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The Hasse Norm Principle For Biquadratic Extensions

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RÉSUMÉ. Nous donnons une formule asymptotique pour le nombre d'extensions biquadratiques du corps des rationnels de discriminant borné qui contredisent le principe de norme de Hasse.

ABSTRACT. We give an asymptotic formula for the number of biquadratic extensions of the rationals of bounded discriminant that fail the Hasse norm principle.

1. Introduction

Let K/k be a Galois extension of number fields with Galois group G. If $\mathbf{A}_K^*$ denotes the set of idèles of K and $N_{K/k}$ the usual norm map then we say that K satisfies the Hasse norm principle if

$$N_{K/k}K^* = N_{K/k}\mathbf{A}_K^* \cap k^*.$$

The Hasse norm theorem states that this principle is satisfied by every cyclic extension (see e.g. [2, p. 185]). Recently, Frei–Loughran–Newton [4] have shown that for every non-cyclic abelian group G there is an extension K/k with Galois group G, for which the Hasse norm principle fails. Moreover, if $G \cong \mathbb{Z}/n\mathbb{Z} \oplus (\mathbb{Z}/Q\mathbb{Z})^r$ where $r \in \mathbb{Z}_{\geqslant 1}$ and Q is the smallest prime dividing n then 0% (when ordered by discriminant) of extensions of k with Galois group G fail the Hasse norm principle. However this density result is not explicit, so there is no estimate for the frequency of extensions with a given Galois group that fail the Hasse norm principle.

In this paper we'll investigate Hasse norm principle failures in the simplest abelian, non-cyclic case where $G = (\mathbb{Z}/2\mathbb{Z})^2$ and $k = \mathbb{Q}$. Let Δ_K denote the discriminant of the field $K = \mathbb{Q}(\sqrt{a}, \sqrt{b})$. Note that for all such K, we have $\Delta_K > 0$ (c.f. (2.3)). We will first give a proof of the number of such fields with discriminant bounded above by X, recovering work of Baily [1].

Theorem 1.1. *Let $S(X)$ denote the number of distinct biquadratic extensions $K/\mathbb{Q}$ such that $\Delta_K \leqslant X$. Then*

$$S(X) = \frac{23}{960} \sqrt{X} \log^2 X \prod_p \left(1 - \frac{1}{p}\right)^3 \left(1 + \frac{3}{p}\right) + O\left(\sqrt{X} \log X\right).$$

There have been several investigations centered on counting the number of extensions of a number field of bounded discriminant with a given Galois group, for various different choices of Galois group (see [3] for a survey). In particular, Baily [1, Theorem 8] produced Theorem 1.1 in 1980, using a simple argument about how many primes could divide the discriminant. Baily's original result did not include information about the error terms although lower order terms are now known [3, Section 2.5]. We use a different approach to Baily and have chosen to include the proof of Theorem 1.1 as the method of computation serves to illustrate the proof of the following theorem which is the main result of this paper.

Theorem 1.2. *Let $\widetilde{S}(X)$ denote the number of distinct biquadratic extensions $K/\mathbb{Q}$ such that $\Delta_K \leqslant X$ and K fails the Hasse norm principle. Then*

$$\widetilde{S}(X) = \frac{1}{3\sqrt{2\pi}} \sqrt{X \log X} \prod_p \left(1 - \frac{1}{p}\right)^{\frac{3}{2}} \left(1 + \frac{3}{2p}\right) + O\left(\sqrt{X}\right).$$

In particular, these two results combined recover the Frei–Loughran–Newton result in this setting (i.e. that 0% of biquadratic extensions of $\mathbb{Q}$ fail the Hasse norm principle).

1.1. Layout of the paper. In Section 2, we will develop specific conditions on the integers a and b that ensure that the extension $K = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ fails the Hasse norm principle. Section 3 is devoted to the proof of the Theorem 1.1. Theorem 1.2 is proven in Section 4 using a similar approach. The main difference is that to apply the criteria developed in Section 2 we must sum a product of Jacobi symbols and to do this we incorporate ideas of Friedlander–Iwaniec [6].

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2. Criteria for Hasse Norm Principle failure

In this section, we'll describe criteria on the integers a and b that determine when the extension $\mathbb{Q}(\sqrt{a}, \sqrt{b})$ fails the Hasse norm principle. We can then sum over the a and b satisfying these criteria to get Theorem 1.2.

First, we describe how to count a and b that define unique biquadratic extensions of $\mathbb{Q}$. Note that K has 3 quadratic subfields

$$k_1 = \mathbb{Q}(\sqrt{a}), \ k_2 = \mathbb{Q}(\sqrt{b}) \text{ and } k_3 = \mathbb{Q}(\sqrt{ab}/(a,b)).$$

Each of these quadratic fields can be uniquely identified by a single square-free integer so fix $k_1 = \mathbb{Q}(\sqrt{a})$ and $k_2 = \mathbb{Q}(\sqrt{b})$. Let $m = (a, b)$ so that

(2.1) $\qquad a = ma_1, b = mb_1 \text{ and } (a_1, m) = (a_1, b_1) = (b_1, m) = 1.$

Then

$$k_1 = \mathbb{Q}(\sqrt{ma_1}), \ k_2 = \mathbb{Q}(\sqrt{mb_1}) \text{ and } k_3 = \mathbb{Q}(\sqrt{a_1 b_1}).$$

It is certainly true that specifying m, a_1, b_1 will determine K. Moreover since K is determined uniquely by its subfields and these subfields by the choice of m, a_1 and b_1, this choice uniquely determines K up to relabelling.

We can write the discriminant of K in terms of (m, a_1, b_1) as follows. By [7, Chapter 8, 7.23] we can express the discriminant of K, denoted Δ_K, in terms of the discriminants of its quadratic subfields by

$$\Delta_K = \Delta_{k_1} \Delta_{k_2} \Delta_{k_3}.$$

Recall that

(2.2) $$\Delta_{\mathbb{Q}(\sqrt{d})} = \begin{cases} d & \text{if } d \equiv 1 \bmod 4 \\ 4d & \text{if } d \equiv 2 \text{ or } 3 \bmod 4. \end{cases}$$

We observe that it is not possible for just one of the integers ma_1, mb_1 and $a_1 b_1$ to be congruent to 2 or 3 $\bmod 4$. For if $ma_1 \equiv 2 \bmod 4$ then either m or $b_1 \equiv 2 \bmod 4$ and hence so is their product with a_1. Moreover $ma_1 \equiv 3 \bmod 4$ if and only if $m \equiv -a_1 \bmod 4$ so either $m \equiv -b_1 \bmod 4$ or $a_1 \equiv -b_1 \bmod 4$. Therefore

(2.3) $$\Delta_K = c^2 m^2 a_1^2 b_1^2$$

where c is either 1 if all the k_i are in the first case of (2.2), 4 if exactly one k_i is in the first case of (2.2) or 8 if all the k_i are in the second case.

We now turn our attention to how to identify Hasse norm principle failures and see that the congruence class of $(m, a_1, b_1) \bmod 4$ again plays a role.

Lemma 2.1. *Let* $(m, a_1, b_1) \equiv (\epsilon_1, \epsilon_2, \epsilon_3) \bmod 4$. *Then*

(1) *When* $\epsilon_1 = \epsilon_2 = \epsilon_3$, K *fails the Hasse norm principle if and only if all of the following hold:*
   (a) $p \mid a_1 \implies \left(\frac{mb_1}{p}\right) = +1$,
   (b) $p \mid b_1 \implies \left(\frac{ma_1}{p}\right) = +1$,
   (c) $p \mid m \implies \left(\frac{a_1 b_1}{p}\right) = +1$.

(2) *When* $\epsilon_1 = \epsilon_2 \neq \epsilon_3$, K *fails the Hasse norm principle if and only if all of the following hold:*

(a) $p \mid a_1 \implies \left(\frac{mb_1}{p}\right) = +1$,

(b) $p \mid b_1 \implies \left(\frac{ma_1}{p}\right) = +1$,

(c) $p \mid m \implies \left(\frac{a_1 b_1}{p}\right) = +1$,

(d) $m \equiv a_1 \bmod 8$.

Similarly for $\epsilon_2 = \epsilon_3 \neq \epsilon_1$ *and* $\epsilon_3 = \epsilon_1 \neq \epsilon_2$.

(3) *If* ϵ_1, ϵ_2 *and* ϵ_3 *are pairwise distinct, then K satisfies the Hasse norm principle.*

Proof. The Hasse norm principle fails in biquadratics if and only if all decomposition groups are cyclic (see e.g. [2, Chapter 7, Section 11.4]). Hence to come up with a criterion for Hasse norm principle failure we need to ensure that all decomposition groups are proper subgroups of the Galois group $(\mathbb{Z}/2\mathbb{Z})^2$. Therefore we need every rational prime to split in K.

A prime splits in K if and only if it splits in at least one of the three quadratic subfields $\mathbb{Q}(\sqrt{ma_1})$, $\mathbb{Q}(\sqrt{mb_1})$ and $\mathbb{Q}(\sqrt{a_1 b_1})$.

(1) If $a_1 \equiv m \equiv b_1 \bmod 4$ then the only primes that ramify in $\mathbb{Q}(\sqrt{ma_1})$ are those dividing ma_1, therefore since m, a_1 and b_1 are pairwise coprime, no prime ramifies in all three quadratic subextensions. A prime p splits in $\mathbb{Q}(\sqrt{a})$ if and only if $\left(\frac{a}{p}\right) = +1$ otherwise it remains inert. This means a prime cannot be inert in all three subfields. Hence we must ensure that all primes that ramify in two of the subfields split in the third.

(2) If $m \equiv a_1 \not\equiv b_1 \bmod 4$ then $ma_1 \equiv 1 \bmod 4$, $mb_1 \equiv 2$ or $3 \bmod 4$ and $a_1 b_1 \equiv 2$ or $3 \bmod 4$. Note that since m and a_1 are coprime they cannot both be congruent to 2, so they must be odd. We see that the rational prime 2 ramifies in the subfields $\mathbb{Q}(\sqrt{mb_1})$ and $\mathbb{Q}(\sqrt{a_1 b_1})$. Therefore we must also ensure that 2 splits in $\mathbb{Q}(\sqrt{ma_1})$ so we must impose the extra condition $ma_1 \equiv 1 \bmod 8$.

(3) In this case we have $ma_1 \equiv 2$ or $3 \bmod 4$, $mb_1 \equiv 2$ or $3 \bmod 4$ and $a_1 b_1 \equiv 2$ or $3 \bmod 4$ so 2 ramifies in all 3 quadratic subextensions hence is totally ramified in K. Therefore the Hasse norm principle holds. $\qquad \square$

3. Proof of Theorem 1.1

We move on to establishing the asymptotic formula for the number of biquadratic extensions $K = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ of bounded discriminant. In Section 2 we saw that this means counting the number of integer triples which are square-free, pairwise coprime and whose product is bounded above in a way that depends on the congruence class of the tuple mod 4. In Section 2, these triples were denoted (m, a_1, b_1) however from here on out we will write them (m_1, m_2, m_3) for simplicity.

Let $\boldsymbol{\delta} = (\delta_2, \delta_3)$ where δ_2 denotes the sign of m_2 and δ_3 the sign of m_3. We note that it would be redundant to keep track of all 3 signs and just 2 will suffice. Observe that the highest power of 2 dividing $m_1 m_2 m_3$ is either 0 or 1. To keep track of this we write

$$m_1 = 2^\mu m_1', \ m_2 = \delta_2 2^\alpha m_2' \ \text{and} \ m_3 = \delta_3 2^\beta m_3',$$

where $2 \nmid m_1' m_2' m_3'$. Finally we denote by $\boldsymbol{\epsilon}$ the congruence class of the m_i' mod 4. Then

$$S(X) = \frac{1}{6} \sum_{\substack{\boldsymbol{\delta} \in \{\pm 1\}^2 \\ \mu, \alpha, \beta \in \{0,1\}}} \sum_{\mu + \alpha + \beta \in \{0,1\}} \sum_{\boldsymbol{\epsilon} \in \{\pm 1\}^3} T(\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta),$$

where $T(\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta)$ counts the number of tuples $(m_1', m_2', m_3') \in \mathbb{N}^3$ such that the following all hold:

(1) $\mu^2(m_1' m_2' m_3') = 1$,
(2) $(m_1', m_2', m_3') \equiv \boldsymbol{\epsilon} \bmod 4$,
(3) $2^{\mu + \alpha + \beta} c_{\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta} m_1' m_2' m_3' \leqslant \sqrt{X}$.

Note that $T(\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta)$ overcounts each triple (m_1', m_2', m_3') by counting every permutation of the components so we divide the sum by 6 to compensate. The constants $c_{\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta}$ correspond to the constant c in (2.3). Recall that $c = 1$ if all the components of (m_1, m_2, m_3) are congruent mod 4, $c = 4$ if two of the components are congruent, and 8 otherwise. We summarise the values below

$$c_{\boldsymbol{\delta}, \boldsymbol{\epsilon}, 0, 0, 0} = \begin{cases} 1 & \text{if } \epsilon_1 = \delta_2 \epsilon_2 = \delta_3 \epsilon_3, \\ 4 & \text{otherwise}; \end{cases} \qquad c_{\boldsymbol{\delta}, \boldsymbol{\epsilon}, 1, 0, 0} = \begin{cases} 4 & \text{if } \delta_2 \epsilon_2 = \delta_3 \epsilon_3, \\ 8 & \text{otherwise}; \end{cases}$$

$$c_{\boldsymbol{\delta}, \boldsymbol{\epsilon}, 0, 1, 0} = \begin{cases} 4 & \text{if } \epsilon_1 = \delta_3 \epsilon_3, \\ 8 & \text{otherwise}; \end{cases} \qquad c_{\boldsymbol{\delta}, \boldsymbol{\epsilon}, 0, 0, 1} = \begin{cases} 4 & \text{if } \epsilon_1 = \delta_2 \epsilon_2, \\ 8 & \text{otherwise}. \end{cases}$$

We'll first tackle the evaluation of $T(\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta)$, for fixed $\boldsymbol{\delta}, \boldsymbol{\epsilon}, \alpha, \beta$ and μ then the remaining summation will be a simple computation.

Lemma 3.1. *Let* $c = c_{\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta}$. *Then we have*

$$T(\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta) = \frac{\sqrt{X} \log^2 X}{160 c 2^{\mu + \alpha + \beta}} \prod_p \left(1 - \frac{1}{p}\right)^3 \left(1 + \frac{3}{p}\right) + O\left(\sqrt{X} \log X\right).$$

Proof. To ease notation, we will denote $N = \sqrt{X} / \left(c 2^{\mu + \alpha + \beta}\right)$. Then,

$$T(\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta) = \sum_{\substack{m_1' m_2' m_3' \leqslant N \\ m_i' \equiv \epsilon_i \bmod 4}} \mu^2(m_1' m_2' m_3').$$

We first remove the congruence condition by applying the character sum

$$\mathbf{1}_{\{m'_i \equiv \epsilon_i \bmod 4\}} = \frac{1}{2} \sum_{\nu_i \in \{1,2\}} \chi^{\nu_i}(m'_i \epsilon_i),$$

where χ is the non-principal character mod 4. Then,

$$T(\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta) = \frac{1}{8} \sum_{m'_1 m'_2 m'_3 \leqslant N} \mu^2(m'_1 m'_2 m'_3) \sum_{\boldsymbol{\nu} \in \{1,2\}^3} \prod_{i \in \{1,2,3\}} \chi^{\nu_i}(m'_i \epsilon_i)$$

$$= \frac{1}{8} \sum_{\boldsymbol{\nu} \in \{1,2\}^3} \prod_i \chi^{\nu_i}(\epsilon_i) \sum_{m'_1 m'_2 m'_3 \leqslant N} \mu^2(m'_1 m'_2 m'_3) \prod_i \chi^{\nu_i}(m'_i).$$

We investigate

$$\sum_{n \leqslant N} a_n := \sum_{n \leqslant N} \mu^2(n) \sum_{m_1 m_2 m_3 = n} \chi^{\nu_1}(m_1) \chi^{\nu_2}(m_2) \chi^{\nu_3}(m_3)$$

by looking at the associated Dirichlet series

$$F(s) := \sum_n \frac{a_n}{n^s}.$$

Comparing this to the product of the L-functions associated to χ^{ν_i} we introduce

$$G(s) := F(s) \prod_{i=1}^{3} L(\chi^{\nu_i}, s)^{-1},$$

where $L(\chi^2, s) = \left(1 - \frac{1}{2^s}\right) \zeta(s)$. For example, when $\nu_i = 2$ for all i we have

$$G(s) = \left(1 - \frac{1}{2^s}\right)^3 \left(1 + \frac{3}{2^s}\right)^{-1} \prod_p \left(1 + \frac{3}{p^s}\right) \left(1 - \frac{1}{p^s}\right)^3.$$

For any ν_i, the Euler product $G(s)$ is absolutely convergent for $\mathrm{Re}(s) > \frac{1}{2}$ and in this region $G(s) \ll 1$. The L-function associated to χ is entire and in the region $\frac{1}{2} < \sigma \leqslant 1$, we have

$$L(\chi, \sigma + it) \ll_\epsilon |t|^{\frac{1-\sigma}{2}},$$

see for example [8, 5.20]. Therefore the Dirichlet series $F(s)$ satisfies the conditions of the Selberg–Delange theorem [9, II.5.2, Theorem 5.2]. When $\nu_i = 2$ for all i, the L-functions each correspond to a copy of $\zeta(s)$ and therefore the sum has order of magnitude $N \log^2 N$. In each other case, there are at most 2 copies of $\zeta(s)$ and hence the contribution is $O(N \log N)$. We conclude that

$$\sum_{n \leqslant N} a_n = CN \log^2 N + O(N \log N),$$

where

$$C = \frac{(1 - \frac{1}{2})^{-3}}{\Gamma(3)} G(1) = \frac{1}{5} \prod_p \left(1 + \frac{3}{p}\right) \left(1 - \frac{1}{p}\right)^3. \qquad \square$$

All that's left is to apply the definition of $c_{\delta,\epsilon,\mu,\alpha,\beta}$ in each case and find that

$$\sum_{\delta \in \{\pm 1\}^2} \sum_{\mu+\alpha+\beta \in \{0,1\}} \sum_{\epsilon \in \{\pm 1\}^3} \frac{1}{c_{\delta,\epsilon,\mu,\alpha,\beta} 2^{\mu+\alpha+\beta}} = 23.$$

4. Proof of Theorem 1.2

Similarly to the previous section, we start by making the change of variables

(4.1) $$m_1 = 2^\mu m_1', \ m_2 = \delta_2 2^\alpha m_2' \ \text{and} \ m_3 = \delta_3 2^\beta m_3',$$

where $\mu^2(2m_1'm_2'm_3') = 1$ and $\mu, \beta, \alpha \in \mathbb{Z}_{\geq 0}$ such that $\mu + \alpha + \beta \leq 1$. We saw in Section 2 that when counting Hasse norm principle failure it is important to keep track of the residue class of $(m_1', m_2', m_3') \bmod 8$ rather than just mod 4 as in Section 3. Recall from Section 2 that if the congruence classes of m_1, m_2 and $m_3 \bmod 4$ are all distinct then K always satisfies the Hasse norm principle. Moreover if exactly two of them are congruent mod 4 then we require that these two are in fact congruent mod 8 to ensure Hasse norm principle failure. Hence we restrict our attention to such classes which we denote $E(\mu, \alpha, \beta)$. Specifically

$$E(\mathbf{0}) = E_1(\mathbf{0}) \cup E_2(\mathbf{0}),$$

where

$$E_1(\mathbf{0}) := \{\boldsymbol{\epsilon} \in ((\mathbb{Z}/8\mathbb{Z})^\times)^3 : \epsilon_1 \equiv \epsilon_2 \equiv \epsilon_3 \bmod 4\};$$

$$E_2(\mathbf{0}) := \bigcup_{\substack{i,j,k \\ \text{pairwise distinct}}} \{\boldsymbol{\epsilon} \in ((\mathbb{Z}/8\mathbb{Z})^\times)^3 : \epsilon_i = \epsilon_j \equiv -\epsilon_k \bmod 4\},$$

and

$$E(1,0,0) = \{\boldsymbol{\epsilon} \in ((\mathbb{Z}/8\mathbb{Z})^\times)^3 : \epsilon_2 = \epsilon_3\};$$

$$E(0,1,0) = \{\boldsymbol{\epsilon} \in ((\mathbb{Z}/8\mathbb{Z})^\times)^3 : \epsilon_1 = \epsilon_3\};$$

$$E(0,0,1) = \{\boldsymbol{\epsilon} \in ((\mathbb{Z}/8\mathbb{Z})^\times)^3 : \epsilon_1 = \epsilon_2\}.$$

Analogously to Section 3, we define the constants $c_{\delta,\epsilon,\mu,\alpha,\beta}$ to account for the different discriminants in each case by setting

$$c_{\delta,\epsilon,\mu,\alpha,\beta} = \begin{cases} 1 & \text{if } (\epsilon_1, \delta_2\epsilon_2, \delta_3\epsilon_3) \in E_1(\mathbf{0}) \text{ and } \mu = 0 = \alpha = \beta \\ 4 & \text{otherwise.} \end{cases}$$

Then

$$(4.2) \quad \widetilde{S}(X) = \frac{1}{6} \sum_{\substack{(\delta_2,\delta_3)\in\{\pm1\}^2 \\ \mu,\alpha,\beta\in\{0,1\}}} \sum_{\mu+\alpha+\beta\in\{0,1\}} \sum_{(\epsilon_1,\delta_2\epsilon_2,\delta_3\epsilon_3)\in E(\mu,\alpha,\beta)} \widetilde{T}(\boldsymbol{\delta},\boldsymbol{\epsilon},\mu,\alpha,\beta),$$

where $\widetilde{T}(\boldsymbol{\delta},\boldsymbol{\epsilon},\mu,\alpha,\beta)$ counts the number of tuples $(m_1',m_2',m_3')\in\mathbb{N}^3$ such that the following all hold:

(1) $\mu^2(m_1'm_2'm_3') = 1$,
(2) $(m_1',m_2',m_3') \equiv \boldsymbol{\epsilon} \bmod 8$,
(3) $2^{\mu+\alpha+\beta} c_{\boldsymbol{\delta},\boldsymbol{\epsilon},\mu,\alpha,\beta} m_1'm_2'm_3' \leqslant \sqrt{X}$,
(4) (m_1',m_2',m_3') satisfies the local conditions for Hasse norm principle failure in Lemma 2.1.

Observe that condition (4) can be detected by the following indicator function

$$\prod_{p|m_1'} \frac{1}{2}\left(1+\left(\frac{m_2m_3}{p}\right)\right) \prod_{p|m_2'} \frac{1}{2}\left(1+\left(\frac{m_1m_3}{p}\right)\right) \prod_{p|m_3'} \frac{1}{2}\left(1+\left(\frac{m_1m_2}{p}\right)\right).$$

Writing $M = \frac{\sqrt{X}}{2^{\mu+\alpha+\beta}c_{\boldsymbol{\delta},\boldsymbol{\epsilon},\mu,\alpha,\beta}}$ this means that we may express $\widetilde{T}(\boldsymbol{\delta},\boldsymbol{\epsilon},\mu,\alpha,\beta)$ as

$$\sum_{\substack{m_1'm_2'm_3'\leqslant M \\ m_i'\equiv\epsilon_i \bmod 8}} \frac{\mu^2(m_1'm_2'm_3')}{\tau(m_1'm_2'm_3')} \prod_p \left(1+\left(\frac{m_1m_2}{p}\right)\right)\left(1+\left(\frac{m_2m_3}{p}\right)\right)\left(1+\left(\frac{m_3m_1}{p}\right)\right),$$

where the product runs over primes that divide $m_1'm_2'm_3'$. By expanding out the product, we clearly have

$$\widetilde{T}(\boldsymbol{\delta},\boldsymbol{\epsilon},\mu,\alpha,\beta) = \sum_{\substack{m_1'm_2'm_3'\leqslant M \\ m_i'\equiv\epsilon_i \bmod 8 \\ m_i'=k_i\tilde{k}_i}} \frac{\mu^2(m_1'm_2'm_3')}{\tau(m_1'm_2'm_3')}\left(\frac{m_1m_2}{k_3}\right)\left(\frac{m_2m_3}{k_1}\right)\left(\frac{m_3m_1}{k_2}\right).$$

Recalling (4.1), we see that

$$\left(\frac{m_1m_2}{k_3}\right) = \left(\frac{2^{\mu+\alpha}k_1k_2\tilde{k}_1\tilde{k}_2}{k_3}\right),$$

$$\left(\frac{m_2m_3}{k_1}\right) = \left(\frac{2^{\alpha+\beta}k_2k_3\tilde{k}_2\tilde{k}_3}{k_1}\right),$$

$$\left(\frac{m_3m_1}{k_2}\right) = \left(\frac{2^{\mu+\beta}k_1k_3\tilde{k}_1\tilde{k}_3}{k_2}\right).$$

We may now repeatedly apply the law of quadratic reciprocity to conclude

$$(4.3) \quad \widetilde{T}(\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta) = \sum_{\substack{m_1' m_2' m_3' \leqslant M \\ m_i' \equiv \epsilon_i \bmod 8 \\ m_i' = k_i \tilde{k}_i}} u(\mathbf{k}) \frac{\mu^2(m_1' m_2' m_3')}{\tau(m_1' m_2' m_3')} \left(\frac{\tilde{k}_1}{k_2 k_3} \right) \left(\frac{\tilde{k}_2}{k_3 k_1} \right) \left(\frac{\tilde{k}_3}{k_1 k_2} \right),$$

where $u(\mathbf{k}) = (-1)^{\nu(k_1)\nu(k_2) + \nu(k_2)\nu(k_3) + \nu(k_3)\nu(k_1)} \left(\frac{2^\mu}{k_2 k_3} \right) \left(\frac{2^\alpha \delta_2}{k_3 k_1} \right) \left(\frac{2^\beta \delta_3}{k_1 k_2} \right)$. Here $\nu(b)$ is defined to be 0 if $b \equiv 1 \bmod 4$ and 1 otherwise, for any odd integer b.

These character sums are strongly reminiscent of the ones studied by Friedlander and Iwaniec in [6], and we will follow their approach to evaluate them, making use of the following results.

Lemma 4.1. *If $q = q_1 q_2$ where $(q_1, q_2) = 1$, $(ad, q) = 1$ and χ_2 is a non-principal character modulo q_2. Then for any $C > 0$ we have*

$$\sum_{\substack{n \leqslant x \\ (n,d)=1 \\ n \equiv a \bmod q_1}} \mu^2(n) \frac{\chi_2(n)}{\tau(n)} \ll_C \tau(d) q x (\log x)^{-C}.$$

Proof. This is the error term in [6, Corollary 2]. $\qquad\qquad\square$

Lemma 4.2. *Let α_m, β_n be any complex numbers supported on odd integers with modulus bounded by one. Then*

$$\sum_{\substack{m,n>V \\ mn \leqslant X}} \alpha_m \beta_n \left(\frac{m}{n} \right) \ll X V^{-\frac{1}{6}} (\log X)^{\frac{7}{6}}.$$

Proof. We break the range of summation into dyadic intervals and then apply [6, Lemma 2] in each interval. Thus

$$\left| \sum_{\substack{m,n>V \\ mn \leqslant X}} \alpha_m \beta_n \left(\frac{m}{n} \right) \right| \leqslant \sum_{\substack{2^i, 2^j > V \\ 2^{i+j} \leqslant X}} \left| \sum_{2^i < m \leqslant 2^{i+1}} \sum_{2^j < n \leqslant 2^{j+1}} \alpha_m \beta_n \left(\frac{m}{n} \right) \right|$$

$$\ll \sum_{\substack{2^i, 2^j > V \\ 2^{i+j} \leqslant X}} 2^{i+j} (2^{-\frac{i}{6}} + 2^{-\frac{j}{6}})(i+j)^{\frac{7}{6}}$$

$$\ll (\log X)^{\frac{7}{6}} \sum_{\substack{2^i, 2^j > V \\ 2^{i+j} \leqslant X}} 2^{i+\frac{5j}{6}}$$

$$= (\log X)^{\frac{7}{6}} \sum_{V \leqslant 2^i \leqslant X/V} 2^i \sum_{V \leqslant 2^j \leqslant X/2^i} 2^{\frac{5j}{6}}$$

$$\ll X V^{-\frac{1}{6}} (\log X)^{\frac{7}{6}}. \qquad\qquad\square$$

The estimate in Lemma 4.1 is only useful when the modulus of the character in our sum is smaller than a power of $\log X$. Set

$$V = (\log X)^B,$$

for a large constant parameter B at our disposal. The moduli of the characters involved in (4.3) can either be thought of as being the k_i or the $\tilde{k}_i$. We get a main term contribution from Lemma 4.1 in the ranges where each of the three characters has small modulus, hence there are three possibilities:

(1) $k_i \leqslant V$ for all i.
(2) $\tilde{k}_i \leqslant V$ for all i.
(3) $\tilde{k}_i, \tilde{k}_j, k_i, k_j \leqslant V$ for some choice of $i \neq j$.

In every other case, we have $k_i, \tilde{k}_j > V$ for some $i \neq j$ and we can apply Lemma 4.2 to exploit cancellation in the $\left(\frac{\tilde{k}_i}{k_j}\right)$ term and get a negligible contribution. To illustrate this, suppose that $k_2, \tilde{k}_1 > V$. Then we can apply Lemma 4.2 to the factor $\left(\frac{\tilde{k}_1}{k_2}\right)$ in (4.3). Let

$$f_1(\tilde{k}_1) = \mathbf{1}_{\left\{\substack{\tilde{k}_1 \equiv \epsilon_1 k_1 \bmod 8 \\ (\tilde{k}_1, 2k_1 \tilde{k}_2 k_3 \tilde{k}_3) = 1}\right\}} \frac{\mu^2(\tilde{k}_1)}{\tau(\tilde{k}_1)} \left(\frac{\tilde{k}_1}{k_3}\right)$$

$$f_2(k_2) = \mathbf{1}_{\left\{\substack{k_2 \equiv \epsilon_2 \tilde{k}_2 \bmod 8 \\ (k_2, 2k_1 \tilde{k}_2 k_3 \tilde{k}_3) = 1}\right\}} \frac{\mu^2(k_2)}{\tau(k_2)} \left(\frac{\tilde{k}_3}{k_2}\right).$$

Then the k_2, $\tilde{k}_1$ sum is given by

$$\sum_{\substack{k_2, \tilde{k}_1 \geqslant V \\ k_2 \tilde{k}_1 \leqslant \frac{\sqrt{X}}{k_1 \tilde{k}_3 k_3 \tilde{k}_2}}} f_1(\tilde{k}_1) f_2(k_2) u(\mathbf{k}) \left(\frac{\tilde{k}_1}{k_2}\right) \ll \sqrt{X} V^{-\frac{1}{6}} (\log X)^{\frac{7}{6}} \frac{1}{\tilde{k}_2 k_1 k_3 \tilde{k}_3}.$$

When summed trivially over the remaining variables, this leads to a contribution of

$$\ll \sqrt{X} V^{-\frac{1}{6}} (\log X)^{\frac{19}{6}}.$$

Of course, this bound applies for all ranges $k_i, \tilde{k}_j > V$ where $i \neq j$.

From here on out then we will restrict to the three different ranges mentioned above. First, consider the range $\tilde{k}_i \leqslant V$ for $i = 1, 2, 3$ and fix $k_1, \tilde{k}_1, \tilde{k}_2$ and $\tilde{k}_3$. In order to evaluate $\widetilde{T}(\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta)$ we need to look at

$$T_{k_1, \tilde{k}_1, \tilde{k}_2, \tilde{k}_3} = \sum\sum_{\substack{k_2 k_3 \leqslant M/m_1' \tilde{k}_2 \tilde{k}_3 \\ k_i \equiv \epsilon_i \tilde{k}_i \bmod 8 \text{ for } i=2,3 \\ (k_2 k_3, 2m_1' \tilde{k}_2 \tilde{k}_3) = 1}} \frac{\mu^2(k_2 k_3)}{\tau(k_2 k_3)} \left(\frac{\tilde{k}_3}{k_2}\right) \left(\frac{\tilde{k}_2}{k_3}\right) \left(\frac{\tilde{k}_1}{k_2 k_3}\right) u(\mathbf{k}),$$

then compute the sum

$$(4.4) \qquad \sum_{\substack{k_1 \tilde{k}_1 \tilde{k}_2 \tilde{k}_3 \leqslant M \\ \tilde{k}_i \leqslant V \\ m_1' \equiv \epsilon_1 \bmod 8}} \sum \sum \frac{\mu^2(2m_1' \tilde{k}_2 \tilde{k}_3)}{\tau(m_1' \tilde{k}_2 \tilde{k}_3)} \left(\frac{\tilde{k}_2 \tilde{k}_3}{k_1} \right) T_{k_1, \tilde{k}_1, \tilde{k}_2, \tilde{k}_3}.$$

Note here that we can pull $u(\mathbf{k})$ out of the expression for $T_{k_1, \tilde{k}_1, \tilde{k}_2, \tilde{k}_3}$. Indeed, $\nu(x)$ only depends on the residue class of $x \bmod 4$, and $\left(\frac{2}{x} \right)$ is determined by the residue class of $x \bmod 8$. Therefore

$$u(\mathbf{k}) = u(\epsilon_1 \tilde{k}_1, \epsilon_2 \tilde{k}_2, \epsilon_3 \tilde{k}_3).$$

Analogously in the range $k_i \leqslant V$ for $i = 1, 2, 3$, we need to evaluate

$$(4.5) \qquad \sum_{\substack{\tilde{k}_1 k_1 k_2 k_3 \leqslant M \\ k_i \leqslant V \\ m_1' \equiv \epsilon_1 \bmod 8}} \sum \sum \frac{\mu^2(2m_1' k_2 k_3)}{\tau(m_1' k_2 k_3)} \left(\frac{\tilde{k}_1}{k_2 k_3} \right) T'_{\tilde{k}_1, k_1, k_2, k_3},$$

where

$$T'_{\tilde{k}_1, k_1, k_2, k_3} = \sum_{\substack{\tilde{k}_2 \tilde{k}_3 \leqslant M/m_1' k_2 k_3 \\ k_i \equiv \epsilon_i \tilde{k}_i \bmod 8 \text{ for } i=2,3 \\ (\tilde{k}_2 \tilde{k}_3, 2m_1' k_2 k_3)=1}} \sum \frac{\mu^2(\tilde{k}_2 \tilde{k}_3)}{\tau(\tilde{k}_2 \tilde{k}_3)} \left(\frac{\tilde{k}_3}{k_2} \right) \left(\frac{\tilde{k}_2}{k_3} \right) \left(\frac{\tilde{k}_2 \tilde{k}_3}{k_1} \right) u(\mathbf{k}).$$

Our approach will be to use Lemma 4.1 to handle the contribution to the sums $T_{k_1, \tilde{k}_1, \tilde{k}_2, \tilde{k}_3}$ and $T'_{\tilde{k}_1, k_1, k_2, k_3}$ arising from non-principal characters, then compute the main terms where each character is principal. In the latter range, the reciprocity factor in the main term is $u(1, 1, 1) = 1$. However in the former, we have the term $u(\boldsymbol{\epsilon})$ which when summed over all possible $\boldsymbol{\epsilon}$ will lead to cancellation (c.f. Lemma 4.6).

Finally, we consider the third type of range. Suppose $k_2, k_3, \tilde{k}_2, \tilde{k}_3 \leqslant V$ then we will fix all four of these variables and evaluate

$$T''_{k_2, \tilde{k}_2, k_3, \tilde{k}_3} = \sum_{\substack{k_1 \tilde{k}_1 \leqslant M/m_2' m_3' \\ k_1 \equiv \epsilon_1 \tilde{k}_1 \bmod 8 \\ (k_1 \tilde{k}_1, 2m_2' m_3')=1}} \sum u(\mathbf{k}) \frac{\mu^2(k_1 \tilde{k}_1)}{\tau(k_1 \tilde{k}_1)} \left(\frac{\tilde{k}_2 \tilde{k}_3}{k_1} \right) \left(\frac{\tilde{k}_1}{k_2 k_3} \right),$$

then compute

$$\sum_{\substack{k_2 \tilde{k}_2 k_3 \tilde{k}_3 \leqslant \min\{M, V\} \\ k_i \equiv \epsilon_i \tilde{k}_i \text{ for } i=2,3}} \frac{\mu^2(k_2 \tilde{k}_2 k_3 \tilde{k}_3)}{\tau(k_2 \tilde{k}_2 k_3 \tilde{k}_3)} \left(\frac{\tilde{k}_2}{k_3} \right) \left(\frac{\tilde{k}_3}{k_2} \right) T''_{k_2, \tilde{k}_2, k_3, \tilde{k}_3}.$$

Before attempting to evaluate $T''_{k_2, \tilde{k}_2, k_3, \tilde{k}_3}$, we wish to remove the $u(\mathbf{k})$ term as in the other ranges. However this is marginally more complicated

than in the other cases. For fixed k_2 and k_3, we may think of $u(\mathbf{k})$ as a character on k_1. Our main term contribution will occur when this character is principal, in particular this requires $k_2 = k_3 = 1$. In that case

$$u(k_1, 1, 1) = \left(\frac{\delta_2 \delta_3 2^{\alpha+\beta}}{k_1} \right).$$

Hence one will only get a main term contribution when $k_2 = k_3 = \tilde{k}_2 = \tilde{k}_3 = 1$ and $\delta_2 \delta_3 2^{\alpha+\beta} = 1$. This gives a further constraint on the outer sum over $\boldsymbol{\delta}, \mu, \alpha$ and β. The following lemma deals with the first two types of ranges.

Lemma 4.3. *If $\tilde{k}_i \leqslant V$ for $i = 1, 2, 3$ then for any $C > 0$, we have*

$$T_{k_1, \tilde{k}_1, \tilde{k}_2, \tilde{k}_3} = \delta_{\tilde{k}_1 = \tilde{k}_2 = \tilde{k}_3 = 1} \frac{u(\boldsymbol{\epsilon})M}{4k_1 \pi^2} \prod_{p | k_1} \left(1 + \frac{1}{p} \right)^{-1} \left\{ 1 + O\left(\tau(k_1) \sqrt{\frac{k_1}{M}} \right) \right\}$$
$$+ O_C(MV^2 \log V (\log M)^{-C}).$$

Analogously, if $k_i \leqslant V$ for $i = 1, 2, 3$ then for any $C > 0$,

$$T'_{\tilde{k}_1, k_1, k_2, k_3} = \delta_{k_1 = k_2 = k_3 = 1} \frac{u(1,1,1)M}{4\tilde{k}_1 \pi^2} \prod_{p | \tilde{k}_1} \left(1 + \frac{1}{p} \right)^{-1} \left\{ 1 + O\left(\tau(\tilde{k}_1) \sqrt{\frac{\tilde{k}_1}{M}} \right) \right\}$$
$$+ O_C(\sqrt{M}V^2 \log V (\log M)^{-C}).$$

Here by $\delta_{\tilde{k}_1 = \tilde{k}_2 = \tilde{k}_3 = 1}$ we simply mean the function that is 1 when all $\tilde{k}_i = 0$ and 0 otherwise. Hence, as expected, our main term corresponds to the case when all characters are principal. The final set of ranges is dealt with by the next lemma.

Lemma 4.4. *If $k_2, \tilde{k}_2, k_3, \tilde{k}_3 \leqslant V$ then for any $C > 0$, we have*

$$T''_{k_2, \tilde{k}_2, k_3, \tilde{k}_3} = \delta_{k_2 = \tilde{k}_2 = k_3 = \tilde{k}_3 = \delta_2 \delta_3 2^{\alpha+\beta} = 1} \frac{M}{\pi^2} \left\{ 1 + O\left(\frac{1}{\sqrt{M}} \right) \right\}$$
$$+ O_C(\sqrt{M}V^2 (\log V)^2 (\log M)^{-C}).$$

The ranges $k_1, \tilde{k}_1, k_2, \tilde{k}_2 \leqslant V$ and $k_1, \tilde{k}_1, k_3, \tilde{k}_3 \leqslant V$ are analogous.

The proof of this follows the proof of Lemma 4.3. Observe that these ranges differ from those in Lemma 4.3 since in those there is a final variable (k_1 and $\tilde{k}_1$ respectively) that needn't be made 1 in our main term. In performing the sum over this last variable we pick up a factor of $\sqrt{\log X}$ which is absent in the ranges considered in Lemma 4.4 hence these ranges give a smaller contribution.

Proof of Lemma 4.3. We will focus on the case when $\tilde{k}_i \leqslant V$. The other case follows the same argument.

The first step will be to remove the congruence conditions on k_2 and k_3 by using a sum of characters mod 8. Performing this and re-arranging, we may write $T_{k_1, \tilde{k}_1, \tilde{k}_2, \tilde{k}_3}$ as

$$\frac{u(\boldsymbol{\epsilon} \tilde{\mathbf{k}})}{16} \sum_{\chi_i \bmod 8} \chi_1(\epsilon_2) \chi_2(\epsilon_3) S,$$

where

$$u(\boldsymbol{\epsilon} \tilde{\mathbf{k}}) = u(\epsilon_1 \tilde{k}_1, \epsilon_2 \tilde{k}_2, \epsilon_3 \tilde{k}_3)$$

and

$$S = \sum_{\substack{k_2 k_3 \leqslant M/k_1 \tilde{k}_1 \tilde{k}_2 \tilde{k}_3 \\ (k_2 k_3, 2m_1' \tilde{k}_2 \tilde{k}_3) = 1}} \frac{\mu^2(k_2 k_3)}{\tau(k_2 k_3)} \left(\frac{\tilde{k}_2}{k_3} \right) \left(\frac{\tilde{k}_3}{k_2} \right) \left(\frac{\tilde{k}_1}{k_2 k_3} \right) \chi_1(k_2) \chi_2(k_3).$$

We gather together the two characters on k_2, k_3 into new characters labelled $\widetilde{\chi}_i$. If either of these characters is non-principal then we can use Lemma 4.1 to get a bound. Suppose that $\widetilde{\chi}_2$ is non-principal. Then for any $C > 0$, the total sum is bounded by

$$\ll \tau(m_1' \tilde{k}_2 \tilde{k}_3) \frac{M}{k_1 \tilde{k}_3} (\log M/\tilde{k}_1 \tilde{k}_2 \tilde{k}_3)^{-C} \sum_{k_2 \leqslant M/\tilde{k}_1 \tilde{k}_2 \tilde{k}_3} \frac{\mu^2(k_2)}{k_2} |\widetilde{\chi}_1(k_2)|$$

$$\ll \tau(m_1' \tilde{k}_2 \tilde{k}_3) \frac{M}{k_1 \tilde{k}_3} (\log M/\tilde{k}_1 \tilde{k}_2 \tilde{k}_3)^{1-C}.$$

When summed trivially over the remaining variables this gives an error term of size

$$\ll \sqrt{X} V^2 \log V (\log X)^{2-C}.$$

This bound is definitely not best possible, but all that matters is we have an arbitrary log power saving over our main term.

We turn now to estimating the main term which is given by

$$\frac{u(\boldsymbol{\epsilon})}{16} \sum_{\substack{k_2 k_3 \leqslant M/k_1 \\ (k_2 k_3, 2k_1) = 1}} \frac{\mu^2(k_2 k_3)}{\tau(k_2 k_3)} = \frac{u(\boldsymbol{\epsilon})}{16} \sum_{\substack{n \leqslant M/k_1 \\ (n, 2k_1) = 1}} \frac{\mu^2(n)}{\tau(n)} \sum_{n = k_2 k_3} 1$$

$$= \frac{u(\boldsymbol{\epsilon})}{16} \sum_{\substack{n \leqslant M/k_1 \\ (n, 2k_1) = 1}} \mu^2(n).$$

The lemma now immediately follows from the simple identity

$$\sum_{\substack{x \leqslant X \\ (x, d) = 1}} \mu^2(x) = \frac{6}{\pi^2} X \prod_{p \mid d} \left(1 + \frac{1}{p} \right)^{-1} + O\left(\tau(d) \sqrt{X} \right). \qquad \square$$

We now sketch the proof of Lemma 4.4 although it follows very similar lines to the above.

Proof of Lemma 4.4. We again perform a character sum mod 8 to express the congruence condition in the definition of $T''_{k_2,\tilde{k}_2,k_3,\tilde{k}_3}$. After re-arranging this means we can express $T''_{k_2,\tilde{k}_2,k_3,\tilde{k}_3}$ as

$$\frac{1}{4} \sum_{\chi \bmod 8} \chi(\epsilon_1) \sum_{\substack{k_1\tilde{k}_1 \leqslant M/k_2\tilde{k}_2\tilde{k}_3 \\ (k_1\tilde{k}_1, 2k_2\tilde{k}_2k_3\tilde{k}_3)=1}} \frac{u(\mathbf{k})\mu^2(k_1\tilde{k}_1)}{\tau(k_1\tilde{k}_1)} \left(\frac{\tilde{k}_2\tilde{k}_3}{k_1}\right)\left(\frac{k_2k_3}{\tilde{k}_1}\right)\chi(k_1)\chi(\tilde{k}_1).$$

As noted before, we may think of $u(\mathbf{k})$ as a character on k_1 which can be combined with the other characters to form a new one, called $\widetilde{\chi}_1$. Similarly, the characters on $\tilde{k}_1$ may be combined to form the new character $\widetilde{\chi}_2$. The error term contribution when either of these new characters is non-principal may again be computed using Lemma 4.1 and it remains to compute the main term

$$\frac{1}{4} \sum_{\substack{k_1\tilde{k}_1 \leqslant M \\ k_1,\tilde{k}_1 \text{ odd}}} \frac{\mu^2(k_1\tilde{k}_1)}{\tau(k_1\tilde{k}_1)}.$$

This is treated exactly as in the earlier proof. $\square$

Finally, to estimate (4.4) we must compute

$$(4.6) \qquad \frac{Mu(\boldsymbol{\epsilon})}{4\pi^2} \sum_{\substack{k_1 \leqslant M \\ k_1 \equiv \epsilon_1 \bmod 8}} \frac{\mu^2(k_1)}{k_1\tau(k_1)} \prod_{p|k_1} \left(1 + \frac{1}{p}\right)^{-1}.$$

We again remove the congruence condition with a character sum, we'll first deal with the non-principal characters. We write the sum involving non-principal character χ as

$$\sum_{k_1 \leqslant M} \frac{\mu^2(k_1)\chi(k_1)}{k_1\tau(k_1)} \prod_{p|k_1} \left(1 + \frac{1}{p}\right)^{-1} = \sum_{k_1 \leqslant M} \frac{\mu^2(k_1)\chi(k_1)}{k_1\tau(k_1)} \sum_{d|k} \mu(d)f(d),$$

where $f(d) = \prod_{p|d} \frac{1}{p+1}$. Then we may re-arrange the sum to

$$\sum_{d \leqslant M} \frac{\mu(d)\chi(d)f(d)}{d\tau(d)} \sum_{\substack{k \leqslant M/d \\ (k,d)=1}} \frac{\mu^2(k)\chi(k)}{k\tau(k)}.$$

For any $C > 0$, it follows from Lemma 4.1 (via partial summation) that the inner sum above is $\ll \tau(d)(\log M)^{-C}$, while the remaining d sum contributes $\ll \log M$. We employ the following general estimate of Friedlander–Iwaniec [5, Theorem A.5] to compute the main term arising from the principal character.

Lemma 4.5. *Suppose g is a multiplicative function supported on squarefree integers such that for some $\kappa > -\frac{1}{2}$ the following hold:*

(1) $\sum_{p \leqslant x} g(p) \log p = \kappa \log x + O(1)$,

(2) $\prod_{w \leqslant p < z} (1 + |g(p)|) \ll \left(\frac{\log z}{\log w} \right)^{|\kappa|}$,

(3) $\sum_p g(p)^2 \log p < \infty$.

Then we have

$$\sum_{n \leqslant x} g(n) = \frac{(\log x)^\kappa}{\Gamma(\kappa + 1)} \prod_p \left[\left(1 - \frac{1}{p} \right)^\kappa (1 + g(p)) \right] + O\left((\log x)^{|\kappa| - 1} \right).$$

That

$$g(n) = \begin{cases} \frac{\mu^2(n)}{\tau(n)n} \prod_{p \mid n} \left(1 + \frac{1}{p} \right)^{-1} & \text{if } n \text{ odd} \\ 0 & \text{otherwise,} \end{cases}$$

satisfies the assumptions with $\kappa = \frac{1}{2}$ is a simple consequence of Mertens' theorems. Therefore

$$\sum_{\substack{k_1 \leqslant M \\ k_1 \equiv \epsilon_1 \bmod 8}} g(k_1) = \frac{\sqrt{\log M}}{2\sqrt{2\pi}} \prod_{p > 2} \left(1 - \frac{1}{p} \right)^{\frac{1}{2}} \left(1 + \frac{1}{2(p+1)} \right) + O\left((\log M)^{-\frac{1}{2}} \right).$$

Hence we see that (4.4) is equal to

$$(4.7) \qquad \frac{3u(\boldsymbol{\epsilon})}{28\pi^2\sqrt{\pi}} M \sqrt{\log M} \prod_p \left(1 - \frac{1}{p} \right)^{\frac{1}{2}} \left(1 + \frac{1}{2p+2} \right) + O\left(M \right).$$

Recall that $M = \frac{\sqrt{X}}{c_{\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta} 2^{\mu + \alpha + \beta}}$ so all that remains is to sum over the possible values of $\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha$ and β. The following lemma shows that in the range where $\tilde{k}_i \leqslant V$ for $i = 1, 2, 3$, the $u(\boldsymbol{\epsilon})$ factor leads to cancellation in the main term.

Lemma 4.6. *We have*

$$\sum_{\substack{\delta_2, \delta_3 \in \{\pm 1\}^2 \\ \mu, \alpha, \beta \in \{0, 1\}}} \sum_{\mu + \alpha + \beta \in \{0, 1\}} 2^{-(\mu + \alpha + \beta)} \sum_{(\epsilon_1, \delta_2 \epsilon_2, \delta_3 \epsilon_3) \in E(\mu, \alpha, \beta)} \frac{u(\boldsymbol{\epsilon})}{c_{\boldsymbol{\delta}, \boldsymbol{\epsilon}, \mu, \alpha, \beta}} = 0.$$

Proof. We start by observing that

$$\left(\frac{2^\mu}{\epsilon_2 \epsilon_3} \right) \left(\frac{2^\alpha}{\epsilon_1 \epsilon_3} \right) \left(\frac{2^\beta}{\epsilon_1 \epsilon_2} \right) = 1.$$

Indeed this is clearly true when $\mu + \alpha + \beta = 0$. If $\mu = 1$ then by the definition of $E(1, 0, 0)$ we must have $\delta_2 \epsilon_2 = \delta_3 \epsilon_3$ therefore

$$\left(\frac{2}{\epsilon_2 \epsilon_3} \right) = 1.$$

The other cases follow similarly.

Therefore

$$u(\boldsymbol{\epsilon}) = (-1)^{\nu(\epsilon_1)\nu(\epsilon_2)+\nu(\epsilon_3)\nu(\epsilon_1)+\nu(\epsilon_2)\nu(\epsilon_3)} \left(\frac{\delta_2}{\epsilon_1\epsilon_3}\right)\left(\frac{\delta_3}{\epsilon_1\epsilon_2}\right)$$

$$= (-1)^{\nu(\epsilon_1)\nu(\epsilon_2)+\nu(\epsilon_3)\nu(\epsilon_1)+\nu(\epsilon_2)\nu(\epsilon_3)+\nu(\delta_2)\nu(\epsilon_1\epsilon_3)+\nu(\delta_3)\nu(\epsilon_1\epsilon_2)}$$

Now we just run through all possible values of $\delta_2, \delta_3, \boldsymbol{\epsilon}, \mu, \alpha$ and β and see what comes out. For simplicity we write $\boldsymbol{\epsilon}' := (\epsilon_1, \delta_2\epsilon_2, \delta_3\epsilon_3)$.

First suppose that $(\delta_2, \delta_3) = (+1, +1)$ then

$$u(\boldsymbol{\epsilon}) = (-1)^{\nu(\epsilon_1)\nu(\epsilon_2)+\nu(\epsilon_3)\nu(\epsilon_1)+\nu(\epsilon_2)\nu(\epsilon_3)}.$$

By Lemma 2.1, we know that at least two components of $\boldsymbol{\epsilon}'$ must be equal therefore for some $\mu \in (\mathbb{Z}/8\mathbb{Z})^{\times}$ we have

$$u(\boldsymbol{\epsilon}) = (-1)^{\mu}$$

Hence the sum over these is 0.

Now suppose $(\delta_2, \delta_3) = (+1, -1)$ so that

$$u(\boldsymbol{\epsilon}) = (-1)^{\nu(\epsilon_1)\nu(\epsilon_2)+\nu(\epsilon_3)\nu(\epsilon_1)+\nu(\epsilon_2)\nu(\epsilon_3)+\nu(\epsilon_1\epsilon_2)}.$$

If $\boldsymbol{\epsilon}' \in E_1(\mathbf{0})$ then $\epsilon_1 \equiv \epsilon_2 \equiv -\epsilon_3 \bmod 4$ so

$$u(\boldsymbol{\epsilon}) = (-1)^{\nu(\epsilon_1)}.$$

So again this sums to 0. Next suppose $\boldsymbol{\epsilon}' \in E_2(\mathbf{0})$ then one of the following cases occurs

   (1) $\epsilon_1 = \epsilon_2 \equiv -\epsilon_3 \bmod 4$ then $u(\boldsymbol{\epsilon}) = (-1)^{\nu(\epsilon_1)+1}$.
   (2) $\epsilon_2 = \epsilon_3 \equiv -\epsilon_1 \bmod 4$ then $u(\boldsymbol{\epsilon}) = (-1)^{\nu(\epsilon_1)+1}$.
   (3) $\epsilon_3 = \epsilon_1 \equiv -\epsilon_2 \bmod 4$ then $u(\boldsymbol{\epsilon}) = (-1)^{\nu(\epsilon_1)}$.

Each of these cases sums to 0.

If $\boldsymbol{\epsilon}' \in E(1, 0, 0)$ then $\epsilon_2 = -\epsilon_3$ so

$$u(\boldsymbol{\epsilon}) = (-1)^{\nu(\epsilon_1)+\nu(\epsilon_1\epsilon_2)}.$$

If $\boldsymbol{\epsilon}' \in E(0, 1, 0)$ then $\epsilon_1 = -\epsilon_3$ so

$$u(\boldsymbol{\epsilon}') = (-1)^{\nu(\epsilon_2)+\nu(\epsilon_1\epsilon_2)}.$$

If $\boldsymbol{\epsilon}' \in E(0, 0, 1)$ then $\epsilon_1 = \epsilon_2$ so

$$u(\boldsymbol{\epsilon}) = (-1)^{\nu(\epsilon_1)}.$$

Again, all of these sum to 0. The case where $(\delta_2, \delta_3) = (-1, +1)$ is similar.

Finally suppose $(\delta_2, \delta_3) = (-1, -1)$, in which case

$$u(\boldsymbol{\epsilon}) = (-1)^{\nu(\epsilon_1)\nu(\epsilon_2)+\nu(\epsilon_3)\nu(\epsilon_1)+\nu(\epsilon_2)\nu(\epsilon_3)+\nu(\epsilon_1\epsilon_2)+\nu(\epsilon_1\epsilon_3)}.$$

Then for $\boldsymbol{\epsilon}' \in E_1(\mathbf{0})$ we have

$$u(\boldsymbol{\epsilon}) = (-1)^{\nu(-\epsilon_1)}.$$

For $\boldsymbol{\epsilon}' \in E(1,0,0)$ we must have $\epsilon_2 = \epsilon_3$ so

$$u(\boldsymbol{\epsilon}) = (-1)^{\nu(\epsilon_2)}.$$

For $\boldsymbol{\epsilon}' \in E(0,1,0)$ we must have $\epsilon_1 = -\epsilon_3$ so

$$u(\boldsymbol{\epsilon}) = (-1)^{\nu(\epsilon_2)+\nu(\epsilon_1\epsilon_2)}.$$

For $\boldsymbol{\epsilon}' \in E(0,0,1)$ we must have $\epsilon_1 = -\epsilon_2$ so

$$u(\boldsymbol{\epsilon}) = (-1)^{\nu(\epsilon_3)+\nu(\epsilon_1\epsilon_3)}.$$

In all of these cases the sum is 0. □

As promised the u term cancels the main term of the sum in this range so the actual main term must be from the range $k_i \leqslant V$. Evaluating the sum T'_{k_1,k_1,k_2,k_3} is identical to evaluating $T_{k_1,\tilde{k}_1,\tilde{k}_2,k_3}$ and therefore in analogy with (4.7), we have that (4.5) is equal to

$$\frac{3}{28\pi^2\sqrt{\pi}} M \sqrt{\log M} \prod_p \Big(1 - \frac{1}{p}\Big)^{\frac{1}{2}}\Big(1 + \frac{1}{2p+2}\Big) + O\left(M\right).$$

It just remains to compute that

$$\sum_{\substack{\delta_2,\delta_3\in\{\pm 1\}^2 \\ \mu,\alpha,\beta\in\{0,1\}}} \sum_{\mu+\alpha+\beta\in\{0,1\}} 2^{-(\mu+\alpha+\beta)} \sum_{(\epsilon_1,\delta_2\epsilon_2,\delta_3\epsilon_3)\in E(\mu,\alpha,\beta)} \frac{1}{c_{\boldsymbol{\delta},\boldsymbol{\epsilon},\mu,\alpha,\beta}} = 112.$$

Therefore

$$\widetilde{S}(X) = \frac{\sqrt{2}}{\pi^{\frac{5}{2}}} \sqrt{X \log X} \prod_p \Big(1 - \frac{1}{p}\Big)^{\frac{1}{2}}\Big(1 + \frac{1}{2p+2}\Big) + O\left(\sqrt{X}\right).$$

The expression in the statement of Theorem 1.2 is achieved by expressing a factor of $\frac{\pi^2}{6}$ as part of the Euler product.

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