JOURNAL de Théorie des Nombres de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 26, n° 3 (2014), p. 673-707.

<http://jtnb.cedram.org/item?id=JTNB_2014__26_3_673_0>

Journal de Théorie des Nombres
de Bordeaux **26** (2014), 673–707

Iwasawa theory for symmetric powers of CM modular forms at non-ordinary primes

par Robert HARRON et Antonio LEI

Résumé. Soit f une forme primitive cuspidale à multiplication complexe (CM) et soit p un nombre premier impair tel que f soit non-ordinaire en p. Nous construisons des fonctions L p-adiques admissibles pour les puissances symétriques de f, vérifiant ainsi des cas particuliers de conjectures de Dabrowski et Panchishkin. À l'aide d'un résultat récent de Benois, nous prouvons la conjecture des zéros triviaux dans notre contexte. De plus, nous construisons des fonctions L p-adiques plus/moins "mixtes" et obtenons une décomposition des fonctions L p-adiques admissibles analogue à celle de Pollack. Du côté arithmétique, nous définissons les groupes de Selmer plus/moins mixtes correspondants et nous énonçons une Conjecture Principale.

Abstract. Let f be a cuspidal newform with complex multiplication (CM) and let p be an odd prime at which f is non-ordinary. We construct admissible p-adic L-functions for the symmetric powers of f, thus verifying conjectures of Dabrowski and Panchishkin in this special case. We combine this with recent work of Benois to prove the trivial zero conjecture in this setting. We also construct "mixed" plus and minus p-adic L-functions and prove an analogue of Pollack's decomposition of the admissible p-adic L-functions. On the arithmetic side, we define corresponding mixed plus and minus Selmer groups and formulate the Main Conjecture of Iwasawa Theory.

1. Introduction

This paper can be considered as a simultaneous sequel to the second author's article [L12] on the main conjecture for the symmetric square of a non-ordinary CM modular form and the first author's article [H13] on the trivial zero conjecture for the symmetric powers of ordinary CM modular

Manuscrit reçu le 13 mars 2013, accepté le 23 octobre 2013.
The first author is partially supported by NSA Young Investigator Grant #H98230-13-1-0223 and by the NSF-RTG grant "Number Theory and Algebraic Geometry at the University of Wisconsin".
The second author was supported by a CRM-ISM postdoctoral fellowship.
Classification math. 11R23, 11F80, 11F67.

forms. Let f be a normalised newform of weight $k \geq 2$, level $\Gamma_1(N)$, and Nebentypus ϵ, and let ρ_f be the p-adic Galois representation attached to f. The study of the Iwasawa theory of the symmetric powers of f began in the 1980s with the symmetric square case in the work of Coates–Schmidt and Hida (e.g. [CSch87, Sch88, Hi88, Hi90]). More precisely, they study the adjoint representation $\mathrm{ad}^0\rho_f \cong \mathrm{Sym}^2\rho_f \otimes \det^{-1}\rho_f$ in the ordinary case, constructing one- and two-variable analytic p-adic L-functions and stating a main conjecture (for f corresponding to an elliptic curve). More recently, still in the ordinary case, Urban ([U06]) has announced a proof of one divisibility in the main conjecture for $\mathrm{ad}^0\rho_f$ under some technical hypotheses. For general higher symmetric powers, even Deligne's conjecture on the algebraicity of special values of L-functions remains open, and consequently the analytic p-adic L-functions interpolating these values have yet to be constructed. The case where f has complex multiplication is however accessible. In the ordinary setting, the analytic p-adic L-functions were constructed by the first author in [H13] and the trivial zero conjecture was proved there as well. At non-ordinary primes, the second author constructed p-adic L-functions and formulated the Main Conjecture of Iwasawa Theory for the symmetric square in [L12]. This article deals with the non-ordinary CM case for all symmetric powers. Specifically, we construct the analytic p-adic L-functions for $\rho_m := \mathrm{Sym}^m \rho_f \otimes \det^{-\lfloor m/2 \rfloor} \rho_f$ and prove the corresponding trivial zero conjecture. We also formulate a Main Conjecture of Iwasawa Theory in this setting and prove it in Theorem 5.9 for sufficiently large p (depending on m and k).

To begin, Section 2 describes our notation and normalizations, while Section 3 develops some basic facts about ρ_m. In Section 4, we construct what we refer to as "mixed" plus and minus p-adic L-functions for ρ_m. The construction of these proceeds as in [H13] taking advantage of the decomposition of ρ_m into a direct sum of Galois representations attached to modular forms and Dirichlet characters. Each modular form that shows up has plus and minus p-adic L-functions constructed by Pollack in [Pol03] and we define $2^{\lceil m/2 \rceil}$ mixed plus and minus p-adic L-functions for ρ_m by independently choosing a sign for each modular form. We then use these mixed p-adic L-functions in Section 5 to formulate the Main Conjecture of Iwasawa Theory for ρ_m. This also involves defining mixed plus and minus Selmer groups which uses the decomposition of ρ_m and the work of the second author in [L11]. In Section 6, we construct $2^{\lceil m/2 \rceil}$ "admissible" p-adic L-functions for ρ_m, verifying a conjecture of Dabrowski [Dab11, Conjecture 1] and Panchishkin [Pan94, Conjecture 6.2]. It should be noted that for $m \geq 2$, none of these p-adic L-functions is determined by its interpolation property. We go on to show that the admissible p-adic L-functions can be decomposed into a linear combination of the mixed p-adic L-functions

and mixed p-adic logarithms analogous to the results of Pollack in [Pol03]. Finally, in Section 6.4, we locate the trivial zeroes of the admissible p-adic L-functions of ρ_m and prove the related trivial zero conjecture showing that the analytic L-invariants are given by Benois's arithmetic L-invariant defined in [Ben11]. This relies on recent work of Benois [Ben13] computing the analytic and arithmetic L-invariants of modular forms at near-central points in the crystalline case. The trivial zero phenomenon presents an interesting feature in our situtation: the order of the trivial zero grows with the symmetric power, thus providing examples of trivial zeroes of order greater than one for motives for $\mathbb{Q}$. This also has ramifications for a question raised by Greenberg in [G94, p. 170]: Is the L-invariant independent of the symmetric power? Our result shows that the answer must be *no*, unless every CM modular form has L-invariant equal to 1. The latter option seems unlikely.

Acknowledgements

We would like to thank Rob Pollack for pointing us to the article [KPZ10]. Thanks are also due to the referee for several suggestions that improved the readability of our article and to Jay Pottharst for catching some errors.

2. Notation and conventions

2.1. Extensions by p-power roots of unity. Throughout this paper, p is a fixed odd prime. If K is a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$, then we write G_K for its absolute Galois group, $\mathcal{O}_K$ for its ring of integers, and χ for the p-adic cyclotomic character of G_K. We fix an embedding $\iota_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and write Frob_∞ for the induced complex conjugation in $G_{\mathbb{Q}}$. We also fix an embedding $\iota_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Let Frob_ℓ denote an arithmetic Frobenius element at ℓ in $G_{\mathbb{Q}}$. If K is a number field, $\mathbb{A}_K$ (resp., $\mathbb{A}_{K,f}$ and $\mathbb{A}_{K,\infty}$) denotes its adele ring (resp., its ring of finite adeles and its ring of infinite adeles).

For an integer $n \geq 0$, we write K_n for the extension $K(\mu_{p^n})$ where μ_{p^n} is the set of p^nth roots of unity, and K_∞ denotes $\bigcup_{n\geq 1} K_n$. When $K = \mathbb{Q}$, we write $k_n = \mathbb{Q}(\mu_{p^n})$ instead, and when $K = \mathbb{Q}_p$, we write $\mathbb{Q}_{p,n} = \mathbb{Q}_p(\mu_{p^n})$. Let G_n denote the Galois group $\mathrm{Gal}(\mathbb{Q}_{p,n}/\mathbb{Q}_p)$ for $0 \leq n \leq \infty$. Then, $G_\infty \cong \Delta \times \Gamma$ where $\Delta = G_1$ is a finite group of order $p-1$ and $\Gamma = \mathrm{Gal}(\mathbb{Q}_{p,\infty}/\mathbb{Q}_{p,1}) \cong \mathbb{Z}_p$. The cyclotomic character χ induces an isomorphism $G_\infty \cong \mathbb{Z}_p^\times$ identifying Δ with μ_{p-1} and Γ with $1 + p\mathbb{Z}_p$. We denote by $g \mapsto \langle g \rangle$ the character that projects G_∞ onto the factor $1 + p\mathbb{Z}_p$. We fix a topological generator γ_0 of Γ.

We identify a Dirichlet character $\eta : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ with a Galois character $\eta : G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_p^\times$ via $\eta(\mathrm{Frob}_\ell) = \iota_p \iota_\infty^{-1} \eta(\ell)$ noting that $\eta(\mathrm{Frob}_\infty) = \eta(-1)$.

2.2. Iwasawa algebras and power series. Given a finite extension E of $\mathbb{Q}_p$, $\Lambda_{\mathcal{O}_E}(G_\infty)$ (resp., $\Lambda_{\mathcal{O}_E}(\Gamma)$) denotes the Iwasawa algebra of G_∞ (resp., Γ) over $\mathcal{O}_E$. We write $\Lambda_E(G_\infty) = \Lambda_{\mathcal{O}_E}(G_\infty) \otimes_{\mathcal{O}_E} E$ and $\Lambda_E(\Gamma) = \Lambda_{\mathcal{O}_E}(\Gamma) \otimes_{\mathcal{O}_E} E$. There is an involution $f \mapsto f^\iota$ on each of these algebras given by sending each group element to its inverse. If M is a finitely generated $\Lambda_E(\Gamma)$-torsion module, we write $\mathrm{Char}_{\Lambda_E(\Gamma)}(M)$ for its characteristic ideal. The Pontryagin dual of M is written as $M^\vee$.

Given a module M over $\Lambda_{\mathcal{O}_E}(G_\infty)$ or $\Lambda_E(G_\infty)$ and a character $\nu : \Delta \to \mathbb{Z}_p^\times$, M^ν denotes the ν-isotypical component of M. Explicitly, there is an idempotent $\pi_\nu \in \Lambda_{\mathcal{O}_E}(G_\infty)$ given by

$$\pi_\nu = \frac{1}{p-1} \sum_{\sigma \in \Delta} \nu(\sigma)\sigma^{-1}$$

that projects M onto M^ν. If $m \in M$, we sometimes write m^ν for $\pi_\nu m$. Note that the character group of Δ is generated by the Teichmüller character $\omega := \chi|_\Delta$.

Let $r \in \mathbb{R}_{\geq 0}$. We define

$$\mathcal{H}_r = \left\{ \sum_{n \geq 0, \sigma \in \Delta} c_{n,\sigma} \cdot \sigma \cdot X^n \in \mathbb{C}_p[\Delta][\![X]\!] : \sup_n \frac{|c_{n,\sigma}|_p}{n^r} < \infty, \text{ for all } \sigma \in \Delta \right\}$$

where $|\cdot|_p$ is the p-adic norm on $\mathbb{C}_p$ such that $|p|_p = p^{-1}$. We write $\mathcal{H}_\infty = \cup_{r \geq 0} \mathcal{H}_r$ and $\mathcal{H}_r(G_\infty) = \{f(\gamma_0 - 1) : f \in \mathcal{H}_r\}$ for $r \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. In other words, the elements of $\mathcal{H}_r$ (respectively $\mathcal{H}_r(G_\infty)$) are the power series in X (respectively $\gamma_0 - 1$) over $\mathbb{C}_p[\Delta]$ with growth rate $O(\log_p^r)$.

Given a subfield E of $\mathbb{C}_p$, we write $\mathcal{H}_{r,E} = \mathcal{H}_r \cap E[\Delta][\![X]\!]$ and similarly for $\mathcal{H}_{r,E}(G_\infty)$. In particular, $\mathcal{H}_{0,E}(G_\infty) = \Lambda_E(G_\infty)$.

Let $n \in \mathbb{Z}$. We define the E-linear map Tw_n from $\mathcal{H}_{r,E}(G_\infty)$ to itself to be the map that sends σ to $\chi(\sigma)^n\sigma$ for all $\sigma \in G_\infty$. It is clearly bijective (with inverse Tw_{-n}).

If $h = \sum_{n \geq 0, \sigma \in \Delta} c_{n,\sigma} \cdot \sigma \cdot (\gamma_0 - 1)^n \in \mathcal{H}_\infty(G_\infty)$ and $\lambda \in \mathrm{Hom}_{\mathrm{cts}}(G_\infty, \mathbb{C}_p^\times)$, we write

$$\lambda(h) = \sum_{n \geq 0, \sigma \in \Delta} c_{n,\sigma} \cdot \lambda(\sigma) \cdot (\lambda(\gamma_0) - 1)^n \in \mathbb{C}_p.$$

2.3. Crystalline representations and Selmer groups. Let $\mathbb{B}_{\mathrm{cris}} \subseteq \mathbb{B}_{\mathrm{dR}}$ be the rings of Fontaine (for $\mathbb{Q}_p$) and φ for the Frobenius acting on $\mathbb{B}_{\mathrm{cris}}$.

Let V be an E-linear p-adic representation of $G_\mathbb{Q}$ for some finite extension $E/\mathbb{Q}_p$. We denote the crystalline Dieudonné module and its dual by $\mathbb{D}_{\mathrm{cris}}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_{\mathbb{Q}_p}]}(V, \mathbb{B}_{\mathrm{cris}})$ and $\widetilde{\mathbb{D}}_{\mathrm{cris}}(V) = (\mathbb{B}_{\mathrm{cris}} \otimes V)^{G_{\mathbb{Q}_p}}$ respectively. We say that V is crystalline if V has the same $\mathbb{Q}_p$-dimension as $\mathbb{D}_{\mathrm{cris}}(V)$.

Fix such a V. If $j \in \mathbb{Z}$, $\mathrm{Fil}^j \mathbb{D}_{\mathrm{cris}}(V)$ denotes the jth de Rham filtration of $\mathbb{D}_{\mathrm{cris}}(V)$.

Let T be a lattice of V which is stable under $G_{\mathbb{Q}}$ and let $A = V/T$. If K is a number field and v is a place of K, we define

$$H^1_f(K_v, V) := \begin{cases} \ker\big(H^1(K_v, V) \to H^1(I_v, V)\big) & \text{if } v \nmid p \\ \ker\big(H^1(K_v, V) \to H^1(K_v, V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}})\big) & \text{if } v \mid p \end{cases}$$

where I_v denotes the inertia subgroup at v. The local condition $H^1_f(K_v, A)$ is defined as the image of the above under the map induced by $V \to A$.

We define the Selmer group of A over K to be

$$H^1_f(K, A) := \ker\left(H^1(K, A) \to \prod_v \frac{H^1(K_v, A)}{H^1_f(K_v, A)}\right)$$

where v runs through the places of K.

We write $V(j)$ for the jth Tate twist of V, i.e. $V(j) = V \otimes Ee_j$ where $G_{\mathbb{Q}_p}$ acts on e_j via χ^j. The Tate dual of V (respectively T) is $V^* := \mathrm{Hom}(V, E(1))$ (respectively $T^* := \mathrm{Hom}(T, \mathcal{O}_E(1))$). We write $A^* = V^*/T^*$.

2.4. Imaginary quadratic fields. Let K be an imaginary quadratic field with ring of integers $\mathcal{O}$. We write ϵ_K for the quadratic character associated to K, i.e. the character on $G_{\mathbb{Q}}$ which sends σ to 1 if $\sigma \in G_K$ and to -1 otherwise.

A Hecke character of K is simply a continuous homomorphism $\psi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ that is trivial on $K^\times$. Its complex L-function is

$$L(\psi, s) = \prod_v (1 - \psi(v)N(v)^{-s})^{-1}$$

where the product runs through the finite places v of K at which ψ is unramified, $\psi(v)$ is the image of the uniformiser of K_v under ψ, and $N(v)$ is the norm of v. We say that ψ is algebraic of type (m, n) where $m, n \in \mathbb{Z}$ if the restriction of ψ to $\mathbb{A}_{K,\infty}^\times$ is of the form $z \mapsto z^m \bar{z}^n$. All Hecke characters are tacitly assumed to be algebraic in what follows.

Let E be a finite extension of $\mathbb{Q}_p$ that contains the image of $\iota_p \iota_\infty^{-1} \psi|_{\mathbb{A}_{K,f}^\times}$. We write V_ψ for the one-dimensional E-linear representation of G_K normalised so that at finite places v relatively prime to p and the conductor of ψ, the action of Frob_v is via multiplication by $\psi(v)$. We write $\widetilde{V}_\psi = \mathrm{Ind}_K^{\mathbb{Q}}(V_\psi)$.

2.5. Modular forms. We will use the term newform to mean a holomorphic modular form of integral weight ≥ 2, level $\Gamma_1(N)$, and Nebentypus character ϵ which is a normalised cuspidal eigenform for the Hecke algebra generated by all Hecke operators T_ℓ ($\ell \nmid N$) and U_ℓ ($\ell \mid N$) and is an element of the new subspace. Let $f = \sum a_n q^n$ be such a newform. Unless

stated otherwise, we assume that f is a CM modular form which is good and non-ordinary at p, i.e.

$$L(f,s) = \sum a_n(f) n^{-s} = L(\psi, s)$$

for some Hecke character ψ of an imaginary quadratic field K with p inert in K and $p \nmid N$. Then, $\epsilon_K(p) = -1$ and we normalise our choice of ψ such that it is of type $(k-1, 0)$. In this case, $a_p(f)$ is always 0 since f is CM.

We write E for the completion in $\overline{\mathbb{Q}}_p$ of the extension of $\mathbb{Q}_p$ generated by the image of $\iota_p \iota_\infty^{-1} \psi|_{\mathbb{A}_{K,f}^\times}$. The coefficient field of f is contained in E.

We write V_f for the dual of the 2-dimensional E-linear representation of $G_\mathbb{Q}$ associated to f inside the cohomology of a Kuga–Sato variety by [D69], so we have a homomorphism $\rho_f : G_\mathbb{Q} \to \mathrm{GL}(V_f)$ with $\det(\rho_f) = \epsilon \chi^{k-1}$. Moreover, $V_f \cong \widetilde{V}_\psi$.

3. The symmetric power of a modular form

Let f be a CM newform as in Section 2.5. Let $m \geq 1$ be an integer. In this section, we collect basic results on the representation $\mathrm{Sym}^m(V_f)$, as well as the main representation of interest in this article, V_m. Throughout this article, we let $r := \lfloor m/2 \rfloor$ and $\widetilde{r} := \lceil m/2 \rceil$.

Proposition 3.1. *If m is even, we have*

$$\mathrm{Sym}^m(V_f) \cong \bigoplus_{i=0}^{\widetilde{r}-1} \left(\widetilde{V}_{\psi^{m-2i}} \otimes (\epsilon_K \det \rho_f)^i \right) \oplus (\epsilon_K \det \rho_f)^r$$

as $G_\mathbb{Q}$-representations. If m is odd, then

$$\mathrm{Sym}^m(V_f) \cong \bigoplus_{i=0}^{\widetilde{r}-1} \left(\widetilde{V}_{\psi^{m-2i}} \otimes (\epsilon_K \det \rho_f)^i \right).$$

Proof. See either [H13, Section 2] or [L12, Proposition 5.1]. □

Definition 3.2. *We define for all $m \geq 1$ the representation*

$$V_m = \mathrm{Sym}^m(V_f) \otimes \det(\rho_f)^{-r}.$$

Note that $V_2 \cong \mathrm{ad}^0(V_f)$.

Corollary 3.3. *As $G_\mathbb{Q}$-representations, we have an isomorphism*

$$V_m \cong \bigoplus_{i=0}^{\widetilde{r}-1} \left(\widetilde{V}_{\psi^{m-2i}} \otimes \left(\epsilon_K^i \epsilon^{i-r} \chi^{(i-r)(k-1)} \right) \right) \oplus \epsilon_K^r$$

when m is even. If m is odd,

$$V_m \cong \bigoplus_{i=0}^{\widetilde{r}-1} \left(\widetilde{V}_{\psi^{m-2i}} \otimes \left(\epsilon_K^i \epsilon^{i-r} \chi^{(i-r)(k-1)} \right) \right).$$

Proof. Recall that $\det(\rho_f) \cong \epsilon\chi^{k-1}$, so this is immediate from Proposition 3.1. □

Proposition 3.4. *For $0 \leq i \leq \widetilde{r}-1$, there exist CM newforms f_i of weight $(m-2i)(k-1)+1$ such that*

$$V_m \cong \bigoplus_{i=0}^{\widetilde{r}-1} \left(V_{f_i} \otimes \chi^{(i-r)(k-1)}\right) \oplus \epsilon_K^r$$

when m is even and

$$V_m \cong \bigoplus_{i=0}^{\widetilde{r}-1} V_{f_i} \otimes \chi^{(i-r)(k-1)}$$

when m is odd. Moreover, these f_i have CM by K, level prime to p, and are non-ordinary at p.

Proof. This is discussed in [H13, Proposition 2.2]. Since ψ^{m-2i} is a Grossencharacter of type $((m-2i)(k-1),0)$, it corresponds to a CM form f_i' of weight $(m-2i)(k-1)+1$ by [Rib77, Theorem 3.4]. Therefore, we may take f_i to be the newform whose Hecke eigenvalues agree with those of $f_i' \otimes \epsilon_K^i \epsilon^{i-r}$ outside of the level of f_i' and the conductor of $\epsilon_K^i \epsilon^{i-r}$. It is CM because it corresponds to the Grossencharacter $\psi^{m-2i} \otimes \epsilon_K^i \epsilon^{i-r}$.

Since p does not divide the conductor of ψ^{m-2i}, we see that the level of f_i' is not divisible by p. The conductor of $\epsilon_K^i \epsilon^{i-r}$ is coprime to p, too, hence the level of f_i is also prime to p. Since p is inert in K, f_i is non-ordinary at p. □

Corollary 3.5. *The complex L-function admits a decomposition*

$$L(V_m, s) = \left(\prod_{i=0}^{\widetilde{r}-1} L\left(f_i, s+(r-i)(k-1)\right)\right) \cdot \begin{cases} L(\epsilon_K^r, s) & m \text{ even}, \\ 1 & m \text{ odd}. \end{cases}$$

Now, we discuss the Hodge theory of V_m at ∞ and describe its critical twists. Recall that the Hodge structure[1] of V_f is of the form $H^{0,k-1} \oplus H^{k-1,0}$, with each summand one-dimensional. Taking symmetric powers, we see that the Hodge structure of $\mathrm{Sym}^m(V_f)$ is of the form

$$\bigoplus_{a=0}^{m} H^{a(k-1),(m-a)(k-1)}$$

with each summand one-dimensional. In particular, if m is even, there is a "middle" term $H^{r(k-1),r(k-1)}$ and to specify the Hodge structure we must

[1]To be precise, when referring to the Hodge structure of a Galois representation, we really mean that of the motive giving the *dual* Galois representation. We do this so that our Hodge structure computations are in line with those of [D79]. Indeed, we define the L-function of V in terms of *arithmetic* Frobenius, the dual of Deligne's definition. These two duals then cancel out.

also say how the "complex conjugation" involution, denoted F_∞, acts on this term: it acts trivially. When we twist by $\det(\rho_f)^{-r}$, this shifts the indices by $-r(k-1)$ and, when m is even, twists the action of F_∞ on the middle term by

$$\begin{aligned}\det(\rho_f)^{-r}(\mathrm{Frob}_\infty) &= \left(\epsilon(-1)(-1)^{(k-1)}\right)^{-r} \\ &= \left((-1)^k(-1)^{(k-1)}\right)^{r} \\ &= (-1)^r.\end{aligned}$$

We have thus proved the following.

Lemma 3.6. *The Hodge structure of V_m is*

$$\bigoplus_{a=0}^{m} H^{-(r-a)(k-1),(\widetilde{r}-a)(k-1)}$$

with each summand one-dimensional. If m is even, then F_∞ acts on $H^{0,0}$ by $(-1)^r$.

Therefore, if $d^\pm$ denotes the dimension of the part of the Hodge structure of V_m on which F_∞ acts by ± 1, then

$$d^+ = \begin{cases} r & \text{if } m \equiv 2 \pmod 4 \\ r+1 & \text{otherwise,} \end{cases} \tag{3.1}$$

$$d^- = \begin{cases} r & \text{if } m \equiv 0 \pmod 4 \\ r+1 & \text{otherwise.} \end{cases} \tag{3.2}$$

From this, we can determine the critical twists of V_m (in the sense of [D79]).

Lemma 3.7. *Let C_m denote the set of all pairs (θ, j) such that $V_m \otimes \theta\chi^j$ is critical, where θ is a Dirichlet character of p-power conductor. Then,*

$$C_m = \{(\theta, j) : -(k-1)+1 \le j \le (k-1) \text{ and } \theta\chi^j(-1) = \mathrm{sgn}(j-1/2)(-1)^r\}$$

if m is even. Otherwise,

$$C_m = \{(\theta, j) : 1 \le j \le k-1\}.$$

Proof. The integers j are forced to lie between

$$\max\{a : H^{a,b} \neq 0 \text{ with } a < b\} + 1 \text{ and } \min\{a : H^{a,b} \neq 0 \text{ with } a > b\}.$$

When there is a non-zero $H^{a,b}$ with $a = b$, there is an additional condition depending on the parity of $\theta\chi^j$ and the sign of $j - \frac{1}{2}$ (i.e. whether j is positive or non-positive) which is determined by the action of F_∞. This occurs when m is even. We leave the details to the reader. □

We remark that this lemma holds without the requirement that θ have p-power conductor.

Recall that the Hodge polygon $P_H(x, V_m)$ (viewed as a function of $x \in [0, \dim(V_m)]$) is the convex piecewise-linear function whose segment of slope a has horizontal length $\dim H^{a,b}$. The vertices of the Hodge polygon of V_m are as follows.

Lemma 3.8. *For $a = 0, 1, \ldots, m$,*

$$P_H(a+1, V_m) = (k-1)\left(\frac{(a+1)(a-2r)}{2}\right).$$

Proof. Since each piece of the Hodge decomposition of V_m has dimension one, the y-coordinate at the end of the ath segment is

$$\sum_{c=0}^{a} -(r-c)(k-1).$$

The result follows from evaluating this sum. □

This whole discussion of the Hodge theory at ∞ holds for any newform (as we have used the term).

We will also require knowledge of the Hodge theory of V_m at p.

Lemma 3.9. *If m is even, the eigenvalues of φ on $\mathbb{D}_{\mathrm{cris}}(V_m)$ are ± 1 with multiplicity $d^{\pm}$. When m is odd, the eigenvalues are $\pm\alpha$ each with multiplicity $d^{\pm} = \widetilde{r}$, where $\pm\alpha$ are the roots of $x^2 + \epsilon(p)p^{k-1}$.*

Proof. Let α and $\overline{\alpha}$ be the roots of $x^2 + \epsilon(p)p^{k-1}$, so that $\overline{\alpha} = -\alpha$. It follows from work of Saito ([Sai97]) that α and $\overline{\alpha}$ are the eigenvalues of φ on $\mathbb{D}_{\mathrm{cris}}(V_f)$. Therefore, there is a basis of $\mathbb{D}_{\mathrm{cris}}(\mathrm{Sym}^m(V_f))$ in which φ is diagonal with entries $\alpha^a\overline{\alpha}^{m-a}$ for $a = 0, 1, \ldots, m$. Twisting by $\det(\rho_f)^{-r}$ translates into multiplying this matrix by $(\alpha\overline{\alpha})^{-r}$. If m is even, the diagonal entries become $\left(\frac{\alpha}{\overline{\alpha}}\right)^a$ for $a = -\widetilde{r}, -r+1, \ldots, r$. For m odd, the diagonal entries become $\alpha\left(\frac{\alpha}{\overline{\alpha}}\right)^a$ for $a = -\widetilde{r}, -r, \ldots, r$. The result follows. □

We remark that this lemma holds without the assumption that f is CM as long as $a_p(f) = 0$.

It is not necessary, but convenient, to use the decomposition of Proposition 3.4 in order to determine the filtration on $\mathbb{D}_{\mathrm{cris}}(V_m)$. We obtain the following description of $\mathbb{D}_{\mathrm{cris}}(V_m)$.

Lemma 3.10. *If m is even, there is a basis $v, v_0, \overline{v}_0, v_1, \overline{v}_1, \ldots, v_{\widetilde{r}-1}, \overline{v}_{\widetilde{r}-1}$ of $\mathbb{D}_{\mathrm{cris}}(V_m)$ such that*

$$\begin{aligned} \varphi(v) &= (-1)^r v, \\ \varphi(v_j) &= v_j, \\ \varphi(\overline{v}_j) &= -\overline{v}_j \end{aligned}$$

and, for $j = -r, -r+1, \ldots, \widetilde{r}$*, the graded piece* $\mathrm{gr}^{j(k-1)}\,\mathbb{D}_{\mathrm{cris}}(V_m)$ *is generated by the image of*

$$\begin{cases} v_{r+j} & \text{if } j < 0, \\ v & \text{if } j = 0, \\ v_{\widetilde{r-j}} + \overline{v}_{\widetilde{r-j}} & \text{if } j > 0. \end{cases}$$

If m *is odd, there is a basis* $v_0, \overline{v}_0, v_1, \overline{v}_1, \ldots, v_{\widetilde{r-1}}, \overline{v}_{\widetilde{r-1}}$ *of* $\mathbb{D}_{\mathrm{cris}}(V_m)$ *such that*

$$\begin{aligned} \varphi(v_j) &= \alpha v_j, \\ \varphi(\overline{v}_j) &= -\alpha \overline{v}_j \end{aligned}$$

and, for $j = -r, -r+1, \ldots, \widetilde{r}$*, the graded piece* $\mathrm{gr}^{j(k-1)}\,\mathbb{D}_{\mathrm{cris}}(V_m)$ *is generated by the image of*

$$\begin{cases} v_{r+j} & \text{if } j \leq 0, \\ v_{\widetilde{r-j}} + \overline{v}_{\widetilde{r-j}} & \text{if } j > 0. \end{cases}$$

Proof. Suppose m is even. The crystalline Dieudonné module of a Dirichlet character η (of prime-to-p conductor) has Hodge–Tate weight 0 and φ acts by $\eta(p)$. Let v be a basis of $\mathbb{D}_{\mathrm{cris}}(\epsilon_K^r)$. For $i = 0, 1, \ldots, \widetilde{r} - 1$, let $v_i', \overline{v}_i'$ be a basis of $\mathbb{D}_{\mathrm{cris}}(V_{f_i})$ in which the action of φ is diagonal. Since f_i has weight $2(r-i)(k-1)+1$, its crystalline Dieudonné module $\mathbb{D}_{\mathrm{cris}}(V_{f_i})$ has non-zero graded pieces in degrees 0 and $2(r-i)(k-1)$. Furthermore, its filtration is determined by the degree $2(r-i)(k-1)$ piece. In order for the Dieudonné module to be weakly admissible, this piece cannot contain either v_i' or $\overline{v}_i'$; therefore, up to rescaling,

$$\mathrm{Fil}^{2(r-i)(k-1)}\,\mathbb{D}_{\mathrm{cris}}(V_{f_i}) = \langle v_i' + \overline{v}_i' \rangle.$$

Twisting by $-(r-i)(k-1)$ shifts the filtration so that it drops into degrees $\pm(r-i)(k-1)$. Let $v_i, \overline{v}_i$ denote the twisted basis. Given what we know from lemma 3.9, $\varphi(v_i) = \pm v_i$ and $\varphi(\overline{v}_i) = \pm \overline{v}_i$. Up to reordering, $\varphi(v_i) = v_i$ and $\varphi(\overline{v}_i) = -\overline{v}_i$, since the Frobenius eigenvalues of V_{f_i} are negatives of each other (since $a_p(f_i) = 0$). The result follows from the decomposition of Proposition 3.4. The proof when m is odd is along the same lines and we leave it to the reader. □

Therefore, the dimension of the de Rham tangent space $t_{\mathrm{dR}}(V_m) := \mathbb{D}_{\mathrm{cris}}(V_m)/\,\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V_m)$ is $d^+ - 1$, unless $m \equiv 2 \pmod 4$, in which case it is d^+. Recall that the Newton polygon $P_N(x, V_m)$ is the piecewise-linear increasing function whose segment of slope λ has length equal to the number of eigenvalues of φ whose valuation is λ. The following lemma follows immediately from the previous lemma.

Lemma 3.11. *If* m *is even, the Newton polygon of* V_m *is the horizontal line from* $(0,0)$ *to* $(m+1, 0)$*. When* m *is odd, it is the straight line from* $(0,0)$ *to* $\left(m+1, \frac{(k-1)(m+1)}{2}\right)$.

4. Plus and minus p-adic L-functions

4.1. Some known p-adic L-functions. In this section, we review some facts concerning the p-adic L-functions of Dirichlet characters and newforms, as well as Pollack's plus and minus p-adic L-functions of newforms.

The following is a consequence of classical results of Kubota–Leopoldt and Iwasawa ([KL64, Iw72]).

Theorem 4.1. *If η is a non-trivial Dirichlet character of conductor $\mathfrak{f}_\eta$ prime to p (resp. the trivial character $\mathbf{1}$) and $E/\mathbb{Q}_p$ contains the values of η, then there exists a unique $L_\eta \in \Lambda_{\mathcal{O}_E}(G_\infty)$ (resp. $L_{\mathbf{1}}$ with $(\gamma_0 - 1)(\gamma_0 - \chi(\gamma_0))L_{\mathbf{1}} \in \Lambda_{\mathcal{O}_E}(G_\infty)$) such that*

$$\theta\chi^j(L_\eta) = e_\eta(\theta, j)\frac{L(\eta\theta^{-1}, j)}{\Omega_\eta(\theta, j)}$$

for any integer j and Dirichlet character θ of conductor p^n satisfying

$$\theta\chi^j(-1) = \operatorname{sgn}\left(j - \frac{1}{2}\right)\eta(-1)$$

where

$$e_\eta(\theta, j) = \left(\frac{p^j}{\eta(p)}\right)^n \cdot \begin{cases} (1 - p^{j-1}(\eta^{-1}\theta)(p)) & \text{if } \theta\chi^j(-1) = \eta(-1), j \geq 1, \\ (1 - p^{-j}(\eta\theta^{-1})(p)) & \text{if } \theta\chi^j(-1) = -\eta(-1), j \leq 0, \end{cases}$$

$$\frac{1}{\Omega_\eta(\theta, j)} = \begin{cases} \dfrac{2\Gamma(j)\mathfrak{f}_\eta^j}{\tau(\theta^{-1})(-2\pi i)^j\theta(\mathfrak{f}_\eta)\tau(\eta)} & \text{if } \theta\chi^j(-1) = \eta(-1), j \geq 1, \\ \left(\dfrac{p^j}{\eta(p)}\right)^{-n} & \text{if } \theta\chi^j(-1) = -\eta(-1), j \leq 0, \end{cases}$$

and τ denotes the Gauss sum of the given character.

Proof. Let us briefly sketch the construction of these elements. The work of Iwasawa in [Iw72, Chapter 6], building upon the main result of [KL64], attaches to a non-trivial *even* Dirichlet character ξ of conductor not divisible by p^2 an Iwasawa function $L_\xi^{\mathrm{KL}} \in \Lambda_{\mathcal{O}_E}(\Gamma)$ (for any E containing the values of ξ) such that

$$(\chi/\omega)^n(L_\xi^{\mathrm{KL}}) = \left(1 - (\xi\omega^{n-1})(p)p^{-n}\right)L(n, \xi\omega^{n-1})$$

for all $n \in \mathbb{Z}_{\leq 0}$. When $\xi = \mathbf{1}$, Iwasawa constructs $L_{\mathbf{1}}^{\mathrm{KL}} \in \mathrm{Frac}(\Lambda_{\mathbb{Z}_p}(G_\infty))$ such that $(\gamma_0 - \chi(\gamma_0))L_{\mathbf{1}}^{\mathrm{KL}} \in \Lambda_{\mathbb{Z}_p}(G_\infty)$ with the same interpolation property. We write

$$L_\eta = \sum_{a \in \mathbb{Z}/(p-1)\mathbb{Z}} \pi_{\omega^a} L_{\eta,a},$$

where π_{ω^a} are the idempotents defined in §2.2 and where

$$L_{\eta,a} := \begin{cases} \mathrm{Tw}_1\left(\left(L^{\mathrm{KL}}_{\eta^{-1}\omega^a}\right)^{\iota}\right) & \text{if } \eta(-1) = (-1)^a, \\ L^{\mathrm{KL}}_{\eta\omega^{1-a}} & \text{if } \eta(-1) = -(-1)^a. \end{cases}$$

□

Next, we recall the existence of p-adic L-functions for newforms.

Theorem 4.2. *Let f be a newform of level prime to p. Let α be a root of $x^2 - a_p(f)x + \epsilon(p)p^{k-1}$ such that $h := \mathrm{ord}_p(\alpha) < k-1$ and let $\overline{\alpha}$ denote the other root. There exist $\Omega_f^{\pm} \in \mathbb{C}^{\times}$ and $L_{f,\alpha} \in \mathcal{H}_{h,E}(G_\infty)$ such that for all integers $j \in [1, k-1]$ and Dirichlet characters θ of conductor p^n, we have*

$$\theta\chi^j(L_{f,\alpha}) = e_{f,\alpha}(\theta, j)\frac{L(f, \theta^{-1}, j)}{\Omega_f(\theta, j)}$$

where

$$e_{f,\alpha}(\theta, j) = \left(\frac{p^j}{\alpha}\right)^n \left(1 - \theta^{-1}(p)\overline{\alpha}p^{-j}\right)\left(1 - \frac{\theta(p)p^{j-1}}{\alpha}\right),$$

and

$$\frac{1}{\Omega_f(\theta, j)} = \frac{\Gamma(j)}{\tau(\theta^{-1})(-2\pi i)^j\Omega_f^{\delta}},$$

where $\delta = (-1)^j\theta(-1)$. Moreover, $L_{f,\alpha}$ is uniquely determined by its values at $\theta\chi^j$ given above.

Proof. This is the main result of [AV75]. □

We now recall Pollack's construction of plus and minus p-adic L-functions in [Pol03]. The plus and minus logarithms are defined as follows.

Definition 4.3. *Let $b \geq 1$ be an integer and define*[2]

$$\log_b^+ = \prod_{a=1}^{b}\prod_{n=1}^{\infty}\frac{\Phi_{2n}(\chi(\gamma_0)^{-a}\gamma_0)}{p},$$

$$\log_b^- = \prod_{a=1}^{b}\prod_{n=1}^{\infty}\frac{\Phi_{2n-1}(\chi(\gamma_0)^{-a}\gamma_0)}{p},$$

where Φ_m denotes the p^mth cyclotomic polynomial.

[2]We take a different normalization than Pollack does. In [Pol03, Corollary 4.2], he attaches to an f of weight k two elements that we denote here $\log^{\pm}_{\mathrm{Pollack}}$ which are related to our logarithms by

$$\log^{\pm}_{k-1} = p^{k-1}\,\mathrm{Tw}_{-1}\log^{\pm}_{\mathrm{Pollack}}.$$

Recall that $\log_b^{\pm} \in \mathcal{H}_{\frac{b}{2}}(G_\infty)$ and that the zeroes of $\log_b^+$ (resp., $\log_b^-$) are simple and located at $\chi^j\theta$ for all integers $j \in [1, b]$ and for all θ of conductor p^{2n+1} (resp., p^{2n}) for positive integers n.

The following is the main result of [Pol03].

Theorem 4.4. *Let f and α be as in Theorem 4.2 with $a_p(f) = 0$. There exist $L_f^{\pm} \in \Lambda_E(G_\infty)$ (independent of α) such that*

$$L_{f,\alpha} = L_f^+ \log_{k-1}^+ + \alpha L_f^- \log_{k-1}^- .$$

In particular, if α and $\overline{\alpha}$ are the two roots to $x^2 + \epsilon(p)p^{k-1} = 0$, we have $\overline{\alpha} = -\alpha$ and

$$\begin{aligned} L_f^+ &= \frac{\overline{\alpha} L_{f,\alpha} - \alpha L_{f,\overline{\alpha}}}{(\overline{\alpha} - \alpha)\log_{k-1}^+}, \\ L_f^- &= \frac{L_{f,\overline{\alpha}} - L_{f,\alpha}}{(\overline{\alpha} - \alpha)\log_{k-1}^-}. \end{aligned} \tag{4.1}$$

Therefore, we can readily combine Theorems 4.2 and 4.4 to obtain the interpolating formulae of $L_f^{\pm}$ as follows.

Lemma 4.5. *Let $j \in [1, k-1]$ be an integer and θ a Dirichlet character of conductor $p^n > 1$. Write δ as in Theorem 4.2. If n is even, then*

$$\theta\chi^j(L_f^+) = \frac{1}{(-\epsilon(p))^{n/2} p^{n(\frac{k-1}{2} - j)} \theta\chi^j(\log_{k-1}^+)} \cdot \frac{L(f, \theta^{-1}, j)}{\Omega_f(\theta, j)},$$

and if $n = 1$,

$$\theta\chi^j(L_f^+) = 0.$$

For all odd n, we have

$$\theta\chi^j(L_f^-) = \frac{1}{(-\epsilon(p))^{(n+1)/2} p^{n(\frac{k-1}{2} - j) + \frac{k-1}{2}} \theta\chi^j(\log_{k-1}^-)} \cdot \frac{L(f, \theta^{-1}, j)}{\Omega_f(\theta, j)}.$$

Moreover,

$$\begin{aligned} \chi^j(L_f^+) &= \frac{1 - p^{-1}}{\chi^j(\log_{k-1}^+)} \cdot \frac{L(f, j)}{\Omega_f(\mathbf{1}, j)}, \\ \chi^j(L_f^-) &= \frac{p^{-j} + \epsilon(p)^{-1} p^{j-k}}{\chi^j(\log_{k-1}^-)} \cdot \frac{L(f, j)}{\Omega_f(\mathbf{1}, j)}. \end{aligned}$$

Proof. As a sample calculation, for L_f^- we have

$$\begin{aligned}
\theta\chi^j(L_f^-) &= (e_{f,\overline{\alpha}}(\theta,j) - e_{f,\alpha}(\theta,j)) \cdot \frac{L(f,\theta^{-1},j)}{(\overline{\alpha}-\alpha)\theta\chi^j(\log_{k-1}^-)\Omega_f(\theta,j)} \\
&= \left(\left(\frac{p^j}{\overline{\alpha}}\right)^n - \left(\frac{p^j}{\alpha}\right)^n\right) \cdot \frac{L(f,\theta^{-1},j)}{(-2\alpha)\theta\chi^j(\log_{k-1}^-)\Omega_f(\theta,j)} \\
&= \left(\frac{2p^{nj}}{(-\alpha)^n}\right) \cdot \frac{L(f,\theta^{-1},j)}{(-2\alpha)\theta\chi^j(\log_{k-1}^-)\Omega_f(\theta,j)} \\
&= \left(\frac{p^{nj}}{(-\alpha)^{n+1}}\right) \cdot \frac{L(f,\theta^{-1},j)}{\theta\chi^j(\log_{k-1}^-)\Omega_f(\theta,j)} \\
&= \left(\frac{p^{nj}}{(-\alpha\overline{\alpha})^{\frac{n+1}{2}}}\right) \cdot \frac{L(f,\theta^{-1},j)}{\theta\chi^j(\log_{k-1}^-)\Omega_f(\theta,j)} \\
&= \left(\frac{p^{nj}}{(-\alpha\overline{\alpha})^{\frac{n+1}{2}}}\right) \cdot \frac{L(f,\theta^{-1},j)}{\theta\chi^j(\log_{k-1}^-)\Omega_f(\theta,j)},
\end{aligned}$$

which gives the result. □

Remark 4.6.

(i) *For a fixed j, $L_f^\pm$ is uniquely determined by $\theta\chi^j(L_f^+)$ for an infinite number of θ since it is in $\Lambda_E(G_\infty)$. Therefore, it is uniquely determined by the interpolating properties given in Lemma 4.5.*

(ii) *The interpolation property for the other n is not known because $\log_{k-1}^\pm$ vanishes there. To distinguish the above characters, $\theta\chi^j$ will be called a* plus-critical twist *(resp., a* minus-critical twist*) of f if $1 \le j \le k-1$ with θ of conductor p^n with $n = 1$ or even (resp., with $n = 0$ or odd).*

(iii) *It is clear from the lemma that L_f^+ vanishes at the plus-critical twist $\theta\chi^j$ if and only if θ has conductor p or the complex L-value vanishes at the central point. Furthermore, L_f^- vanishes at the minus-critical twist $\theta\chi^j$ if and only if k is even, $\epsilon(p) = -1$, $\theta = \mathbf{1}$, and $j = \frac{k}{2}$ (the central point) or the complex L-value vanishes at the central point.*

For notational simplicity later on, we define the following.

Definition 4.7. *Let $j \in [1, k-1]$ be an integer and θ a Dirichlet character of conductor p^n with $n \ge 0$. We define*

$$e_f^+(\theta,j) = \begin{cases} \dfrac{1-\theta(p)p^{-1}}{(-\epsilon(p))^{n/2}p^{n(\frac{k-1}{2}-j)}\theta\chi^j(\log_{k-1}^+)} & \text{if } n \text{ is even,} \\ 0 & \text{if } n = 1. \end{cases}$$

Similarly, we define

$$e_f^-(\theta,j)=\begin{cases}\dfrac{1}{(-\epsilon(p))^{(n+1)/2}p^{n(\frac{k-1}{2}-j)+\frac{k-1}{2}}\theta\chi^j(\log_{k-1}^-)} & \text{if } n \text{ is odd},\\ \dfrac{p^{-j}+\epsilon(p)^{-1}p^{j-k}}{\chi^j(\log_{k-1}^-)} & \text{if } n=0.\end{cases}$$

4.2. Mixed plus and minus p-adic L-functions for V_m. We now define mixed plus and minus p-adic L-functions for V_m with the decomposition of Corollary 3.5 in mind.

Let $\mathfrak{S}$ denote the set of all choices of $\pm$ for $i=0,1,\dots,\widetilde{r}-1$, so that $\#\mathfrak{S}=2^{\widetilde{r}}$. For $\mathfrak{s}\in\mathfrak{S}$, let $\mathfrak{s}_i$ denote the ith choice and let $\mathfrak{s}^\pm$ denote the number of plusses and minuses, respectively.

Definition 4.8. *Let f_i be the modular forms as given in Proposition 3.4. For an $\mathfrak{s}\in\mathfrak{S}$, define*

$$L_{V_m}^{\mathfrak{s}}=\left(\prod_{i=0}^{\widetilde{r}-1}\mathrm{Tw}_{(r-i)(k-1)}\left(L_{f_i}^{\mathfrak{s}_i}\right)\right)\cdot\begin{cases}L_{\epsilon_K^r} & m \text{ even},\\ 1 & m \text{ odd}.\end{cases}$$

Remark 4.9. *We have that $L_{V_m}^{\mathfrak{s}}\in\Lambda_E(G_\infty)$ for m odd; whereas if m is even, then $L_{V_m}^{\mathfrak{s}}\in\Lambda_E(G_\infty)$ unless $4|m$, in which case $(\gamma_0-1)(\gamma_0-\chi(\gamma_0))L_{V_m}^{\mathfrak{s}}\in\Lambda_E(G_\infty)$.*

Definition 4.10. *If θ is a Dirichlet character of conductor p^n, $j\in\mathbb{Z}$, and $\mathfrak{s}\in\mathfrak{S}$, then the character $\theta\chi^j$ will be called $\mathfrak{s}$-critical (for V_m) if $(\theta,j)\in C_m$ and*

$$\begin{cases}n=1 \text{ or is even} & \text{if } \mathfrak{s}^-=0,\\ n=0 \text{ or is odd} & \text{if } \mathfrak{s}^+=0,\\ n=0 \text{ or } 1 & \text{otherwise.}\end{cases}$$

For $\mathfrak{s}\in\mathfrak{S}$ and for $\mathfrak{s}$-critical $\theta\chi^j$, define

$$e_{V_m}^{\mathfrak{s}}(\theta,j):=\left(\prod_{i=0}^{\widetilde{r}-1}e_{f_i}^{\mathfrak{s}_i}(\theta,j+(r-i)(k-1))\right)\cdot\begin{cases}e_{\epsilon_K^r}(\theta,j) & m \text{ even},\\ 1 & m \text{ odd}.\end{cases}$$

For all $(\theta,j)\in C_m$, let

$$\Omega_m(\theta,j):=\left(\prod_{i=0}^{\widetilde{r}-1}\Omega_{f_i}(\theta,j+(r-i)(k-1))\right)\cdot\begin{cases}\Omega_{\epsilon_K^r}(\theta,j) & m \text{ even},\\ 1 & m \text{ odd}.\end{cases}$$

Theorem 4.11. *Let $\mathfrak{s}\in\mathfrak{S}$. If $\theta\chi^j$ is $\mathfrak{s}$-critical, then*

$$\theta\chi^j\left(L_{V_m}^{\mathfrak{s}}\right)=e_{V_m}^{\mathfrak{s}}(\theta,j)\frac{L(V_m,\theta^{-1},j)}{\Omega_m(\theta,j)}.$$

Proof. This follows from the definitions together with Theorem 4.1 and Lemma 4.5. □

Let ω^a be a character on Δ. If $L^{\mathfrak{s}}_{V_m} = \sum_{\sigma\in\Delta, n\geq 0} a_{\sigma,n}\sigma(\gamma_0 - 1)^n$, the ω^a-isotypical component of $L^{\mathfrak{s}}_{V_m}$ is given by

$$L^{\mathfrak{s},\omega^a}_{V_m} := \pi_{\omega^a} \cdot L^{\mathfrak{s}}_{V_m} = \pi_{\omega^a} \sum_{n\geq 0} \left(\sum_{\sigma\in\Delta} a_{\sigma,n}\eta(\sigma) \right) (\gamma_0 - 1)^n.$$

Therefore,

$$\theta\chi^j \left(L^{\mathfrak{s},\omega^a}_{V_m} \right) = \begin{cases} \theta\chi^j \left(L^{\mathfrak{s}}_{V_m} \right) & \text{if } \theta\chi^j|_\Delta = \omega^a, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.12. *Let ω^a be a character on Δ. We have $L^{\mathfrak{s},\omega^a}_{V_m} \neq 0$. Moreover, if either of $\mathfrak{s}^+$ or $\mathfrak{s}^-$ vanishes, it is uniquely determined by its interpolation property given by Theorem 4.11.*

Proof. The proof of [L11, Lemma 6.5] shows that $L^{\pm,\omega^a}_{f_i} \neq 0$ for all i. Furthermore, it is known that $L^{\omega^a}_{\epsilon^r_K} \neq 0$. The same then follows for $L^{\mathfrak{s},\omega^a}_{V_m}$ by definition. The uniqueness is immediate since $L^{\mathfrak{s},\omega^a}_{V_m} \in \pi_{\omega^a}\Lambda_E(G_\infty)$ (or possibly its field of fractions) and when $\mathfrak{s}$ consists of only plusses or minuses, there is at least one j for which there are infinitely many θ such that $\theta\chi^j$ is $\mathfrak{s}$-critical. □

5. Selmer groups and the Main Conjecture of Iwasawa Theory

In this section, we formulate the Main Conjecture of Iwasawa Theory for V_m. Since we are in the non-ordinary setting, this involves a relation between the mixed plus and minus p-adic L-functions introduced in the previous section and the mixed plus and minus Selmer groups that we define below. We once again take advantage of the decomposition of Proposition 3.4 to define these Selmer groups of V_m as the direct sum of the plus and minus Selmer groups of the ϵ^r_K and V_{f_i}. The latter were defined by the second author in [L11, Section 4]. We then apply the results of Mazur–Wiles [MzW84] and [L11, Section 7].

5.1. Selmer groups. Fix a $G_{\mathbb{Q}}$-stable $\mathcal{O}_E$-lattice T_ψ of V_ψ. It gives rise to natural $\mathcal{O}_E$-lattices $T_{\psi^{m-2i}}$ for $0 \leq i \leq \widetilde{r} - 1$, which in turn give rise to natural $\mathcal{O}_E$-lattices T_{f_i} in V_{f_i} which are stable under $G_{\mathbb{Q}}$. Write $V_{\epsilon^r_K}$ (resp. $T_{\epsilon^r_K}$) for the representation over E (resp. $\mathcal{O}_E$) associated to ϵ^r_K when m is even. Let $A_{f_i} = V_{f_i}/T_{f_i}$, and $A_{\epsilon^{m/2}_K} = V_{\epsilon^{m/2}_K}/T_{\epsilon^{m/2}_K}$. We briefly remind the reader about the Selmer groups of $A^*_{\mathbf{1}}$, $A^*_{\epsilon_K}$, and $A^*_{f_i}$.

Let θ denote $\mathbf{1}$ or ϵ_K and let $F = \mathbb{Q}$ or K, respectively. Classical Iwasawa theory studies the modules $X_\infty := \mathrm{Gal}(M_\infty/F_\infty)$, where M_∞

is the maximal p-abelian extension of F_∞ unramified outside of p, and $Y_\infty := \mathrm{Gal}(L_\infty/F_\infty)$, where L_∞ is the maximal p-abelian extension of F_∞ everywhere unramified. The arguments of [G89, Section 1] show that, as $\mathrm{Gal}(F_\infty/\mathbb{Q})$-modules,

$$H^1_f(F_\infty, A^*_\theta) \cong \mathrm{Hom}_{\mathrm{cts}}(X_\infty, A^*_\theta) \tag{5.1}$$

and

$$H^1_f(F_\infty, A_\theta) \cong \mathrm{Hom}_{\mathrm{cts}}(Y_\infty, A_\theta).^3 \tag{5.2}$$

In order to get the Selmer group over k_∞ when $\theta = \epsilon_K$, we take the ϵ_K-isotypic piece (for the action of $\mathrm{Gal}(F_\infty/k_\infty)$). Thus,

$$H^1_f(k_\infty, A^*_\theta) \cong \mathrm{Hom}_{\mathrm{cts}}(X_\infty, A^*_\theta)^\theta. \tag{5.3}$$

and

$$H^1_f(k_\infty, A_\theta) \cong \mathrm{Hom}_{\mathrm{cts}}(Y_\infty, A_\theta)^\theta. \tag{5.4}$$

We define

$$\mathrm{Sel}_{k_\infty}(A^*_\theta) := \pi^+ \cdot H^1_f(k_\infty, A^*_\theta(-1))(1) + \pi^- \cdot H^1_f(k_\infty, A^*_\theta), \tag{5.5}$$

where

$$\pi^\pm = \sum_{\substack{a \in \mathbb{Z}/(p-1) \\ \theta(-1) = \pm(-1)^a}} \pi_{\omega^a}.$$

For comments on this definition, see Remark 5.4

We now make an additional hypothesis.

Hypothesis 5.1. *For $0 \le i \le \widetilde{r} - 1$, we have $(p+1) \nmid (m-2i)(k-1)$.*

Lemma 5.2. *If Hypothesis 5.1 holds, then $\left(A^*_{f_i}(j)\right)^{G_{\mathbb{Q}_{p,n}}} = 0$ for all $0 \le i \le \widetilde{r} - 1$, $j \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\ge 0}$.*

Proof. Recall that f_i is of weight $(m-2i)(k-1)+1$, so Hypothesis 5.1 implies [L11, Assumption (2)]. Therefore, the result follows from [L11, Lemma 4.4]. □

As in [L11, Section 4], the restriction map $H^1(\mathbb{Q}_{p,\nu}, T^*_{f_i}) \to H^1(\mathbb{Q}_{p,n}, T^*_{f_i})$ is injective for any integers $n \ge \nu \ge 0$. On identifying the former as a subgroup of the latter, we define

$$H^{1,\pm}_f(\mathbb{Q}_{p,n}, T^*_{f_i}) = \Big\{ x \in H^1_f(\mathbb{Q}_{p,n}, T^*_{f_i}) : \mathrm{cor}_{n/\nu+1}(x) \in H^1_f(\mathbb{Q}_{p,\nu}, T^*_{f_i}) \text{ for all } \nu \in S^\pm_n \Big\} \tag{5.6}$$

[3] More precisely, Greenberg relates the right-hand side to his Selmer group. Appealing to Proposition 4.2 of [Och00] then shows that the left-hand side has finite index in the right-hand side. It is an old result of Iwasawa's that the latter has no proper finite index submodules.

where cor denotes the corestriction map and

$$S_n^+ = \{\nu \in [0, n-1] : \nu \text{ even}\},$$
$$S_n^- = \{\nu \in [0, n-1] : \nu \text{ odd}\}.$$

Let

$$H_f^{1,\pm}(\mathbb{Q}_{p,n}, A_{f_i}^*) = H_f^{1,\pm}(\mathbb{Q}_{p,n}, T_{f_i}^*) \otimes E/\mathcal{O}_E \subset H_f^1(\mathbb{Q}_{p,n}, A_{f_i}^*).$$

The plus and minus Selmer groups over $k_n = \mathbb{Q}(\mu_{p^n})$ are defined by

$$H_f^{1,\pm}(k_n, A_{f_i}^*) = \ker\left(H_f^1(k_n, A_{f_i}^*) \to \frac{H^1(\mathbb{Q}_{p,n}, A_{f_i}^*)}{H_f^{1,\pm}(\mathbb{Q}_{p,n}, A_{f_i}^*)}\right)$$

and those over $k_\infty = \mathbb{Q}(\mu_{p^\infty})$ are defined by

$$H_f^{1,\pm}(k_\infty, A_{f_i}^*) = \varinjlim H_f^{1,\pm}(k_n, A_{f_i}^*),$$

which can be identified as a subgroup of $H^1(k_\infty, A_{f_i}^*)$. For $j = 0, 1, \dots, (m-2i)(k-1)-1$, we may identify $H_f^1(k_\infty, A_{f_i}^*)$ with $H_f^1(k_\infty, A_{f_i}^*(j))$ by multiplying a cocycle of $H_f^1(k_\infty, A_{f_i}^*)$ by e_j (note that this does not give us a $\Lambda_{\mathbb{Z}_p}(\Gamma)$-isomorphism since the actions of $\Lambda_{\mathbb{Z}_p}(\Gamma)$ differ by Tw_j).

We use this to define the Tate twisted plus and minus Selmer groups of $A_{f_i}^*(j)$ by

$$H_f^{1,\pm}(k_\infty, A_{f_i}^*(j)) := H_f^{1,\pm}(k_\infty, A_{f_i}^*) \cdot e_j. \tag{5.7}$$

Now,

$$T_m := \left(\bigoplus_{i=0}^{\widetilde{r}-1} T_{f_i}((i-r)(k-1))\right) \oplus \begin{cases} T_{\epsilon_K^r} & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$$

is a $G_{\mathbb{Q}}$-stable $\mathcal{O}_E$-lattice in V_m. Let $A_m = V_m/T_m$. Since Selmer groups decompose under direct sums, we make the following definition for the mixed plus and minus Selmer groups of A_m^*.

Definition 5.3. *Let $\mathfrak{s} \in \mathfrak{S}$. Under Hypothesis 5.1, we define $\mathrm{Sel}_{k_\infty}^{\mathfrak{s}}(A_m^*)$ to be*

$$\left(\bigoplus_{i=0}^{\widetilde{r}-1} H_f^{1,\mathfrak{s}_i}\left(k_\infty, A_{f_i}^*\left((r-i)(k-1)\right)\right)\right) \oplus \begin{cases} \mathrm{Sel}_{k_\infty}(A_{\epsilon_K^r}^*) & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$$

Remark 5.4. *The definition of the Selmer group in the Dirichlet character case may seem odd at first and, when m is even, it may seem strange that the modular forms are not given the same treatment in the definition of $\mathrm{Sel}_{k_\infty}^{\mathfrak{s}}(A_m^*)$. We make a few comments to clarify this situation. First, the complications arise because we insist on defining our Selmer groups over k_∞ as opposed to the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. While this is commonly done when discussing modular forms, it is quite rarely done for*

Dirichlet characters exactly because of the parity issues we circumvent with our definitions (both of the Selmer groups in this section and the analytic p-adic L-functions of Theorem 4.1). Second, concerning the modular forms when m is even, the definition in (5.7) *shows that*

$$H_f^{1,\mathfrak{s}_i}\left(k_\infty, A_{f_i}^*\left((r-i)(k-1)\right)(-1)\right)(1) \cong H_f^{1,\mathfrak{s}_i}\left(k_\infty, A_{f_i}^*\left((r-i)(k-1)\right)\right),$$

so that defining the Selmer groups of the f_i using the $\pi^\pm$ as in the Dirichlet character case would actually yield isomorphic Selmer groups.

5.2. Main conjectures. We begin by recalling the results of [MzW84, L11] on the main conjectures for Dirichlet characters and CM newforms.

Write

$$L_{\mathbf{1},a} = \frac{F_a}{G_a}$$

with $F_a, G_a \in \Lambda_{\mathbb{Z}_p}(\Gamma)$ and

$$G_a = \begin{cases} \gamma_0 - 1 & a = 0, \\ \gamma_0 - \langle\gamma_0\rangle & a = 1, \\ 1 & \text{otherwise.} \end{cases}$$

To streamline the statement of the main conjecture, let

$$\widetilde{L}_{\mathbf{1}} := \sum_{a \in \mathbb{Z}/(p-1)\mathbb{Z}} \pi_{\omega^a} F_a$$

and $\widetilde{L}_{\epsilon_K} := L_{\epsilon_K}$. The results of Mazur and Wiles are equivalent to the following theorem.

Theorem 5.5 ([MzW84])**.** *Let $\theta = \mathbf{1}$ or ϵ_K. Then, for all $a \in \mathbb{Z}/(p-1)\mathbb{Z}$,*

$$\mathrm{Char}_{\Lambda_{\mathbb{Z}_p}(\Gamma)}\left(\left(\mathrm{Sel}_{k_\infty}(A_\theta^*)^\vee\right)^{\omega^a}\right) = \left(\left(\mathrm{Tw}_1(\widetilde{L}_\theta)\right)^{\omega^a}\right).$$

Proof. Since $\left(\mathrm{Tw}_1(\widetilde{L}_\theta)\right)^{\omega^a} = \mathrm{Tw}_1(\widetilde{L}_{\theta,a+1})$, the definition of $\widetilde{L}_\theta$ together with the results of Mazur–Wiles show that the right-hand is

$$\begin{cases} \mathrm{Tw}_1\left(\mathrm{Char}_{\Lambda_{\mathbb{Z}_p}(\Gamma)}\left(Y_\infty^{\theta\omega^{a+1}}\right)\right) & \text{if } \theta(-1) = (-1)^a \\ \mathrm{Tw}_1\left(\mathrm{Char}_{\Lambda_{\mathbb{Z}_p}(\Gamma)}\left(X_\infty^{\theta\omega^{a+1}}\right)\right) & \text{if } \theta(-1) = -(-1)^a. \end{cases}$$

We then use that fact that $(M(j))^{\omega^a} \cong (M^{\omega^{a-j}})(j)$ for G_∞-modules M and that $\mathrm{Char}_{\Lambda_{\mathbb{Z}_p}(\Gamma)}(M(j)) = \mathrm{Tw}_{-j}(\mathrm{Char}_{\Lambda_{\mathbb{Z}_p}(\Gamma)}(M))$ for Γ-modules M, together with equations (5.3) and (5.4), to confirm that we have simply rephrased the results of [MzW84]. □

Let

$$\widetilde{L}_{f_i}^\pm := \sum_{a \in \mathbb{Z}/(p-1)\mathbb{Z}} \frac{\pi_{\omega^a} L_{f_i}^\pm}{\mathrm{Tw}_{-1}\left(\mathfrak{l}_{f_i,a}^\pm\right)},$$

where $\mathfrak{l}^{\pm}_{f_i,a}$ generate the images of the plus and minus Coleman maps, as explained in the appendix. The following is a consequence of the work of the second author combined with that of [KPZ10].

Theorem 5.6 ([L11, Section 6])**.** *Assume Hypothesis 5.1. Then, for all $a \in \mathbb{Z}/(p-1)\mathbb{Z}$, the module $\left(H^{1,\pm}_f\left(k_\infty, A^*_{f_i}((r-i)(k-1))\right)^\vee\right)^{\omega^a}$ is $\Lambda_E(\Gamma)$-torsion and*

$$\mathrm{Char}_{\Lambda_E(\Gamma)}\left(\left(H^{1,\pm}_f\left(k_\infty, A^*_{f_i}((r-i)(k-1))\right)^\vee\right)^{\omega^a}\right) = \left((\mathrm{Tw}_{(r-i)(k-1)+1}(\widetilde{L}^{\pm}_{f_i}))^{\omega^a}\right).$$

In particular, Kato's main conjecture for f_i holds.

Remark 5.7. *In both* [PolRu04] *and* [L11]*, the main conjecture was only proved for the case where $a = 0$ and f_i is defined over $\mathbb{Q}$ with K having class number* 1*, but using the techniques developed in* [KPZ10]*, the same proof in fact works for general a and f_i.*

For $\mathfrak{s} \in \mathfrak{S}$, define

$$\widetilde{L}^{\mathfrak{s}}_{V_m} = \left(\prod_{i=0}^{\widetilde{r}-1} \mathrm{Tw}_{(r-i)(k-1)}\left(\widetilde{L}^{\mathfrak{s}_i}_{f_i}\right)\right) \cdot \begin{cases} \widetilde{L}_{\epsilon_K^r} & m \text{ even}, \\ 1 & m \text{ odd}. \end{cases}$$

We are now ready to formulate the Main Conjecture of Iwasawa Theory for V_m.

Conjecture 5.8. *For all $\mathfrak{s} \in \mathfrak{S}$ and $a \in \mathbb{Z}/(p-1)\mathbb{Z}$, we have equality*

$$\mathrm{Char}_{\Lambda_E(\Gamma)}\left((\mathrm{Sel}^{\mathfrak{s}}_{k_\infty}(A^*_m)^\vee)^{\omega^a}\right) = \left((\mathrm{Tw}_1(\widetilde{L}^{\mathfrak{s}}_{V_m}))^{\omega^a}\right).$$

We have the following result towards this conjecture.

Theorem 5.9. *Assume Hypothesis 5.1. Then, for all $\mathfrak{s} \in \mathfrak{S}$ and $a \in \mathbb{Z}/(p-1)\mathbb{Z}$, $\left(H^{1,\mathfrak{s}}_f(k_\infty, A^*_m)^\vee\right)^{\omega^a}$ is $\Lambda_E(\Gamma)$-torsion and*

$$\mathrm{Char}_{\Lambda_E(\Gamma)}\left((\mathrm{Sel}^{\mathfrak{s}}_{k_\infty}(A^*_m)^\vee)^{\omega^a}\right) = \left((\mathrm{Tw}_1(\widetilde{L}^{\mathfrak{s}}_{V_m}))^{\omega^a}\right).$$

Proof. Since

$$\mathrm{Char}_{\Lambda_E(\Gamma)}(M_1 \oplus M_2) = \mathrm{Char}_{\Lambda_E(\Gamma)}(M_1) \cdot \mathrm{Char}_{\Lambda_{\mathbb{Q}_p}(\Gamma)}(M_2),$$

the theorem follows from Theorems 5.5 and 5.6. □

6. The admissible p-adic L-functions

In this section, we construct admissible[4] p-adic L-functions for the symmetric powers of CM newforms for non-ordinary primes, thus verifying the conjecture of Dabrowski–Panchishkin ([Dab11, Conjecture 1],[Pan94, Conjecture 6.2]). In [L12], the second author used elliptic units to construct two p-adic L-functions for the symmetric square lying in $\mathcal{H}_{k-1,E}(G_\infty)$, but only proved that they interpolate the L-values $L(\mathrm{Sym}^2(V_f), \theta^{-1}, 2k-2)$ (or equivalently $L(V_2, \theta^{-1}, k-1)$). Here, we construct $2^{\widetilde{r}}$ p-adic L-functions for V_m and we show that they have the expected interpolation property at all critical twists and the expected growth rate. We also relate them to the mixed plus and minus p-adic L-functions $L^{\mathfrak{s}}_{V_m}$ introduced in Definition 4.8, thus providing an interesting analogue of Pollack's decomposition of p-adic L-functions of non-ordinary modular forms. As in [H13], these p-adic L-functions are defined as products of p-adic L-functions of newforms and Dirichlet characters as given by the decomposition in Corollary 3.5. This approach was also taken by Dabrowski in [Dab93] for $\mathrm{Sym}^m(V_f)$. We then go on to prove the trivial zero conjecture for V_m using a recent result of Denis Benois ([Ben13]).

6.1. Definition and basic properties. Recall that each newform f_i in the decomposition of Corollary 3.5 has weight $k_i := (m-2i)(k-1)+1$ and let ϵ_i denote its Nebentypus. It follows from the proof of Lemma 3.10 that the roots of $x^2 + \epsilon_i(p)p^{(m-2i)(k-1)}$ are $\pm p^{(r-i)(k-1)}$ if m is even and $\pm\alpha p^{(r-i)(k-1)}$ if m is odd. Accordingly, we let

$$\alpha_{i,\pm} := \begin{cases} \pm p^{(r-i)(k-1)} & \text{if } m \text{ is even,} \\ \pm\alpha p^{(r-i)(k-1)} & \text{if } m \text{ is odd.} \end{cases} \tag{6.1}$$

Since $h_i := \mathrm{ord}_p(\alpha_{i,\pm}) < k_i - 1$, the result of Amice–Vélu (Theorem 4.2) provides for each choice of i and $\pm$ a p-adic L-function $L_{f_{i,\pm}} \in \mathcal{H}_{h_i,E}(G_\infty)$ (which should not be confused with the notation for Pollack's p-adic L-functions $L^{\pm}_{f_i}$). Recall the notation concerning elements of $\mathfrak{S}$ as introduced in section §4.2. Corollary 3.5 suggests the following definition.

[4]The qualifier "admissible" here is what is commonly used to distinguish the more standard p-adic L-functions constructed in this section from the plus/minus p-adic L-functions constructed above. The terminology refers to the fact that, in general, the usual p-adic L-function conjecturally attached to a motive is not expected to be given by the Mellin transform of a p-adic measure, but rather by that of an unbounded p-adic distribution; such distributions are still expected to be so-called "admissible measures". The prototypical examples are the two p-adic L-functions of an elliptic curve over $\mathbb{Q}$ with supersingular reduction at p. In this case, Rob Pollack showed that one could "change basis" to obtain two p-adic measures (giving the plus and minus p-adic L-functions).

Definition 6.1. *For each $\mathfrak{s} \in \mathfrak{S}$, define*

$$L_{V_m,\mathfrak{s}} = \left(\prod_{i=0}^{\widetilde{r}-1} \mathrm{Tw}_{(r-i)(k-1)}\left(L_{f_i,\mathfrak{s}_i}\right)\right) \cdot \begin{cases} L_{\epsilon_K^r} & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases} \tag{6.2}$$

This gives $2^{\widetilde{r}}$ p-adic L-functions for V_m. Note that these $2^{\widetilde{r}}$ functions come from fixing a root of $x^2 + \epsilon_i(p)p^{(m-2i)(k-1)}$ for each i; this is different from the $2^{\widetilde{r}}$ choices of §4.2.

Theorem 6.2. *For each $\mathfrak{s} \in \mathfrak{S}$ and each $(\theta, j) \in C_m$, where θ has conductor p^n, we have*

$$\theta\chi^j(L_{V_m,\mathfrak{s}}) = e_{m,\mathfrak{s}}(\theta, j)\frac{L(V_m, \theta^{-1}, j)}{\Omega_m(\theta, j)}, \tag{6.3}$$

where $e_{m,\mathfrak{s}}(\theta, j)$ is given by

$$\left(\prod_{i=0}^{\widetilde{r}-1} e_{f_i,\alpha_{i,\mathfrak{s}_i}}(\theta, j + (r-i)(k-1))\right) \cdot e_{\epsilon_K^r}(\theta, j) =$$
$$\frac{1}{p^{n(k-1)\frac{r(r+1)}{2}}} \left((-1)^n\left(1 - \theta^{-1}(p)p^{-j}\right)\left(1 + \theta(p)p^{j-1}\right)\right)^{\mathfrak{s}^-}$$
$$\times \left(\left(1 + \theta^{-1}(p)p^{-j}\right)\left(1 - \theta(p)p^{j-1}\right)\right)^{\mathfrak{s}^+} e_{\epsilon_K^r}(\theta, j)$$

when m is even and

$$\left(\prod_{i=0}^{\widetilde{r}-1} e_{f_i,\alpha_{i,\mathfrak{s}_i}}(\theta, j + (r-i)(k-1))\right) =$$
$$\frac{1}{\left(\alpha^{r+1}p^{(k-1)\frac{r(r+1)}{2}}\right)^n} \left((-1)^n\left(1 - \theta^{-1}(p)\alpha p^{-j}\right)\left(1 + \frac{\theta(p)p^{j-1}}{\alpha}\right)\right)^{\mathfrak{s}^-}$$
$$\times \left(\left(1 + \theta^{-1}(p)\alpha p^{-j}\right)\left(1 - \frac{\theta(p)p^{j-1}}{\alpha}\right)\right)^{\mathfrak{s}^+}$$

when m is odd and $\Omega_m(\theta, j)$ is as in Definition 4.10.

Furthermore,

$$L_{V_m,\mathfrak{s}} \in \mathcal{H}_{(k-1)\frac{d^+d^-}{2},E}(G_\infty) \tag{6.4}$$

unless $4|m$, in which case

$$(\gamma_0 - 1)(\gamma_0 - \chi(\gamma_0))L_{V_m,\mathfrak{s}} \in \mathcal{H}_{(k-1)\frac{d^+d^-}{2},E}(G_\infty).$$

Remark 6.3. *When $m \geq 2$, $L_{V_m,\mathfrak{s}}$ is not uniquely determined by its interpolation property. Indeed, an element of $\mathcal{H}_a$ needs to satisfy an interpolation property at $\theta\chi^j$ for at least $\lfloor a+1 \rfloor$ choices of j. When $m \geq 4$ is even, $(k-1)\frac{d^+d^-}{2} \geq 3(k-1)$ is greater than the number of distinct j such that $(\theta, j) \in C_m$ (this number being $2(k-1)$). Similarly, when $m \geq 3$ is odd, the latter number is only $k-1$ which is less than $(k-1)\frac{d^+d^-}{2} \geq 2(k-1)$. When $m = 2$, $(k-1)\frac{d^+d^-}{2} = 2(k-1)$, but the parity condition on (θ, j) implies that only half of the j's can be used.*

Proof. One can easily verify that C_m is a subset of all the pairs (θ, j) at which the elements $\mathrm{Tw}_{(r-i)(k-1)}(L_{f_i,\mathfrak{s}_i})$ and $L_{\epsilon_K^r}$ satisfy the interpolation properties of Theorems 4.2 and 4.1, respectively. The interpolation property for $L_{V_m,\mathfrak{s}}$ then follows immediately from its definition.

As for the growth condition, Theorems 4.2 and 4.1 tell us that

$$\mathrm{Tw}_{(r-i)(k-1)}(L_{f_i,\mathfrak{s}_i}) \in \mathcal{H}_{(\frac{m}{2}-i)(k-1),E}(G_\infty)$$

(as the twisting operation does not affect the growth) and $L_{\epsilon_K^r} \in \mathrm{Frac}(\mathcal{H}_0)$. When taking a product, the growth rates are additive; hence, the growth of $L_{V_m,\mathfrak{s}}$ is

$$\sum_{i=0}^{\widetilde{r}-1} \left(\frac{m}{2} - i\right)(k-1).$$

A simple summation then gives the stated growth rate. The statements about being in $\mathcal{H}_{(k-1)\frac{d^+d^-}{2},E}(G_\infty)$ rather than its fraction field follow by the corresponding statements in Theorems 4.2 and 4.1. □

Proposition 6.4. *Let ω^a be a character of Δ. For each $\mathfrak{s}$, the ω^a-isotypical component of $L_{V_m,\mathfrak{s}}$ is non-zero.*

Proof. This follows from the same proof as Proposition 4.12. □

6.2. A conjecture of Dabrowski–Panchishkin. We begin this section by computing the generalized Hasse invariant $h_p(V_m)$ of V_m. Recall that $h_p(V_m) := \max(P_N(d^\pm, V_m) - P_H(d^\pm, V_m))$ (see e.g. [Pan94, §5]). We obtain the following from lemmas 3.8 and 3.11.

Lemma 6.5. *For all $m \geq 2$,*

$$h_p(V_m) = (k-1)\frac{d^+d^-}{2}.$$

Proof. If m is even, $P_N(x, V_m)$ is identically zero, so

$$h_p(V_m) = \max(-P_H(d^\pm, V_M)).$$

A simple computation shows that

$$h_p(V_m) = -P_H(r+1, V_m) = (k-1)\frac{r(r+1)}{2},$$

as desired. When m is odd, we have $d^+ = d^- = r+1$. Thus,

$$h_p(V_m) = P_N(r+1, V_m) - P_H(r+1, V_m) = (r+1)\frac{k-1}{2} - (k-1)\frac{(r+1)(-r)}{2},$$

which yields the result. □

This lemma, together with (6.4) of Theorem 6.2, shows that $L_{V_m,\mathfrak{s}}$ has the growth property predicted by [Dab11, Conjecture 1(iv)] and [Pan94, Conjecture 6.2(iv)], and (6.3) of Theorem 6.2 shows that it satisfies the expected interpolation property.

6.3. Decomposition into mixed plus and minus *p*-adic *L*-functions. We generalize the decomposition of Theorem 4.4, due to Pollack, to our setting. In other words, we decompose each of the $2^{\widetilde{r}}$ p-adic L-functions $L_{V_m,\mathfrak{s}}$ introduced in Definition 6.1 as a linear combination of the $2^{\widetilde{r}}$ p-adic L-functions $L^{\mathfrak{s}}_{V_m}$ given in Definition 4.8.

Let

$$\ell_i^+ := \log_{k_i-1}^+ \text{ and } \ell_i^- := \alpha_{i,+}\log_{k_i-1}^-.$$

Then, (4.1) becomes

$$L_{f_i}^{\mathfrak{s}_i} = \frac{L_{f_i,+} + \mathfrak{s}_i L_{f_i,-}}{2\ell_i^{\mathfrak{s}_i}}, \tag{6.5}$$

and we have the following result.

Lemma 6.6. *For all $\mathfrak{s} \in \mathfrak{S}$, we have*

$$2^{\widetilde{r}}\ell_{V_m}^{\mathfrak{s}} L_{V_m}^{\mathfrak{s}} = \sum_{\mathfrak{t}\in\mathfrak{S}} a_{\mathfrak{s},\mathfrak{t}} L_{V_m,\mathfrak{t}} \tag{6.6}$$

where

$$\ell_{V_m}^{\mathfrak{s}} = \prod_{i=0}^{\widetilde{r}-1} \mathrm{Tw}_{(r-i)(k-1)}\left(\ell_i^{\mathfrak{s}_i}\right)$$

and $a_{\mathfrak{s},\mathfrak{t}} \in \{+1,-1\}$. Moreover, $a_{\mathfrak{s},\mathfrak{t}}$ is given by $(-1)^{b_{\mathfrak{s},\mathfrak{t}}}$ where $b_{\mathfrak{s},\mathfrak{t}}$ is the number of $i \in [0, \widetilde{r}-1]$ such that $\mathfrak{s}_i = \mathfrak{t}_i = -$.

Proof. On substituting (6.5) into the definition of $L^{\mathfrak{s}}_{V_m}$, we have

$$L^{\mathfrak{s}}_{V_m} = \left(\prod_{i=0}^{\widetilde{r}-1} \mathrm{Tw}_{(r-i)(k-1)}\left(\frac{L_{f_i,+} + \mathfrak{s}_i L_{f_i,-}}{2\ell_i^{\mathfrak{s}_i}}\right)\right) \cdot \begin{cases} L_{\epsilon_K^r} & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases}$$

Hence, upon expanding, we obtain (6.6). To find $a_{\mathfrak{s},\mathfrak{t}}$, we observe that the sign in front of $L_{V_m,\mathfrak{t}}$ is given by

$$\prod_{\substack{i=0 \\ \mathfrak{t}_i=-}}^{\widetilde{r}-1} \mathfrak{s}_i,$$

so we are done. □

Let $\mathfrak{A}$ be the $2^{\widetilde{r}} \times 2^{\widetilde{r}}$ matrix $(a_{\mathfrak{s},\mathfrak{t}})_{(\mathfrak{s},\mathfrak{t})\in\mathfrak{S}\times\mathfrak{S}}$.

Proposition 6.7. *The matrix $\mathfrak{A}$ is symmetric and has orthogonal columns and rows.*

Proof. It is clear from the description of $a_{\mathfrak{s},\mathfrak{t}}$ in Lemma 6.6 that $\mathfrak{A}$ is symmetric. For the orthogonality, we show that for $\mathfrak{t} \neq \mathfrak{u}$,

$$\#\{\mathfrak{s} \in \mathfrak{S} : a_{\mathfrak{s},\mathfrak{t}} = a_{\mathfrak{s},\mathfrak{u}}\} = \#\{\mathfrak{s} \in \mathfrak{S} : a_{\mathfrak{s},\mathfrak{t}} = -a_{\mathfrak{s},\mathfrak{u}}\} = 2^{\widetilde{r}-1}, \tag{6.7}$$

which would imply that the $(\mathfrak{t},\mathfrak{u})$-entry

$$\sum_{\mathfrak{s}\in\mathfrak{S}} a_{\mathfrak{s},\mathfrak{t}} a_{\mathfrak{s},\mathfrak{u}}$$

of $\mathfrak{A}\mathfrak{A}^{\mathrm{T}}$ is zero

Let $I = \{i : \mathfrak{t}_i \neq \mathfrak{u}_i\}$. By definition, $\#I \geq 1$. Let $\mathfrak{s} \in \mathfrak{S}$ be such that $a_{\mathfrak{s},\mathfrak{t}} = a_{\mathfrak{s},\mathfrak{u}}$. By Lemma 6.6,

$$b_{\mathfrak{s},\mathfrak{t}} \equiv b_{\mathfrak{s},\mathfrak{u}} \pmod 2. \tag{6.8}$$

For each $i \in I$, either $(\mathfrak{t}_i, \mathfrak{u}_i) = (+,-)$ or $(-,+)$. Hence, each $i \in I$ such that $\mathfrak{s}_i = -$ contributes a "one" to one side of the equation in (6.8), but not to the other side. But for $i \notin I$, each $\mathfrak{s}_i = -$ would contribute a "one" to both sides. Therefore, in order for (6.8) to hold, $\#\{i \in I : \mathfrak{s}_i = -\}$ must be even, but for $i \notin I$, no such condition is required. Hence, the number of choices coming from $i \in I$ is given by

$$\sum_{n \text{ even}} \binom{\#I}{n} = 2^{\#I-1}$$

and for $i \notin I$, there are $2^{\widetilde{r}-\#I}$ choices. This gives $2^{\widetilde{r}-1}$ choices for $\mathfrak{s}$ such that $a_{\mathfrak{s},\mathfrak{t}} = a_{\mathfrak{s},\mathfrak{u}}$. Since this is half of $\mathfrak{S}$, the other half must have $a_{\mathfrak{s},\mathfrak{t}} = -a_{\mathfrak{s},\mathfrak{u}}$. □

Corollary 6.8. *The matrix $\mathfrak{A}$ is invertible with inverse $2^{-\widetilde{r}}\mathfrak{A}$.*

Proof. Since all entries of $\mathfrak{A}$ are $+1$ or -1, the fact that $\mathfrak{A}$ is symmetric and has orthogonal columns and rows implies that $\mathfrak{A}^2$ is a diagonal matrix. The $(\mathfrak{t},\mathfrak{t})$-entry of $\mathfrak{A}^2$ is

$$\sum_{\mathfrak{s}\in\mathfrak{S}} a_{\mathfrak{s},\mathfrak{t}}^2 = 2^{\widetilde{r}},$$

hence the result. □

Corollary 6.9. *For all $\mathfrak{t} \in \mathfrak{S}$, we have an expansion*

$$L_{V_m,\mathfrak{t}} = \sum_{s\in\mathfrak{S}} a_{\mathfrak{s},\mathfrak{t}} \ell_{V_m}^{\mathfrak{s}} L_{V_m}^{\mathfrak{s}}.$$

Proof. This follows immediately from Lemma 6.6 and Corollary 6.8. □

6.4. Trivial zeros and L-invariants. In this section, we will use the main result of [Ben13] to determine the (analytic) L-invariants of V_m. It follows that these L-invariants are given by Benois's (arithmetic) L-invariant (as defined in [Ben11]). We also discuss a case that does not fit into the standard situation of trivial zeroes namely when $4|m$ we can say something at a couple of non-critical twists.

From now on, if m and k are both odd and $\alpha = \pm p^{\frac{k-1}{2}}$, then we fix the choice $\alpha = p^{\frac{k-1}{2}}$. We can read off the following from Theorems 4.1 and 4.2.

Lemma 6.10.

(i) *The p-adic L-function $L_{\epsilon_K^r}$ has no trivial zeroes.*
(ii) *If m is even, then $\mathrm{Tw}_{(r-i)(k-1)}(L_{f_i,+})$ (resp., $\mathrm{Tw}_{(r-i)(k-1)} L_{f_i,-}$) has a trivial zero at the critical twist (θ, j) if and only if $j = 1$ (resp., $j = 0$) and $\theta = \mathbf{1}$.*
(iii) *If m is odd, then $\mathrm{Tw}_{(r-i)(k-1)}(L_{f_i,+})$ (resp., $\mathrm{Tw}_{(r-i)(k-1)} L_{f_i,-}$) has a trivial zero at the critical twist (θ, j) if and only if $j = \frac{k+1}{2}$ (resp., $j = \frac{k-1}{2}$), $\theta = \mathbf{1}$, k is odd, and $\alpha = p^{\frac{k-1}{2}}$.*

Proof. In all cases, θ must be trivial otherwise $\theta(p) = 0$.

For $L_{\epsilon_K^r}$, a trivial zero must occur at $j = 0$ or 1 and requires that $\epsilon_K^r(p) = 1$. Since p is inert in K, we have $\epsilon_K(p) = -1$, so r needs to be even. But for even r, $(\mathbf{1}, j) \notin C_m$ for $j = 0, 1$.

For the f_i, in all cases,

$$e_{f_i,\alpha_{i,\pm}}(\mathbf{1}, j) = \left(1 - \alpha_{i,\mp} p^{-j}\right)\left(1 - \frac{p^{j-1}}{\alpha_{i,\pm}}\right). \tag{6.9}$$

For m even, (6.1) tells us that this is equal to

$$\left(1 \pm p^{\frac{k_i-1}{2}-j}\right)\left(1 \mp p^{j-\frac{k_i+1}{2}}\right)$$

since $(r-i)(k-1) = \frac{k_i-1}{2}$. Hence, $L_{f_i,-}$ has a trivial zero at $j = \frac{k_i-1}{2}$ and $L_{f_i,+}$ has one at $j = \frac{k_i+1}{2}$. The twist operation simply shifts these to the left by $\frac{k_i-1}{2}$. When m is odd, the proof is similar. In this case, we can write $\alpha = \zeta p^{\frac{k-1}{2}}$ where ζ is a root of unity. Then, using (6.1) shows that (6.9) becomes

$$\left(1 \pm \zeta p^{\frac{k_i-1}{2}-j}\right)\left(1 \mp \zeta p^{j-\frac{k_i+1}{2}}\right)$$

since $(r-i)(k-1) + \frac{k-1}{2} = \frac{k_i-1}{2}$. If k is even, then k_i is odd, so $\frac{k_i \pm 1}{2} \notin \mathbb{Z}$, so no trivial zeroes can occur. Assume that k is odd. Then, ζ must be ± 1, in which case we have fixed the choice $\zeta = 1$. In this case, $L_{f_i,-}$ has a trivial zero at $j = \frac{k_i-1}{2}$ and $L_{f_i,+}$ has one at $j = \frac{k_i+1}{2}$. The twist operation shifts these to the left by $\frac{k_i-1}{2} - \frac{k-1}{2}$. □

For the character ω^a of Δ and an element $L \in \mathrm{Frac}(\mathcal{H}_\infty(G_\infty))$, we write

$$L_a(s) := \langle \pi_{\omega^a} L \rangle^s$$

for the ath branch of L. This is a meromorphic function of $s \in \mathbb{Z}_p$ and for $j \in \mathbb{Z}$, $L_a(j) = \omega^{a-j}\chi^j(L)$.

The following is a special case of the main theorem of [Ben13].

Theorem 6.11 (Theorem 4.3.2 of [Ben13])**.** *Let f' be a newform of prime-to-p level, Nebentypus ϵ', and odd weight k' with $a_p(f') = 0$. Suppose $\alpha' = p^{\frac{k'-1}{2}}$ is a root of $x^2 + \epsilon'(p)p^{k'-1}$. Then, for $a = \frac{k'+1}{2}$ or $\frac{k'-1}{2}$, Benois' arithmetic L-invariant $\mathcal{L}_{V_{f'},\alpha'}(\mathbf{1}, a) \in \mathbb{C}_p$ of (V'_f, α') at $(\mathbf{1}, a)$ (as defined in* [Ben11]*) satisfies*

$$\lim_{s \to a} \frac{L_{f',\alpha',a}(s)}{s-a} = \mathcal{L}_{V_{f'},\alpha'}(\mathbf{1}, a)\left(1 + \frac{1}{p}\right)\frac{L(f', a)}{\Omega_{f'}(\mathbf{1}, a)}.$$

Proof. Here are some remarks that explain how the above statement follows from [Ben13, Theorem 4.3.2]. First, what we denote $\mathcal{L}_{V_{f'},\alpha'}(\mathbf{1}, \frac{k'+1}{2})$ is denoted $-\mathscr{L}_{\alpha'}(f')$ by Benois. Since we assume that $a_p(f') = 0$, we know that the other root of $x^2 + \epsilon'(p)p^{k-1}$ is $\overline{\alpha}' = -\alpha'$, hence $\epsilon'(p) = -1$ and φ acts semisimply on $\mathbb{D}_{\mathrm{cris}}(V_f)$. What we write $L_{f',\alpha',a}(s)$ is denoted $L_{p,\alpha'}(f, \omega^a, s)$ in [Ben13]. Finally, to obtain the statement for $a = \frac{k'-1}{2}$ from [Ben13, Theorem 4.3.2], one uses the compatibility of Benois' L-invariant with the p-adic functional equation as discussed in [Ben13, §0.3, Remark 3]. □

Note that for $a \in \mathbb{Z}/(p-1)\mathbb{Z}$, the p-adic L-function $L_{V_m,\mathfrak{s},a}(s)$ is an analytic function of $s \in \mathbb{Z}_p$ unless $4|m$, in which case it follows from Theorem 6.2 that simple poles arise for $L_{V_m,\mathfrak{s},1}(s)$ at $s = 1$ and for $L_{V_m,\mathfrak{s},0}(s)$ at $s = 0$. Translating C_m into this setting, we obtain the following lemma.

Lemma 6.12. *An integer j is critical for $L_{V_m,\mathfrak{s},a}$ if and only if*

$$\begin{cases} -(k-1)+1 \le j \le 0 & \text{if } m \text{ is even and } a \text{ and } r \text{ have opposite parities,} \\ 1 \le j \le k-1 & \text{otherwise.} \end{cases}$$

In particular, note that for even m the integers 0 and 1 are only critical when r is odd.

Proof. It follows from the fact that evaluating $L_{V_m,\mathfrak{s},a}(s)$ at $s = j$ is the same as evaluating $\omega^{a-j}\chi^j(L_{V_m,\mathfrak{s}})$. □

Recall from Proposition 3.4 that when m is even the f_i all have odd weight, and when m is odd their weight has the same parity as the weight of f. This allows us to appeal to Benois' theorem to define the L-invariant

of $(V_m, \mathfrak{s})$ as a product of the L-invariants of the f_i. If m is even, then for $a = 0$ or 1, let $\sigma_a = \operatorname{sgn}(a - \frac{1}{2})$ and

$$\mathcal{L}_{V_m,\mathfrak{s}}(\mathbf{1}, a) := \prod_{\substack{i=0 \\ \mathfrak{s}_i = \sigma_a}}^{\widetilde{r}-1} \mathcal{L}_{V_{f_i,\alpha_{i,\mathfrak{s}_i}}}\left(\mathbf{1}, a + \frac{k_i - 1}{2}\right). \tag{6.10}$$

If m is odd, with k odd and $\alpha = p^{\frac{k-1}{2}}$, then for $a = 0$ or 1, let

$$\mathcal{L}_{V_m,\mathfrak{s}}\left(\mathbf{1}, a + \frac{k-1}{2}\right) := \prod_{\substack{i=0 \\ \mathfrak{s}_i = \sigma_a}}^{\widetilde{r}-1} \mathcal{L}_{V_{f_i,\alpha_{i,\mathfrak{s}_i}}}\left(\mathbf{1}, a + \frac{k_i - 1}{2}\right). \tag{6.11}$$

Theorem 6.13. *Suppose j is critical for $L_{V_m,\mathfrak{s},a}(s)$.*

(i) *If m is even, then*

$$\operatorname{ord}_{s=j} L_{V_m,\mathfrak{s},a}(s) \text{ is } \begin{cases} \geq \mathfrak{s}^+ & \text{if } a = j = 1 \text{ (and } r \text{ is odd)}, \\ \geq \mathfrak{s}^- & \text{if } a = j = 0 \text{ (and } r \text{ is odd)}, \\ = 0 & \text{otherwise.} \end{cases}$$

Furthermore, for r odd,

$$\lim_{s\to 1} \frac{L_{V_m,\mathfrak{s},1}(s)}{(s-1)^{\mathfrak{s}^+}} = 2^{\mathfrak{s}^-+1} \cdot \left(1 - \frac{1}{p}\right)^{\mathfrak{s}^-} \cdot \left(1 + \frac{1}{p}\right)^{\mathfrak{s}^+} \cdot \mathcal{L}_{V_m,\mathfrak{s}}(\mathbf{1}, 1) \cdot \frac{L(V_m, 1)}{\Omega_m(\mathbf{1}, 1)}$$

and

$$\lim_{s\to 0} \frac{L_{V_m,\mathfrak{s},0}(s)}{s^{\mathfrak{s}^-}} = 2^{\mathfrak{s}^++1} \cdot \left(1 - \frac{1}{p}\right)^{\mathfrak{s}^+} \cdot \left(1 + \frac{1}{p}\right)^{\mathfrak{s}^-} \cdot \mathcal{L}_{V_m,\mathfrak{s}}(\mathbf{1}, 0) \cdot \frac{L(V_m, 0)}{\Omega_m(\mathbf{1}, 0)}.$$

(ii) *If m is odd, then*

$$\operatorname{ord}_{s=j} L_{V_m,\mathfrak{s},a}(s) \text{ is } \begin{cases} \geq \mathfrak{s}^+ & \text{if } k \text{ odd, } \alpha = p^{\frac{k-1}{2}}, \text{ and } a = j = \dfrac{k+1}{2}, \\ \geq \mathfrak{s}^- & \text{if } k \text{ odd, } \alpha = p^{\frac{k-1}{2}}, \text{ and } a = j = \dfrac{k-1}{2}, \\ = ? & \text{if } k \text{ even, } j = \dfrac{k}{2}, \text{ and } L\left(V_m, \omega^{\frac{k}{2}-a}, \dfrac{k}{2}\right) = 0 \\ = 0 & \text{otherwise.} \end{cases}$$

Furthermore, in the first two cases,

$$\lim_{s\to\frac{k+1}{2}} \frac{L_{V_m,\mathfrak{s},\frac{k+1}{2}}(s)}{\left(s - \frac{k+1}{2}\right)^{\mathfrak{s}^+}} =$$

$$2^{\mathfrak{s}^-} \cdot \left(1 - \frac{1}{p}\right)^{\mathfrak{s}^-} \cdot \left(1 + \frac{1}{p}\right)^{\mathfrak{s}^+} \cdot \mathcal{L}_{V_m,\mathfrak{s}}\left(\mathbf{1}, \frac{k+1}{2}\right) \cdot \frac{L\left(V_m, \frac{k+1}{2}\right)}{\Omega_m\left(\mathbf{1}, \frac{k+1}{2}\right)}$$

and

$$\lim_{s\to\frac{k-1}{2}} \frac{L_{V_m,\mathfrak{s},\frac{k-1}{2}}(s)}{\left(s-\frac{k-1}{2}\right)^{\mathfrak{s}^-}} = 2^{\mathfrak{s}^+}\cdot\left(1-\frac{1}{p}\right)^{\mathfrak{s}^+}\cdot\left(1+\frac{1}{p}\right)^{\mathfrak{s}^-}\cdot\mathcal{L}_{V_m,\mathfrak{s}}\left(\mathbf{1},\frac{k-1}{2}\right)\cdot\frac{L\left(V_m,\frac{k-1}{2}\right)}{\Omega_m\left(\mathbf{1},\frac{k-1}{2}\right)}.$$

In the third case, we have that the p-adic interpolation factor does not vanish.

In all cases, $\mathcal{L}_{V_m,\mathfrak{s}}(\mathbf{1},a)$ equals Benois' arithmetic L-invariant of $(V_m,\mathfrak{s})$ at $(\mathbf{1},a)$ as defined in [Ben11].

Remark 6.14. *The inequalities in the theorem would become equalities if the corresponding L-invariants of the newforms f_i were known to be non-zero. When m is odd, k is even, and $j=\frac{k}{2}$, we are at a central point which complicates matters since the order of vanishing of the archimedean L-function is a subtle point. Since the p-adic interpolation factor does not vanish, it would be natural to conjecture that*

$$\operatorname{ord}_{s=\frac{k}{2}} L_{V_m,\mathfrak{s},a}(s) = \operatorname{ord}_{s=\frac{k}{2}} L\left(V_m,\omega^{\frac{k}{2}-a},s\right).$$

Proof. Let $L_{i,\mathfrak{s}_i,a}(s)=\mathrm{Tw}_{(r-i)(k-1)}L_{f_i,\mathfrak{s}_i,a}(s)$. Suppose m is even. The central point of the L-function of V_m is at $s=1/2$, so none of the values we are considering are central. Therefore, none of the archimedean L-values we deal with vanish and all zeroes must come from the p-adic interpolation factor. Indeed, away from the near-central points, the non-vanishing of the archimedean L-functions at critical integers follows from their definition as an Euler product to the right of the critical strip $0<s<1$ and by the functional equation to the left of the critical strip (since we are only considering critical integers). At the near-central points, the non-vanishing is a classical theorem for $L(\epsilon_K,s)$ and a result of Jacquet–Shalika for newforms ([JS76]). We have that

$$L_{V_m,\mathfrak{s},a}(s) = L_{\epsilon_K^r,a}(s)\cdot\prod_{i=0}^{\widetilde{r-1}} L_{i,\mathfrak{s}_i,a}(s).$$

We know from Lemma 6.10 that $L_{\epsilon_K^r,a}(s)$ has no trivial zeroes at critical j and similarly for $L_{i,\mathfrak{s}_i,a}(s)$ when $(\mathfrak{s}_i,a,j)\neq(+,1,1)$ or $(-,0,0)$. This shows that the order of vanishing of $L_{V_m,\mathfrak{s},a}(j)$ is zero away from these two exceptional cases. Note that the exceptional cases do not correspond to critical j when r is even, so we may assume r is odd from now on. For

$a = 0, 1$, we get from Benois' result (Theorem 6.11) that near $s = a$,

$$L_{i,\sigma_a,a}(s) = (s-a)\mathcal{L}_{V_{f_i},\alpha_{i,\sigma_a}}\left(\mathbf{1}, a + \frac{k_i - 1}{2}\right)\left(1 + \frac{1}{p}\right)\frac{L(f_i, a)}{\Omega_{f_i}(\mathbf{1}, a)} + \text{higher order terms.}$$

Since

$$L_{\epsilon_K,a}(a) = 2\frac{L(\epsilon_K, a)}{\Omega_{\epsilon_K}(\mathbf{1}, a)}$$

and

$$L_{i,-\sigma_a,a}(a) = 2\left(1 - \frac{1}{p}\right)\frac{L\left(f_i, a + \frac{k_i-1}{2}\right)}{\Omega_{f_i}(\mathbf{1}, a)},$$

the statements in part (i) hold.

Now, suppose m is odd. If k is even, we know from Lemma 6.10 that no trivial zeroes occur. But $j = \frac{k}{2} \in \mathbb{Z}$ is the central point and the order of vanishing of the archimedean L-function at j can be positive. If it is, we have no knowledge of the order of vanishing of the p-adic L-function, but if it isn't, we know the latter is non-vanishing at the central point. Outside of this case, we are at non-central points so the archimedean L-values are non-zero as explained above. We may therefore assume k is odd. Using Lemma 6.10 again, we know that the order of vanishing of $L_{V_m,\mathfrak{s},a}$ is zero away from the two exceptional cases $(\alpha, \mathfrak{s}_i, a, j) = \left(p^{\frac{k-1}{2}}, +, \frac{k+1}{2}, \frac{k+1}{2}\right)$ and $\left(p^{\frac{k-1}{2}}, -, \frac{k-1}{2}, \frac{k-1}{2}\right)$. The rest of the proof proceeds as in the case of even m.

We know from [Ben11] that the $\mathcal{L}_{V_{f_i},\alpha_{i,\pm}}(\mathbf{1}, a)$ are equal to Benois' arithmetic L-invariants. That the $\mathcal{L}_{V_m,\mathfrak{s}}(\mathbf{1}, a)$ are as well follows from the fact that Benois' L-invariant is multiplicative on direct sums of representations of $G_{\mathbb{Q}}$. □

An interesting phenomenon occurs for $4|m$. In this case, each of the twisted p-adic L-functions of the f_i has a trivial zero at $(\mathbf{1}, 1)$ or $(\mathbf{1}, 0)$, but V_m itself is not critical at these characters because of the presence of the p-adic L-function of $\epsilon_K^r = \mathbf{1}$. However, this latter p-adic L-function is given by $\zeta_p(s)$ (resp. $\zeta_p(1-s)$) at the corresponding branches. Here $\zeta_p(s)$ is the p-adic Riemann zeta function of Kubota–Leopoldt and we know its residue at $s = 1$.

Theorem 6.15. *If $4|m$ and $j = 0, 1$, then*

$$\operatorname{ord}_{s=j} L_{V_m,\mathfrak{s},a}(s) \text{ is } \begin{cases} \geq \mathfrak{s}^+ - 1 & \text{if } a = j = 1, \\ \geq \mathfrak{s}^- - 1 & \text{if } a = j = 0, \\ = 0 & \text{otherwise.} \end{cases}$$

Furthermore,

$$\lim_{s\to 1}\frac{L_{V_m,\mathfrak{s},1}(s)}{(s-1)^{\mathfrak{s}^+-1}}=2^{\mathfrak{s}^-}\cdot\left(1-\frac{1}{p}\right)^{\mathfrak{s}^-+1}\cdot\mathcal{L}_{V_m,\mathfrak{s}}(\mathbf{1},1)\cdot\frac{L(V_m,1)}{\Omega_m(\mathbf{1},1)}$$

and

$$\lim_{s\to 0}\frac{L_{V_m,\mathfrak{s},0}(s)}{s^{\mathfrak{s}^--1}}=-2^{\mathfrak{s}^+}\cdot\left(1-\frac{1}{p}\right)^{\mathfrak{s}^++1}\cdot\mathcal{L}_{V_m,\mathfrak{s}}(\mathbf{1},0)\cdot\frac{L(V_m,0)}{\Omega_m(\mathbf{1},0)}.$$

The inequalities would become equalities if the corresponding L-invariants of the f_i were known to be non-zero.

Proof. The proof is along the same lines as the previous theorem noting that near $s=1$

$$\zeta_p(s)=\left(1-\frac{1}{p}\right)\cdot\frac{1}{s-1}+\text{ higher order terms.}$$

□

Appendix A. Plus and minus Coleman maps

In [L11, Section 5], we describe the images of plus and minus Coleman up to isomorphism. Here, we will give a more explicit description using ideas in [LLZ11].

Let us first review the definition of the Coleman maps in [L11]. Let f be a modular form as in Section 2.5 with $a_p=0$ but not necessarily CM. Let $H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V_f)=\mathbb{Q}_p\otimes\varprojlim H^1(\mathbb{Q}_{p,n},T_f)$ (note that V_f in this paper is defined to be the Tate dual of the V_f in [L11]). The plus and minus Coleman maps are $\Lambda_E(G_\infty)$-homomorphisms

$$\mathrm{Col}^{\pm}:H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V_f)\to\Lambda_E(G_\infty),$$

which are defined by the relation

$$\mathrm{Tw}_1(\log^{\pm}_{k-1})\,\mathrm{Col}^{\pm}=\mathcal{L}_{\eta^{\pm}},\tag{A.1}$$

where

$$\mathcal{L}_{\eta^{\pm}}:H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V_f)\to\mathcal{H}_{(k-1)/2,E}(G_\infty)$$

are the Perrin-Riou pairing associated to $\eta^{\pm}\in\widetilde{\mathbb{D}}_{\mathrm{cris}}(V_f^*(-1))$. Here, $\eta^-\in\mathrm{Fil}^1\,\widetilde{\mathbb{D}}_{\mathrm{cris}}(V_f^*(-1))$ and $\eta^+=\varphi(\eta^-)$.

Let $\mathbf{z}\in H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V_f)$. We write $z_{-j,n}$ for the natural image of $\mathbf{z}$ in $H^1(\mathbb{Q}_{p,n},V_f(-j))$. For an integer $j\in[0,k-2]$ and a Dirichlet character θ

of conductor $p^n > 1$, we have

$$\chi^j(\mathcal{L}_{\eta^\pm}(\mathbf{z})) = j!\left[\left(1-\frac{\varphi^{-1}}{p}\right)(1-\varphi)^{-1}(\eta^\pm_{j+1}), \exp^*(z_{-j,0})\right]_0, \tag{A.2}$$

$$\theta\chi^j(\mathcal{L}_{\eta^\pm}(\mathbf{z})) = \frac{j!}{\tau(\theta^{-1})}\sum_{\sigma\in G_n}\theta^{-1}(\sigma)\left[\varphi^{-n}(\eta^\pm_{j+1}), \exp^*(z^\sigma_{-j,n})\right]_n. \tag{A.3}$$

Here, $\eta^\pm_{j+1}$ denotes the natural image of $\eta^\pm$ in $\widetilde{\mathbb{D}}_{\mathrm{cris}}(V_f^*(j))$ and $[,]_n$ is the natural pairing on

$$\mathbb{Q}_{p,n}\otimes\widetilde{\mathbb{D}}_{\mathrm{cris}}(V_f^*(j))\times\mathbb{Q}_{p,n}\otimes\widetilde{\mathbb{D}}_{\mathrm{cris}}(V_f(-j))\to\mathbb{Q}_{p,n}\otimes E.$$

Proposition A.1. *The $\Lambda_E(G_\infty)$-homomorphism*

$$\begin{aligned}\underline{\mathrm{Col}} : H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f) &\to \Lambda_E(G_\infty)^{\oplus 2}\\ \mathbf{z} &\mapsto (\mathrm{Col}^+(\mathbf{z}), \mathrm{Col}^-(\mathbf{z}))\end{aligned}$$

has determinant $\prod_{j=0}^{k-2}(\chi^{-j}(\gamma_0)\gamma_0 - 1)$.

Proof. The $\delta(V)$-conjecture of Perrin-Riou in [PR94, Conjecture 3.4.7], which is a consequence of [Col98, Théorème IX.4.5], says that the determinant of the $\Lambda_E(G_\infty)$-homomorphism

$$\begin{aligned}H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f) &\to \mathcal{H}_{(k-1)/2,E}(G_\infty)^{\oplus 2}\\ \mathbf{z} &\mapsto (\mathcal{L}_{\eta^+}(\mathbf{z}), \mathcal{L}_{\eta^-}(\mathbf{z}))\end{aligned}$$

is given by $\prod_{j=0}^{k-2}\log_p(\chi^{-j}(\gamma_0)\gamma_0)$. But we have

$$\prod_{j=0}^{k-2}\log_p(\chi^{-j}(\gamma_0)\gamma_0) = \log^+_{k-1}\times\log^-_{k-1}\times\prod_{j=0}^{k-2}(\chi^{-j}(\gamma_0)\gamma_0 - 1)$$

by definition. Hence the result follows from (A.1). □

Theorem A.2. *The image of* $\underline{\mathrm{Col}}$ *is given by*

$$\begin{aligned}S := \{(F,G)\in\Lambda_E(G)^{\oplus 2} : (\epsilon(p)^{-1}p^{1+j-k}+p^{-j-1})\chi^j(F) = (1-p^{-1})\chi^j(G),\\ \theta\chi^j(F) = 0 \text{ for all integers } 0\le j\le k-2 \text{ and all } \theta \text{ of conductor } p\}.\end{aligned}$$

Proof. Since the set S has determinant $\prod_{j=0}^{k-2}(\chi^{-j}(\gamma_0)\gamma_0 - 1)$, in view of Proposition A.1, we only need to show that $\underline{\mathrm{Col}}(\mathbf{z})$ is contained in S for all $\mathbf{z}\in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f)$.

Fix an integer $j\in[0,k-2]$. If θ is a Dirichlet character of conductor p, (A.3) says that

$$\theta\chi^j(\mathcal{L}_{\eta^+}(\mathbf{z})) = \frac{j!}{\tau(\theta^{-1})}\sum_{\sigma\in G_n}\theta^{-1}(\sigma)\left[\varphi^{-1}(\eta^+_{j+1}), \exp^*(z^\sigma_{-j,n})\right]_n.$$

But

$$\varphi^{-1}(\eta_{j+1}^{+}) = p^{j+1}\eta_{j+1}^{-} \in \mathrm{Fil}^0\,\widetilde{\mathbb{D}}_{\mathrm{cris}}(V_f^*(j))$$

and

$$\left[\eta_{j+1}^{-}, \exp^*(z_{-j,n}^{\sigma})\right]_n = 0. \tag{A.4}$$

Hence $\theta\chi^j(\mathcal{L}_{\eta^+}(\mathbf{z})) = 0$.

On $\widetilde{\mathbb{D}}_{\mathrm{cris}}(V_f^*(j))$, we have $\varphi^2 + \epsilon(p)p^{k-2j-3} = 0$, so [LLZ11, Lemma 5.6] implies that

$$\left(1 - \frac{\varphi^{-1}}{p}\right)(1-\varphi)^{-1} = \frac{(1+\epsilon(p)p^{k-2j-2})\varphi + \epsilon(p)p^{k-2j-3}(p-1)}{\epsilon(p)p^{k-2j-2}(1+\epsilon(p)p^{k-2j-3})}.$$

We write $\delta_j = \epsilon(p)p^{k-2j-2}(1+\epsilon(p)p^{k-2j-3})$, then (A.2) and (A.4) imply that

$$\chi^j(\mathcal{L}_{\eta^+}(\mathbf{z})) = \frac{j!}{\delta_j} \times \epsilon(p)p^{k-2j-3}(p-1) \times \left[\eta_{j+1}^{+}, \exp^*(z_{-j,0})\right]_0,$$

$$\chi^j(\mathcal{L}_{\eta^-}(\mathbf{z})) = \frac{j!}{\delta_j} \times \frac{1+\epsilon(p)p^{k-2j-2}}{p^{j+1}} \times \left[\eta_{j+1}^{+}, \exp^*(z_{-j,0})\right]_0.$$

Therefore, we indeed have

$$(\epsilon(p)^{-1}p^{1+j-k} + p^{-j-1})\chi^j(\mathcal{L}_{\eta^+}(\mathbf{z})) = (1-p^{-1})\chi^j(\mathcal{L}_{\eta^-}(\mathbf{z}))$$

as required. □

Note that $\epsilon(p)^{-1}p^{1+j-k} + p^{-j-1} \neq 0$ unless $\epsilon(p) = -1$ and $j = k/2 - 1$. We therefore deduce that

$$\mathrm{im}(\mathrm{Col}^+) = \{F \in \Lambda_E(G_\infty) : \theta\chi^j(F) = 0 \text{ for all } j \in [0, k-2] \text{ and Dirichlet characters } \theta \text{ of conductor } p\}$$

and

$$\mathrm{im}(\mathrm{Col}^-) = \begin{cases} \{F \in \Lambda_E(G_\infty) : \chi^{\frac{k}{2}-1}(F) = 0\} & \text{if } \epsilon(p) = -1 \text{ and } k \text{ even,} \\ \Lambda_E(G_\infty) & \text{otherwise.} \end{cases}$$

If $a \in \mathbb{Z}/(p-1)\mathbb{Z}$, let

$$\mathfrak{l}_{f,a}^{+} = \prod_{\substack{0 \le j \le k-2 \\ j \not\equiv a \pmod{p-1}}} (\chi^{-j}(\gamma_0)\gamma_0 - 1)$$

and

$$\mathfrak{l}_{f,a}^{-} = \begin{cases} \chi^{-\frac{k}{2}+1}(\gamma_0)\gamma_0 - 1 & \text{if } \epsilon(p) = -1,\ k \text{ even}, a \equiv \frac{k}{2} - 1 \pmod{p-1}, \\ 1 & \text{otherwise.} \end{cases}$$

Then,

$$\mathrm{im}(\mathrm{Col}^{\pm})^{\omega^a} = \Lambda_E(\Gamma)\mathfrak{l}_{f,a}^{\pm}.$$

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