G}_{\mathfrak{p}})$) is the class of the trivial $\underline{G}$-torsor $\underline{G}$ (resp. trivial G-torsor G, trivial $\underline{G}_{\mathfrak{p}}$-torsor $\underline{G}_{\mathfrak{p}}$). There exists a canonical map of pointed-sets (mapping the distinguished point to the distinguished point):

$$\lambda : H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \to H^1(K, G) \times \prod_{\mathfrak{p} \notin S} H^1_{\text{ét}}(\hat{\mathcal{O}}_{\mathfrak{p}}, \underline{G}_{\mathfrak{p}}) \tag{1.1}$$

which is defined by mapping a class in $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ represented by X to the class represented by $(X \otimes_{\mathcal{O}_S} \operatorname{Spec} K) \times \prod_{\mathfrak{p} \notin S} X \otimes_{\mathcal{O}_S} \operatorname{Spec} \hat{\mathcal{O}}_{\mathfrak{p}}$. Let $[\xi_0] := \lambda([\underline{G}])$. The *principal genus* of $\underline{G}$ is then $\lambda^{-1}([\xi_0])$, i.e., the classes of $\underline{G}$-torsors over $\operatorname{Spec} \mathcal{O}_S$ that are generically and locally trivial at all points of $\mathcal{O}_S$. More generally, a *genus* of $\underline{G}$ is any fiber $\lambda^{-1}([\xi])$ where $[\xi] \in \operatorname{Im}(\lambda)$. The *set of genera* of $\underline{G}$ is then:

$$\operatorname{gen}(\underline{G}) := \{\lambda^{-1}([\xi]) \ : \ [\xi] \in \operatorname{Im}(\lambda)\},$$

whence $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ is a disjoint union of its genera.

Given a representative P of a class in $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$, by referring also to $\underline{G}$ as a $\underline{G}$-torsor acting on itself by conjugations, the quotient of $P \times_{\mathcal{O}_S} \underline{G}$ by the $\underline{G}$-action $(p, g) \mapsto (ps^{-1}, sgs^{-1})$ is an affine $\mathcal{O}_S$-group scheme ${}^P\underline{G}$, called the *twist* of $\underline{G}$ by P. It is an inner form of $\underline{G}$, thus is locally isomorphic to $\underline{G}$ in the étale topology, namely, every fiber of it at a prime of $\mathcal{O}_S$ is isomorphic to $\underline{G}_{\mathfrak{p}} := \underline{G} \otimes_{\mathcal{O}_S} \hat{\mathcal{O}}_{\mathfrak{p}}$ over some finite étale extension of $\hat{\mathcal{O}}_{\mathfrak{p}}$. The map $\underline{G} \mapsto {}^P\underline{G}$ defines a bijection of pointed-sets $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \to H^1_{\text{ét}}(\mathcal{O}_S, {}^P\underline{G})$ (e.g., [31, §2.2, Lem. 2.2.3, Ex. 1 and 2]).

A group scheme defined over $\operatorname{Spec} \mathcal{O}_S$ is said to be *reductive* if it is affine and smooth over $\operatorname{Spec} \mathcal{O}_S$, and each geometric fiber of it at a prime $\mathfrak{p}$ is (connected) reductive over $k_{\mathfrak{p}}$ ([15, Exp. XIX, Def. 2.7]). It is *semisimple* if it is reductive, and the rank of its root system equals that of its lattice of weights ([15, Exp. XXI, Def. 1.1.1]). Suppose $\underline{G}$ is semisimple and that its fundamental group $\underline{F}$ is of order prime to $\mathrm{char}(K)$. Being finite, of multiplicative type ([15, XXII, Cor. 4.1.7]), commutative and smooth, $\underline{F}$ decomposes into finitely many factors of the form $\mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$ or $\mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m)$ where $\underline{\mu}_m := \mathrm{Spec}\,\mathcal{O}_S[t]/(t^m-1)$ and R is some finite (possibly trivial) étale extension of $\mathcal{O}_S$. Consequently, $H^r_{\text{ét}}(\mathcal{O}_S,\underline{F})$ are abelian groups for all $r \geq 0$. The following two $\mathcal{O}_S$-invariants of $\underline{F}$ will play a major role in the description of $H^1_{\text{ét}}(\mathcal{O}_S,\underline{G})$:

Definition 1.1. Let R be a finite étale extension of $\mathcal{O}_S$. We define:

$$i(\underline{F}) := \begin{cases} \mathrm{Br}(R)[m] & \underline{F} = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m) \\ \ker(\mathrm{Br}(R)[m] \xrightarrow{N^{(2)}} \mathrm{Br}(\mathcal{O}_S)[m]) & \underline{F} = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m) \end{cases}$$

where $N^{(2)}$ is induced by the norm map $N_{R/\mathcal{O}_S}$ and for a group $*$, $*[m]$ stands for its m-torsion part. For $\underline{F} = \prod_{k=1}^r \underline{F}_k$ where each $\underline{F}_k$ is one of the above, $i(\underline{F})$ is the direct product $\prod_{k=1}^r i(\underline{F}_k)$.

We also define for such R:

$$j(\underline{F}) := \begin{cases} \mathrm{Pic}(R)/m & \underline{F} = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m) \\ \ker(\mathrm{Pic}(R)/m \xrightarrow{N^{(1)}/m} \mathrm{Pic}(\mathcal{O}_S)/m) & \underline{F} = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m) \end{cases} \tag{1.2}$$

where $N^{(1)}$ is induced by $N_{R/\mathcal{O}_S}$, and again $j(\prod_{k=1}^r \underline{F}_k) := \prod_{k=1}^r j(\underline{F}_k)$.

Definition 1.2. We call $\underline{F}$ *admissible* if it is a finite direct product of the following factors:

(1) $\mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$,

(2) $\mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m)$, $[R:\mathcal{O}_S]$ is prime to m,

where R is any finite étale extension of $\mathcal{O}_S$.

After computing in Section 2 the cohomology sets of some related $\mathcal{O}_S$-groups, we observe in Section 3 Proposition 3.1, that if $\underline{F}$ is admissible then there exists an exact sequence of pointed sets:

$$1 \to \mathrm{Cl}_S(\underline{G}) \overset{h}{\hookrightarrow} H^1_{\text{ét}}(\mathcal{O}_S,\underline{G}) \xrightarrow{w_{\underline{G}}} i(\underline{F}) \to 1.$$

We deduce in Corollary 3.2 that $\mathrm{gen}(\underline{G})$ bijects to $i(\underline{F})$. In Section 4, Theorem 4.1, we show that $\mathrm{Cl}_S(\underline{G})$ surjects onto $j(\underline{F})$. If $G_S := \prod_{s\in S} G(\hat{K}_s)$ is non-compact, then this is a bijection. This leads us to formulate in Corollary 4.4 a necessary and sufficient condition for the *Hasse local-global principle* to hold for $\underline{G}$. In Section 5, we use the above results to express in Theorem 5.2 the Tamagawa number $\tau(G)$ of an almost simple K-group G with an admissible fundamental group F, using the (restricted) Euler–Poincaré characterstic of some $\mathcal{O}_S$-model of F and a local invariant, and show how this new description facilitates the computation of $\tau(G)$ when G is a twisted group.

2. Étale cohomology

2.1. The class set. Consider the ring of S-integral adèles $\mathbb{A}_S := \prod_{\mathfrak{p}\in S} \hat{K}_\mathfrak{p} \times \prod_{\mathfrak{p}\notin S} \hat{\mathcal{O}}_\mathfrak{p}$, being a subring of the adèles $\mathbb{A}$. The *S-class set* of an affine and of finite type $\mathcal{O}_S$-group $\underline{G}$ is the set of double cosets:

$$\mathrm{Cl}_S(\underline{G}) := \underline{G}(\mathbb{A}_S)\backslash \underline{G}(\mathbb{A})/G(K)$$

(when over each $\hat{\mathcal{O}}_\mathfrak{p}$ the above local model $\underline{G}_\mathfrak{p}$ is taken). It is finite (cf. [8, Prop. 3.9]), and its cardinality, called the *S-class number* of $\underline{G}$, is denoted by $h_S(\underline{G})$. According to Nisnevich ([26, Thm. I.3.5]) if $\underline{G}$ is smooth, the map λ introduced in (1.1) applied to it forms the following exact sequence of pointed-sets (when the trivial coset is considered as the distinguished point in $\mathrm{Cl}_S(\underline{G})$):

$$1 \to \mathrm{Cl}_S(\underline{G}) \to H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \xrightarrow{\lambda} H^1(K, G) \times \prod_{\mathfrak{p}\notin S} H^1_{\text{ét}}(\hat{\mathcal{O}}_\mathfrak{p}, \underline{G}_\mathfrak{p}). \tag{2.1}$$

The left exactness reflects the fact that $\mathrm{Cl}_S(\underline{G})$ can be identified with the principal genus of $\underline{G}$.

If, furthermore, $\underline{G}$ has the property:

$$\forall\, \mathfrak{p} \notin S: \ \ H^1_{\text{ét}}(\hat{\mathcal{O}}_\mathfrak{p}, \underline{G}_\mathfrak{p}) \hookrightarrow H^1_{\text{ét}}(\hat{K}_\mathfrak{p}, G_\mathfrak{p}), \tag{2.2}$$

then sequence (2.1) is simplified to (cf. [26, Cor. 3.6]):

$$1 \to \mathrm{Cl}_S(\underline{G}) \to H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \xrightarrow{\lambda_K} H^1(K, G), \tag{2.3}$$

which indicates that any two $\underline{G}$-torsors share the same genus if and only if they are K-isomorphic. If $\underline{G}$ has connected fibers, then by Lang's Theorem $H^1_{\text{ét}}(\hat{\mathcal{O}}_\mathfrak{p}, \underline{G}_\mathfrak{p})$ vanishes for any prime $\mathfrak{p}$ (see [30, Ch. VI, Prop. 5] and recall that all residue fields are finite), thus $\underline{G}$ has property (2.2).

Remark 2.1. The multiplicative $\mathcal{O}_S$-group $\underline{\mathbb{G}}_m$ admits property (2.2) thus sequence (2.3), in which the rightmost term vanishes by Hilbert 90 Theorem. Hence the class set $\mathrm{Cl}_S(\underline{\mathbb{G}}_m)$, being finite as previously mentioned, is bijective as a pointed-set to $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_m)$, which is identified with $\mathrm{Pic}(\mathcal{O}_S)$ (cf. [24, Ch. III, §4]) thus being finite too. This holds true for any finite étale extension R of $\mathcal{O}_S$.

Remark 2.2. If $\underline{G}$ (locally of finite presentation) is disconnected but its connected component $\underline{G}^0$ is reductive and $\underline{G}/\underline{G}^0$ is a finite representable group, then it admits again property (2.2) (see the proof of Proposition 3.14 in [12]), thus sequence (2.3) as well. If, furthermore, for any $[\underline{G}'] \in \mathrm{Cl}_S(\underline{G})$, the map $G'(K) \to (G'/(G')^0)(K)$ is surjective, then $\mathrm{Cl}_S(\underline{G}) = \mathrm{Cl}_S(\underline{G}^0)$ (cf. [5, Lem. 3.2]).

Lemma 2.3. *Let $\underline{G}$ be a smooth and affine $\mathcal{O}_S$-group scheme with connected fibers. Suppose that its generic fiber G is almost simple, simply connected and G_S is non-compact. Then $H^1_{ét}(\mathcal{O}_S, \underline{G}) = 1$.*

Proof. The proof, basically relying on the strong approximation property related to G, is the one of Lemma 3.2 in [4], replacing $\{\infty\}$ by S. □

2.2. The fundamental group: the quasi-split case. The following is the Shapiro Lemma for the étale cohomology:

Lemma 2.4. *Let $f : R \to S$ be a finite étale extension of schemes and Γ a smooth R-module. Then $\forall\, p : H^p_{ét}(S, \mathrm{Res}_{R/S}(\Gamma)) \cong H^p_{ét}(R, \Gamma)$.*

(See [2, VIII, Cor. 5.6] in which the Leray spectral sequence for R/S degenerates, whence the edge morphism $H^p_{\text{ét}}(S, \mathrm{Res}_{R/S}(\Gamma)) \to H^p_{\text{ét}}(R, \Gamma)$ is an isomorphism.)

Remark 2.5. As C is smooth, $\mathrm{Spec}\, \mathcal{O}_S$ is normal, i.e., is integrally closed locally everywhere, thus any finite étale covering of $\mathcal{O}_S$ arises by its normalization in some separable unramified extension of K (e.g., [22, Thm. 6.13]).

Assume $\underline{F} = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$, R is finite étale over $\mathcal{O}_S$. Then the Shapiro Lemma (2.4) with $p = 2$ gives $H^2_{\text{ét}}(\mathcal{O}_S, \underline{F}) \cong H^2_{\text{ét}}(R, \underline{\mu}_m)$. Étale cohomology applied to the Kummer sequence over R

$$1 \to \underline{\mu}_m \to \underline{\mathbb{G}}_m \xrightarrow{x \mapsto x^m} \underline{\mathbb{G}}_m \to 1 \tag{2.4}$$

gives rise to the exact sequences of abelian groups:

$$\begin{aligned} &1 \to H^0_{\text{ét}}(R, \underline{\mu}_m) \to R^\times \xrightarrow{\times m} (R^\times)^m \to 1, \\ &1 \to R^\times/(R^\times)^m \to H^1_{\text{ét}}(R, \underline{\mu}_m) \to \mathrm{Pic}(R)[m] \to 1, \\ &1 \to \mathrm{Pic}(R)/m \to H^2_{\text{ét}}(R, \underline{\mu}_m) \xrightarrow{i_*} \mathrm{Br}(R)[m] \to 1, \end{aligned} \tag{2.5}$$

in which as above $\mathrm{Pic}(R)$ is identified with $H^1_{\text{ét}}(R, \underline{\mathbb{G}}_m)$, and the Brauer group $\mathrm{Br}(R)$ (classifying Azumaya R-algebras) is identified with $H^2_{\text{ét}}(R, \underline{\mathbb{G}}_m)$ (cf. [24, Ch. IV, §2]).

2.3. The fundamental group: the non quasi-split case. The group $\underline{F} = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m)$ fits into the short exact sequence of smooth $\mathcal{O}_S$-groups (recall $\underline{\mu}_m$ is assumed to be smooth as m is prime to $\mathrm{char}(K)$):

$$1 \to \underline{F} \to \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m) \xrightarrow{N_{R/\mathcal{O}_S}} \underline{\mu}_m \to 1$$

which yields by étale cohomology together with Shapiro's isomorphism the long exact sequence:

$$\cdots \to H^r_{\text{ét}}(\mathcal{O}_S,\underline{F}) \xrightarrow{I^{(r)}} H^r_{\text{ét}}(R,\underline{\mu}_m) \xrightarrow{N^{(r)}} H^r_{\text{ét}}(\mathcal{O}_S,\underline{\mu}_m) \to H^{r+1}_{\text{ét}}(\mathcal{O}_S,\underline{F}) \to \cdots . \tag{2.6}$$

Notation 2.6. For a group homomorphism $f : A \to B$, we denote by $f/m : A/m \to B/m$ and $f[m] : A[m] \to B[m]$ the canonical maps induced by f.

Lemma 2.7. *If $[R:\mathcal{O}_S]$ is prime to m, then $N^{(r)}, N^{(r)}[m]$ and $N^{(r)}/m$ are surjective for all $r \geq 0$. In particular, if $\underline{F} = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m)$, then sequence* (2.6) *induces an exact sequence of abelian groups for every $r \geq 0$:*

$$1 \to H^r_{ét}(\mathcal{O}_S,\underline{F}) \xrightarrow{I^{(r)}} H^r_{ét}(R,\underline{\mu}_m) \xrightarrow{N^{(r)}} H^r_{ét}(\mathcal{O}_S,\underline{\mu}_m) \to 1. \tag{2.7}$$

Proof. The composition of the induced norm $N_{R/\mathcal{O}_S}$ with the diagonal morphism coming from the Weil restriction

$$\underline{\mu}_{m,\mathcal{O}_S} \to \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_{m,R}) \xrightarrow{N_{R/\mathcal{O}_S}} \underline{\mu}_{m,\mathcal{O}_S} \tag{2.8}$$

is the multiplication by $n := [R:\mathcal{O}_S]$. It induces for every $r \geq 0$ the maps:

$$H^r_{\text{ét}}(\mathcal{O}_S,\underline{\mu}_m) \to H^r_{\text{ét}}(R,\underline{\mu}_m) \xrightarrow{N^{(r)}} H^r_{\text{ét}}(\mathcal{O}_S,\underline{\mu}_m) \tag{2.9}$$

whose composition is again the multiplication by n on $H^r_{\text{ét}}(\mathcal{O}_S,\underline{\mu}_m)$, being an automorphism when n is prime to m. Hence $N^{(r)}$ is surjective for all $r \geq 0$.

Replacing $\underline{\mu}_m$ with $\underline{\mathbb{G}}_m$ in sequence (2.8) and taking the m-torsion subgroups of the resulting cohomology sets, we get the group maps:

$$H^r_{\text{ét}}(\mathcal{O}_S,\underline{\mathbb{G}}_m)[m] \to H^r_{\text{ét}}(R,\underline{\mathbb{G}}_m)[m] \xrightarrow{N^{(r)}[m]} H^r_{\text{ét}}(\mathcal{O}_S,\underline{\mathbb{G}}_m)[m]$$

whose composition is multiplication by n on $H^r_{\text{ét}}(\mathcal{O}_S,\underline{\mathbb{G}}_m)[m]$, being an automorphism again as n is prime to m, whence $N^{(r)}[m]$ is an epimorphism for every $r \geq 0$. The same argument applied to $N^{(r)}/m$ shows it is surjective for every $r \geq 0$ as well. □

Back to the general case ($[R:\mathcal{O}_S]$ does not have to be prime to m), applying the Snake lemma to the exact and commutative diagram of abelian groups:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Pic}(R)/m & \longrightarrow & H^2_{\text{ét}}(R,\underline{\mu}_m) & \xrightarrow{i_*} & \mathrm{Br}(R)[m] & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle N^{(1)}/m} & & \downarrow{\scriptstyle N^{(2)}} & & \downarrow{\scriptstyle N^{(2)}[m]} & & \\ 1 & \longrightarrow & \mathrm{Pic}(\mathcal{O}_S)/m & \longrightarrow & H^2_{\text{ét}}(\mathcal{O}_S,\underline{\mu}_m) & \longrightarrow & \mathrm{Br}(\mathcal{O}_S)[m] & \longrightarrow & 1 \end{array} \tag{2.10}$$

yields an exact sequence of m-torsion abelian groups:

$$
\begin{aligned}
(2.11)\quad 1 \to \ker(\mathrm{Pic}(R)/m \xrightarrow{N^{(1)}/m} \mathrm{Pic}(\mathcal{O}_S)/m) \to \ker(N^{(2)}) \\
\xrightarrow{i'_*} \ker(\mathrm{Br}(R)[m] \xrightarrow{N^{(2)}[m]} \mathrm{Br}(\mathcal{O}_S)[m]) \\
\to \mathrm{coker}(\mathrm{Pic}(R)/m \xrightarrow{N^{(1)}/m} \mathrm{Pic}(\mathcal{O}_S)/m),
\end{aligned}
$$

where i'_* is the restriction of i_* to $\ker(N^{(2)})$. Together with the surjection $I^{(2)} : H^2_{\text{ét}}(\mathcal{O}_S, \underline{F}) \twoheadrightarrow \ker(N^{(2)})$ coming from sequence (2.6), we get the commutative diagram:

$$
(2.12)\qquad
\begin{array}{ccc}
 & & \ker\left(\mathrm{Pic}(R)/m \to \mathrm{Pic}(\mathcal{O}_S)/m\right) \\
 & & \big\downarrow \\
H^2_{\text{ét}}(\mathcal{O}_S, \underline{F}) & \xrightarrow{I^{(2)}} & \ker\left(H^2_{\text{ét}}(R, \underline{\mu}_m) \xrightarrow{N^{(2)}} H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_m)\right) \\
 & \searrow^{i^{(1)}_*} & \big\downarrow i'_* \\
 & & \ker\left(\mathrm{Br}(R)[m] \to \mathrm{Br}(\mathcal{O}_S)[m]\right).
\end{array}
$$

Proposition 2.8. *If $[R : \mathcal{O}_S]$ is prime to m, then there exists a canonical exact sequence of abelian groups*

$$
\begin{aligned}
1 \to \ker\left(\mathrm{Pic}(R)/m \to \mathrm{Pic}(\mathcal{O}_S)/m\right) \to \ker(N^{(2)}) \\
\xrightarrow{i'_*} \ker\left(\mathrm{Br}(R)[m] \to \mathrm{Br}(\mathcal{O}_S)[m]\right) \to 1.
\end{aligned}
$$

Proof. This sequence is the column in diagram (2.12) since Lemma 2.7 shows the surjectivity of $N^{(1)}/m$, which in turn implies the surjectivity of i'_* by the exactness of sequence (2.11). □

Recall the definition of $i(\underline{F})$ (Definition 1.1), and of the maps i_* and $i^{(1)}_*$ (sequences (2.5) and (2.12)).

Definition 2.9. Let $\underline{F}$ be one of the basic factors of an admissible fundamental group (see Def. 1.2). The map $\bar{i}_* : H^2_{\text{ét}}(\mathcal{O}_S, \underline{F}) \to i(\underline{F})$ is defined as:

$$
\bar{i}_* := \begin{cases} i_* & \underline{F} = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m), \\ i^{(1)}_* & \underline{F} = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m) \text{ and } ([R : \mathcal{O}_S], m) = 1. \end{cases}
$$

More generally, if $\underline{F} = \prod_{k=1}^r \underline{F}_k$ where each $\underline{F}_k$ is one of the above, we set it to be the composition:

$$
\bar{i}_* : H^2_{\text{ét}}(\mathcal{O}_S, \underline{F}) \xrightarrow{\sim} \bigoplus_{k=1}^r H^2_{\text{ét}}(\mathcal{O}_S, \underline{F}_k) \xrightarrow{\bigoplus_{k=1}^r (\bar{i}_*)_k} i(\underline{F}) = \prod_{k=1}^r i(\underline{F}_k).
$$

Corollary 2.10. *If $\underline{F}$ is admissible, then there exists a short exact sequence*

$$1 \to j(\underline{F}) \to H^2_{ét}(\mathcal{O}_S, \underline{F}) \xrightarrow{\bar{i}_*} i(\underline{F}) \to 1. \tag{2.13}$$

Proof. If $\underline{F} = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$ then the sequence of the corollary is simply a restatement of the last sequence in (2.5) by the definitions of $i(\underline{F})$ and $j(\underline{F})$ (see Definition 1.1). On the other hand, if $\underline{F} = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m)$ with $[R : \mathcal{O}_S]$ is prime to m, then $I^{(2)}$ induces an isomorphism of abelian groups $H^2_{\text{ét}}(\mathcal{O}_S, \underline{F}) \cong \ker(N^{(2)})$ by the exactness of (2.7) for $r = 2$. Thus the sequence of the corollary is isomorphic to the sequence in Proposition 2.8 again by the definitions of $j(\underline{F})$ and $i(\underline{F})$. The two cases considered above suffice to establish the corollary by the definition of admissible (see Definition 1.2) and the definition of $\bar{i}_*$ (see Definition 2.9). ☐

Definition 2.11. Let $\underline{X}$ be a constructible sheaf defined over $\mathrm{Spec}\,\mathcal{O}_S$ and let $h_i(\underline{X}) := |H^i_{\text{ét}}(\mathcal{O}_S, \underline{X})|$. The (restricted) *Euler–Poincaré characteristic* of $\underline{X}$ is defined to be (cf. [25, Ch. II, §2]):

$$\chi_S(\underline{X}) := \prod_{i=0}^{2} h_i(\underline{X})^{(-1)^i}.$$

Definition 2.12. Let R be a finite étale extension of $\mathcal{O}_S$. We define:

$$l(\underline{F}) := \begin{cases} \frac{|R^\times[m]|}{[R^\times : (R^\times)^m]} & \underline{F} = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m) \\ \frac{|\ker(N^{(0)}[m])|}{|\ker(N^{(0)}/m)|} & \underline{F} = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m). \end{cases}$$

As usual, for $\underline{F} = \prod_{k=1}^r \underline{F}_k$ where each $\underline{F}_k$ is one of the above, we put $l(\underline{F}) = \prod_{k=1}^r l(\underline{F}_k)$.

Lemma 2.13. *If $\underline{F}$ is admissible then $\chi_S(\underline{F}) = l(\underline{F}) \cdot |i(\underline{F})|$.*

Proof. It is sufficient to check the assertion for the two basic types of (direct) factors:

Suppose $\underline{F} = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$. Then sequences (2.5) together with Shapiro's Lemma give

$$h_i(\underline{F}) = |H^i_{\text{ét}}(R, \underline{\mu}_m)| = \begin{cases} |R^\times[m]|, & i = 0 \\ [R^\times : (R^\times)^m] \cdot |\mathrm{Pic}(R)[m]|, & i = 1 \\ |\mathrm{Pic}(R)/m| \cdot |\mathrm{Br}(R)[m]| & i = 2. \end{cases}$$

So as $\mathrm{Pic}(R)$ is finite (see Remark 2.1), $|\mathrm{Pic}(R)[m]| = |\mathrm{Pic}(R)/m|$ and we get:

$$\chi_S(\underline{F}) := \frac{h_0(\underline{F}) \cdot h_2(\underline{F})}{h_1(\underline{F})} = \frac{|R^\times[m]| \cdot |\mathrm{Pic}(R)/m| \cdot |\mathrm{Br}(R)[m]|}{[R^\times : (R^\times)^m] \cdot |\mathrm{Pic}(R)[m]|} = l(\underline{F}) \cdot |i(\underline{F})|.$$

Now suppose $\underline{F} = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m)$ such that $[R : \mathcal{O}_S]$ is prime to m. By Lemma 2.7 $N^{(r)}, N^{(r)}[m]$ and $N^{(r)}/m$ are surjective for all $r \geq 0$, so the long sequence (2.6) is cut into short exact sequences:

$$\forall\, r \geq 0: \; 1 \to H^r_{\text{ét}}(\mathcal{O}_S, \underline{F}) \xrightarrow{I^{(r)}} H^r_{\text{ét}}(R, \underline{\mu}_m) \xrightarrow{N^{(r)}} H^r_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_m) \to 1 \tag{2.14}$$

from which we see that (notice that $N^{(0)}[m]$ coincides with $N^{(0)}$):

$$h_0(\underline{F}) = |\ker(R^\times[m] \xrightarrow{N^{(0)}[m]} \mathcal{O}_S^\times[m])|. \tag{2.15}$$

The Kummer exact sequences for $\underline{\mu}_m$ defined over both $\mathcal{O}_S$ and R yield the exact diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R^\times/(R^\times)^m & \longrightarrow & H^1_{\text{ét}}(R, \underline{\mu}_m) & \longrightarrow & \mathrm{Pic}(R)[m] & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle N^{(0)}/m} & & \downarrow{\scriptstyle N^{(1)}} & & \downarrow{\scriptstyle N^{(1)}[m]} & & \\ 1 & \longrightarrow & \mathcal{O}_S^\times/(\mathcal{O}_S^\times)^m & \longrightarrow & H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_m) & \longrightarrow & \mathrm{Pic}(\mathcal{O}_S)[m] & \longrightarrow & 1 \end{array} \tag{2.16}$$

from which we see together with sequence (2.14) that:

$$h_1(\underline{F}) = |\ker(N^{(1)})| = \left|\ker(R^\times/(R^\times)^m \xrightarrow{N^{(0)}/m} \mathcal{O}_S^\times/(\mathcal{O}_S^\times)^m)\right| \cdot \left|\ker(\mathrm{Pic}(R)[m] \xrightarrow{N^{(1)}[m]} \mathrm{Pic}(\mathcal{O}_S)[m])\right|.$$

Similarly, by sequence (2.14) and Proposition 2.8 we find that:

$$h_2(\underline{F}) = |\ker(N^{(2)})| = \left|\ker(\mathrm{Pic}(R)/m \xrightarrow{N^{(1)}/m} \mathrm{Pic}(\mathcal{O}_S)/m)\right| \cdot \left|\ker(\mathrm{Br}(R)[m] \xrightarrow{N^{(2)}[m]} \mathrm{Br}(\mathcal{O}_S)[m])\right|.$$

Altogether we get:

$$\chi_S(\underline{F}) = \frac{h_0(\underline{F}) \cdot h_2(\underline{F})}{h_1(\underline{F})} = \frac{|\ker(N^{(0)}[m])|}{|\ker(N^{(0)}/m)|} \cdot \frac{|\ker(N^{(1)}[m])|}{|\ker(N^{(1)}/m)|} \cdot |\ker(N^{(2)}[m])|.$$

The group of units $R^\times$ is a finitely generated abelian group (cf. [29, Prop. 14.2]), thus the quotient $R^\times/(R^\times)^m$ is a finite group. Since $\mathrm{Pic}(R)[m]$ is also finite, $\ker(N^{(1)})$ in diagram (2.16) is finite, thus $|\ker(N^{(1)})[m]| = |\ker(N^{(1)})/m|$, and we are left with:

$$\chi_S(\underline{F}) = \frac{|\ker(N^{(0)}[m])|}{|\ker(N^{(0)}/m)|} \cdot |\ker(N^{(2)}[m])| = l(\underline{F}) \cdot |i(\underline{F})|. \qquad \square$$

Remark 2.14. The computation of $l(\underline{F})$, for specific choices of R, $\mathcal{O}_S$ and m, is an interesting (and probably open) problem. For example, when $\underline{F}$ is not quasi-split, the denominator of this number is the order of the group of units of R whose norm down to $\mathcal{O}_S$ is an m-th power of a unit in $\mathcal{O}_S$, modulo $(R^\times)^m$. Such computations are hard to find in the literature, if they exist at all.

3. The set of genera

From now and on we assume $\underline{G}$ is semisimple and that its fundamental group $\underline{F}$ is of order prime to $\mathrm{char}(K)$, thus smooth. Étale cohomology applied to the universal covering of $\underline{G}$

$$1 \to \underline{F} \to \underline{G}^{\mathrm{sc}} \to \underline{G} \to 1, \tag{3.1}$$

gives rise to the exact sequence of pointed-sets:

$$H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{sc}}) \to H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \xrightarrow{\delta_{\underline{G}}} H^2_{\text{ét}}(\mathcal{O}_S, \underline{F}) \tag{3.2}$$

in which the co-boundary map $\delta_{\underline{G}}$ is surjective, as the domain $\mathcal{O}_S$ is of Douai-type, implying that $H^2_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{sc}}) = 1$ (see [17, Def. 5.2 and Ex. 5.4 (iii)]).

Proposition 3.1. *There exists an exact sequence of pointed-sets:*

$$1 \to \mathrm{Cl}_S(\underline{G}) \xrightarrow{h} H^1_{\acute{e}t}(\mathcal{O}_S, \underline{G}) \xrightarrow{w_{\underline{G}}} i(\underline{F})$$

in which h is injective. If $\underline{F}$ is admissible, then $w_{\underline{G}}$ is surjective.

Proof. It is shown in [26, Thm. 2.8 and proof of Thm. 3.5] that there exist a canonical bijection $\alpha_{\underline{G}} : H^1_{\mathrm{Nis}}(\mathcal{O}_S, \underline{G}) \cong \mathrm{Cl}_S(\underline{G})$ and a canonical injection $i_{\underline{G}} : H^1_{\mathrm{Nis}}(\mathcal{O}_S, \underline{G}) \hookrightarrow H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ of pointed-sets (as Nisnevich's covers are étale). Then the map h of the statement is the composition $i_{\underline{G}} \circ \alpha_{\underline{G}}^{-1}$.

Assume $\underline{F} = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$. The composition of the surjective map $\delta_{\underline{G}}$ from (3.2) with Shapiro's isomorphism and the surjective morphism i_* from (2.5), is a surjective R-map:

$$w_{\underline{G}} : H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \overset{\delta_{\underline{G}}}{\twoheadrightarrow} H^2_{\text{ét}}(\mathcal{O}_S, \underline{F}) \xrightarrow{\sim} H^2_{\text{ét}}(R, \underline{\mu}_m) \overset{i_*}{\twoheadrightarrow} \mathrm{Br}(R)[m]. \tag{3.3}$$

On the generic fiber, since $G^{\mathrm{sc}} := \underline{G}^{\mathrm{sc}} \otimes_{\mathcal{O}_S} K$ is simply connected, $H^1(K, G^{\mathrm{sc}})$ vanishes due to Harder (cf. [19, Satz A]), as well as its other K-forms (this would not be true, however, if K were a number field with real places). So Galois cohomology applied to the universal K-covering

$$1 \to F \to G^{\mathrm{sc}} \to G \to 1 \tag{3.4}$$

yields an embedding of pointed-sets $\delta_G : H^1(K, G) \hookrightarrow H^2(K, F)$, which is also surjective as K is of Douai-type as well. The extension R of $\mathcal{O}_S$ arises from an unramified Galois extension L of K by Remark 2.5, and Galois cohomology applied to the Kummer exact sequence of L-groups

$$1 \to \mu_m \to \mathbb{G}_m \xrightarrow{x \mapsto x^m} \mathbb{G}_m \to 1$$

yields, together with Shapiro's Lemma $H^2(K,F) \cong H^2(L,\underline{\mu}_m)$ and Hilbert 90 Theorem, the identification $(i_*)_L : H^2(K,F) \cong \mathrm{Br}(L)[m]$, whence the composition $(i_*)_L \circ \delta_G$ is an injective L-map:

$$w_G : H^1(K,G) \xhookrightarrow{\delta_G} H^2(K,F) \overset{(i_*)_L}{\cong} \mathrm{Br}(L)[m].$$

Now we know due to Grothendieck that $\mathrm{Br}(R)$ is a subgroup of $\mathrm{Br}(L)$ (see [18, Prop. 2.1] and [24, Ex. 2.22, case (a)]). Altogether we retrieve the commutative diagram of pointed-sets:

(3.5)
$$\begin{array}{ccc} H^1_{\text{ét}}(\mathcal{O}_S,\underline{G}) & \xrightarrow{w_{\underline{G}}} & \mathrm{Br}(R)[m] \\ \downarrow{\scriptstyle \lambda_K} & & \downarrow{\scriptstyle j} \\ H^1(K,G) & \xhookrightarrow{w_G} & \mathrm{Br}(L)[m], \end{array}$$

from which, together with sequence (2.3) (recall $\underline{G}$ has connected fibers), we may observe that:

$$\mathrm{Cl}_S(\underline{G}) = \ker(\lambda_K) = \ker(w_{\underline{G}}).$$

When $\underline{F} = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m)$, we define the map $w_{\underline{G}}$ using diagram 2.12 to be the composition

(3.6)
$$w_{\underline{G}} : H^1_{\text{ét}}(\mathcal{O}_S,\underline{G}) \xrightarrow{\delta_{\underline{G}}} H^2_{\text{ét}}(\mathcal{O}_S,\underline{F}) \xrightarrow{i_*^{(1)}} \ker\Big(\mathrm{Br}(R)[m] \xrightarrow{N^{(2)}[m]} \mathrm{Br}(\mathcal{O}_S)[m]\Big)$$

being surjective by Corollary 2.10 given that $[R:\mathcal{O}_S]$ is prime to m. On the generic fiber, Galois cohomology with Hilbert 90 Theorem give:

$$w_G : H^1(K,G) \xhookrightarrow{\delta_G} H^2(K,F) \overset{(i_*^{(1)})_K}{\cong} \ker\Big(\mathrm{Br}(L)[m] \xrightarrow{N_L^{(2)}[m]} \mathrm{Br}(K)[m]\Big).$$

This time we get the commutative diagram of pointed sets:

(3.7)
$$\begin{array}{ccc} H^1_{\text{ét}}(\mathcal{O}_S,\underline{G}) & \xrightarrow{w_{\underline{G}}} & \ker\Big(\mathrm{Br}(R)[m] \xrightarrow{N^{(2)}[m]} \mathrm{Br}(\mathcal{O}_S)[m]\Big) \\ \downarrow{\scriptstyle \lambda_K} & & \downarrow{\scriptstyle j} \\ H^1(K,G) & \xhookrightarrow{w_G} & \ker\Big(\mathrm{Br}(L)[m] \xrightarrow{(N^{(2)}[m])_L} \mathrm{Br}(K)[m]\Big), \end{array}$$

from which we may deduce again that:

$$\mathrm{Cl}_S(\underline{G}) \overset{(2.3)}{=} \ker(\lambda_K) = \ker(w_{\underline{G}}).$$

More generally, if $\underline{F}$ is a direct product of such basic factors, then as the cohomology sets commute with direct products, the target groups of $w_{\underline{G}}$

and w_G become the product of the target groups of their factors, and the same argument gives the last assertion. □

Corollary 3.2. *There is an injection of pointed sets $w'_{\underline{G}} : \mathrm{gen}(\underline{G}) \hookrightarrow i(\underline{F})$. If $\underline{F}$ is admissible then $w'_{\underline{G}}$ is a bijection. In particular if $\underline{F}$ is split, then $|\mathrm{gen}(\underline{G})| = |F|^{|S|-1}$.*

Proof. The commutativity of diagrams (3.5) and (3.7) and the injectivity of the map j in them show that $w_{\underline{G}}$ is constant on each fiber of λ_K, i.e., on the genera of $\underline{G}$. Thus w_G induces a map (see Proposition 3.1):

$$w'_{\underline{G}} : \mathrm{gen}(\underline{G}) \to \mathrm{Im}(w_{\underline{G}}) \subseteq i(\underline{F}).$$

These diagrams commutativity together with the injectivity of w_G imply the injectivity of $w'_{\underline{G}}$.

If $\underline{F}$ is admissible then $\mathrm{Im}(w_{\underline{G}}) = i(\underline{F})$. In particular if $\underline{F}$ is split, then

$$\mathrm{gen}(\underline{G}) \cong \prod_{i=1}^{r} \mathrm{Br}(\mathcal{O}_S)[m_i].$$

It is shown in the proof of [4, Lem. 2.2] that

$$\mathrm{Br}(\mathcal{O}_S) = \ker\left(\mathbb{Q}/\mathbb{Z} \xrightarrow{\sum_{\mathfrak{p}\in S} \mathrm{Cor}_{\mathfrak{p}}} \mathbb{Q}/\mathbb{Z}\right)$$

where $\mathrm{Cor}_{\mathfrak{p}}$ is the corestriction map at $\mathfrak{p}$. So $|\mathrm{Br}(\mathcal{O}_S)[m_i]| = m_i^{|S|-1}$ for all i and the last assertion follows. □

The following table refers to absolutely almost simple and adjoint $\mathcal{O}_S$-groups whose fundamental group is split. The right column is Corollary 3.2:

Type of $\underline{G}$	$\underline{F}$	$\#\ \mathrm{gen}(\underline{G})$
${}^1\mathrm{A}_{n-1}$	$\underline{\mu}_n$	$n^{\vert S\vert-1}$
$\mathrm{B}_n, \mathrm{C}_n, \mathrm{E}_7$	$\underline{\mu}_2$	$2^{\vert S\vert-1}$
${}^1\mathrm{D}_n$	$\underline{\mu}_4,\ n = 2k+1$ $\underline{\mu}_2 \times \underline{\mu}_2,\ n = 2k$	$4^{\vert S\vert-1}$
${}^1\mathrm{E}_6$	$\underline{\mu}_3$	$3^{\vert S\vert-1}$
$\mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$	1	1

Lemma 3.3. *Let $\underline{G}$ be a semisimple and almost simple $\mathcal{O}_S$-group not of (absolute) type A, then $H^1_{\acute{e}t}(\mathcal{O}_S, \underline{G})$ bijects as a pointed-set to the abelian group $H^2_{\acute{e}t}(\mathcal{O}_S, \underline{F})$.*

Proof. Since G^{sc} is not of (absolute) type A, it is locally isotropic everywhere ([9, 4.3 and 4.4]), whence $\ker(\delta_{\underline{G}}) \subseteq H^1_{\acute{e}t}(\mathcal{O}_S, \underline{G}^{\mathrm{sc}})$ vanishes due to Lemma 2.3. Moreover, for any $\underline{G}$-torsor P, the base-point change: $\underline{G} \mapsto {}^P\underline{G}$

defines a bijection of pointed-sets: $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \to H^1_{\text{ét}}(\mathcal{O}_S, {}^P\underline{G})$ (see in Section 1). But ${}^P\underline{G}$ is an inner form of $\underline{G}$, thus not of type A as well, hence also $H^1_{\text{ét}}(\mathcal{O}_S, ({}^P\underline{G})^{\text{sc}}) = 1$. We get that all fibers of $\delta_{\underline{G}}$ in (3.2) are trivial, which together with the surjectivity of $\delta_{\underline{G}}$ amounts to the asserted bijection. □

In other words, the fact that $\underline{G}$ is not of (absolute) type A guarantees that not only G^{sc}, but also the universal covering of the generic fiber of inner forms of $\underline{G}$ of other genera are locally isotropic everywhere. This provides $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ the structure of an abelian group.

Corollary 3.4. *If $\underline{G}$ is not of (absolute) type A, then all its genera share the same cardinality.*

Proof. The map $w_{\underline{G}}$ factors through $\delta_{\underline{G}}$ (see (3.3) and (3.6)) which is a bijection of pointed-sets in this case by Lemma 3.3. So writing: $w_{\underline{G}} = \overline{w}_{\underline{G}} \circ \delta_{\underline{G}}$. we get due to Proposition 3.1 the exact sequence of pointed-sets (a-priory, abelian groups):

$$1 \to \text{Cl}_S(\underline{G}) \to H^2_{\text{ét}}(\mathcal{O}_S, \underline{F}) \xrightarrow{\overline{w}_{\underline{G}}} i(\underline{F})$$

in which all genera, corresponding to the fibers of $w_{\underline{G}}$, are of the same cardinality. □

Following E. Artin in [1], we shall say that a Galois extension L of K is *imaginary* if no prime of K is decomposed into distinct primes in L.

Remark 3.5. If $\underline{G}$ is of (absolute) type A, but $S = \{\infty\}$, G is $\hat{K}_\infty$-isotropic, and F splits over an imaginary extension of K, then $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ still bijects as a pointed-set to $H^2_{\text{ét}}(\mathcal{O}_S, \underline{F})$.

Proof. As aforementioned, removing one closed point of a projective curve, the resulting Hasse domain has a trivial Brauer group. Thus $\text{Br}(\mathcal{O}_S = \mathcal{O}_{\{\infty\}}) = 1$, and as F splits over an imaginary extension $L = \mathbb{F}_q(C')$, corresponding to an étale extension $R = \mathbb{F}_q[C' - \{\infty'\}]$ of $\mathcal{O}_{\{\infty\}}$ (see Remark 2.5) where ∞' is the unique prime of L lying above ∞, $\text{Br}(R)$ remains trivial. This implies by Corollary 3.2 that $\underline{G}$ has only one genus, namely, the principal one, in which the generic fibers of all representatives (being K-isomorphic to G) are isotropic at ∞. Then the resulting vanishing of $\ker(\delta_{\underline{G}}) \subseteq H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\text{sc}})$ due to Lemma 2.3 is equivalent to the injectivity of $\delta_{\underline{G}}$. □

The following general framework due to Giraud (see [10, §2.2.4]), gives an interpretation of the $\underline{G}$-torsors which may help us describe $w_{\underline{G}}$ more concretely.

Proposition 3.6. *Let R be a scheme and X_0 be an R-form, namely, an object of a fibered category of schemes defined over R. Let $\mathbf{Aut}_{X_0}$ be its R-group of automorphisms. Let $\mathfrak{Forms}(X_0)$ be the category of R-forms that are locally isomorphic for some topology to X_0 and let $\mathfrak{Tors}(\mathrm{Aut}_{X_0})$ be the category of Aut_{X_0}-torsors in that topology. The functor*

$$\varphi : \mathfrak{Forms}(X_0) \to \mathfrak{Tors}(\mathbf{Aut}_{X_0}) : \ X \mapsto \mathbf{Iso}_{X_0,X}$$

is an equivalence of fibered categories.

Example 3.7. Let (V, q) be a regular quadratic $\mathcal{O}_S$-space of rank $n \geq 3$ and let $\underline{G}$ be the associated *special orthogonal group* $\underline{\mathbf{SO}}_q$ (see [13, Def. 1.6]). It is smooth and connected (cf. [13, Thm. 1.7]), and its generic fiber is of type B_n if $\mathrm{rank}(V)$ is odd, and of type ${}^1\mathrm{D}_n$ otherwise. In both cases $\underline{F} = \underline{\mu}_2$, so we assume $\mathrm{char}(K)$ is odd. Any such quadratic regular $\mathcal{O}_S$-space (V', q') of rank n gives rise to a $\underline{G}$-torsor P by

$$V' \mapsto P = \mathbf{Iso}_{V,V'}$$

where an isomorphism $A : V \to V'$ is a *proper* q-isometry, i.e., such that $q' \circ A = q$ and $\det(A) = 1$. So $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ properly classifies regular quadratic $\mathcal{O}_S$-spaces that are locally isomorphic to (V, q) in the étale topology. Then $\delta_{\underline{G}}([P])$ is the *second Stiefel–Whitney* class of P in $H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2)$, classifying $\mathcal{O}_S$-Azumaya algebras with involutions (see [6, Def. 1, Rem. 3.3 and Prop. 4.5]), and

$$w_{\underline{G}}([\underline{\mathbf{SO}}_{q'}]) = \begin{cases} [\mathbf{C}_0(q')] - [\mathbf{C}_0(q)] \in \mathrm{Br}(\mathcal{O}_S)[2] & n \text{ is odd} \\ [\mathbf{C}(q')] - [\mathbf{C}(q)] \in \mathrm{Br}(\mathcal{O}_S)[2] & n \text{ is even,} \end{cases}$$

where $\mathbf{C}(q)$ and $\mathbf{C}_0(q)$ are the Clifford algebra of q and its even part, respectively.

Example 3.8. Let $\underline{G} = \underline{\mathbf{PGL}}_n$ for $n \geq 2$. It is smooth and connected ([14, Lem. 3.3.1]) with $\underline{F} = \underline{\mu}_n$, so we assume $(\mathrm{char}(K), n) = 1$. For any projective $\mathcal{O}_S$-space of rank n, by the Skolem–Noether Theorem for unital rings (see [21, p. 145]) $\underline{\mathbf{PGL}}(V) = \mathbf{Aut}(\mathrm{End}_{\mathcal{O}_S}(V))$. It is an inner form of $\underline{G}$ obtained for $V = \mathcal{O}_S^n$. So the pointed-set $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ classifies the projective $\mathcal{O}_S$-modules of rank n up to invertible $\mathcal{O}_S$-modules. Given such a projective $\mathcal{O}_S$-module V, the Azumaya $\mathcal{O}_S$-algebra $A = \mathrm{End}_{\mathcal{O}_S}(V)$ of rank n^2 corresponds to a $\underline{G}$-torsor by (see [16, V, Rem. 4.2]):

$$A \mapsto P = \mathbf{Iso}_{\underline{M}_n, A}$$

where $\underline{M}_n$ is the $\mathcal{O}_S$-sheaf of $n \times n$ matrices. Here $w_{\underline{G}}([P]) = [A]$ in $\mathrm{Br}(\mathcal{O}_S)[n]$.

4. The principal genus

In this section, we study the structure of the principal genus $\mathrm{Cl}_S(\underline{G})$.

Theorem 4.1. *If $\underline{F}$ is admissible then there exists a surjection of pointed-sets*

$$\psi_{\underline{G}} : \mathrm{Cl}_S(\underline{G}) \twoheadrightarrow j(\underline{F}),$$

being a bijection provided that G_S is non-compact (e.g., G is not anisotropic of type A).

Proof. Combining the two epimorphisms, $w_{\underline{G}}$ defined in Prop. 3.1 and $\delta_{\underline{G}}$ described in Section 3, together with the exact sequence (2.13), yields the exact and commutative diagram:

$$\begin{array}{ccccccccc} & & 1 & \longrightarrow & H^1_{\text{ét}}(\mathcal{O}_S,\underline{G}) & = & H^1_{\text{ét}}(\mathcal{O}_S,\underline{G}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \delta_{\underline{G}}} & & \downarrow{\scriptstyle w_{\underline{G}}} & & \\ 1 & \longrightarrow & j(\underline{F}) & \xrightarrow{\partial} & H^2_{\text{ét}}(\mathcal{O}_S,\underline{F}) & \xrightarrow{\bar{i}_*} & i(\underline{F}) & \longrightarrow & 1 \end{array} \tag{4.1}$$

in which $\ker(w_{\underline{G}}) = \mathrm{Cl}_S(\underline{G})$. We imitate the Snake Lemma argument (the diagram terms are not necessarily all groups): for any $[H] \in \mathrm{Cl}_S(\underline{G})$ one has $\bar{i}_*(\delta_{\underline{G}}([H])) = [0]$, i.e., $\delta_{\underline{G}}([H])$ has a ∂-preimage in $j(\underline{F})$ which is unique as ∂ is a monomorphism of groups. This constructed map denoted $\psi_{\underline{G}}$ gives rise to an exact sequence of pointed-sets:

$$1 \to \mathfrak{K} \to \mathrm{Cl}_S(\underline{G}) \xrightarrow{\psi_{\underline{G}}} j(\underline{F}) \to 1.$$

If G_S is non-compact, then for any $[\underline{H}] \in \mathrm{Cl}_S(\underline{G})$ the generic fiber H is K-isomorphic to G thus H_S is non-compact as well, thus $\ker(H^1_{\text{ét}}(\mathcal{O}_S,\underline{H}) \xrightarrow{\delta_{\underline{H}}} H^2_{\text{ét}}(\mathcal{O}_S,\underline{F})) \subseteq H^1_{\text{ét}}(\mathcal{O}_S,\underline{H}^{\text{sc}})$ vanishes by Lemma 2.3. This means that $\delta_{\underline{G}}$ restricted to $\mathrm{Cl}_S(\underline{G})$ is an embedding, so $\mathfrak{K} = 1$ and $\psi_{\underline{G}}$ is a bijection. $\square$

Remark 4.2. The description of $\mathrm{Cl}_S(\underline{G})$ in Theorem 4.1 holds true also for a disconnected group $\underline{G}$ (where $\underline{F}$ is the fundamental group of $\underline{G}^0$), under the hypotheses of Remark 2.2.

Definition 4.3. We say that the *local-global Hasse principle* holds for $\underline{G}$ if $h_S(\underline{G}) = 1$.

This property means (when $\underline{G}$ is connected) that a $\underline{G}$-torsor is $\mathcal{O}_S$-isomorphic to $\underline{G}$ if and only if its generic fiber is K-isomorphic to G. Recall the definition of $j(\underline{F})$ from Definition 1.1.

Corollary 4.4. *Suppose $\underline{F} \cong \prod_{i=1}^r \mathrm{Res}_{R_i/\mathcal{O}_S}(\underline{\mu}_{m_i})$ where R_i are finite étale extensions of $\mathcal{O}_S$. If G_S is non-compact, then the Hasse principle holds for $\underline{G}$ if and only if $\forall\ i : (|\mathrm{Pic}(R_i)|, m_i) = 1$. Otherwise ($G_S$ is compact), this principle holds for $\underline{G}$ only if $\forall\ i : (|\mathrm{Pic}(R_i)|, m_i) = 1$. More generally, if $\underline{F}$ is* *admissible and G_S is non-compact, then this principle holds for $\underline{G}$ provided that for each factor of the form* $\mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$ *or* $\mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m)$ *one has:* $(|\mathrm{Pic}(R)|, m) = 1$.

Example 4.5. If C^{af} is an affine non-singular $\mathbb{F}_q$-curve of the form $y^2 = x^3 + ax + b$, i.e., obtained by removing some $\mathbb{F}_q$-rational point ∞ from an elliptic (projective) $\mathbb{F}_q$-curve C, then $\mathrm{Pic}(C^{\mathrm{af}}) = \mathrm{Pic}(\mathcal{O}_{\{\infty\}}) \cong C(\mathbb{F}_q)$ (cf. e.g., [4, Ex. 4.8]). Let again $\underline{G} = \underline{\mathbf{PGL}}_n$ such that $(\mathrm{char}(K), n) = 1$. As $|S| = 1$ and $\underline{F}$ is split, $\underline{G}$ admits a single genus (Corollary 3.2), which means that all projective $\mathcal{O}_{\{\infty\}}$-modules of rank n are K-isomorphic. If $\underline{G}$ is K-isotropic, according to Theorem 4.1, there are exactly $|C^{\mathrm{af}}(\mathbb{F}_q)/2|$ $\mathcal{O}_{\{\infty\}}$-isomorphism classes of such modules, so the Hasse principle fails for $\underline{G}$ if and only if $|C^{\mathrm{af}}(\mathbb{F}_q)|$ is even. This occurs exactly when C^{af} has at least one $\mathbb{F}_q$-point on the x-axis (thus of order 2).

On the other hand, take $\mathcal{O}_S = \mathbb{F}_3[t, t^{-1}]$ obtained by removing $S = \{t, t^{-1}\}$ from the projective $\mathbb{F}_3$-line, and $\underline{G} = \underline{\mathbf{PGL}}_n$ to be rationally isotropic over $\mathcal{O}_S$: for example for $n = 2$, it is isomorphic to the special orthogonal group of the standard split $\mathcal{O}_S$-form $q_3(x_1, x_2, x_3) = x_1x_2 + x_3^2$. Then as q_3 is rationally isotropic over $\mathcal{O}_S$ (e.g., $q_3(1, 2, 1) = 0$) and $\mathcal{O}_S$ is a UFD, according to Corollary 4.4 the Hasse-principle holds for $\underline{G}$ and there are two genera as $|F| = |S| = 2$ (see Cor. 3.2).

Example 4.6. Let (V, q) be an $\mathcal{O}_S$-regular quadratic form of even rank $n = 2k \geq 4$ and let $\underline{G} = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mathbf{SO}}_q)$ where R is finite étale over $\mathcal{O}_S$. Then $\underline{F} = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_2)$, whence according to Corollary 3.2, $\mathrm{gen}(\underline{G}) \cong \mathrm{Br}(R)[2]$. As G and its twisted K-forms are K-isotropic (e.g., [28, p. 352]), each genus of q contains exactly $\mathrm{Pic}(R)/2$ elements.

Example 4.7. Let C' be an elliptic $\mathbb{F}_q$-curve and $(C')^{\mathrm{af}} := C' - \{\infty'\}$. Then $R := \mathbb{F}_q[(C')^{\mathrm{af}}]$ is a quadratic extension of $\mathcal{O}_{\{\infty\}} = \mathbb{F}_q[x]$ where $\infty = (1/x)$ and ∞' is the unique prime lying above ∞, thus $L := R \otimes_{\mathcal{O}_{\{\infty\}}} K$ is imaginary over K. Let $\underline{G} = \mathrm{Res}_{R/\mathcal{O}_{\{\infty\}}}(\underline{\mathbf{PGL}}_m)$, m is odd and prime to q. Then $\underline{F} = \mathrm{Res}^{(1)}_{R/\mathcal{O}_{\{\infty\}}}(\underline{\mu}_m)$ is smooth, and $\underline{G}$ is smooth and quasi-split as well as its generic fiber, thus is K-isotropic. By Remark 3.5 and sequence (2.13), we get (notice that $\mathcal{O}_{\{\infty\}}$ is a PID and that $\mathrm{Br}(R) = 1$):

$$
\begin{aligned}
\mathrm{Cl}_S(\underline{G}) = H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{G}) &\cong H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{F}) \\
&\cong \ker(\mathrm{Pic}(R)/m \to \mathrm{Pic}(\mathcal{O}_{\{\infty\}})/m) = \mathrm{Pic}(R)/m.
\end{aligned}
$$

Hence the Hasse-principle holds for $\underline{G}$ if and only if $|\mathrm{Pic}(R)| = |C'(\mathbb{F}_q)|$ is prime to m.

5. The Tamagawa number of twisted groups

In this section we start with the generic fiber. Let G be a semisimple group defined over a global field $K = \mathbb{F}_q(C)$ with fundamental group F. The *Tamagawa number* $\tau(G)$ of G is defined as the covolume of the group $G(K)$ in the adelic group $G(\mathbb{A})$ (embedded diagonally as a discrete subgroup), with respect to the Tamagawa measure (see [33]). T. Ono has established in [27] a formula for the computation of $\tau(G)$ in case K is an algebraic number field, which was later proved by Behrend and Dhillon in [3, Thm. 6.1] also in the function field case:

$$\tau(G) = \frac{|\widehat{F}^{\mathfrak{g}}|}{|Ш^1(\widehat{F})|} \tag{5.1}$$

where $\widehat{F} := \mathrm{Hom}(F \otimes K^s, \mathbb{G}_m)$, $\mathfrak{g}$ is the absolute Galois group $\mathrm{Gal}(K^s/K)$, and $Ш^1(\widehat{F})$ is the first Shafarevitch–Tate group assigned to $\widehat{F}$ over K. As a result, if F is split, then $\tau(G) = |F|$. So our main innovation, based on the above results and the following ones, would be simplifying the computation of $\tau(G)$ in case F is not split, as may occur when G is a twisted group.

The following construction, as described in [7] and briefly revised here, expresses the global invariant $\tau(G)$ using some local data. Suppose G is almost simple defined over the above $K = \mathbb{F}_q(C)$, not anisotropic of type A, such that $(|F|, \mathrm{char}(K)) = 1$. We remove one arbitrary closed point ∞ from C and refer as above to the integral domain $\mathcal{O}_S = \mathcal{O}_{\{\infty\}}$. At any prime $\mathfrak{p} \neq \infty$, we consider the Bruhat–Tits $\mathcal{O}_\mathfrak{p}$-model of $G_\mathfrak{p}$ corresponding to some special vertex in its associated building. Patching all these $\mathcal{O}_\mathfrak{p}$-models along the generic fiber results in an affine and smooth $\mathcal{O}_{\{\infty\}}$-model $\underline{G}$ of G (see [7, §5]). It may be locally disconnected only at places that ramify over a minimal splitting field L of G (cf. [9, 4.6.22]).

Denote $\mathbb{A}_\infty := \mathbb{A}_{\{\infty\}} = \hat{K}_\infty \times \prod_{\mathfrak{p} \neq \infty} \hat{\mathcal{O}}_\mathfrak{p} \subset \mathbb{A}$. Then $\underline{G}(\mathbb{A}_\infty)G(K)$ is a normal subgroup of $\underline{G}(\mathbb{A})$ (cf. [32, Thm. 3.2 3]). The set of places Ram_G that ramify in L is finite, thus by the Borel density theorem (e.g., [11, Thm. 2.4, Prop. 2.8]), $\underline{G}(\mathcal{O}_{\{\infty\} \cup \mathrm{Ram}_G})$ is Zariski-dense in $\prod_{\mathfrak{p} \in \mathrm{Ram}_G \setminus \{\infty\}} \underline{G}_\mathfrak{p}$. This implies that $\underline{G}(\mathbb{A}_\infty)G(K) = \underline{G}^0(\mathbb{A}_\infty)G(K)$, where $\underline{G}^0$ is the connected component of $\underline{G}$.

Since all fibers of the natural epimorphism

$$\varphi : \underline{G}(\mathbb{A})/G(K) \twoheadrightarrow \underline{G}(\mathbb{A})/\underline{G}(\mathbb{A}_\infty)G(K)$$

are isomorphic to $\ker(\varphi) = G(\mathbb{A}_\infty)G(K)/G(K)$, we get a bijection of measure spaces

$$
\begin{aligned}
(5.2)\ G(\mathbb{A})/G(K) &\cong \mathrm{Im}(\varphi) \times \ker(\varphi) \\
&= (G(\mathbb{A})/G(\mathbb{A}_\infty)G(K)) \times (G(\mathbb{A}_\infty)/G(\mathbb{A}_\infty) \cap G(K)) \\
&= (\underline{G}^0(\mathbb{A})/\underline{G}^0(\mathbb{A}_\infty)G(K)) \times (\underline{G}^0(\mathbb{A}_\infty)/\underline{G}^0(\mathbb{A}_\infty) \cap G(K)) \\
&\cong \mathrm{Cl}_{\{\infty\}}(\underline{G}^0) \times (\underline{G}^0(\mathbb{A}_\infty)/\underline{G}^0(\mathbb{A}_\infty) \cap G(K))
\end{aligned}
$$

in which the left factor cardinality is the finite index $h_\infty(G) := h_{\{\infty\}}(\underline{G}^0)$ (see Section 2), and in the right factor $\underline{G}^0(\mathbb{A}_\infty) \cap G(K) = \underline{G}^0(\mathcal{O}_{\{\infty\}})$. Due to the Weil conjecture stating that $\tau(G^{\mathrm{sc}}) = 1$, as was recently proved in the function field case by Gaistgory and Lurie (see [23, (2.4)]), applying the Tamagawa measure τ on these spaces results in the Main Theorem in [7]:

Theorem 5.1. *Let* $\mathfrak{g}_\infty = \mathrm{Gal}(\hat{K}_\infty^s/\hat{K}_\infty)$ *be the Galois absolute group,* $F_\infty := \ker(G_\infty^{sc} \to G_\infty)$*,* $\underline{F} := \ker(\underline{G}^{sc} \to \underline{G})$ *whose order is prime to* $\mathrm{char}(K)$*, and* $\widehat{F_\infty} := \mathrm{Hom}(F_\infty \otimes \hat{K}_\infty^s, \mathbb{G}_{m,\hat{K}_\infty^s})$*. Then*

$$\tau(G) = h_\infty(G) \cdot \frac{t_\infty(G)}{j_\infty(G)},$$

where $t_\infty(G) = |\widehat{F_\infty}^{\mathfrak{g}_\infty}|$ *is the number of types in one orbit of a special vertex, in the Bruhat–Tits building associated to* $G_\infty(\hat{K}_\infty)$*, and* $j_\infty(G) = h_1(\underline{F})/h_0(\underline{F})$*.*

We adopt Definition 1.2 of being admissible to F, with a Galois extension L/K replacing $R/\mathcal{O}_S$. If $\underline{G}$ is not of (absolute) type A and F is admissible, then due to the above results Theorem 5.1 can be reformulated involving the fundamental group data only:

Theorem 5.2. *Let* G *be an almost-simple group not of (absolute) type A defined over* $K = \mathbb{F}_q(C)$ *with an admissible fundamental group* F *whose order is prime to* $\mathrm{char}(K)$*. Then for any choice of a prime* ∞ *of* K *one has:*

$$\tau(G) = \frac{\chi_{\{\infty\}}(\underline{F})}{|i(\underline{F})|} \cdot |\widehat{F_\infty}^{\mathfrak{g}_\infty}| = l(\underline{F}) \cdot |\widehat{F_\infty}^{\mathfrak{g}_\infty}|,$$

where $\chi_{\{\infty\}}(\underline{F})$ *is the (restricted) Euler–Poincaré characteristic (cf. Definition 2.11),* $i(\underline{F})$ *and* $l(\underline{F})$ *are as in Definitions 1.1 and 2.12, respectively, and the right factor is a local invariant.*

Proof. If G is not of (absolute) type A, according to Corollary 3.4 all genera of $\underline{G}$ have the same cardinality. By Lemma 3.3 and Corollary 3.2 ($\underline{F}$ is admissible as F is, see Remark 2.5) we then get

$$h_\infty(G) = |\mathrm{Cl}_{\{\infty\}}(\underline{G})| = \frac{|H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \underline{G})|}{|\,\mathrm{gen}(\underline{G})|} = \frac{h_2(\underline{F})}{|i(\underline{F})|}.$$

Now the first asserted equality follows from Theorem 5.1 together with Definition 2.11:

$$\tau(G) = 1/j_\infty(\underline{G}) \cdot h_\infty(\underline{G}) \cdot t_\infty(G)$$
$$= \frac{h_0(\underline{F})}{h_1(\underline{F})} \cdot \frac{h_2(\underline{F})}{|i(\underline{F})|} \cdot |\widehat{F_\infty}^{\mathfrak{g}_\infty}| = \frac{\chi_{\{\infty\}}(\underline{F})}{|i(\underline{F})|} \cdot |\widehat{F_\infty}^{\mathfrak{g}_\infty}|.$$

The rest is Lemma 2.13. □

Remark 5.3. By the geometric version of Čebotarev's density theorem (see in [20]), there exists a closed point ∞ on C at which G_∞ is split. We shall call such a point a *splitting point* of G.

Corollary 5.4. *Let G be an adjoint group defined over $K = \mathbb{F}_q(C)$ with fundamental group F whose order is prime to* char(K) *and whose splitting field is L. Choose some splitting point ∞ of G on C and let R be a minimal étale extension of $\mathcal{O}_{\{\infty\}} := \mathbb{F}_q[C - \{\infty\}]$ such that $R \otimes_{\mathcal{O}_{\{\infty\}}} K = L$. Let $N^{(0)} : R^\times \to \mathcal{O}_{\{\infty\}}^\times$ be the induced norm. Then:*

1. *If G is of type ${}^2D_{2k}$ then $\tau(G) = \frac{|R^\times[2]|}{[R^\times:(R^\times)^2]} \cdot |F|$.*
2. *If G is of type ${}^{3,6}D_4$ or ${}^2E_6$ then $\tau(G) = \frac{|\ker(N^{(0)}[m])|}{|\ker(N^{(0)}/m)|} \cdot |F|$ (see Notation 2.6).*

In both cases if L is imaginary over K, then $\tau(G) = |F|$.

Proof. All groups under consideration are almost simple. When G is adjoint of type ${}^2\mathrm{D}_{2k}$ then F is quasi-split, and when it is adjoint both of type ${}^{3,6}\mathrm{D}_4$ or ${}^2\mathrm{E}_6$ then $F = \mathrm{Res}^{(1)}_{L/K}(\mu_m)$ where m is prime to $[L : K]$ (e.g., [28, p. 333]), thus F is admissible. So the assertions (1), (2) are just Theorem 5.2 in which as F_∞ splits, $|\widehat{F_\infty}^{\mathfrak{g}_\infty}| = |F_\infty| = |F|$.

As C is projective, removing a single point ∞ from it implies that $\mathcal{O}_{\{\infty\}}^\times = \mathbb{F}_q^\times$ (an element of $\mathcal{O}_{\{\infty\}}$ is regular at ∞^{-1}, thus its inverse is irregular there, hence not invertible in $\mathcal{O}_{\{\infty\}}$, unless it is a unit). If L is imaginary, then in particular $R = \mathbb{F}_q[C' - \{\infty'\}]$ where C' is a finite étale cover of C and ∞' is the unique point lying over ∞, thus still $R^\times = \mathbb{F}_q^\times$ being finite, whence $|R^\times[2]| = [R^\times : (R^\times)^2]$. In the cases F is not quasi-split the equality $R^\times = \mathcal{O}_{\{\infty\}}^\times = \mathbb{F}_q^\times$ means that $N^{(0)}$ is trivial, and we are done. □

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References

[1] E. Artin, "Quadratische Körper im Gebiete der höheren Kongruenzen", *Math. Z.* **19** (1927), p. 153-206.

[2] M. Artin, A. Grothendieck & J.-L. Verdier (eds.), *Théorie des Topos et Cohomologie Étale des Schémas (SGA 4)*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer, 1972/1973.

[3] K. Behrend & A. Dhillon, "Connected components of moduli stacks of torsors via Tamagawa numbers", *Can. J. Math.* **61** (2009), no. 1, p. 3-28.

[4] R. A. Bitan, "The Hasse principle for bilinear symmetric forms over a ring of integers of a global function field", *J. Number Theory* **168** (2016), p. 346-359.

[5] ———, "Between the genus and the Γ-genus of an integral quadratic Γ-form", *Acta Arith.* **181** (2017), no. 2, p. 173-183.

[6] ———, "On the classification of quadratic forms over an integral domain of a global function field", *J. Number Theory* **180** (2017), p. 26-44.

[7] R. A. Bitan & R. Köhl, "A building-theoretic approach to relative Tamagawa numbers of semisimple groups over global function fields", *Funct. Approximatio, Comment. Math.* **53** (2015), no. 2, p. 215-247.

[8] A. Borel & G. Prasad, "Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups", *Publ. Math., Inst. Hautes Étud. Sci.* **69** (1989), p. 119-171.

[9] F. Bruhat & J. Tits, "Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée", *Publ. Math., Inst. Hautes Étud. Sci.* **60** (1984), p. 197-376.

[10] B. Calmès & J. Fasel, "Groupes Classiques", in *On group schemes*, Panoramas et Synthèses, vol. 46, Société Mathématique de France, 2015, p. 1-133.

[11] P.-E. Caprace & N. Monod, "Isometry groups of non-positively curved spaces: discrete subgroups", *J. Topol.* **2** (2009), p. 701-746.

[12] V. Chernousov, P. Gille & A. Pianzola, "A classification of torsors over Laurent polynomial rings", *Comment. Math. Helv.* **92** (2017), no. 1, p. 37-55.

[13] B. Conrad, "Math 252. Properties of orthogonal groups", `http://math.stanford.edu/~conrad/252Page/handouts/O(q).pdf`.

[14] ———, "Math 252. Reductive group schemes", `http://math.stanford.edu/~conrad/252Page/handouts/luminysga3.pdf`.

[15] M. Demazure & A. Grothendieck (eds.), *Séminaire de Géométrie Algébrique du Bois Marie 1962-64 (SGA 3). Schémas en groupes, Tome II*, Documents Mathématiques, Société Mathématique de France, 2011.

[16] J. Giraud, *Cohomologie non abélienne*, Grundlehren der Mathematischen Wissenschaften, vol. 179, Springer, 1971.

[17] C. D. González-Avilés, "Quasi-abelian crossed modules and nonabelian cohomology", *J. Algebra* **369** (2012), p. 235-255.

[18] A. Grothendieck, "Le groupe de Brauer III: Exemples et compléments", in *Dix Exposes Cohomologie Schemas*, Advanced Studies Pure Math., vol. 3, American Mathematical Society, 1968, p. 88-188.

[19] G. Harder, "Über die Galoiskohomologie halbeinfacher algebraischer Gruppen, III", *J. Reine Angew. Math.* **274/275** (1975), p. 125-138.

[20] M. Jarden, "The Čebotarev density theorem for function fields: An elementary approach", *Math. Ann.* **261** (1982), no. 4, p. 467-475.

[21] M.-A. Knus, *Quadratic and hermitian forms over rings*, Grundlehren der Mathematischen Wissenschaften, vol. 294, Springer, 1991.

[22] H. W. Lenstra, "Galois theory for schemes", `http://websites.math.leidenuniv.nl/algebra/GSchemes.pdf`.

[23] J. Lurie, "Tamagawa Numbers of Algebraic Groups Over Function Fields".

[24] J. S. Milne, *Étale Cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, 1980.

[25] ———, *Arithmetic Duality Theorems*, 2nd ed., BookSurge, 2006.

[26] Y. Nisnevich, "Étale Cohomology and Arithmetic of Semisimple Groups", PhD Thesis, Harvard University (USA), 1982.

[27] T. Ono, "On the Relative Theory of Tamagawa Numbers", *Ann. Math.* **82** (1965), p. 88-111.

[28] V. Platonov & A. Rapinchuk, *Algebraic Groups and Number Theory*, Pure and Applied Mathematics, vol. 139, Academic Press Inc., 1994.

[29] M. Rosen, *Number Theory in Function Fields*, Graduate Texts in Mathematics, vol. 210, Springer, 2000.

[30] J.-P. Serre, *Algebraic Groups and Class Fields*, Graduate Texts in Mathematics, vol. 117, Springer, 1988.

[31] A. N. Skorobogatov, *Torsors and Rational Points*, Cambridge Tracts in Mathematics, vol. 144, Cambridge University Press, 2001.

[32] N. Q. Thắńg, "A Norm Principle for class groups of reductive group schemes over Dedekind rings", *Vietnam J. Math.* **43** (2015), no. 2, p. 257-281.

[33] A. Weil, *Adèles and Algebraic Groups*, Progress in Mathematics, vol. 23, Birkhäuser, 1982.

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