JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 30, n^{o} 3 (2018), p. 917-945.

<http://jtnb.cedram.org/item?id=JTNB_2018__30_3_917_0>

Journal de Théorie des Nombres
de Bordeaux **30** (2018), 917–945

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par Roland BACHER

Résumé. Le nombre p_d de classes de similitude de réseaux parfaits en dimension d vérifie asymptotiquement les inégalités $e^{d^{1-\epsilon}} < p_d < e^{d^{3+\epsilon}}$ pour $\epsilon > 0$ arbitrairement petit.

Abstract. We show that the number p_d of non-similar perfect d-dimensional lattices satisfies eventually the inequalities $e^{d^{1-\epsilon}} < p_d < e^{d^{3+\epsilon}}$ for arbitrary small strictly positive ϵ.

1. Main result

We denote by $\Lambda_{\min}$ the set of vectors having minimal non-zero length in an Euclidean lattice (discrete subgroup of an Euclidean vector space). An Euclidean lattice Λ of rank $d = \dim(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ is *perfect* if the set $\{v \otimes v\}_{v \in \Lambda_{\min}}$ spans the full $\binom{d+1}{2}$-dimensional vector space of all symmetric elements in $(\Lambda \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} (\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$. In the sequel, a lattice will always denote an Euclidean lattice of finite rank (henceforth called the dimension of the lattice). Every perfect lattice is similar to an integral lattice and the number of similarity classes of perfect lattices of given dimension is finite, cf. for example [9]. Similarity classes of perfect lattices are in one-to-one correspondence with isomorphism classes of primitive integral perfect lattices. (A lattice is primitive integral if the set of all possible scalar-products is a coprime set of integers.) For general information on lattices, the reader can consult [4].

The main result of this paper can be resumed as follows:

Theorem 1.1. *For every strictly positive ϵ, the number p_d of isomorphism classes of perfect d-dimensional primitive integral lattices satisfies eventually the inequalities*

$$e^{d^{1-\epsilon}} < p_d < e^{d^{3+\epsilon}} .$$

Otherwise stated, there exist a largest real number $\alpha \in [1, 3]$ and a smallest real number $\beta \in [\alpha, 3]$ such that we have eventually $e^{d^{\alpha-\epsilon}} < p_d < e^{d^{\beta+\epsilon}}$. I suspect that $1 < \alpha = \beta \leq 2$. The inequality $1 < \alpha$ is suggested

Manuscrit reçu le 22 juin 2017, révisé le 1er février 2018, accepté le 5 avril 2018.
2010 *Mathematics Subject Classification.* 11H55, 11T06, 20K01, 05B30, 05E30.
Mots-clefs. Perfect lattice.
This work has been partially supported by the LabEx PERSYVAL-Lab (ANR–11-LABX-0025). The author is a member of the project-team GALOIS supported by this LabEx.

by the large number of known perfect forms in dimension 8 and 9 (where we lack a complete classification), see [5]. I present a few non-rigorous arguments for $\beta \leq 2$ in Section 18.

The two inequalities of Theorem 1.1 are proved by completely different methods which give actually explicit lower and upper bounds for the numbers p_d.

Lower bounds are obtained by describing an explicit family of non-isomorphic primitive integral perfect lattices of minimum 4. (The minimum of a lattice is the squared Euclidean norm of a shortest non-zero element.) Proving perfection of the family is easy. Showing that it consists only of non-isomorphic elements is somewhat tedious. Our proof is based on the fact that "error correction" is possible for a symmetric and reflexive relation obtained by adding a few "errors" to an equivalence relation on a finite set. Finally, we compute the number of lattices of dimension d in the family and show that this number grows exponentially fast with d. This first part is essentially a refined sequel of [1]. Similar methods and constructions have been used in [3] and [2].

Constructions used in this paper and in [1] yield scores of perfect integral lattices with minimum 4. A complete understanding or classification of such lattices is probably a task doomed to failure. Integral perfect lattices of minimum 2 are root lattices of type A, D or E. They are thus rare and very well understood. Perfect integral lattices of minimum 3 sit between these two worlds. Is there some hope for a (at least partial) classification or are there already too many of them?

The upper bound for p_d is also explicit, see Theorem 10.1. We prove it by elementary geometric and combinatorial arguments. Somewhat weaker bounds (amounting to the eventual inequality $p_d < e^{d^{4+\epsilon}}$) were obtained by C. Soulé in Section 1 of [12].

A slight modification of the proof of Theorem 10.1 gives an upper bound for the number of $\mathrm{GL}_d(\mathbb{Z})$-orbits of d-dimensional symmetric lattice polytopes containing no non-zero lattice points in their interior, see Section 11.

We have tried to make this paper as elementary and self-contained as possible. We apologise for the resulting redundancies with the existing literature.

2. Perfection

A subset $\mathcal{P}$ of a vector space V of finite dimension d over a field $\mathbb{K}$ of characteristic $\neq 2$ is *perfect* if the set $\{v \otimes v\}_{v \in \mathcal{P}}$ spans the full $\binom{d+1}{2}$-dimensional vector space $\sum_{v,w \in V} v \otimes w + w \otimes v$ of all symmetric tensor-products in $V \otimes_{\mathbb{K}} V$. Perfection belongs to the realm of linear algebra: every perfect set $\mathcal{P}$ contains a perfect subset $\mathcal{S}$ of $\binom{d+1}{2}$ elements giving rise to a basis $\{v \otimes v\}_{v \in \mathcal{S}}$ of symmetric tensor products. Moreover, every subset of d linearly independent elements in a perfect set $\mathcal{P}$ can be extended to a perfect subset of $\binom{d+1}{2}$ elements in $\mathcal{P}$.

An example of a perfect set is given by $b_1, \dots, b_d, b_i + b_j, 1 \leq i < j \leq d$ where $b_1, \dots, b_d$ is a basis of V.

Other interesting examples over subfields of real numbers are given by 4-designs. (We recall that a t-design is a finite subset $\mathcal{S}$ of the Euclidean $1-$sphere $\mathbb{S}^{d-1}$ such that the mean value over $\mathcal{S}$ of any polynomial of degree at most t equals the mean value of the polynomial over the unit sphere). See [11] for a survey of B. Venkov's work on relations between 4-designs and perfect lattices.

Given a perfect set $\mathcal{P}$ with $\binom{d+1}{2}$ elements in a d-dimensional vector space over a field $\mathbb{K}$ of characteristic $\neq 2$, any map $\nu : \mathcal{P} \longrightarrow \mathbb{K}$ corresponds to a unique bilinear product $\langle \cdot, \cdot \rangle$ on $V \times V$ such that $\nu(v) = \langle v, v \rangle$ for all $v \in \mathcal{P}$. Such a bilinear product is in general not positively defined over a real field. Equivalently, a homogeneous quadratic form $q : V \longmapsto \mathbb{K}$ is completely defined by its restriction to a perfect set.

Perfect lattices are lattices whose set of minimal vectors (shortest non-zero elements) determines the Euclidean structure completely, up to similarity. Examples of perfect lattices are given e.g. by lattices whose minimal vectors form a 4-design. This shows perfection of root lattices of type A, D, E, of the Leech lattice and of many other interesting lattices with large automorphism groups.

A useful tool for proving perfection of a set is the following easy lemma (see [9, Proposition 3.5.5] or [1, Proposition 1.1] for a proof):

Lemma 2.1. *Suppose that a set $\mathcal{S}$ of a vector space V intersects a hyperplane $\mathcal{H}$ (linear subspace of codimension 1) in a perfect subset $\mathcal{S}' = S \cap \mathcal{H}$ of $\mathcal{H}$ and that $\mathcal{S} \setminus \mathcal{S}'$ spans V. Then $\mathcal{S}$ is perfect in V.*

The condition on the span of $\mathcal{S} \setminus \mathcal{S}'$ in Lemma 2.1 is necessary as shown by the following easy result whose proof is left to the reader:

Lemma 2.2. *Given a linear hyperplane $\mathcal{H}$ of a vector space V, any perfect set of V contains a basis of V not intersecting $\mathcal{H}$.*

Proposition 2.3. *Let $\mathcal{S}$ be a subset of a vector space V intersecting two distinct hyperplanes $\mathcal{H}', \mathcal{H}''$ in two perfect subsets $\mathcal{S}' = \mathcal{S} \cap \mathcal{H}', \mathcal{S}'' = \mathcal{S} \cap \mathcal{H}''$, of $\mathcal{H}'$, respectively $\mathcal{H}''$. Suppose moreover that $\mathcal{S} \setminus (\mathcal{S}' \cup \mathcal{S}'')$ is non-empty. Then $\mathcal{S}$ is perfect in V.*

Proof. Let $d+1$ be the dimension of V. Lemma 2.2 and perfection of $\mathcal{S}''$ in $\mathcal{H}''$ imply that $\mathcal{S}'' \setminus \mathcal{S}'$ contains a basis $b_1, \dots, b_d$ of $\mathcal{H}''$. Adding an element $b_0 \in \mathcal{S} \setminus \{\mathcal{S}' \cup \mathcal{S}''\}$ to such a basis, we get a basis $b_0, \dots, b_d \subset \mathcal{S} \setminus \mathcal{S}'$. Lemma 2.1 implies perfection of $\mathcal{S}$ in V. □

Proposition 2.3 implies immediately the following result for lattices:

Corollary 2.4. *Let Λ be a lattice of minimum m and rank d containing two distinct perfect sublattices Λ' and Λ'' which are both of minimum m and rank $d-1$. Suppose moreover that $\Lambda'+\Lambda''$ is of rank d (or, equivalently, of finite index in Λ) and that Λ contains a minimal vector v such that $\mathbb{Q}v\cap\Lambda'=\mathbb{Q}v\cap\Lambda''=\{0\}$. Then Λ is perfect.*

3. The lattices $L_d(h_1,h_2,\dots)$: Construction and Results

Given a strictly increasing sequence $1\le h_1<h_2<\dots$ of integers, we denote by $L_d(h_1,h_2,\dots)$ the lattice of all integral vectors in $\mathbb{Z}^{d+2}$ which are orthogonal to $c=(1,1,1,\dots,1,1)\in\mathbb{Z}^{d+2}$ and to the vector $h=(1,2,\dots,h_1-1,\widehat{h_1},h_1+1,\dots,\widehat{h_2},\dots)\in\mathbb{Z}^{d+2}$ with strictly increasing coordinates given by the $(d+2)$ smallest elements of $\{1,2,3,4,\dots\}\setminus\{h_1,h_2,h_3,\dots\}$. We think of the missing coefficients $h_1,h_2,\dots$ as "holes" or "forbidden indices". Indeed, the lattice $L_d=L_d(h_1,\dots)$ is also the set of all vectors $(x_1,x_2,\dots)$ with finite support and zero coordinate-sum $\sum_{i=1,2,\dots}x_i=0$ (defining the enumerably infinite-dimensional root lattice A_∞ of type A) such that $\sum ix_i=0$ and non-zero coordinates have indices among the first $d+2$ elements of $\{1,2,\dots\}\setminus\{h_1,h_2,\dots\}=\{1,2,\dots,\widehat{h_1},\dots\}$. Equivalently, L_d is the sublattice of $\mathbb{Z}^{\{1,2,\dots\}}$ supported by the $d+2$ smallest possible indices such that L_d is orthogonal to $c=(1,1,1,1,\dots)$ and $h=(1,2,3,4,\dots)$ (the elements c and h belong of course only to the "dual lattice" $\mathbb{Z}^{\{1,2,\dots\}}$ of the "infinite dimensional lattice" $\mathbb{Z}^\infty$ generated by an enumerable orthogonal basis) and L_d is also orthogonal to $b_{h_1},b_{h_2},\dots$ with b_i denoting the i-th element $(0,0,\dots,0,1,0,\dots)$ of the standard basis of $\mathbb{Z}^\infty$.

In the sequel, we will often use the notation $\sum_i\alpha_ib_i$ with indices $i\in\{1,2,\dots\}\setminus\{i_1,i_2,\dots\}$ corresponding to coefficients of $h=(1,2,\dots,h_1-1,\widehat{h_1},h_1+1,\dots)\in\mathbb{Z}^{d+2}$ when working with elements of $L_d(h_1,\dots)$.

The lattice $L_d(h_1,h_2,\dots)$ is even integral of dimension d with no elements of (squared euclidean) norm 2. It is the sublattice of $\mathbb{Z}^{d+2}$ orthogonal to $\mathbb{Z}c+\mathbb{Z}h$ which is a full two-dimensional sublattice of $\mathbb{Z}^{d+2}$ except if $\{1,2,\dots\}\setminus\{h_1,h_2,\dots\}$ is an arithmetic progression (a case which will henceforth always be excluded). The squared volume $\mathrm{vol}((L_d(h_1,\dots)\otimes_{\mathbb{Z}}\mathbb{R})/L_d(h_1,\dots)^2)$ of a fundamental domain, also called the determinant of $L_d(h_1,h_2,\dots)$, equals thus $\langle c,c\rangle\langle h,h\rangle-\langle c,h\rangle^2$.

Theorem 3.1. *$L_d(h_1,h_2,\dots)$ is perfect of minimum 4 if $d\ge 10$ and $h_{i+1}-h_i\ge 6$ for all i.*

Given a lattice $L_d(h_1,h_2,\dots)\subset\mathbb{Z}^{d+2}$ orthogonal to $\mathbb{Z}c+\mathbb{Z}h$ (with $c=(1,1,\dots,1)$ and $h=(1,2,3,\dots,\widehat{h_1},\dots,\omega-2,\omega-1,\omega)$) we get an isometric lattice of the same form by considering $L_d(h_1',h_2',\dots)$ where $\{h_1',h_2',\dots\}=\{\omega+1-h_1,\omega+1-h_2,\dots\}\cap\{1,2,3,\dots\}$. Indeed, we have $h'=(\omega+1)\times(1,1,1,\dots,1)-h=(1,2,\dots,\widehat{h_1'},\dots)$, up to a permutation of coordinates.

We call two such lattices *essentially isomorphic*. Essentially isomorphic lattices are related by a suitable affine reflection of their holes. An example of two essentially isomorphic lattices is given by

$$L_6(2,5,6,9) = \mathbb{Z}^8 \cap \left(\mathbb{Z}(1,1,1,1,1,1,1,1) + \mathbb{Z}(1,3,4,7,8,10,11,12)\right)^{\perp}$$

and

$$L_6(4,7,8,11) = \mathbb{Z}^8 \cap \left(\mathbb{Z}(1,1,1,1,1,1,1,1) + \mathbb{Z}(1,2,3,5,6,9,10,12)\right)^{\perp}.$$

The following result yields a large family of non-isomorphic lattices:

Theorem 3.2. *If two lattices $L_d(h_1,\dots,h_k = d+k+1)$ and $L_d(h'_1,\dots,h'_{k'} = d+k'+1)$ of the same dimension $d \geq 46$ satisfy the conditions $h_1, h'_1 \geq 7,\ h_k = d+k+1,\ h'_{k'} = d+k'+1$ and $h_{i+1}-h_i, h'_{i+1}-h'_i \geq 4$ for all i, then they are isomorphic if and only if $k' = k$ and $h'_i = h_i$ for all i.*

4. Proof of Theorem 3.1

The proof for perfection is an induction. The induction step is the following special case of Corollary 2.4:

Proposition 4.1. *Let $h_1, h_2, \dots \subset \{2,3,4,\dots\}$ be a strictly increasing sequence of natural integers ≥ 2 such that $L_d(h_1,h_2,\dots)$ and $L_d(h_1-1, h_2-1, h_3-1,\dots)$ are perfect lattices of minimum 4 and such that $L_{d+1}(h_1, h_2,\dots)$ contains a minimal vector $(1,x_2,\dots,x_{d+1},1)$ starting and ending with a coefficient 1. Then $L_{d+1}(h_1,h_2,\dots)$ is perfect.*

Proof of Theorem 3.1. We assume perfection of every d-dimensional lattice $L_d(h_1,h_2,\dots)$ with $h_{i+1}-h_i \geq 6$ for all i.

We consider a $(d+1)$-dimensional lattice $L_{d+1}(h_1,h_2,\dots)$ with $h_1 > 1$ (otherwise we remove h_1 and shift all holes $h_2,\dots$ by 1, i.e. we consider $L_{d+1}(h_2-1,h_3-1,\dots)$).

By induction, the two d-dimensional sublattices $L_d(h_1,h_2,\dots)$ (consisting of all elements in $L_{d+1}(h_1,h_2,\dots)$ with last coordinate 0) and $L_d(h_1-1, h_2-1,\dots)$ (consisting of all elements in $L_{d+1}(h_1,h_2,\dots)$ with first coordinate 0) are perfect. In order to apply Proposition 4.1, we have only to show that $L_{d+1}(h_1,\dots)$ contains a minimal vector of the form $(1,\dots,1)$, i.e. starting and ending with a coordinate 1. We denote by $h = (1,a,b,c,\dots,w,x,y,z)$ the $(d+3)$-dimensional vector $(1,2,\dots,h_1-1,\widehat{h_1},h_1+1,\dots,)$. The condition $h_{i+1}-h_i \geq 6 \geq 3$ ensures that both sets $\{2,3,4\}$ and $\{z-1,z-2,z-3\}$ contain at most one element in the set $\{h_1,\dots\}$ of holes. There exist thus $s \in \{a,b,c\}$ and $t \in \{w,x,y\}$ such that $s+t = 1+z$ ensuring existence of a minimal vector $b_1-b_s-b_t+b_z$ of the form $(1,\dots,1)$ in $L_{d+1}(h_1,h_2,\dots)$. Proposition 4.1 implies now perfection of $L_{d+1}(h_1,\dots)$.

We have yet to check the initial conditions. It turns out that Theorem 3.1 holds almost in dimension 9. Indeed, all lattices $L_9(h_1,\dots)$ with

$h_{j+1} - h_j \geq 6$ for all j are perfect (as can be checked by a machine computation) except the lattice $L_9(4,10)$ (given by all elements of $\mathbb{Z}^{11}$ orthogonal to $(1,1,1,1,1,1,1,1,1,1,1)$ and $(1,2,3,5,6,7,8,9,11,12,13)$). Fortunately, the two essentially isomorphic lattices $L_{10}(4,10) = L_{10}(5,11)$ (which are the two only possible ways to extend $L_9(4,10)$ into a 10 dimensional lattice of the form $L_{10}(h_1,\dots)$ with $h_{i+1} - h_i \geq 6$) are perfect. All other 10-dimensional lattices $L_{10}(h_1,h_2,\dots)$ satisfying the conditions of Theorem 3.1 are associated to two lattices $L_9(h_1,h_2,\dots)$ and $L_9(h_1-1,h_2-1,\dots)$ which are both non-isomorphic to $L_9(4,10)$. They are thus perfect by Proposition 4.1. $\square$

We display below the list of all 9-dimensional lattices $L_9(h_1,\dots)$ with $h_{i+1} - h_i \geq 6$, up to essential isomorphism. The last entry is devoted to the 10-dimensional lattice $L_{10}(4,10)$. Columns have hopefully understandable meanings (the last column, labelled d_2, gives the rank of the vector space of symmetric tensors spanned by all vectors $v \otimes v$ for v belonging to the set $\Lambda_{\min}$ of minimal vectors):

dim	h_j	coordinates of h	det	$\lvert\Lambda_{\min}\rvert/2$	d_2
9		$1,2,3,4,5,6,7,8,9,10,11$	1210	70	45
9	2	$1,3,4,5,6,7,8,9,10,11,12$	1330	66	45
9	3	$1,2,4,5,6,7,8,9,10,11,12$	1426	63	45
9	4	$1,2,3,5,6,7,8,9,10,11,12$	1498	61	45
9	5	$1,2,3,4,6,7,8,9,10,11,12$	1546	60	45
9	6	$1,2,3,4,5,7,8,9,10,11,12$	1570	60	45
9	$2,8$	$1,3,4,5,6,7,9,10,11,12,13$	1700	56	45
9	$2,9$	$1,3,4,5,6,7,8,10,11,12,13$	1674	57	45
9	$2,10$	$1,3,4,5,6,7,8,9,11,12,13$	1624	57	45
9	$2,11$	$1,3,4,5,6,7,8,9,10,12,13$	1550	60	45
9	$2,12$	$1,3,4,5,6,7,8,9,10,11,13$	1452	62	45
9	$3,9$	$1,2,4,5,6,7,8,10,11,12,13$	1778	55	45
9	$3,10$	$1,2,4,5,6,7,8,9,11,12,13$	1726	56	45
9	$3,11$	$1,2,4,5,6,7,8,9,10,12,13$	1650	58	45
9	$4,10$	$1,2,3,5,6,7,8,9,11,12,13$	1804	54	**44**
10	$4,10$	$1,2,3,5,6,7,8,9,11,12,13,14$	2507	75	55

5. Isomorphic lattices

The proof of Theorem 3.2 is based on the fact that metric properties of minimal vectors in a suitable lattice $L_d(h_1,h_2,\dots)$ determine the sequence $h_1,h_2,\dots$ up to the essential isomorphism.

Our main tool for proving this assertion is a graph-theoretical result of independent interest described in the next Section. It gives lower bounds on the amount of "tampering" which does not destroy large equivalence classes of an equivalence relation on a finite set.

5.1. α-quasi-equivalence classes. Consider an equivalence relation on some finite set which has been slightly "tampered with" and transformed into a symmetric and reflexive relation which is in general no longer transitive. This section describes sufficient (but not necessarily optimal) conditions on the amount of tampering which allow the recovery of suitable equivalence classes.

Symmetric and reflexive relations on a set V are in one-to-one correspondence with simple graphs with V as their set of vertices. (Recall that a simple graph has only undirected edges without multiplicities joining distinct vertices.) Given a symmetric and reflexive relation R, two distinct elements u, v of V are adjacent (joined by an undirected edge) if and only if u and v are related by R. Equivalence relations correspond to disjoint unions of complete graphs. We use this graph-theoretical framework until the end of this Section.

We denote by $\mathcal{N}_\Gamma(v)$ the set of neighbours (adjacent vertices) of a vertex v in a simple graph Γ and we denote by $A\Delta B = (A \cup B) \setminus (A \cap B)$ the symmetric difference of two sets A, B.

Given a real positive number α in $[0, 1/3)$, a subset $\mathcal{C}$ of vertices of a finite simple graph Γ is an *α-quasi-equivalence class of* Γ if

$$|(\mathcal{N}_\Gamma(v) \cup \{v\})\Delta\mathcal{C}| \leq \alpha|\mathcal{C}|$$

for every vertex v of $\mathcal{C}$ and

$$|\mathcal{N}_\Gamma(v) \cap \mathcal{C}| < (1 - 3\alpha)|\mathcal{C}|$$

for every vertex v which is not in $\mathcal{C}$.

Example 5.1. Let $\mathcal{C}$ be a set of at least 29 vertices of a simple graph Γ. Suppose that $|(\mathcal{N}_\Gamma(v) \cup \{v\})\Delta\mathcal{C}| \leq 8$ for $v \in \mathcal{C}$ and $|\mathcal{N}_\Gamma(v) \cap \mathcal{C}| \leq 4$ for $v \notin \mathcal{C}$. Then $\mathcal{C}$ is a $\frac{2}{7}$-quasi-equivalence class.

0-quasi-equivalence classes in Γ are (vertex-sets of) maximal complete subgraphs (also called maximal cliques) of Γ. Given a small strictly positive real ϵ, a large ϵ-quasi-equivalence class $\mathcal{C}$ induces almost a maximal complete subgraph: only very few edges (a proportion of at most ϵ) between a fixed vertex $v \in \mathcal{C}$ and the remaining vertices of $\mathcal{C}$ can be missing and v can only be adjacent to at most $\lfloor \epsilon|\mathcal{C}| \rfloor$ vertices outside $\mathcal{C}$. Notice however that a vertex $w \notin \mathcal{C}$ can be adjacent to a very large proportion (strictly smaller than $(1 - 3\epsilon)$) of vertices in $\mathcal{C}$.

On the other hand, given $\alpha = \frac{1}{3} - \epsilon$ (for small $\epsilon > 0$), an α-quasi-equivalence class $\mathcal{C}$ can have many missing edges between elements of $\mathcal{C}$ and it can have a rather large amount of edges joining an element of $\mathcal{C}$ with elements in the complement of $\mathcal{C}$. An element $w \notin \mathcal{C}$ is however connected only to a very small proportion (of at most 3ϵ) of vertices in $\mathcal{C}$. Such a class corresponds to a secret organisation whose existence is difficult to discover for non-members. It displays also some aspects of a connected component.

Proposition 5.2. *Distinct α-quasi-equivalence classes of a finite graph are disjoint.*

Proof. A common vertex v of two intersecting α-quasi-equivalence classes $\mathcal{C}_1$ and $\mathcal{C}_2$ with $|\mathcal{C}_1| \geq |\mathcal{C}_2|$ gives rise to the inequalities

$$\begin{aligned}\alpha|\mathcal{C}_1| &\geq |(\mathcal{N}_\Gamma(v) \cup \{v\})\Delta\mathcal{C}_1| \\ &\geq |(\mathcal{C}_1 \setminus \mathcal{C}_2) \setminus \mathcal{N}_\Gamma(v)|\end{aligned}$$

and

$$\begin{aligned}\alpha|\mathcal{C}_1| &\geq \alpha|\mathcal{C}_2| \\ &\geq |(\mathcal{N}_\Gamma(v) \cup \{v\})\Delta\mathcal{C}_2| \\ &\geq |(\mathcal{C}_1 \setminus \mathcal{C}_2) \cap \mathcal{N}_\Gamma(v)|.\end{aligned}$$

Thus we get

$$2\alpha|\mathcal{C}_1| \geq |\mathcal{C}_1 \setminus \mathcal{C}_2|.$$

The trivial equality

$$|C_1 \cap C_2| + |C_1 \setminus C_2| = |C_1|$$

implies now

$$|\mathcal{C}_1 \cap \mathcal{C}_2| \geq |C_1| - 2\alpha|C_1| = (1 - 2\alpha)|\mathcal{C}_1|. \tag{5.1}$$

Assuming $\mathcal{C}_1 \neq \mathcal{C}_2$ (and $|\mathcal{C}_1| \geq |\mathcal{C}_2|$) we can choose an element $w \in \mathcal{C}_1 \setminus \mathcal{C}_2$. We have

$$\begin{aligned}\alpha|\mathcal{C}_1| &\geq |(\mathcal{N}_\Gamma(w) \cup \{w\})\Delta\mathcal{C}_1| \\ &\geq |(\mathcal{C}_1 \cap \mathcal{C}_2) \setminus \mathcal{N}_\Gamma(w)|\end{aligned}$$

and

$$\begin{aligned}(1-3\alpha)|\mathcal{C}_1| &\geq (1-3\alpha)|\mathcal{C}_2| \\ &> |\mathcal{N}_\Gamma(w) \cap \mathcal{C}_2| \\ &\geq |(\mathcal{C}_1 \cap \mathcal{C}_2) \cap \mathcal{N}_\Gamma(w)|\end{aligned}$$

which imply

$$(1-2\alpha)|\mathcal{C}_1| > |\mathcal{C}_1 \cap \mathcal{C}_2|$$

in contradiction with inequality (5.1). $\square$

5.2. Error-graphs. The *error-graph* $\mathcal{E}(\cup_i \mathcal{C}_i \subset \Gamma)$ of a simple graph Γ with a vertex partition $V = \cup_i \mathcal{C}_i$ into disjoint subsets $\mathcal{C}_i$ is defined as follows: $V = \cup_i \mathcal{C}_i$ is also the vertex set of $\mathcal{E} = \mathcal{E}(\cup_i \mathcal{C}_i \subset \Gamma)$. Two vertices $u_{\mathcal{E}}, v_{\mathcal{E}}$ of $\mathcal{E}$ are adjacent in $\mathcal{E}$ if there exists either an index i such that the corresponding vertices u_Γ and v_Γ are non-adjacent (in Γ) and belong to a common subset $\mathcal{C}_i$, or u_Γ and v_Γ are adjacent in Γ and belong to two different subsets $\mathcal{C}_i, \mathcal{C}_j$.

Otherwise stated, $\mathcal{E}$ is obtained from Γ by exchanging adjacency and non-adjacency in every induced subgraph with vertices $\mathcal{C}_i$.

Edges of the error-graph $\mathcal{E}(\cup_i \mathcal{C}_i \subset \Gamma)$ are thus "errors" of the symmetric relation on V encoded by (edges of) Γ with respect to the equivalence relation with equivalence classes $\mathcal{C}_i$.

5.3. Neighbourhoods. We consider the set $\Lambda_{\min}$ of all minimal vectors in a fixed lattice $\Lambda = L_d(h_1, h_2, \dots)$ with minimum 4. Two minimal vectors $v, w \in \Lambda_{\min}$ are *neighbours* if $\langle v, w\rangle = 2$. The set $\mathcal{N}(v)$ of neighbours of a given element $v \in \Lambda_{\min}$ can be partitioned into six disjoint subsets

$$\mathcal{N}(v) = \mathcal{F}_{**00}(v) \cup \mathcal{F}_{*0*0}(v) \cup \mathcal{F}_{*00*}(v) \cup \mathcal{F}_{0**0}(v) \cup \mathcal{F}_{0*0*}(v) \cup \mathcal{F}_{00**}(v)$$

with stars, respectively zeros, indicating coordinates ± 1, respectively 0, in the support $\{i, i+k, j-k, j\}$, $i < i+k < j-k < j$ of $v = b_i - b_{i+k} - b_{j-k} + b_k$ where the elements b_i are the standard basis vectors indexed by elements of the set $\{1, 2, \dots\} \setminus \{h_1, h_2, \dots\}$.

The involution $\iota : w \longrightarrow v - w$ induces a one-to-one correspondence between the two subsets of the three pairs

$$\{\mathcal{F}_{**00}(v), \mathcal{F}_{00**}(v)\}, \{\mathcal{F}_{*0*0}(v), \mathcal{F}_{0*0*}(v)\}, \{\mathcal{F}_{*00*}(v), \mathcal{F}_{0**0}(v)\} \ .$$

We call such pairs *complementary* and denote them using the hopefully self-explanatory notations

$$\begin{aligned}
\mathcal{F}_{aabb}(v) &= \mathcal{F}_{**00}(v) \cup \mathcal{F}_{00**}(v),\\
\mathcal{F}_{abab}(v) &= \mathcal{F}_{*0*0}(v) \cup \mathcal{F}_{0*0*}(v),\\
\mathcal{F}_{abba}(v) &= \mathcal{F}_{*00*}(v) \cup \mathcal{F}_{0**0}(v).
\end{aligned}$$

We write $\overline{\mathcal{F}_{aabb}(v)}, \overline{\mathcal{F}_{abab}(v)}, \overline{\mathcal{F}_{abba}(v)}$ for the orbits under ι of the three sets $\mathcal{F}_{aabb}(v), \mathcal{F}_{abab}(v), \mathcal{F}_{abba}(v)$. Either $\mathcal{F}_{**00}(v)$ or $\mathcal{F}_{00**}$ represent all elements of $\overline{\mathcal{F}_{aabb}(v)}$. The same statement holds for $\overline{\mathcal{F}_{abab}(v)}, \overline{\mathcal{F}_{abba}(v)}$. Henceforth, we identify often a vector $w \in \mathcal{N}(v)$ with its class in $\overline{\mathcal{N}(v)}$.

The following table lists all 6 elements of the set $\mathcal{N}(0, 1, 0, -1, -1, 0, 1)$ in $L_5(\emptyset) = (\mathbb{Z}(1, 1, 1, 1, 1, 1, 1) + \mathbb{Z}(1, 2, 3, 4, 5, 6, 7))^\perp \subset \mathbb{Z}^7$ together with the sets $\mathcal{F}_{**00}, \mathcal{F}_{*0*0}, \mathcal{F}_{*00*}, \mathcal{F}_{0**0}, \mathcal{F}_{0*0*}, \mathcal{F}_{00**}$ and $\mathcal{F}_{aabb}, \mathcal{F}_{abab}, \mathcal{F}_{abba}$ (with dropped common argument $v = (0, 1, 0, -1, -1, 0, 1) \in \Lambda_{\min}$) containing them:

0	1	0	−1	−1	0	1		
−1	1	1	−1	0	0	0	$\in \mathcal{F}_{**00}$	$\subset \mathcal{F}_{aabb}$
1	0	−1	0	−1	0	1	$\in \mathcal{F}_{00**}$	$\subset \mathcal{F}_{aabb}$
0	1	−1	0	−1	1	0	$\in \mathcal{F}_{*0*0}$	$\subset \mathcal{F}_{abab}$
0	0	1	−1	0	−1	1	$\in \mathcal{F}_{0*0*}$	$\subset \mathcal{F}_{abab}$
0	1	−1	0	0	−1	1	$\in \mathcal{F}_{*00*}$	$\subset \mathcal{F}_{abba}$
0	0	1	−1	−1	1	0	$\in \mathcal{F}_{0**0}$	$\subset \mathcal{F}_{abba}$

Since $\langle u, v-w\rangle = 2 - \langle u,w\rangle$, the parity of the scalar product is well-defined on $\overline{\mathcal{N}(v)}$. We get thus a map $\overline{\mathcal{N}(v)} \times \overline{\mathcal{N}(v)} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ where $\overline{\mathcal{N}(v)} = \overline{\mathcal{F}_{aabb}(v)} \cup \overline{\mathcal{F}_{abab}(v)} \cup \overline{\mathcal{F}_{abba}(v)}$. We say that two classes represented by u, w have a generic scalar product if $\langle u, v\rangle \equiv 0 \pmod 2$ if and only if u, v belong both to the same subset $\overline{\mathcal{F}_{aabb}(v)}, \overline{\mathcal{F}_{abab}(v)}, \overline{\mathcal{F}_{abba}(v)}$ of $\overline{\mathcal{N}(v)}$. Generic values are given by the table

	$\overline{\mathcal{F}_{aabb}}$	$\overline{\mathcal{F}_{abab}}$	$\overline{\mathcal{F}_{abba}}$
$\overline{\mathcal{F}_{aabb}}$	**0**	**1**	**1**
$\overline{\mathcal{F}_{abab}}$	**1**	**0**	**1**
$\overline{\mathcal{F}_{abba}}$	**1**	**1**	**0**

Non-generic values, often called errors in the sequel, occur at most 8 times with a given first element in $\overline{\mathcal{N}(v)}$. More precisely, the table

	$\overline{\mathcal{F}_{aabb}}$	$\overline{\mathcal{F}_{abab}}$	$\overline{\mathcal{F}_{abba}}$
$\overline{\mathcal{F}_{aabb}}$	≤ 2	≤ 4	≤ 2
$\overline{\mathcal{F}_{abab}}$	≤ 4	≤ 2	≤ 2
$\overline{\mathcal{F}_{abba}}$	≤ 2	≤ 2	0

displays upper bounds for the number of errors (of the map $\overline{\mathcal{N}(v)} \times \overline{\mathcal{N}(v)} \longrightarrow \mathbb{Z}/2\mathbb{Z}$) occurring with a fixed element of $\overline{\mathcal{N}(v)}$.

We illustrate the first line by considering $v = (0,0,0,0,1,-1,1,-1,0,0,\dots,0) \in L_{d\geq 12}(h_1 \geq 15,\dots)$. Vectors $e_1,\dots,e_8$ representing all errors within $\overline{\mathcal{F}(v)}$ occurring with the class of $w = (0,0,0,0,1,-1,0,0,0,0,-1,1,$

$0,\dots,0)$ in $\overline{\mathcal{F}_{aabb}(v)}$ are given by

v	0	0	0	0	+	−	−	+	0	0	0	0	0	0	$\langle w, e_j\rangle$
$w = e_0$					+	−					−	+			4
e_1					+	−				−	+				1
e_2					+	−						−	+		1
e_3					+		−		−		+				0
e_4					+		−			−		+			2
e_5					+		−				−		+		2
e_6					+		−					−		+	0
e_7		−			+			+			−				2
e_8	−				+			+				−			0

where we keep only the signs of non-zero coefficients. The elements e_1, e_2 realise the maximal number of two errors occurring within $\overline{\mathcal{F}_{aabb}(v)}$, the vectors $e_3, \dots, e_6$ realise the maximal number of 4 errors occurring between the class of w in $\overline{\mathcal{F}_{aabb}(v)}$ and $\overline{\mathcal{F}_{abab}(v)}$ and e_7, e_8 realise the maximal number of 2 errors between the class of w and $\overline{\mathcal{F}_{abba}(v)}$.

We consider $\overline{\mathcal{N}(v)}$ as the vertex-set of a graph with edges given by pairs of different vertices with an even scalar product among representatives.

Proposition 5.3. *If at least two of the three classes $\overline{\mathcal{F}_{aabb}(v)}, \overline{\mathcal{F}_{abab}(v)}$, $\overline{\mathcal{F}_{abba}(v)}$ contain at least 29 elements then all three classes are uniquely defined in terms of the graph-structure on $\overline{\mathcal{N}(v)}$.*

Proof. Example 5.1 and the above bounds for the maximal number of errors (non-generic values of scalar products) show that two such classes with at least 29 elements define $\frac{2}{7}$-quasi-equivalence relations in $\overline{\mathcal{N}(v)}$. They are thus well-defined by Proposition 5.2. The third class is given by the remaining elements. □

5.4. The path $\mathcal{P}$ associated to $(1, -1, -1, 1, 0, \dots, 0)$.

Proposition 5.4. *Let v denote the minimal vector $b_1 - b_2 - b_3 + b_4 = (1, -1, -1, 1, 0, 0, \dots, 0)$ of a fixed lattice $L_d(h_1, \dots, h_k = d+1+k)$ satisfying the conditions of Theorem 3.2.*

The set $h_1, \dots, h_k$ of holes is uniquely determined by the graph-structure on the set of equivalence classes $\overline{\mathcal{N}(v)}$ of neighbours of v.

Proof. We keep the notations of Section 5.3 except for dropping the argument $v = (1, -1, -1, 1, 0, \dots)$ for subsets of the sets $\mathcal{N} = \mathcal{N}(v)$ or $\overline{\mathcal{N}} = \overline{\mathcal{N}(v)}$.

The set $\overline{\mathcal{F}_{abba}}$ is empty and the conditions $h_1 \geq 7, h_k = d + k + 1$ and $h_{i+1} - h_i \geq 4$ imply that $\mathcal{A} = \overline{F_{aabb}}$ and $\mathcal{B} = \overline{\mathcal{F}_{abab}}$ have both exactly $d-3-k$ elements. Indeed, among the $k+d-3$ vectors $b_1 - b_2 - b_{i-1} + b_i$, $i = 6, \dots, k+d+2$, exactly $2k$ vectors of the form $b_1 - b_2 - b_{h_i-1} + b_{h_i}, b_1 - b_2 - b_{h_i} + b_{h_i+1}$

are not orthogonal to all elements $b_{h_1},\dots,b_{h_k}$. A similar argument works for $\mathcal{B}$.

We construct an oriented path $\mathcal{P}$ as follows: We start with the error-graph $\mathcal{E}=\mathcal{E}(\mathcal{A}\cup\mathcal{B}\subset\overline{\mathcal{N}})$ of $\overline{\mathcal{N}}$. We denote by $\mathcal{E}'$ the subgraph of $\mathcal{E}$ obtained by removing all vertices of $\mathcal{B}=\overline{\mathcal{F}_{abab}}$ which are involved in a triangle with the two remaining vertices in $\mathcal{A}=\overline{F_{aabb}}$. Such triangles are given by three vectors $b_1-b_3,b_{l-2}+b_l\in\mathcal{B},\ b_1-b_2-b_{l-2}+b_{l-1},\ b_1-b_2-b_{l-1}+b_l\in\mathcal{A}$ with $\{l-2,l-1,l\}\subset\{5,\dots,d+k\}$ not intersecting $\{h_1,\dots,h_k\}$.) We denote by $\mathcal{H}$ the remaining set of k vertices of $\mathcal{B}$. Elements of $\mathcal{H}$ are of the form $b_1-b_3-b_{h_i-1}+b_{h_i+1}$ and correspond to the k holes $h_1,h_2,\dots$. The graph $\mathcal{E}'$ is a path-graph (or segment, or Dynkin graph of type A) having one leaf (vertex of degree one) in $\mathcal{A}$ and one leaf in $\mathcal{H}\subset\mathcal{B}$. We orient its edges in order to get an oriented path

$$\begin{gathered}
b_1-b_2-b_5+b_6,b_1-b_2-b_6+b_7,\dots,b_1-b_2-b_{h_1-2}+b_{h_1-1},\\
b_1-b_3-b_{h_1-1}+b_{h_1+1},\\
b_1-b_2-b_{h_1+1}+b_{h_1+2},\dots,b_1-b_2-b_{h_2-2}+b_{h_2-1},\\
\vdots\\
b_1-b_3-b_{h_{k-1}-1}+b_{h_{k-1}+1},\\
b_1-b_2-b_{h_{k-1}+1}+b_{h_{k-1}+2},\dots,b_1-b_2-b_{d+k-1}+b_{d+k},\\
b_1-b_3-b_{d+k}+b_{d+k+2}
\end{gathered}$$

starting at the vertex $b_1-b_2-b_5+b_6$ of $\mathcal{A}$ and ending at the vertex $b_1-b_3-b_{d+k}+b_{d+k+2}$ of $\mathcal{H}\subset\mathcal{B}$.

The final path $\mathcal{P}$ is obtained from $\mathcal{E}'$ as follows: We add k additional vertices to the vertex-set $\mathcal{A}$ by considering the midpoints of the k oriented edges $(b_1-b_2-b_{h_i-2}+b_{h_i-1},b_1-b_3-b_{h_i-1}+b_{h_i+1})$ which start in $\mathcal{A}$ and end in $\mathcal{H}$. These oriented edges are well-defined since two distinct vertices $b_1-b_3-b_{h_i-1}+b_{h_i+1},b_1-b_3-b_{h_j-1}+b_{h_j+1}$ of $\mathcal{H}$ are never adjacent in $\mathcal{E}'$ (they are always separated by at least two vertices $b_1-b_2-b_{h_i+1}+b_{h_i+2},b_1-b_2-b_{h_i+2}+b_{h_i+3}\in\mathcal{A}$ if $i<j$) and since the initial vertex of $\mathcal{E}'$ does not belong to $\mathcal{H}$. We label now the $d-3+k=(d-k-3+k)+k$ vertices of $\mathcal{P}$ increasingly by $5,6,\dots,d+k+1$. The set of labels of the k vertices in $\mathcal{H}$ defines now the sequence $h_1,\dots,h_k=d+k+1$. $\square$

5.5. Admissible vectors.

Lemma 5.5. *Both sets $\mathcal{A}=\overline{\mathcal{F}_{aabb}(v)}$ and $\mathcal{B}=\overline{\mathcal{F}_{abab}(v)}$ have errors for $v=b_1-b_2-b_3+b_4=(1,-1,-1,1,0,\dots)$ in a lattice $L_d(h_1,\dots,h_k=d+1+k)$ satisfying the conditions of Theorem 3.2.*

Recall that an error in $\mathcal{A}$ (respectively $\mathcal{B}$) is given by two minimal vectors s,t in $\mathcal{A}$ (respectively in $\mathcal{B}$) having an odd scalar product $\langle s,t\rangle\equiv 1\pmod 2$.

Proof of Lemma 5.5. An error in $\mathcal{A}$ is realised by $b_1-b_2-b_{d+k-2}+b_{d+k-1}$, $b_1-b_2-b_{d+k-1}+b_{d+k}$.

An error in $\mathcal{B}$ is realised by $b_1 - b_3 - b_{d+k-2} + b_{d+k}, b_1 - b_3 - b_{d+k} + b_{d+k+2}$. □

Lemma 5.6. *We have the inequality $k \leq \lfloor (d-2)/3 \rfloor$ for the number k of holes in a d-dimensional lattice $L_d(h_1, \ldots, h_k = d+k+1)$ satisfying the conditions of Theorem 3.2.*

Equality is achieved for $h_i = 3 + 4i$ for $i = 1, 2, \ldots$.

We leave the easy proof (based on the pigeon-hole principle) to the reader.

A minimal vector u of a lattice $L_d(h_1, \ldots, h_k = d+k+1)$ satisfying the conditions of Theorem 3.2 is called *admissible* if the following conditions hold:

(1) $\overline{N(u)}$ has an even number of at least $2\lceil (2d-7)/3 \rceil$ elements defining two $\frac{2}{7}$-quasi-equivalence classes $\mathcal{A}, \mathcal{B}$ of equal size.
(2) Both classes $\mathcal{A}, \mathcal{B}$ contain errors (i.e. do not define complete subgraphs of $\overline{\mathcal{N}(u)}$).
(3) The error graph $\mathcal{E}(\mathcal{A} \cup \mathcal{B} \subset \overline{\mathcal{N}(u)})$ contains a triangle with two vertices in $\mathcal{A}$ and one vertex in $\mathcal{B}$.
(4) The construction of the oriented path $\mathcal{P}$ explained in the proof of Proposition 5.4 works and yields an oriented path with $2d - 6 - |\overline{\mathcal{N}(u)}|/2$ vertices.

The proof of Theorem 3.2 follows now from the following result:

Proposition 5.7. *A lattice $L_d(h_1, \ldots, h_k = d+k+1)$ satisfying the conditions of Theorem 3.2 contains a unique pair $\pm v$ of admissible minimal vectors.*

Proof. We show first that $v = v_c = b_1 - b_2 - b_3 + b_4 = (1, -1, -1, 1, 0, 0, \ldots, 0)$ is admissible. The sets $\mathcal{A} = \overline{\mathcal{F}_{aabb}(v)}$ and $\mathcal{B} = \overline{\mathcal{F}_{abab}(v)}$ contain both $d-3-k$ elements. Lemma 5.6 yields $d - 3 - k \geq \lceil (2d-7)/3 \rceil \geq 29$ for $d \geq 46$. The sets $\mathcal{A}$ and $\mathcal{B}$ are thus $\frac{2}{7}$-quasi equivalence classes by Example 5.1. (See also the proof of Proposition 5.3.)

Lemma 5.5 shows that both classes $\mathcal{A}$ and $\mathcal{B}$ contain errors.

A triangle in the error graph $\mathcal{E}(\mathcal{A} \cup \mathcal{B} \subset \overline{\mathcal{N}(v)})$ with two vertices in $\mathcal{A}$ and a last vertex in $\mathcal{B}$ is defined by $b_1 - b_2 - b_{d+k-2} + b_{d+k-1}, b_1 - b_2 - b_{d+k-1} + b_{d+k}, b_1 - b_3 - b_{d+k-2} + b_{d+k}$.

The last condition is fulfilled as shown by the proof of Proposition 5.4 and yields the path $\mathcal{P}$ with $d + k - 3 = 2d - 6 - (d - k - 3)$ vertices.

We consider now an admissible minimal vector u of $L_d(h_1, \ldots, h_k = d+k+1)$. Condition (1) for admissibility shows that $\overline{\mathcal{N}(u)}$ consists of two $\frac{2}{7}$-quasi-equivalence classes $\mathcal{A}, \mathcal{B}$ containing both at least $\lceil (2d-7)/3 \rceil \geq \lceil (2 \cdot 46 - 7)/3 \rceil = 29$ elements. By Proposition 5.3, they define thus two of the three classes $\overline{\mathcal{F}_{aabb}}, \overline{\mathcal{F}_{abab}}, \overline{\mathcal{F}_{abba}}$.

Since $\overline{\mathcal{F}_{abba}}$ does not contain internal errors, these classes are $\overline{\mathcal{F}_{aabb}}$ and $\overline{\mathcal{F}_{abab}}$.

Condition (3) for admissibility implies that $\mathcal{A} = \overline{\mathcal{F}_{aabb}}$, $\mathcal{B} = \overline{\mathcal{F}_{abab}}$ and that u is of the form $b_a - b_{a+p} - b_{a+2p} + b_{a+3p}$ for some strictly positive integers a and p. We can exclude $p \geq 3$: Indeed, both sets $\{a+p-2, a+p-1, a+p+1\}$ and $\{a+2p-1, a+2p+1, a+2p+2\}$ intersect $\{h_1, \dots, h_k = d+k+1\}$ in at most a single element. For $p \geq 3$, there exists thus $\epsilon \in \{2, 1, -1\}$ such that the lattice $L_d(h_1, \dots)$ contains the minimal vector $b_a - b_{a+p-\epsilon} - b_{a+2p+\epsilon} + b_{a+3p}$ in contradiction with $\overline{\mathcal{F}_{abba}} = \emptyset$. In order to exclude $p = 2$ we use condition (4): Suppose indeed that $p = 2$. If there exists two elements $w_i = b_a - b_{a+2} - b_{a_i-2} + b_{a_i}, i = 1, 2$ in $\overline{\mathcal{F}_{aabb}}$ with $a_1 \not\equiv a_2 \pmod 2$, then the construction of the oriented path of Proposition 5.4 fails since w_1 and w_2 define vertices in two different connected components of the intermediary graph $\mathcal{E}'$. Two such elements w_1, w_2 do not necessarily exist if the sequence of holes consist of at most two arithmetic progressions of step 4. In this case, there exist two elements $t_i = b_a - b_{a+4} - b_{c_i-4} + b_{c_i}, i = 1, 2$ in $\overline{\mathcal{F}_{abab}}$ with $c_1 \not\equiv c_2 \pmod 4$, also leading to a disconnected graph $\mathcal{E}'$. (The case $p = 2$ can also be excluded by showing that it cannot lead to a path $\mathcal{P}$ having the correct length.)

We have thus $p = 1$ and u is of the form $u = b_a - b_{a+1} - b_{a+2} + b_{a+3}$. The intermediary graph $\mathcal{E}'$ constructed in the proof of Proposition 5.4 is connected if an only if $a \in \{1, 2, d+k-3\}$. The case $a = 2$ and $h_1 = 7$ leads to a graph $\mathcal{E}'$ with both leaves in $\mathcal{H} \subset \overline{\mathcal{F}_{abab}}$ and is thus excluded. The case $a = 2$ and $h > 7$ leads to a sequence of holes defining the lattice $L_{d-1}(h_1 - 1, h_2 - 1, \dots, h_k + d)$ which is only $(d-1)$-dimensional. The case $a = d + k - 3$ leads to a graph $\mathcal{E}'$ having both leaves in $\mathcal{A}$ except if $h_{k-1} = d + k - 5$. This last case leads also to a sequence defining a lattice of dimension $d - 1$ as follows: The corresponding sets $\overline{\mathcal{F}_{aabb}}$ and $\overline{\mathcal{F}_{abab}}$ have both $d-3-k$ elements. They lead to a path of length $d-3-k+2(k-1) = (d-1)+(k-1)-3$ defining a set $h_1, \dots, h_{k-1}$ of $k-1$ holes for a lattice of dimension $d - 1$. □

5.6. Proof of Theorem 3.2.

Proof of Theorem 3.2. We construct the set of minimal vectors (of a lattice as in Theorem 3.2) and use it for determining all minimal admissible vectors. They form a unique pair by Proposition 5.7. This pair corresponds thus to the vectors $\pm(1, -1, -1, 1, 0, \dots)$ which determine the set $h_1, \dots$ of holes uniquely by Proposition 5.4. □

6. A linear recursion

We denote by $\alpha(n)$ the number of strictly increasing integer-sequences of length n which start with 1 and which have no missing integers at distance strictly smaller than 6. A sequence $s_1 = 1, s_2, \ldots, s_n$ contributes thus 1 to α_n if and only if $s_{i+1} - s_i \in \{1, 2\}$ for $i = 1, \ldots, n-1$ and $s_{i+1} - s_i = 2, s_{j+1} - s_j = 2$ for $1 \leq i < j \leq n-1$ implies $s_j - s_i \geq 5$.

Proposition 6.1. *We have $\alpha(i) = i$ for $i = 1, \ldots, 6$. For $n \geq 6$ we have the recursion*

$$\alpha(n) = \alpha(n-1) + \alpha(n-5).$$

In particular, the sequence $\alpha(1), \alpha(2), \ldots$ grows exponentially fast.

The first few terms of $\alpha(1), \ldots$ are given by

$$1, 2, 3, 4, 5, 6, 8, 11, 15, 20, 26, \ldots.$$

Proof of Proposition 6.1. The sequence $1, 2, 3, 4, \ldots, n$ is the unique contribution to $\alpha(n)$ without holes (missing integers in the set $\{1 = s_1, s_2, \ldots, s_n\}$) and there are $n-1$ sequences (contributing 1 to $\alpha(n)$) with exactly 1 hole. Since contributions to $\alpha(n)$ for $n \leq 6$ have at most one hole, we get the formula $\alpha(i) = i$ for $i = 1, \ldots, 6$.

For $n \geq 6$, a contribution of 1 to $\alpha(n)$ is either given by $1, \ldots, s_{n-1} = \omega - 1, s_n = \omega$ for some integer $\omega \geq n$ and such contributions are in one-to-one correspondence with contributions to $\alpha(n-1)$ (erase the last term) or it is of the form

$$1, \ldots, s_{n-5} = \omega - 6, \omega - 5, \omega - 4, s_{n-2} = \omega - 3, s_{n-1} = \omega - 2, s_n = \omega$$

with $s_1 = 1, \ldots, s_{n-5} = \omega - 6$ an arbitrary contribution to $\alpha(n-5)$.

The sequence $\alpha(1), \ldots$ satisfies the linear recursion $\alpha(n) = \alpha(n-1) + \alpha(n-5)$ with characteristic polynomial $P = z^5 - z^4 - 1 = (z^2 - z + 1)(z^3 - z - 1)$. Straightforward (but tedious) computations show

$$\alpha(n) = \frac{1}{7}\pi(n) + \frac{1}{161} \sum_{\rho, \rho^3 - \rho = 1} (45\rho^n + 61\rho^{n+1} + 36\rho^{n+2})$$

where $\pi(n) = (-1 - 2/\sqrt{-3})e^{in\pi/3} + (-1 + 2/\sqrt{-3})e^{-in\pi/3}$ is 6-periodic given by

$n =$	$6k$	$6k+1$	$6k+2$	$6k+3$	$6k+4$	$6k+5$
$\pi(n) =$	-2	-3	-1	2	3	1

and where the sum is over the three roots of the polynomial $z^3 - z - 1$. The sequence α_n satisfies thus $\lim_{n\to\infty} \frac{\alpha(n)}{\rho^n} = \frac{45 + 61\rho + 36\rho^2}{161}$ where $\rho \sim 1.324718$ is the unique real root of $z^3 - z - 1$. $\square$

Remark 6.2.

(1) Setting $\tilde{\alpha}(i) = 1 + (i-1)t$ for $i = 1, \ldots, 6$ and $\tilde{\alpha}(n) = \tilde{\alpha}(n-1) + t\tilde{\alpha}(n-5), n \geq 6$ we get a sequence $\tilde{\alpha}(1), \tilde{\alpha}(2), \ldots$ specialising to $\alpha(n)$ at $t = 1$ with coefficients counting contributions to $\alpha(n)$ according to their number of holes (i.e. the coefficient of t^k in

$\tilde{\alpha}(n)$ is the number of sequences contributing to $\alpha(n)$ which are of the form $s_1 = 1, s_2, \dots, s_n = n + k$).

(2) A trivial exponential lower bound on $\alpha(n)$ can be obtained as follows: Remove either the 6-th or the 7-th element in every set $\{7l + 1, 7l + 2, \dots, 7l + 6, 7l + 7\}$. This shows $\alpha(n) \geq 2^{\lfloor n/6 \rfloor}$.

7. An exponentially large family of perfect lattices

Proposition 7.1. *The number of d-dimensional lattices satisfying the conditions of Theorem 3.1 and of Theorem 3.2 equals $\alpha(d-8)$ for $d \geq 46$ where $\alpha(1), \dots$ is the sequence of Section 6 recursively defined by $\alpha(n) = n$ for $n = 1, \dots, 5$ and by $\alpha(n) = \alpha(n-1) + \alpha(n-5)$ for $n \geq 6$.*

Proof. Such a lattice $L_d(h_1, \dots)$ corresponds to a vector

$$h = (1, 2, 3, 4, 5, s_1{+}5, s_2{+}5, \dots, \omega{-}6 = s_{d-8}{+}5, \omega{-}5, \omega{-}4, \omega{-}3, \omega{-}2, \omega)$$

where $s_1 = 1, s_2, \dots, s_{d-8} = \omega - 11$ is a sequence contributing to $\alpha(d-8)$. This construction is one-to-one for $d \geq 46$: every sequence $s_1, \dots, s_{d-8}$ contributing to $\alpha(d-8)$ corresponds to a lattice satisfying the conditions of Theorem 3.1 and of Theorem 3.2. $\square$

8. Maximal indices for pairs of well-rounded lattices with the same minimum

A lattice Λ is *well-rounded* if its minimal vectors span the ambient vector space $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

We denote by I_d the smallest integer such that every well-rounded d-dimensional sublattice Λ' with minimal vectors contained in the set of minimal vectors of a well-rounded d-dimensional lattice Λ has index at most I_d in Λ. The integers I_d (and refinements describing all possible group-structures of the quotient group Λ/Λ' in small dimensions) have been studied by several authors, see for example [7].

The maximal index I_d is of course realised by a sublattice generated by d suitable minimal vectors of a suitable well-rounded d-dimensional lattice.

Since perfect lattices are well-rounded, the definition of I_d implies immediately:

Proposition 8.1. *A sublattice $\sum_{i=1}^{d} \mathbb{Z} v_d$ generated by d linearly independent minimal elements of a perfect d-dimensional lattice Λ is of index at most I_d in Λ.*

We give now upper bounds for I_d.

8.1. An upper bound for I_d from Minkowski's inequality.

Proposition 8.2. *We have*

$$I_d \leq \left\lfloor \left(\frac{4}{\pi}\right)^{d/2} \left(\frac{d}{2}\right)! \right\rfloor . \tag{8.1}$$

The proof of Proposition 8.2 uses Minkowski's inequality (see [10, Chapter 3]) stating that a centrally symmetric convex subset $\mathcal{C} = -\mathcal{C}$ of $\mathbb{R}^d$ has volume at most 2^d if it contains no non-zero elements of $\mathbb{Z}^d$ in its interior.

Proof of Proposition 8.2. Without loss of generality, we can consider a sublattice $\Lambda' = \sum_{i=1}^d \mathbb{Z}b_i$ generated by d linearly independent minimal vectors $b_1, \ldots, b_d$ of norm 1 of a d-dimensional well-rounded lattice Λ with minimum 1. We have the inequality $V' \leq 1$ for the volume $V' = \mathrm{vol}(\mathbb{R}^d/\Lambda')$ of a fundamental domain for Λ'. (The equality $V' = 1$ holds if and only if $b_1, \ldots, b_d$ is an orthonormal basis. The lattice Λ' is then the standard lattice $\mathbb{Z}^d$.)

Applying Minkowski's inequality $\mathrm{vol}(\mathbb{B}^d) \leq 2^d V$ where $V = \mathrm{vol}(\mathbb{R}^d/\Lambda)$ to the d-dimensional Euclidean unit ball $\mathbb{B}^d$ of volume $\frac{\pi^{d/2}}{(d/2)!}$ (see for example [4, Formula (17) of Chapter 1]) we get the inequality

$$V' \leq 1 = \frac{(d/2)!}{\pi^{d/2}} \mathrm{vol}(\mathbb{B}^d) \leq \frac{(d/2)!}{\pi^{d/2}} 2^d V$$

showing $\frac{V'}{V} \leq \left(\frac{4}{\pi}\right)^{d/2} \left(\frac{d}{2}\right)!$ for the index $V'/V \in \mathbb{N}$ of the lattice Λ' in Λ. $\square$

8.2. Upper bounds for I_d in terms of Hermite's constants. I thank Jacques Martinet for drawing my attention to the ideas of this section which is essentially identical with Section 2 of [7]. The scope of the following lines is to provide the lazy reader with an armchair.

We denote by $\min(\Lambda) = \min_{v \in \Lambda \setminus \{\mathbf{0}\}} \langle v, v \rangle$ the (squared Euclidean) norm of a shortest non-zero element in a d-dimensional Euclidean lattice Λ. The *determinant* $\det(\Lambda)$ of Λ is the determinant of a Gram matrix (with coefficients $\langle b_i, b_j \rangle$ for a basis $b_1, \ldots, b_d$ of $\Lambda = \sum_{i=1}^d \mathbb{Z}b_i$) for Λ. It equals the squared volume $\mathrm{Vol}(\mathbb{R}^d/\Lambda)^2$ of the flat d-dimensional torus $\mathbb{R}^d/\Lambda$. The Hermite invariant of Λ is given by $\gamma(\Lambda) = \frac{\min(\Lambda)}{\det(\Lambda)^{1/d}}$. It is invariant under rescalings and its $d/2$-th power is proportional to the packing-density of Λ. Hermite's constant $\gamma_d = \max_{\Lambda, \dim(\Lambda) = d} \gamma(\Lambda)$ is the Hermite constant of a densest d-dimensional lattice-packing. Equivalently, γ_d is equal to four times the squared maximal injectivity radius of a suitable flat d-dimensional torus with volume 1.

We have now (see [7, Proposition 2.1]):

Proposition 8.3. *We have*

$$I_d \leq \lfloor \gamma_d^{d/2} \rfloor \ .$$

Proof. We consider a sublattice $\Lambda' = \sum_{i=1}^d \mathbb{Z}b_i$ generated by d linearly independent minimal vectors $b_1, \dots, b_d$ of minimum 1 in a d-dimensional well-rounded lattice Λ with minimum 1. We have thus $[\Lambda : \Lambda']^2 \det(\Lambda) = \det(\Lambda')$. Since Λ' is generated by vectors of norm 1 we have $\det(\Lambda') \leq 1$ showing $[\Lambda : \Lambda']^2 \det(\Lambda) \leq 1$. Since Λ has minimum 1 we have $\gamma(\Lambda) = \frac{1}{\det(\Lambda)^{1/d}} \leq \gamma_d$ implying $\det(\Lambda) \geq \gamma_d^{-d}$. We have thus $[\Lambda : \Lambda'] \leq \gamma_d^{d/2}$ and we get the inequality for I_d by considering a suitable pair $\Lambda' \subset \Lambda$ achieving the natural integer I_d. □

The upper bound $\gamma_d \leq (4/3)^{(d-1)/2}$ (see for example [8, p. 36]) gives the very bad upper bound $I_d \leq (4/3)^{d(d-1)/4}$.

Blichfeldt's upper bound

$$\gamma_d \leq \frac{2}{\pi}\Gamma(2 + d/2)^{2/d} \tag{8.2}$$

(see for example [8, p. 42]) is more interesting and leads to

$$I_d \leq \left(\frac{2}{\pi}\right)^{d/2} \Gamma(2 + d/2) \tag{8.3}$$

which is roughly $2^{1+d/2}/d$ times better than (8.1). The proof of (8.1) is however much more elementary than the proof of (8.3).

The first few values of the upper bounds (8.1) and (8.3) are

d	1	2	3	4	5	6	7	8	9	10
(8.1)	1	1	1	3	6	12	27	63	155	401
(8.3)	1	1	1	2	3	6	10	19	37	75
I_d	1	1	1	2	2	4	8	16	16	?

The last row contains the correct values $I_1, \dots, I_9$ which follow from Theorem 1.1 in [7].

9. Enumerating pairs of lattices

We denote by $\sigma_d(N)$ the number of distinct lattices Λ containing $\mathbb{Z}^d$ as a sublattice of index N. Considering dual lattices, we see that the number $\sigma_d(N)$ is also equal to the number of distinct sublattices of index N in $\mathbb{Z}^d$.

Lemma 9.1. *We have*

$$\sigma_d(N) \leq N^d$$

for the number $\sigma_d(N)$ of lattices containing $\mathbb{Z}^d$ as a sublattice of index N.

Proof. Every such lattice Λ contains sublattices $\Lambda_0 = \mathbb{Z}^d \subset \Lambda_1 \subset \cdots \subset \Lambda_k = \Lambda$ with Λ_i/Λ_{i-1} cyclic of prime order p_i such that $p_1 \leq p_2 \leq \cdots \leq p_k$ are all prime-divisors of $N = \prod_{i=1}^k p_i$, with multiplicities taken into account. Since $\mathbb{Z}^d$ is contained with prime index p in exactly $(p^d-1)/(p-1)$ overlattices, we have $\sigma_d(N) \leq \prod_{i=1}^k \frac{p_i^d-1}{p_i-1} \leq \prod_{i=1}^k p_i^d = N^d$. $\square$

Remark 9.2. A nice exact formula for the number $\sigma_d(N)$ of subgroups of index N in $\mathbb{Z}^d$ is given for example in [13] or [6] and equals

$$\sigma_d(N) = \prod_{p|N} \binom{e_p+d-1}{d-1}_p \tag{9.1}$$

where $\prod_{p|N} p^{e_p} = N$ is the factorization of N into prime-powers and where

$$\binom{e_p+d-1}{d-1}_p = \prod_{j=1}^{d-1} \frac{p^{e_p+j}-1}{p^j-1}$$

is the evaluation of the q-binomial

$$\begin{bmatrix} e_p+d-1 \\ d-1 \end{bmatrix}_q = \frac{[e_p+d-1]_q!}{[e_p]_q!\ [d-1]_q!}$$

(with $[k]_q! = \prod_{j=1}^k \frac{q^j-1}{q-1}$) at the prime-divisor p of N.

Lemma 9.1 follows of course also from (9.1) and from the inequality

$$\binom{k+d-1}{d-1}_q \leq q^{kd}$$

which holds by induction on $k \in \mathbb{N}$ for $d \geq 1,\ q \geq 2$ (and which is asymptotically exact for $k=1$ since $\binom{d}{d-1}_2 = \frac{2^d-1}{2-1}$).

Proposition 9.3. *A d-dimensional lattice Λ is contained with index $\leq h$ in at most h^{d+1} different d-dimensional overlattices.*

Proof. Lemma 9.1 implies that the number of such overlattices equals at most $\sum_{n=1}^h \sigma_d(n) \leq \sum_{n=1}^h n^d \leq h \cdot h^d$. $\square$

10. An upper bound for the number of perfect lattices

Theorem 10.1. *Up to similarities, there exist at most*

$$I_d^{d+1}\binom{3^d I_d}{\binom{d}{2}}$$

d-dimensional perfect lattices where I_d is as in Section 8.

Theorem 10.1 becomes effective after replacing I_d for example with the upper bound (8.1) or (8.3).

The main ingredient for proving Theorem 10.1 is the following easy observation which is of independent interest:

Lemma 10.2. *Let $v_1, \dots, v_d \in \Lambda_{\min}$ be d linearly independent minimal elements in a d-dimensional perfect lattice Λ such that the index I of the sublattice $\Lambda' = \sum_{i=1}^d \mathbb{Z} v_i$ in Λ is maximal.*

If $\mathcal{C} = \{w \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \mid w = \sum_{i=1}^d x_i v_i,\ x_i \in [-1,1]\}$ denotes the centrally symmetric d-dimensional parallelogram of all elements having coordinates in $[-1,1]$ with respect to the basis $v_1, \dots, v_d$, then all minimal vectors of Λ belong to $\mathcal{C}$.

Observe that the convex set $\mathcal{C}$ of Lemma 10.2 is simply the 1-ball of the $\| \cdot \|_\infty$-norm $\| \sum_{i=1}^d x_i v_i \|_\infty = \max_i |x_i|$ with respect to the basis $v_1, \dots, v_d$.

Proof of Lemma 10.2. Otherwise there exists $w \in \Lambda_{\min}$ such that $w = \sum_{i=1}^d \beta_i v_i$ with $|\beta_j| > 1$ for some index $j \in \{1, \dots, d\}$. Exchanging v_j with w leads then to a sublattice (generated by the d linearly independent minimal vectors $v_1, \dots, \hat{v_j}, \dots, v_d, w$) of index strictly larger than I in Λ. $\square$

Proof of Theorem 10.1. Given a perfect d-dimensional lattice Λ, we choose a set $v_1, \dots, v_d$ of d linearly independent minimal elements satisfying the condition of Lemma 10.2. We denote by I the index of the sublattice $\Lambda' = \sum_{i=1}^d \mathbb{Z} v_i$ of maximal index I in Λ. We consider the convex set $\mathcal{C} = \{w \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \mid w = \sum_{i=1}^d x_i v_i,\ x_i \in [-1,1]\}$ containing $\Lambda_{\min}$ according to Lemma 10.2.

We can now extend $v_1, \dots, v_d$ to a perfect subset $v_1, \dots, v_d, v_{d+1}, \dots, v_{\binom{d+1}{2}}$ of $\Lambda_{\min} \cap \mathcal{C}$. Since every element of Λ has at most 3^d representatives modulo Λ' in $\mathcal{C}$, the $\binom{d}{2}$ elements $v_{d+1}, \dots, v_{\binom{d+1}{2}}$ belong to the finite subset $\mathcal{C} \cap \Lambda$ containing at most $3^d I$ elements. There are thus at most $\binom{3^d I}{\binom{d}{2}}$ possibilities for the euclidean structure of Λ (which is determined up to a scalar by the $\binom{d+1}{2}$ minimal vectors of the perfect set $v_1, \dots, v_{\binom{d+1}{2}}$).

Proposition 9.3 gives the upper bound I_d^{d+1} for the number of all overlattices Λ containing $\sum_{i=1}^d \mathbb{Z} v_i$ as a sublattice of index at most I_d.

This yields the upper bound $I_d^{d+1} \binom{3^d I_d}{\binom{d}{2}}$ for the number of different perfect d-dimensional lattices (up to similarity). $\square$

Remark 10.3. Minimal vectors of a perfect lattice Λ do in general generate a perfect sublattice of Λ. This does not invalidate Theorem 10.1. The aim of the vectors $v_1, \dots, v\binom{d+1}{2}$ is only to pin down the Euclidean metric. The lattice Λ is determined by one of the I_d^{d+1} possible choices of a suitable overlattice of $\sum_{i=1}^d \mathbb{Z}v_i$.

The reader should also be aware that $\mathcal{C}$ does in general not contain the Euclidean unit ball defined by the perfect set $v_1, \dots, v_{\binom{d+1}{2}}$ of Λ.

The bound of Theorem 10.1 can be improved to

$$I_d^{d+1} \binom{\lfloor (3^d I_d - (2d+1))/2 \rfloor}{\binom{d}{2}}.$$

It is indeed enough to choose $\binom{d}{2}$ suitable pairs of opposite elements in $(\mathcal{C} \cap \Lambda) \setminus \{\mathbf{0}, \pm v_1, \dots, \pm v_d\}$ which contains at most $\lfloor (3^d I_d - (2d+1))/2 \rfloor$ such pairs.

11. Digression: Symmetric lattice polytopes

A lattice polytope is the convex hull of a finite number of lattice-points in $\mathbb{Z}^d$. Its dimension is the dimension of its interior. We call such a polytope P symmetric if $P = -P$. The group $\mathrm{GL}_d(\mathbb{Z})$ acts on the set of d-dimensional symmetric lattice polytopes. A slight variation of the proof of Theorem 10.1 shows the following result:

Theorem 11.1. *There are at most*

$$(d!)^{d+1} 2^{3^d d!}$$

different $\mathrm{GL}_d(\mathbb{Z})$*-orbits of d-dimensional symmetric lattice polytopes containing no non-zero elements of $\mathbb{Z}^d$ in their interior.*

Theorem 11.1 gives also an upper bound on the number of perfect lattices since convex hulls of minimal vectors of perfect lattices are obviously symmetric lattice polytopes containing no interior non-zero lattice points (non-similar perfect d-dimensional lattices give obviously rise to polytopes in different $\mathrm{GL}_d(\mathbb{Z})$-orbits). The bound of Theorem 11.1 is of course much worse than the bound given by Theorem 10.1.

11.1. The maximal index of hollow sublattices. A sublattice Λ' of a d-dimensional lattice Λ is called *hollow* if Λ' is generated by d linearly independent elements $v_1, \dots, v_d$ of Λ such that the interior of the convex hull spanned by $\pm v_1, \dots, \pm v_d$ contains no non-zero elements of Λ.

Hollowness is defined only in terms of convexity and is independent of metric properties.

We denote H_d the maximal index of a hollow sublattice of a d-dimensional lattice.

Proposition 11.2. *We have $H_d \leq d!$.*

Proof of Proposition 11.2. We can work without loss of generality with $\Lambda = \mathbb{Z}^d \subset \mathbb{R}^d$. We denote by $\mathcal{C}$ the convex hull of generators $\pm v_1, \ldots, \pm v_d$ (satisfying the condition of hollowness) of a hollow sublattice $\Lambda' = \sum_{i=1}^d \mathbb{Z} v_i$ of $\mathbb{Z}^d$. The volume $\mathrm{vol}(\mathcal{C})$ of $\mathcal{C}$ equals $\frac{2^d}{d!} I$ where I is the index of Λ' in $\mathbb{Z}^d$. Minkowski's inequality $\mathrm{vol}(\mathcal{C}) \leq 2^d$ implies the result. □

Remark 11.3. Since sublattices generated by d linearly independent minimal vectors of a d-dimensional well-rounded lattice are always hollow, we have $I_d \leq H_d$ giving the bad upper bound $I_d \leq d!$ for the integers I_d introduced in Section 8.

11.2. Proof of Theorem 11.1.

Proof. We consider such a lattice polytope P with vertices in $\mathbb{Z}^d$. We choose a set $v_1, \ldots, v_d$ of d linearly independent vertices of P generating a sublattice $\Lambda' = \sum_{i=1}^d \mathbb{Z} v_i$ of maximal index I in $\mathbb{Z}^d$. An obvious analogue of Lemma 10.2 holds and shows that all vertices of P are contained in $\mathcal{C} = \{w \in \mathbb{R}^d \mid w = \sum_{i=1}^d x_i v_i, x_i \in [-1, 1]\}$. Since $\mathcal{C}$ intersects $\mathbb{Z}^d$ in at most $3^d I$ elements there are at most $2^{3^d I}$ possibilities for choosing the vertices of P. Since $I \leq H_d \leq d!$ by Proposition 11.2 and since there are at most $H_d^{d+1} \leq (d!)^{d+1}$ possibilities (see Lemma 9.1) for superlattices containing Λ' with index at most $H_d \leq d!$, we get the result. □

Remark 11.4. The upper bound in Theorem 11.1 can easily be improved to $(d!)^{d+1} 2^{\lfloor (3^d d! - (2d+1))/2 \rfloor}$, see Remark 10.3.

11.3. Lattice-polytopes defined by short vectors. d-dimensional polytopes defined as convex hulls of minimal vectors in Euclidean lattices define of course symmetric lattice polytopes with no non-zero lattice-points in their interior. The following result is a slight generalization of this construction:

Proposition 11.5.

(1) *Let Λ be a d-dimensional lattice of minimum 1. We have $\mathring{P} \cap \Lambda = \{\mathbf{0}\}$ if $\mathring{P}$ is the interior of a d-dimensional polytope P defined as the convex hull of a set $\mathcal{V} = -\mathcal{V} \subset \Lambda \setminus \{\mathbf{0}\}$ of pairs of opposite non-zero elements of squared Euclidean length ≤ 2 in Λ.*

(2) *We have moreover $P \cap \Lambda = \mathcal{V} \cup \{\mathbf{0}\}$ and $\mathcal{V}$ is the set of vertices of P if the inequality is strict (ie. if all elements of $\mathcal{V}$ are of squared Euclidean length strictly smaller than 2).*

Taking $\mathcal{V} = \{\pm b_i, \pm b_i \pm b_j\}$ for $b_1, \ldots, b_d$ an orthogonal basis of the standard lattice $\mathbb{Z}^d$ shows that the bound 2 is sharp in part (2).

Proof. Suppose that $u \notin \mathcal{V}$ belongs to $P \cap \Lambda$. There exists thus d elements $v_1, \ldots, v_d \in \mathcal{V}$ such that u is contained in the simplex Σ spanned by the origin $\mathbf{0}$ and by $v_1, \ldots, v_d$. Since Λ has minimum 1, the element u belongs to the subset $\sigma \subset \Sigma$ of all elements of Σ which are at distance ≥ 1 from $v_1, \ldots, v_d$. The norm of u is thus at most equal to an element w of maximal norm in σ. For such an element w there exists an index i such that w is at distance exactly 1 from v_i and such that $\langle w, v_i - w \rangle \geq 0$. This implies that w is of norm at most 1 with equality if and only if w is on the boundary ∂P of P. □

Remark 11.6. The constant 2 in part (1) of Proposition 11.5 is perhaps not optimal. The sequence $c_1 = 4, c_2, \ldots$ of upper bounds c_d on the set of all possible constants for assertion (1) of Proposition 11.5) is decreasing to a limit ≥ 2.

12. Proof of Theorem 1.1

Proof. Proposition 7.1 counts the number $\alpha(d-8)$ of d-dimensional lattices $L_d(h_1, \ldots)$ satisfying the conditions of Theorems 3.1 and 3.2. All these lattices are perfect by Theorem 3.1 and non-isomorphic (and thus also non-similar since they have the same minimum) by Theorem 3.2. Their number $\alpha(d-8)$ grows exponentially fast by Proposition 6.1. This implies eventually $p_d > e^{e^{d-\epsilon}}$.

The eventual upper bound on p_d follows from Theorem 10.1. □

13. Upper bounds for cells

Similarity classes of d-dimension perfect lattices correspond to cells of maximal dimension of the Voronoï complex, a finite $\binom{d+1}{2}$-dimensional cellular complex encoding information on $\mathrm{GL}_d(\mathbb{Z})$.

The proof of Theorem 10.1 can easily be modified to show that this complex has at most

$$I_d^{d+1} \binom{3^d I_d}{k+1}$$

(with I_d replaced by the upper bound (8.1) or (8.3)) different cells of dimension k. (Cells are in general defined by affine subspaces. This explains the necessity of choosing $k+1$ elements.) Slight improvements of this result are possible by choosing carefully the initial d linearly independent minimal elements $v_1, \ldots, v_d$.

Without error on my behalf, these bounds are somewhat smaller than the bounds given in Proposition 2 of [12].

14. Algorithmic aspects

The proof of Theorem 10.1 can be made into a naive algorithm as follows:

For a fixed dimension d, consider the list $\mathcal{L}$ of all overlattices Λ of $\mathbb{Z}^d$ such that $\Lambda/\mathbb{Z}^d$ has at most I_d elements.

Given a lattice $\Lambda \in \mathcal{L}$, construct the finite set $\mathcal{C} = \Lambda \cap [-1,1]^d$. For every subset $\mathcal{S}$ of $\binom{d}{2}$ elements in $\mathcal{C}$, check if $\mathcal{S} \cup \{b_1, \dots, b_d\}$ (with $b_1, \dots, b_d$ denoting the standard basis of $\mathbb{Z}^d$) is perfect. If this is the case, check if the quadratic form q defined by $q(v_i) = 1$ for $i = 1, \dots, \binom{d+1}{2}$ is positive definite.

Check finally that the lattice Λ (endowed with this quadratic form) has no non-zero elements of length shorter than 1 and add the resulting perfect lattice to your list $\mathcal{P}$ of perfect d-dimensional lattices if this is the case and if $\mathcal{P}$ does lack it (up to similarity).

This algorithm can be accelerated using the following facts:

(1) It is enough to consider overlattices Λ containing $b_1, \dots, b_d$ as a hollow set. We can reduce the list $\mathcal{L}$ of overlattices by taking only one overlattice in each orbit of the group $\mathcal{S}_d \ltimes \{\pm 1\}^d$ acting linearly (by coordinate permutations and sign changes) on the set $\pm b_1, \dots, \pm b_d$.

(2) The list $\mathcal{C}$ can be made smaller: First of all, its size can be divided by 2 by considering only elements with strictly positive first non-zero coordinate. Secondly, an element v can only lead to a perfect set if the convex hull of $\pm b_1, \dots, \pm b_d, \pm v$ intersects $\Lambda \setminus \{0\}$ only in its vertices. This idea can be refined since the same property holds for the convex hull of $\pm b_1, \dots, b_d, \pm v_{d+1}, \dots, \pm v_{d+k}$ for the subset $b_1, \dots, b_d, v_{d+1}, \dots, v_{d+k}$ of the first $d+k$ elements of the perfect set $\{b_1, \dots, b_d\} \cup \{v_{d+1}, \dots, v_{\binom{d+1}{2}}\}$.

(3) Writing $v_i = \sum_{j=1}^{i} \alpha_{i,j} b_j$ we get a $\binom{d+1}{2} \times d$ matrix whose minors are all in $[-1,1]$ (otherwise we get a contradiction with maximality of the index I of the sublattice $\mathbb{Z}^d$ generated by $v_1 = b_1, \dots, v_d = b_d$).

15. Improving I_d?

The upper bounds (8.1) and (8.3) for I_d are not tight. They can however not be improved too much as shown below.

First of all, let us observe that the maximal index I (of sublattices generated by d linearly independent minimal vectors of a perfect lattice) can be as small as 1:

Proposition 15.1. *Every set of d linearly independent roots of the root lattice A_d generates A_d.*

I ignore if there exist other perfect lattices generated by any maximal set of linearly independent minimal vectors.

Proof of Proposition 15.1. We consider the usual realisation of A_d as the sublattice of all elements in $\mathbb{Z}^{d+1}$ with coordinate sum 0. Roots are given by $b_i - b_j$ where $b_1, \dots, b_{d+1}$ is the standard orthogonal basis of $\mathbb{Z}^{d+1}$. Let $\mathcal{B} = \{f_1, \dots, f_d\}$ be a set of d linearly independent roots of A_d. We associate to $\mathcal{B}$ a graph with vertices $1, \dots, d+1$. An element $b_i - b_j \in \mathcal{B}$ yields an edge joining the vertices i, j. The resulting graph has d edges and it has to be connected (otherwise we get a sublattice of rank $d+1-k$ in $\mathbb{Z}^d$ where k denotes the number of connected components). It is thus a tree $T_{\mathcal{B}}$. Roots of A_d correspond to the $\binom{d+1}{2}$ shortest paths joining two distinct vertices of $T_{\mathcal{B}}$. It follows that $\mathcal{B}$ generates the root lattice A_d. $\square$

The corresponding situation is fairly different for root lattice D_{2d} of even dimension:

Proposition 15.2. *The maximal index of a sublattice generated by* $2d$ *linearly independent roots of* D_{2d} *equals* 2^{d-1}.

Proof of Proposition 15.2. We consider $f_1 = b_1 + b_2, f_2 = b_1 - b_2, f_3 = b_3 + b_4, f_4 = b_3 - b_4, \dots, f_{2d-1} = b_{2d-1} + b_{2d}, f_{2d} = b_{2d-1} - b_{2d}$. These elements generate a sublattice of index 2^{d-1}. Since they are orthogonal, this index is maximal. $\square$

Proposition 15.2 dashes hope for big improvements on the upper bound I_d: Any such improvement has necessarily exponential growth. We have however no candidate (of a sublattice generated by d linearly minimal elements of a d-dimensional perfect lattice) such that the quotient group has elements of very large order.

Remark 15.3. An analogue of Proposition 15.2 holds also for root lattices D_{2d+1} but the proof is slightly more involved. Indeed, such a lattice contains a sublattice generated by roots of index 2^{d-1} corresponding to the root system $A_1^{2d-2}A_3$. For proving that the index 2^{d-1} is maximal, show first that D_n cannot contain a root system of type E. It follows that $2d+1$ linearly independent roots of D_{2d+1} generate a root system with connected components of type A and D whose ranks sum up to $2d+1$. Each such root system is possible and easy computations show that $A_1^{2d-2}A_3$ maximises the index among all possibilities.

16. Perfect lattices of small dimensions

Since $I_2 = I_3 = 1$, the classification of perfect lattices of dimension 2 and 3 can easily be done by hand.

16.1. Dimension 2. There is essentially (i.e. up to action of the dihedral group of isometries of the square with vertices $\pm b_1, \pm b_2$) only one way to extend $v_1 = b_1 = (1,0), v_2 = b_2 = (0,1)$ to a perfect set. It is given by choosing $v_3 = b_1 + b_2 = (1,1)$ and leads to the root lattice A_2.

16.2. Dimension 3. There are only three essentially different ways (up to the obvious action of the group $\mathcal{S}_3 \ltimes \{\pm 1\}^3$) for enlarging $v_1 = b_1, v_2 = b_2, v_3 = b_3$ to a perfect set in $\{0, \pm 1\}^3$.

The first one, given by $v_4 = b_1 + b_2 + b_3 = (1,1,1), v_5 = b_1 + b_2 = (1,1,0), v_6 = b_2 + b_3 = (0,1,1)$ (with Gram matrix $\frac{1}{2}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$) leads to the root lattice A_3 in its standard realisation.

The second one, given by $v_4 = b_1 + b_2, v_5 = b_2 + b_3, v_6 = b_1 + b_3$ leads to the degenerate quadratic form $\frac{1}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$.

The third one, $v_4 = b_1 + b_2, v_5 = b_2 + b_3, v_6 = b_1 - b_3$ defines also A_3. It leads to the quadratic form $\frac{1}{2}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ and corresponds to

$$\begin{array}{rcl} v_1 & = & (1 \quad -1 \quad 0 \quad 0) \\ v_2 & = & (0 \quad 1 \quad -1 \quad 0) \\ v_3 & = & (0 \quad -1 \quad 0 \quad 1) \\ v_4 & = & (1 \quad 0 \quad -1 \quad 0) \\ v_5 & = & (0 \quad 0 \quad -1 \quad 1) \\ v_6 & = & (1 \quad 0 \quad 0 \quad -1) \end{array}$$

with respect to the standard realisation

$$A_3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}$$

of the root lattice A_3.

16.3. Dimension 4. We give no complete classification in dimension 4 but we describe briefly how the two 4-dimensional perfect lattices fit into our framework.

In dimension 4 we have to consider overlattices containing $\mathbb{Z}^4$ with index 1 or 2. The known classification shows that the root lattices A_4 and D_4 are (up to similarities) the only perfect lattices in dimension 4. Every linearly independent set of 4 roots of A_4 generates A_4 by Proposition 15.1. The root lattice A_4 is thus obtained by considering (for example) the $10 = \binom{5}{2}$ vectors of $\{0,1\}^4$ with consecutive coefficients 1. The first 4 vectors can be chosen as the standard basis $v_1 = (1,0,0,0), v_2 = (0,1,0,0), v_3 = (0,0,1,0), v_4 = (0,0,0,1)$. The remaining six vectors are

$$(1,1,0,0), (1,1,1,0), (1,1,1,1), (0,1,1,0), (0,1,1,1), (0,0,1,1).$$

Proposition 15.2 shows that four linearly independent roots of D_4 generate a sublattice of index at most 2 in D_4. The index is exactly 2 if and only if the four roots are pairwise orthogonal. Given four such orthogonal roots $v_1, \dots, v_4$, the remaining eight pairs of roots are given by $\pm\frac{1}{2}(v_1 \pm v_2 \pm v_3 \pm v_4)$.

17. Bases of small height for perfect lattices

Theorem 17.1. *A d-dimensional perfect lattice has a $\mathbb{Z}$-basis $f_1, \dots, f_d$ such that the coordinates $\beta_i \in \mathbb{Z}$ (with respect to the basis $f_1, \dots, f_d$) of a minimal element $v = \sum_{i=1}^d \beta_i f_i$ satisfy the inequalities $|\beta_i| \leq 2^{i-1} I_d$, $i = 1, \dots, d$ with I_d as in Section 8.*

The main ingredient of the proof is the following result:

Lemma 17.2. *Let $v_1, \dots, v_d$ be d linearly independent elements generating a sublattice $\sum_{i=1}^d \mathbb{Z} v_i$ of index I in a d-dimensional lattice Λ. There exist a $\mathbb{Z}$-basis $f_1, \dots, f_d$ of $\Lambda = \sum_{i=1}^d \mathbb{Z} f_i$ such that we have $v_i = \sum_{j=i}^d \alpha_{i,j} f_j$ and $|\alpha_{i,j}| \leq 2^{\max(0,j-i-1)} I$ for all i and all $j \geq i$.*

The proof of Lemma 17.2 shows in fact slightly more since it constructs such a basis $f_1, \dots, f_d$ of Λ in the fundamental domain $\sum_{i=1}^d [0,1] v_i$ of the lattice $\sum_{i=1}^d \mathbb{Z} v_i$.

Proof of Theorem 17.1. We choose minimal elements $v_1, \dots, v_d$ generating a sublattice $\sum_{i=1}^d \mathbb{Z} v_i$ of maximal index $I \leq I_d$ in a perfect d-dimensional lattice Λ. Lemma 10.2 shows that we have $v = \sum_{i=1}^d \lambda_i v_i$ with $\lambda_i \in [-1, 1]$ for every minimal vector v of Λ. With respect to a basis $f_1, \dots, f_d$ of Λ as in Lemma 17.2 we get

$$v = \sum_{i=1}^d \lambda_i v_i = \sum_{i=1}^d \lambda_i \sum_{j=i}^d \alpha_{i,j} f_j .$$

We have thus $v = \sum_{j=1}^d \beta_j f_j$ with $\beta_j = \sum_{i=1}^j \lambda_i \alpha_{i,j}$. Since $\lambda_i \in [-1, 1]$ and since $|\alpha_{i,j}| \leq 2^{\max(0,j-i-1)} I$ by Lemma 17.2, we get $\beta_j \leq I \sum_{i=1}^j 2^{\max(0,j-i-1)}$ $= 2^{j-1} I$. We apply now the inequality $I \leq I_d$ of Proposition 8.1. □

Proof of Lemma 17.2. The result clearly holds for $d = 1$.

Consider $d+1$ linearly independent elements $v_0, \dots, v_d$ generating a sublattice of index I in a $(d+1)$-dimensional lattice Λ. We denote by π the linear form defined by $\pi(v_0) = 1$ and $\pi(v_1) = \dots = \pi(v_d) = 0$. The set $\pi(\Lambda)$ is of the form $\frac{1}{a}\mathbb{Z}$ with a dividing I such that $\Lambda' = \ker(\pi) \cap \Lambda$ contains $\sum_{i=1}^d \mathbb{Z} v_i$ as a sublattice of index I/a. By induction, there exists a basis $f_1, \dots, f_d$ of Λ' such that every element $v_1, \dots, v_i = \sum_{j=i}^d \alpha_{i,j} f_j, \dots, v_d$ involves only coordinates $\alpha_{i,j}$ of absolute value at most $2^{\max(0,j-i-1)} I/a$. We choose now an element $f_0 \in \Lambda$ with $\pi(f_0) = \frac{1}{a}$ in the fundamental domain $\sum_{i=0}^d [0,1] v_i$ of $\mathbb{R}^{d+1} / \sum_{i=0}^d \mathbb{Z} v_i$. We have thus $f_0 = \sum_{i=0}^d \lambda_i v_i$ with $\lambda_0 = \frac{1}{a}$

and $\lambda_1, \ldots, \lambda_d \in [0,1]$. We get thus

$$\begin{aligned} af_0 &= v_0 + a\sum_{i=1}^{d} \lambda_i v_i \\ &= v_0 + a\sum_{i=1}^{d}\sum_{j=i}^{d} \lambda_i \alpha_{i,j} f_j. \end{aligned}$$

This shows that $v_0 = \sum_{j=0}^{d} \beta_j f_j$ has coordinates β_j given by $\beta_0 = a$ and by

$$\beta_j = -a\sum_{i=1}^{j} \lambda_i \alpha_{i,j}$$

for $j = 1, \ldots, d$. We have obviously $|\beta_0| = |a| \le I$. Since $\lambda_i \in [0,1]$ and since $|\alpha_{i,j}| \le 2^{\max(0,j-i-1)} I/a$ we get $|\beta_j| \le I \sum_{i=1}^{j} 2^{\max(0,j-i-1)} = 2^{\max(0,j-1)} I = 2^{\max(0,j-0-1)} I$ for $j = 1, \ldots, d$. □

18. Heuristic arguments for an improved upper bound

We present a few non-rigorous thoughts suggesting an eventual upper bound of $e^{d^{2+\epsilon}}$ (for arbitrarily small strictly positive ϵ) for the numbers $p(d)$ of d-dimensional perfect lattices, up to similarities.

Consider an increasing function $\alpha : \mathbb{N} \to \mathbb{N}$ such that $\lim_{d\to\infty} \frac{\log\log \alpha(d)}{\log d} = 0$ (e.g. $\alpha(d) = \lceil d^{k(1+\log(d))} \rceil$ for some positive constant k). We denote by $p_\alpha(d)$ the set of all similarity classes of d-dimensional integral perfect lattices having a basis involving only elements of (squared euclidean) norm at most $\alpha(d)$. For any $\epsilon > 0$, we have clearly $p_\alpha(d) < e^{d^{2+\epsilon}}$ for almost all $d \in \mathbb{N}$ since such lattices have Gram matrices in the set of all $(1+2\alpha(d))^{\binom{d+1}{2}}$ symmetric matrices with coefficients in $\{-\alpha(d), \ldots, \alpha(d)\}$. In order to have $p(d) < e^{d^{2+\epsilon}}$ for any $\epsilon > 0$ and for any $d > N(\epsilon)$, it is now sufficient to have

$$\lim_{d\to\infty} \frac{p_\alpha(d)}{p(d)} e^{d^{2+\epsilon}} = \infty \tag{18.1}$$

for all $\epsilon > 0$.

In other terms, $p(d) = o\left(e^{d^{2+\epsilon}}\right)$ would imply that the proportion of similarity classes of integral perfect lattices of dimension d generated by, say, vectors shorter than $d^{100\log d}$ (with respect to all $p(d)$ perfect lattices) decays extremely fast.

Equivalently, we can consider the set $\tilde{p}_\alpha(d)$ of all similarity classes of d-dimensional integral perfect lattices having d linearly independent minimal elements generating a sublattice of index at most α. The proof of Theorem 10.1 shows that (18.1) with $\tilde{p}_\alpha(d)$ replacing $p_\alpha(d)$ implies also the eventual inequalities $p_d < e^{d^{2+\epsilon}}$.

Acknowledgements. I thank A. Ash, E. Bayer, M. Decauwert, P. Elbaz-Vincent, L. Fukshansky, J. Martinet and the anonymous referee for interesting discussions, remarks or comments.

References

[1] R. Bacher, "Constructions of some perfect integral lattices with minimum 4", *J. Théor. Nombres Bordx* **27** (2015), no. 3, p. 655-687.

[2] A. Böttcher, S. Eisenbarth, L. Fukshansk, S. R. Garcia & H. Maharaj, "Spherical 2-designs and lattices from Abelian groups", *Discrete Comput. Geom.* (2017).

[3] A. Böttcher, L. Fukshansky, S. R. Garcia & H. Maharaj, "On lattices generated by finite abelian groups", *SIAM J. Discrete Math.* **29** (2015), no. 1, p. 382-404.

[4] J. H. Conway & N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Grundlehren der Mathematischen Wissenschaften, vol. 290, Springer, 1999.

[5] M. Dutour Sikirić, A. Schürmann & F. Vallentin, "Classification of eight-dimensional perfect forms", *Electron. Res. Announc. Am. Math. Soc.* **13** (2007), p. 21-32.

[6] B. Gruber, "Alternative formulae for the number of sublattices", *Acta Crystallogr.* **53** (1997), no. 6, p. 807-808.

[7] W. Keller, J. Martinet & A. Schürmann, "On classifying Minkowskian sublattices. With an Appendix by Mathieu Dutour Sikirić", *Math. Comput.* **81** (2012), no. 278, p. 1063-1092.

[8] Y. Kitaoka, *Arithmetic of quadratic forms*, Cambridge Tracts in Mathematics, vol. 106, Cambridge University Press, 1993.

[9] J. Martinet, *Les réseaux parfaits des espaces euclidiens*, Masson, 1996 (an English translation has been published by Springer).

[10] Minkowski, *Geometrie der Zahlen*, Teubner, 1896.

[11] G. Nebe, "Boris Venkov's theory of lattices and spherical designs", in *Diophantine methods, lattices, and arithmetic theory of quadratic forms (BIRS, 2011)*, Contemporary Mathematics, vol. 587, American Mathematical Society, 2013, p. 1-19.

[12] C. Soulé, "Perfect forms and the Vandiver conjecture", *J. Reine Angew. Math.* **517** (1999), p. 209-221.

[13] Y. M. Zou, "Gaussian binomials and the number of sublattices", *Acta Crystallogr.* **62** (2006), no. 5, p. 409-410.

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