JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 26, n° 2 (2014), p. 531-558.

<http://jtnb.cedram.org/item?id=JTNB_2014__26_2_531_0>

Journal de Théorie des Nombres
de Bordeaux **26** (2014), 531–558

An explicit computation of *p*-stabilized vectors

par Michitaka MIYAUCHI et Takuya YAMAUCHI

Résumé. Nous donnons une méthode concrète pour calculer les vecteurs p-stables dans l'espace des éléments fixés par un sous-groupe parahorique d'un groupe réductif p-adique. Nous discutons d'une application globale et, en particulier, nous donnons un exemple explicite d'un relèvement de Saito-Kurokawa p-stable.

Abstract. In this paper, we give a concrete method to compute p-stabilized vectors in the space of parahori-fixed vectors for connected reductive groups over p-adic fields. An application to the global setting is also discussed. In particular, we give an explicit p-stabilized form of a Saito-Kurokawa lift.

1. Introduction

Let F be a non-archimedean local field, $\mathfrak{o}$ the ring of integers of F, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, ϖ a uniformizer of F, and $\mathbb{F} = \mathfrak{o}/\mathfrak{p}$ the residue field of F. We normalize the valuation $|\cdot|$ of F so that $|\varpi| = q^{-1}$, where q is the cardinality of $\mathbb{F}$. Let G be a connected reductive group defined over F, B its standard Borel subgroup and $K = G(\mathfrak{o})$ a maximal compact subgroup of $G(F)$ whenever it is defined. Let P be a parabolic subgroup of G which contains B and $K_P = \{g \in K |\ g \bmod \mathfrak{p} \in P(\mathbb{F})\}$ the parahoric subgroup of K associated to P.

Let π be an irreducible smooth representation of $G(F)$ such that the space π^{K_P} of K_P-fixed vectors in π is non-trivial. Then the Hecke algebra $\mathcal{H}_{K_P}$ of $G(F)$ associated to K_P acts on π^{K_P}. In this paper, we first give a method to compute the eigenvalues for the special elements of $\mathcal{H}_{K_P}$ on π^{K_P}, which are called "U_p-operators". We next give an explicit construction of simultaneous eigenvectors for these U_p-operators, which are called "p-stabilized vectors".

An idea to compute eigenvalues of U_p-operators is to consider the Jacquet module of π associated to P. When P is the standard Borel subgroup of $G(F)$, this has been a well-known method for experts. We extend a result of Casselman [6] which is proved only for the standard Borel subgroup to any parabolic subgroup P and any compact open subgroup of $G(F)$ which has an Iwahori factorization relative to P in the sense of [6]. Then it can

Manuscrit reçu le 30 novembre 2012, accepté le 3 septembre 2013.
Classification math. 11F85, 22E50.

be reduced a calculation of the eigenvalues for U_p-operators to the same problem for the actions of the elements corresponding to those operators on the Jacquet modules (Proposition 2.3).

Our results might give a potential tool to study an arithmetic investigation of automorphic forms in Iwasawa theory, Hida theory, or deformation theory of Galois representations [14],[21],[20], though we do not discuss about this in this paper. In particular, we will know that what kind of p-stabilized forms can be embedded into a Hida family with respect a specific parabolic subgroup of G.

This paper is organized as follows. In Section 2, we study the action of Hecke algebras on the space of parahori-fixed vectors by using Jacquet modules. In Section 3, we introduce the notion of U_p-operators and p-stabilized vectors. In Section 4, we outline a method to construct p-stabilized vectors when π has a non-zero K-fixed vector. Without this assumption on π, it seems to be difficult to check the non-triviality of the vector which we will construct. In Section 5, 6, and 7, we make up a list of all U_p-eigenvalues and p-stabilized vectors for $GL_2, U(2,1)$, and GSp_4. A relation to the global setting is discussed in Section 8 and then the global p-stabilized forms are given in the final section in cases of GL_2 and GSp_4. In particular, we will give them for Saito-Kurokawa lifts where the existence has been already discussed in Proposition 4.2.2, p.688 of [20] (see (9.2) and (9.3)).

The authors would like to thank Professor Tomonori Moriyama for helpful conversations. The second author is partially supported by JSPS Grant-in-Aid for Scientific Research No.23740027 and JSPS Postdoctoral Fellowships for Research Abroad No.378.

2. The action of Hecke operators via Jacquet modules

We keep the notation in Section 1. In this section, we study parahori-fixed vectors of smooth representations of $G(F)$. We fix a maximal torus T of G and a minimal parabolic subgroup B of G which contains T. Then we have the Levi decomposition $B = TU$, where U is the unipotent radical of B. Let P be a parabolic subgroup of G containing B with Levi decomposition $P = MN$. We denote by $\overline{N}$ the unipotent radical of the parabolic subgroup opposite to P.

Henceforth, for any algebraic group H, we sometimes denote by H the group of F-valued points of H for the sake of simplicity. This should cause no confusion in the remainder of the paper.

For any smooth representation (π, V) of $G(F)$, we define its Jacquet module (π_N, V_N) as follows (cf. Section 3 of [6]): Set $V(N) = \langle \pi(n)v - v \mid n \in N,\ v \in V\rangle$ and $V_N = V/V(N)$. We define a representation (π_N, V_N) of M by

$$\pi_N(m)r_N(v) = \delta_P^{-\frac{1}{2}}(m)r_N(\pi(m)v),\ m \in M,\ v \in V,$$

where r_N is the natural projection from V to V_N and δ_P is the modulus character of $P(F)$.

Let $I := \{g \in K \mid g \bmod \mathfrak{p} \in B(\mathfrak{o}/\mathfrak{p})\}$ be the standard Iwahori subgroup of $G(F)$. If a smooth representation (π, V) of $G(F)$ is admissible, then by p.7, Theorem of [8], the canonical projection $r_U : V \to V_U$ induces a $\mathbb{C}$-linear isomorphism

$$V^I \xrightarrow{\sim} (V_U)^{I \cap T(F)}. \tag{2.1}$$

The following theorem generalizes this isomorphism to any parabolic subgroups.

Theorem 2.1. *Let (π, V) be an admissible representation of $G(F)$. Suppose that a compact subgroup J of $G(F)$ contains I and it has an Iwahori factorization with respect to P in the sense of* [6] *(see before Proposition 1.4.4 in loc.cit.). Then the canonical projection $r_N : V \to V_N$ induces a linear isomorphism*

$$V^J \xrightarrow{\sim} (V_N)^{J \cap M}.$$

Proof. By Theorem 3.3.3 of [6], the map $r_N : V^J \longrightarrow (V_N)^{J \cap M}$ is surjective. We now prove the injectivity of this map. By (2.1), we get $V^I \cap V(U) = \{0\}$. Since $I \subset J$ and $U \supset N$, we have $V^I \supset V^J$ and $V(U) \supset V(N)$. This gives us that $V^J \cap V(N) \subset V^I \cap V(U) = \{0\}$. This implies that $r_N : V^J \longrightarrow (V_N)^{J \cap M}$ is injective. □

As in Theorem 2.1, let J be a compact subgroup of $G(F)$ which contains I. Assume that J has an Iwahori factorization with respect to P.

Definition 2.2. ([5] Definition 6.5) We say that an element $m \in M$ is *positive relative to* (P, J) if the following conditions are fulfilled:

$$m(J \cap N)m^{-1} \subset J \cap N, \ m^{-1}(J \cap \overline{N})m \subset J \cap \overline{N}.$$

We denote by M^+ the set of all positive elements in M. We say that an element m in M is *negative relative to* (P, J) if m^{-1} is positive. We write M^- for the set of all negative elements in M.

Given a compact open subgroup J of $G(F)$, we define the Hecke algebra $\mathcal{H}_J := \mathcal{H}[G(F)//J]$ of $G(F)$ associated to J to be the space of all compactly supported functions $f : G(F) \longrightarrow \mathbb{C}$ which satisfy $f(j_1 g j_2) = f(g)$, for $j_1, j_2 \in J$ and $g \in G(F)$. Then $\mathcal{H}_J$ becomes an algebra under the convolution with respect to the Haar measure on $G(F)$ normalized so that the volume of J is one. For any $g \in G(F)$, we denote by $f_g = [JgJ] \in \mathcal{H}_J$ the characteristic function of JgJ. Since the algebra $\mathcal{H}_J$ is generated by $f_g, \ g \in G(F)$, if we define $\mathcal{H}_{J,\mathbb{Q}} := \mathbb{Q}[f_g \mid g \in G(F)]$, then we have $\mathcal{H}_J = \mathcal{H}_{J,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. For any $\mathbb{Q}$-algebra A, we put $\mathcal{H}_{J,A} := \mathcal{H}_{J,\mathbb{Q}} \otimes_{\mathbb{Q}} A$.

If (π, V) is a smooth representation of $G(F)$, then the Hecke algebra $\mathcal{H}_J$ acts on V^J (cf. [6]). We denote by Z_M the center of M. We consider Hecke operators associated to positive elements in Z_M.

Proposition 2.3. *Let J be as in Definition 2.2 and (π, V) an admissible representation of $G(F)$. Then for any $\zeta \in Z_M \cap M^+$, we have*

$$r_N(\pi(f_\zeta)v) = \delta_P^{-\frac{1}{2}}(\zeta)\pi_N(\zeta)r_N(v), \ v \in V^J,$$

where $r_N : V^J \xrightarrow{\sim} (V_N)^{J \cap M}$ is the isomorphism given in Theorem 2.1.

Proof. For $\zeta \in Z_M \cap M^+$ and $v \in V^J$, we have

$$\pi(f_\zeta)v = \int_{J\zeta J} \pi(g)vdg = \sum_{k \in J/J\cap\zeta J\zeta^{-1}} \pi(k\zeta)v.$$

Since we assume that J has an Iwahori factorization, we get $J = (J \cap \overline{N})(J \cap M)(J \cap N)$. Because ζ is positive and it belongs to Z_M, we have that $J \cap \zeta J \zeta^{-1} = (J \cap \overline{N})(J \cap M)\zeta(J \cap N)\zeta^{-1}$, so that $J/J \cap \zeta J \zeta^{-1} = (J \cap N)/\zeta(J \cap N)\zeta^{-1}$. Therefore we obtain

$$\pi(f_\zeta)v = \sum_{k \in (J\cap N)/\zeta(J\cap N)\zeta^{-1}} \pi(k\zeta)v.$$

Since $r_N(\pi(k\zeta)v) = r_N(\pi(\zeta)v) = \delta_P^{\frac{1}{2}}(\zeta)\pi_N(\zeta)r_N(v)$, $k \in J \cap N$ and $[J \cap N : \zeta(J \cap N)\zeta^{-1}] = \delta_P^{-1}(\zeta)$, we have

$$r_N(\pi(f_\zeta)v) = \delta_P^{-\frac{1}{2}}(\zeta)\pi_N(\zeta)r_N(v),$$

as required. □

Proposition 2.4. *Suppose that a compact open subgroup J of $G(F)$ has an Iwahori factorization relative to P. Then f_{ζ_1} and f_{ζ_2} are commutative, for any ζ_1, $\zeta_2 \in Z_M \cap M^+$.*

Proof. We shall claim that $f_{\zeta_1} * f_{\zeta_2} = f_{\zeta_1\zeta_2}$. Then we obtain

$$f_{\zeta_1} * f_{\zeta_2} = f_{\zeta_1\zeta_2} = f_{\zeta_2\zeta_1} = f_{\zeta_2} * f_{\zeta_1}$$

because $\zeta_1\zeta_2 = \zeta_2\zeta_1$. It follows from [5] (6.6) that

$$[J\zeta_1 J : J] = [J \cap \overline{N} : \zeta_1^{-1}(J \cap \overline{N})\zeta_1][J \cap M : \zeta_1^{-1}(J \cap M)\zeta_1 \cap (J \cap M)].$$

Hence we get

$$[J\zeta_1 J : J] = [J \cap \overline{N} : \zeta_1^{-1}(J \cap \overline{N})\zeta_1]$$

since ζ_1 lies in the center of M. Similarly, we obtain $[J\zeta_2 J : J] = [J \cap \overline{N} : \zeta_2^{-1}(J \cap \overline{N})\zeta_2]$ and $[J\zeta_1\zeta_2 J : J] = [J \cap \overline{N} : (\zeta_1\zeta_2)^{-1}(J \cap \overline{N})\zeta_1\zeta_2]$. Since ζ_1 and ζ_2 are both positive, we have

$$J \cap \overline{N} \supset \zeta_2^{-1}(J \cap \overline{N})\zeta_2 \supset (\zeta_1\zeta_2)^{-1}(J \cap \overline{N})\zeta_1\zeta_2.$$

So we obtain

$$[J\zeta_1\zeta_2 J : J] = [J\zeta_1 J : J][J\zeta_2 J : J],$$

and hence $f_{\zeta_1} * f_{\zeta_2} = f_{\zeta_1\zeta_2}$ by Proposition 2.2 in Chapter 3 of [10]. □

3. *p*-stabilized vectors

For simplicity, we assume that the dimension of the center of G is at most one. Let π be an irreducible smooth representation of $G(F)$. Then π is admissible by [12]. In this section, we give a notion of p-stabilized vectors (or of p-stabilization) for parahori-fixed vectors in π.

Let Δ be the set of all simple roots of (G, T) which is a subset of the character group $X^*(T) := \mathrm{Hom}_{\mathrm{alg}}(T, GL_1)$. Let P be a parabolic subgroup of G containing B, $P = MN$ its Levi decomposition, and K_P the parahoric subgroup which corresponds to P. Let Δ_P be the subset of Δ corresponding to P. We define T_P^- to be the semi-group consisting of the elements t in $T(F)/T(\mathfrak{o})$ such that

(3.1) $\quad |\alpha(t)| \leq 1$ for all $\alpha \in \Delta$ and $t(K_P \cap N)t^{-1} \subset K_P \cap N$.

We can choose a complete system of representatives for T_P^- as elements in $Z_M \cap M^+$. Put $m_P = \sharp(\Delta \setminus \Delta_P)$. Note that $\Delta_B = \emptyset$. For each $\alpha \in \Delta$, there exists $t_\alpha \in T_B^-$ such that $\mathrm{ord}_\varpi \alpha(t_\alpha) = 1$ and $\mathrm{ord}_\varpi \alpha(t_\beta) = 0$ for all $\beta \in \Delta \setminus \{\alpha\}$. Put

$$t_{m_P+1} := \begin{cases} \varpi^{-1}\mathrm{Id}, & \text{if } Z_M \supset F^\times \\ \mathrm{Id}, & \text{otherwise}, \end{cases}$$

where Id is the identity element of $G(F)$. We write $\Delta \setminus \Delta_P = \{\alpha_1, \ldots, \alpha_{m_P}\}$ and $t_i = t_{\alpha_i}$ for $i \in \{1, \ldots, m_P\}$. Then the semi-group T_P^- is generated by $t_1, \ldots, t_{m_P}, t_{m_P+1}$. For any $\mathbb{Q}$-algebra A which is contained in $\mathbb{C}$, we consider the subalgebra

$$\mathcal{U}_{P,A} := A[[K_P t K_P] \mid t \in T_P^-]$$

of the Hecke algebra $\mathcal{H}_{K_P,A}$ over A.

Lemma 3.1. *Put* $U_{\varpi,i}^P := [K_P t_i K_P] \in \mathcal{H}_{K_P}$, *for* $i \in \{1, \ldots, m_P+1\}$. *Then the ring* $\mathcal{U}_{P,A}$ *is a commutative* A*-algebra generated by* $U_{\varpi,1}^P, \ldots, U_{\varpi,m_P+1}^P$.

Proof. Recall that we take a complete system of representatives for T_P^- as elements in $Z_M \cap M^+$. Then the commutativity follows from Proposition 2.4. The later claim follows from the fact that T_P^- is generated by $t_1, \ldots, t_{m_P}, t_{m_P+1}$. □

Definition 3.2. Let (π, V) be an irreducible smooth representation of $G(F)$ such that $V^{K_P} \neq \{0\}$. We say that a non-zero vector v in V^{K_P} is *a p-stabilized vector with respect to* $\widehat{P}$ if it is a simultaneous eigenvector for all $U_{\varpi,1}^P, \ldots, U_{\varpi,m_P}^P$. Here $\widehat{P}$ is the Langlands dual of P (cf. [3]).

Remark 3.3. *The condition (3.1) on T_P^- is crucial to get the commutativity of $\mathcal{U}_{P,A}$. In general, this property does not hold for $\mathcal{H}_{K_P,A}$.*

4. Construction of p-stabilized vectors

Let (π, V) be an irreducible smooth representation of $G(F)$ which has a non-zero K-fixed vector. In this section, we give a method to produce p-stabilized vectors for π. Let $P = MN$ be a parabolic subgroup of $G(F)$ containing B. Then by Theorem 2.1, the Jacquet functor r_N induces an isomorphism $r_N : V^{K_P} \simeq (V_N)^{K_P \cap M}$. We set $W = (V_N)^{K_P \cap M}$. Let H denote the subgroup of Z_M generated by $t_1, \ldots, t_{m_P} \in T_P^-$. As we have seen before, we may assume $H \subset Z_M \cap M^+$. For a quasi-character χ of H and $n \in \mathbb{N}$, we define

$$W_{\chi,n} = \{w \in W \mid (\pi_N(t) - \chi(t))^n w = 0 \text{ for any } t \in H\}$$

and put $W_{\chi,\infty} = \bigcup_{n \in \mathbb{N}} W_{\chi,n}$. Similarly, we define $(V_N)_{\chi,\infty}$ for V_N. Let $\mathcal{S}$ denote the set of quasi-characters χ of H such that $W_{\chi,\infty} \neq \{0\}$. Since W is a finite-dimensional H-module, we have

$$W = \bigoplus_{\chi \in \mathcal{S}} W_{\chi,\infty}.$$

For an element w in W, w is a simultaneous eigenvector for $t_1, \ldots, t_{m_P}$ if and only if w lies in $W_{\chi,1}$, for some $\chi \in \mathcal{S}$.

Let ϕ_K be a non-zero K-fixed vector in V. By the Iwasawa decomposition $G = PK$, the element $v = r_N(\phi_K)$ generates V_N as an M-module, so does W. Since H is contained in the center of M, we have

$$V_N = \bigoplus_{\chi \in \mathcal{S}} (V_N)_{\chi,\infty}$$

as an M-module. We claim that the $W_{\chi,\infty}$-component of v is not zero, for any $\chi \in \mathcal{S}$. If the $W_{\chi,\infty}$-component of v is zero, then v lies in the proper M-submodule $\bigoplus_{\chi' \neq \chi} (V_N)_{\chi',\infty}$ of V_N. This contradicts the fact that v generates V_N as an M-module. So the claim follows.

We fix a character χ of H in $\mathcal{S}$. For any $\chi' \in \mathcal{S}$ which is different from χ, there exists an integer $1 \leq i(\chi') \leq m_P$ such that $\chi(t_{i(\chi')}) \neq \chi'(t_{i(\chi')})$. Put $n(\chi') = \dim W_{\chi',\infty}$. Then

$$v' = \prod_{\chi' \neq \chi} (\pi_N(t_{i(\chi')}) - \chi'(t_{i(\chi')}))^{n(\chi')} v$$

is a non-zero vector in $W_{\chi,\infty}$. Therefore, there exist non-negative integers $n(\chi, i)$ for $1 \leq i \leq m_P$ such that $v'' := \prod_{1 \leq i \leq m_P} (\pi_N(t_i) - \chi(t_i))^{n(\chi,i)} v'$ is a non-zero vector in $W_{\chi,1}$. By Proposition 2.3,

$$\phi = \prod_{1\le i\le m_P} (\delta_P^{\frac{1}{2}}(t_i)\pi(U^P_{\varpi,i}) - \chi(t_i))^{n(\chi,i)} \times \prod_{\chi'\neq\chi} (\delta_P^{\frac{1}{2}}(t_{i(\chi')})\pi(U^P_{\varpi,i(\chi')}) - \chi'(t_{i(\chi')}))^{n(\chi')}\phi_K$$

is a p-stabilized vector with respect to $\widehat{P}$, which satisfies

$$\pi(U^P_{\varpi,i})\phi = \delta_P(t_i)^{-\frac{1}{2}}\chi(t_i)\phi,$$

for all $i \in \{1, \dots, m_P\}$.

In the following series of sections, we give examples of p-stabilized vectors in various settings.

5. GL_2-case

Let α be the simple root of GL_2 such that $\alpha : T \longrightarrow F^\times,\ \mathrm{diag}(a, b) \mapsto ab^{-1}$. We have a U_p-operator $U^B_{\varpi,1} = [It_1I]$, where $t_1 = \mathrm{diag}(1, \varpi^{-1})$. Let $\pi = \pi(\chi)$ be an unramified principal series representation of $\mathrm{GL}_2(F)$ where $\chi = \chi_1 \otimes \chi_2$ and χ_1, χ_2 are unramified quasi-characters of $F^\times$. Then π has a non-zero K-fixed vector ϕ_0, where $K = \mathrm{GL}_2(\mathfrak{o})$. We shall give an explicit p-stabilized vector for π. The semisimplification of π_U is

$$\chi_1 \otimes \chi_2 + \chi_2 \otimes \chi_1.$$

The element t_1 acts on each irreducible component of π_U by $\chi_2(\varpi^{-1})$ and $\chi_1(\varpi^{-1})$ respectively. If $\chi_1(\varpi^{-1}) \neq \chi_2(\varpi^{-1})$, then we have $\pi_U = (\chi_1 \otimes \chi_2) \oplus (\chi_2 \otimes \chi_1)$. It follows from Proposition 2.3 and the results in Section 4 that

$$f_1 := (\delta_B^{\frac{1}{2}}(t_1)U^B_{\varpi,1} - \chi_2(\varpi^{-1}))\phi_0,\ \ f_2 := (\delta_B^{\frac{1}{2}}(t_1)U^B_{\varpi,1} - \chi_1(\varpi^{-1}))\phi_0$$

are p-stabilized vectors with respect to B with the eigenvalues $q^{\frac{1}{2}}\chi_1(\varpi^{-1})$, $q^{\frac{1}{2}}\chi_2(\varpi^{-1})$ respectively.

If $\chi_1(\varpi^{-1}) = \chi_2(\varpi^{-1})$, then we have $\chi_1 = \chi_2$ since χ_1 and χ_2 are unramified. In this case, any irreducible component of π_U is isomorphic to $\chi_1 \otimes \chi_1$, but π_U is not decomposed as $(\chi_1 \otimes \chi_1)^{\oplus 2}$. This follows from the fact that $\mathbb{C} \simeq \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi, \pi) = \mathrm{Hom}_T(\pi_U, \chi_1 \otimes \chi_1)$ (Schur's lemma and Frobenius reciprocity). In this case, $f_3 := (\delta_B^{\frac{1}{2}}(t_1)U^B_{\varpi,1} - \chi_1(\varpi^{-1}))\phi_0$ is a p-stabilized vector with respect to B with the eigenvalue $q^{\frac{1}{2}}\chi_1(\varpi^{-1})$.

We can express f_i in terms of Iwahori fixed vectors as follows. Let ϕ be a generator of π^K. Choose a basis $\{\phi,\ \phi' = \pi(t_1^{-1})\phi\}$ of π^I. Then we have

$$U^B_{\varpi,1}(\phi, \phi') = (\phi, \phi') \begin{pmatrix} a(\phi) & q \\ -\chi_1(\varpi^{-1})\chi_2(\varpi^{-1}) & 0 \end{pmatrix}$$

where $a(\phi) = \chi_1(\varpi^{-1}) + \chi_2(\varpi^{-1})$. Therefore we have

$$\chi_1(\varpi)f_1 = \phi - q^{-\frac{1}{2}}\chi_2(\varpi^{-1})\phi',\ \chi_2(\varpi)f_2 = \phi - q^{-\frac{1}{2}}\chi_1(\varpi^{-1})\phi'$$

(see Section 9.1 for the relation to the global setting). It is the same for f_3.

6. $U(2,1)$-case

Let $U(2,1)$ be the quasi-split unitary group in three variables associated to an quadratic extension E/F. Put

$$\Phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We denote by $^-$ the conjugate for the non-trivial element in $\mathrm{Gal}(E/F)$. We realize $\mathrm{U}(2,1)(F)$ as the subgroup of $\mathrm{GL}_3(E)$ consisting of all g satisfying ${}^t\overline{g}\Phi g = \Phi$. Let B be the upper triangular Borel subgroup of $U(2,1)$, T the diagonal subgroup of B and $K = \mathrm{U}(2,1)(F) \cap \mathrm{GL}_3(\mathfrak{o}_E)$, where $\mathfrak{o}_E$ is the ring of integers in E. We write E^1 for the norm-one subgroup of E/F. Then T is isomorphic to $E^\times \times E^1$. Let χ be an unramified quasi-character of $E^\times$ and let $\mathbf{1}_{E^1}$ denote the trivial character of E^1. Due to [13], the corresponding parabolically induced representation $\mathrm{Ind}_{B(F)}^{\mathrm{U}(2,1)(F)}(\chi \otimes \mathbf{1}_{E^1})$ is irreducible except for the following cases:

(i) $\chi = |\cdot|_E^{\pm}$, where $|\cdot|_E$ denotes the normalized absolute value of E;
(ii) $\chi|_{F^\times} = \omega_{E/F}|\cdot|^{\pm}$, where $\omega_{E/F}$ is the non-trivial character of $F^\times$ which is trivial on $N_{E/F}(E^\times)$;
(iii) $\chi|_{F^\times}$ is trivial and χ is not trivial.

Suppose that $\pi = \mathrm{Ind}_{B(F)}^{\mathrm{U}(2,1)(F)}(\chi \otimes \mathbf{1}_{E^1})$ is irreducible. Then π has a non-zero K-fixed vector ϕ_0. We shall produce an explicit p-stabilized vector for π.

We fix a uniformizer ϖ_E of E and set

$$t_1 = \begin{pmatrix} \varpi_E & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{\varpi}_E^{-1} \end{pmatrix}.$$

Let U be the unipotent radical of B. Then t_1 is positive relative to (B,U). The semisimplification of π_U is $\chi \otimes \mathbf{1}_{E^1} + \overline{\chi}^{-1} \otimes \mathbf{1}_{E^1}$, where $\overline{\chi}$ denotes the quasi-character of $E^\times$ defined by $\overline{\chi}(x) = \chi(\overline{x})$, for $x \in E^\times$. The element t_1 acts on $\chi \otimes \mathbf{1}_{E^1}$ and $\overline{\chi}^{-1} \otimes \mathbf{1}_{E^1}$ by $\chi(\varpi_E)$ and $\overline{\chi}^{-1}(\varpi_E)$ respectively. As in the GL_2-case, by Proposition 2.3 and the results in Section 4,

$$(\delta_B^{\frac{1}{2}}(t_1)U_{\varpi,1}^B - \overline{\chi}^{-1}(\varpi_E))\phi_0,\ (\delta_B^{\frac{1}{2}}(t_1)U_{\varpi,1}^B - \chi(\varpi_E))\phi_0$$

are p-stabilized vectors with respect to B with the eigenvalues $q_E\chi(\varpi_E)$ and $q_E\overline{\chi}^{-1}(\varpi_E)$ respectively.

7. GSp_4-case

Hereafter we follows the notations in [16]. Put

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

We realize $\mathrm{GSp}_4(F)$ as the subgroup of $\mathrm{GL}_4(F)$ consisting of all g such that ${}^t gJg = \lambda J$, for some $\lambda \in F^\times$. Let B be the Borel subgroup of GSp_4 consisting of the upper triangular elements, T the diagonal subgroup of B and U the unipotent radical of B. Let P (respectively Q) be the Siegel (respectively Klingen) parabolic subgroup of GSp_4 containing B. Let $P = M_P N_P$ and $Q = M_Q N_Q$ be Levi decompositions. Let α_1, α_2 be simple roots defined by $\alpha_1(t) = ab^{-1}$ and $\alpha_2(t) = b^2c^{-1}$ for $t = \mathrm{diag}(a, b, cb^{-1}, ca^{-1})$, $a, b, c \in F^\times$. Note that $\Delta_P = \{\alpha_1\}$ and $\Delta_Q = \{\alpha_2\}$. Thus there exist two U_p-operators $U^B_{\varpi,i} = [It_iI]$, $i = 1, 2$ where $t_1 = \mathrm{diag}(1, 1, \varpi^{-1}, \varpi^{-1})$ and $t_2 = \mathrm{diag}(1, \varpi^{-1}, \varpi^{-1}, \varpi^{-2})$. Note that t_1, t_2 are positive elements in T^-_B relative to (B, I), and t_1 (respectively t_2) is a positive element in T^-_P (respectively T^-_Q) relative to (P, K_P) (respectively (Q, K_Q)). In this section, we give p-stabilized vectors according to the classification of the parahori-spherical representations of $\mathrm{GSp}_4(F)$ by Roberts and Schmidt [16].

7.1. Iwahori case. In this subsection, we study p-stabilized vectors with respect to $\widehat{B} = B$. The strategy taking here is as follows:

(i) Let (π, V) be an irreducible admissible representation of $\mathrm{GSp}_4(F)$ admitting a non-zero Iwahori-fixed vector. Then π is an irreducible constituent of some unramified principal series representation $\chi_1 \times \chi_2 \rtimes \sigma$. We use the classification of such representations in Table A.15 of [16].

(ii) We make up a list of the simultaneous eigenvalues for U_p-operators $U^B_{\varpi,1}, U^B_{\varpi,2}$ in terms of the following Satake parameters of $\chi_1 \times \chi_2 \rtimes \sigma$:

$$\alpha = \chi_1\chi_2\sigma(\varpi^{-1}),\ \beta = \chi_1\sigma(\varpi^{-1}),\ \gamma = \chi_2\sigma(\varpi^{-1}),\ \delta = \sigma(\varpi^{-1}). \tag{7.1}$$

By Proposition 2.3, the problem is reduced to the computation of the simultaneous eigenvalues for t_1, t_2 on $(V_U)^{I\cap T}$. We note that $V_U = (V_U)^{I\cap T}$. Table A.3 of [16] gives the semisimplification of the Jacquet module π_{N_P} of π associated to the Siegel parabolic subgroup P. So we can easily get the semisimplification π_U^{ss} of π_U by using the transitivity of Jacquet functors. The elements t_1, t_2 act on each irreducible component of π_U^{ss} by its central character. Thus we can obtain the set S' of pairs of simultaneous eigenvalues for (t_1, t_2) on $V_U = (V_U)^{I\cap T}$. It will turns out that S' is contained in S, where S is the set of pairs of simultaneous eigenvalues for (t_1, t_2) on $(\chi_1 \times \chi_2 \rtimes \sigma)_U = (\chi_1 \times \chi_2 \rtimes \sigma)_U^{I\cap T}$, that is,

$$S = \{(\delta, \delta\gamma), (\delta, \delta\beta), (\alpha, \alpha\beta), (\alpha, \alpha\gamma), (\gamma, \gamma\delta), (\gamma, \gamma\alpha), (\beta, \beta\delta), (\beta, \beta\alpha)\}.$$

(iii) Suppose that S' contains just $\dim V_U$-elements. Since t_1, t_2 generates T/Z_G, this implies that $(\pi_U)^{I\cap T}$ is a semisimple and multiplicity-free T-module. We further assume that π has a non-zero K-fixed vector ϕ_K. In this case, given an element (s,t) in S', the vector

$$\phi_{s,t} := \prod_{\substack{(s',t')\in S' \\ s'\neq s}} (\delta_B^{\frac{1}{2}}(t_1)U^B_{\varpi,1} - s') \prod_{\substack{(s',t')\in S' \\ t'\neq t}} (\delta_B^{\frac{1}{2}}(t_2)U^B_{\varpi,2} - t')\phi_K$$

is a p-stabilized vector with respect to $\widehat{B} = B$ with the eigenvalues $(\delta_B^{-\frac{1}{2}}(t_1)s, \delta_B^{-\frac{1}{2}}(t_2)t)$ because of Proposition 2.3 and the results in Section 4. We note that Table A. 15 of [16] gives a list of the K-spherical representations of $\mathrm{GSp}_4(F)$.

7.1.1. ***Case I.*** Let χ_1, χ_2, σ be unramified quasi-characters of $F^\times$. Then the corresponding parabolically induced representation $\pi = \chi_1 \times \chi_2 \rtimes \sigma$ is irreducible if and only if $\chi_1 \neq \nu^{\pm 1}$, $\chi_2 \neq \nu^{\pm 1}$ and $\chi_1 \neq \nu^{\pm 1}\chi_2^{\pm 1}$. Here let us put $\nu = |\cdot|$. Due to Table A. 15 of [16], π has a non-zero K-fixed vector. The semisimplification of π_{N_P} is given in Table A.3 of [16]. We use the notation in section A.3 of [16]. Let r_{T,M_P} denote the Jacquet functor from the category of the smooth representations of M_P to that of B. Note that the semisimplification of $r_{T,M_P}((\chi_1\times\chi_2)\otimes\sigma)$ is $(\chi_1\otimes\chi_2\otimes\sigma)+(\chi_2\otimes\chi_1\otimes\sigma)$. Since $r_U = r_{T,M_P} \circ r_{N_P}$, the semisimplification of π_U is

$$\begin{aligned}&(\chi_1 \otimes \chi_2 \otimes \sigma) + (\chi_2 \otimes \chi_1 \otimes \sigma) + (\chi_1^{-1} \otimes \chi_2^{-1} \otimes \chi_1\chi_2\sigma)\\ &+ (\chi_2^{-1} \otimes \chi_1^{-1} \otimes \chi_1\chi_2\sigma) + (\chi_1 \otimes \chi_2^{-1} \otimes \chi_2\sigma) + (\chi_2^{-1} \otimes \chi_1 \otimes \chi_2\sigma)\\ &+ (\chi_2 \otimes \chi_1^{-1} \otimes \chi_1\sigma) + (\chi_1^{-1} \otimes \chi_2 \otimes \chi_1\sigma).\end{aligned}$$

The elements t_1, t_2 act on each component of the semisimplification of π_U by the following pairs of scalars:

$$(\delta, \delta\gamma), (\delta, \delta\beta), (\alpha, \alpha\beta), (\alpha, \alpha\gamma), (\gamma, \gamma\delta), (\gamma, \gamma\alpha), (\beta, \beta\delta), (\beta, \beta\alpha).$$

The pairs above are pairwise distinct if and only if $\chi_1 \neq 1$, $\chi_2 \neq 1$ and $\chi_1 \neq \chi_2^{\pm 1}$.

7.1.2. ***Case II.*** Let χ, σ be unramified quasi-characters of $F^\times$ such that $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm\frac{3}{2}}$. Put $\chi_1 = \nu^{\frac{1}{2}}\chi$ and $\chi_2 = \nu^{-\frac{1}{2}}\chi$. In what follows, we consider the irreducible constituents π of $\chi_1 \times \chi_2 \rtimes \sigma$.

Case IIa. Put $\pi = \chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$. Noting that $r_{T,M_P}(\chi\mathrm{St}_{\mathrm{GL}(2)} \otimes \sigma) = \nu^{\frac{1}{2}}\chi\otimes\nu^{-\frac{1}{2}}\chi\otimes\sigma$. It follows from Table A.3 of [16] that the semi-simplification of π_U is

$$(\chi_1 \otimes \chi_2 \otimes \sigma) + (\chi_2^{-1} \otimes \chi_1^{-1} \otimes \chi_1\chi_2\sigma) + (\chi_1 \otimes \chi_2^{-1} \otimes \chi_2\sigma) + (\chi_2^{-1} \otimes \chi_1 \otimes \chi_2\sigma).$$

Hence the elements t_1, t_2 act on each irreducible component of π_U by

$$(\delta, \delta\gamma), (\alpha, \alpha\gamma), (\gamma, \gamma\delta), (\gamma, \gamma\alpha)$$

respectively. The pairs above are pairwise distinct if and only if $\chi^2 \neq 1$.

Case IIb. If $\pi = \chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$, then π has a non-zero K-fixed vector. Since $r_{T,M_P}(\chi\mathbf{1}_{\mathrm{GL}(2)} \otimes \sigma) = \nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi \otimes \sigma$, it follows from Table A.3 of [16] that the semisimplification of π_U is

$$(\chi_2 \otimes \chi_1 \otimes \sigma) + (\chi_1^{-1} \otimes \chi_2^{-1} \otimes \chi_1\chi_2\sigma) + (\chi_2 \otimes \chi_1^{-1} \otimes \chi_1\sigma) + (\chi_1^{-1} \otimes \chi_2 \otimes \chi_1\sigma).$$

The elements t_1, t_2 act on each component by

$$(\delta, \delta\beta), (\alpha, \alpha\beta), (\beta, \beta\delta), (\beta, \beta\alpha)$$

respectively. The pairs above are pairwise distinct if and only if $\chi^2 \neq 1$.

7.1.3. ***Case III.*** Let χ and ρ be unramified quasi-characters of $F^\times$. We assume that $\chi \neq 1$ and $\chi \neq \nu^{\pm 2}$. Put $\chi_1 = \chi,\ \chi_2 = \nu$, and $\sigma = \nu^{-\frac{1}{2}}\rho$. Next, we consider the irreducible constituents π of $\chi_1 \times \chi_2 \rtimes \sigma$.

Case IIIa. If $\pi = \chi \rtimes \rho\mathrm{St}_{\mathrm{GSp}(2)}$, then the semisimplification of π_U is

$$(\chi_1 \otimes \chi_2 \otimes \sigma) + (\chi_2 \otimes \chi_1 \otimes \sigma) + (\chi_2 \otimes \chi_1^{-1} \otimes \chi_1\sigma) + (\chi_1^{-1} \otimes \chi_2 \otimes \chi_1\sigma).$$

The actions of $t_i,\ i = 1, 2$ on each irreducible component of π_U are just

$$(\delta, \delta\gamma), (\delta, \delta\beta), (\beta, \beta\delta), (\beta, \beta\alpha)$$

respectively. The pairs above are pairwise distinct if and only if $\chi \neq \nu^{\pm 1}$.

Case IIIb. Suppose that $\pi = \chi \rtimes \rho\mathbf{1}_{\mathrm{GSp}(2)}$. Then π admits a non-zero K-fixed vector and the semisimplification of π_U is

$$(\chi_1^{-1} \otimes \chi_2^{-1} \otimes \chi_1\chi_2\sigma) + (\chi_2^{-1} \otimes \chi_1^{-1} \otimes \chi_1\chi_2\sigma) + (\chi_1 \otimes \chi_2^{-1} \otimes \chi_2\sigma) + (\chi_2^{-1} \otimes \chi_1 \otimes \chi_2\sigma).$$

Hence the actions of $t_i,\ i = 1, 2$ on each component are

$$(\alpha, \alpha\beta), (\alpha, \alpha\gamma), (\gamma, \gamma\delta), (\gamma, \gamma\alpha)$$

respectively. The pairs above are pairwise distinct if and only if $\chi \neq \nu^{\pm 1}$.

7.1.4. ***Case IV.*** Let ρ be an unramified quasi-character of $F^\times$. Put $\chi_1 = \nu^2,\ \chi_2 = \nu$, and $\sigma = \nu^{-\frac{3}{2}}\rho$. We will consider the irreducible constituents π of $\chi_1 \times \chi_2 \rtimes \sigma$. In this case, $\alpha, \beta, \gamma, \delta$ are different from each other. So π_U is a semisimple and multiplicity-free T-module.

Case IVa. Suppose that $\pi = \rho\mathrm{St}_{\mathrm{GSp}(4)}$. Then we have $\pi_U = \chi_1 \otimes \chi_2 \otimes \sigma$, and the elements t_1, t_2 act on it by $(\delta, \delta\gamma)$. Since $\dim \pi^I = 1$, any non-zero I-fixed vector is itself a p-stabilized vector with the eigenvalues $(\delta_B^{-\frac{1}{2}}(t_1)\delta, \delta_B^{-\frac{1}{2}}(t_2)\delta\gamma)$.

Case IVb. If $\pi = L(\nu^2, \nu^{-1}\rho\mathrm{St}_{\mathrm{GSp}(2)})$, then π_U is isomorphic to

$$(\chi_2 \otimes \chi_1 \otimes \sigma) \oplus (\chi_2 \otimes \chi_1^{-1} \otimes \chi_1\sigma) \oplus (\chi_1^{-1} \otimes \chi_2 \otimes \chi_1\sigma).$$

Hence $t_i,\ i = 1, 2$ act on each component by

$$(\delta, \delta\beta), (\beta, \beta\delta), (\beta, \beta\alpha)$$

respectively.

Case IVc. Suppose that $\pi = L(\nu^{\frac{3}{2}}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-\frac{3}{2}}\rho)$. Then π_U is isomorphic to

$$(\chi_2^{-1} \otimes \chi_1^{-1} \otimes \chi_1\chi_2\sigma) \oplus (\chi_1 \otimes \chi_2^{-1} \otimes \chi_2\sigma) \oplus (\chi_2^{-1} \otimes \chi_1 \otimes \chi_2\sigma),$$

and the elements t_1, t_2 act on each irreducible component by

$$(\alpha, \alpha\gamma), (\gamma, \gamma\delta), (\gamma, \gamma\alpha)$$

respectively.

Case IVd. If $\pi = \rho\mathbf{1}_{\mathrm{GSp}(4)}$, then we have $\dim \pi^I = \dim \pi^K = 1$, and hence $\pi^I = \pi^K$. We also get $\pi_U = \chi_1^{-1} \otimes \chi_2^{-1} \otimes \chi_1\chi_2\sigma$, and the elements t_1, t_2 act on π_U by $(\alpha, \alpha\beta)$. Thus any non-zero K-fixed vector is a p-stabilized vector with the eigenvalues $(\delta_B^{-\frac{1}{2}}(t_1)\alpha, \delta_B^{-\frac{1}{2}}(t_2)\alpha\beta)$.

7.1.5. ***Case V.*** Let ξ and ρ be unramified quasi-characters of $F^\times$. We assume that $\xi^2 = 1$ and $\xi \neq 1$. Put $\chi_1 = \nu\xi,\ \chi_2 = \xi$, and $\sigma = \nu^{-\frac{1}{2}}\rho$. We consider the irreducible constituents π of $\chi_1 \times \chi_2 \rtimes \sigma$. In this case, $\alpha, \beta, \gamma, \delta$ are different from each other. This means that π_U is a semisimple and multiplicity-free T-module.

Case Va. If $\pi = \delta([\xi, \nu\xi], \nu^{-\frac{1}{2}}\rho)$, then we have

$$\pi_U = (\chi_1 \otimes \chi_2 \otimes \sigma) \oplus (\chi_1 \otimes \chi_2^{-1} \otimes \chi_2\sigma).$$

The elements t_1, t_2 act on each component by

$$(\delta, \delta\gamma), (\gamma, \gamma\delta)$$

respectively.

Case Vb. Suppose that $\pi = L(\nu^{\frac{1}{2}}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-\frac{1}{2}}\rho)$. Then we have

$$\pi_U = (\chi_2^{-1} \otimes \chi_1^{-1} \otimes \chi_1\chi_2\sigma) \oplus (\chi_2^{-1} \otimes \chi_1 \otimes \chi_2\sigma),$$

and hence the elements t_1, t_2 act on each irreducible component by

$$(\alpha, \alpha\gamma), (\gamma, \gamma\alpha)$$

respectively.

Case Vc. In the case when $\pi = L(\nu^{\frac{1}{2}}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-\frac{1}{2}}\rho)$, we have

$$\pi_U = (\chi_2 \otimes \chi_1 \otimes \sigma) \oplus (\chi_2 \otimes \chi_1^{-1} \otimes \chi_1\sigma).$$

The elements t_1, t_2 act on each component by

$$(\delta, \delta\beta), (\beta, \beta\delta)$$

respectively.

Case Vd. If $\pi = L(\nu\xi, \xi \rtimes \nu^{-\frac{1}{2}}\rho)$, then π has a non-zero K-fixed vector. We have

$$\pi_U = (\chi_1^{-1} \otimes \chi_2^{-1} \otimes \chi_1\chi_2\sigma) \oplus (\chi_1^{-1} \otimes \chi_2 \otimes \chi_1\sigma).$$

Hence the elements t_1, t_2 act on each component by

$$(\alpha, \alpha\beta), (\beta, \beta\alpha)$$

respectively.

7.1.6. ***Case VI.*** Let ρ be an unramified quasi-character of $F^\times$. Put $\chi_1 = \nu$, $\chi_2 = \mathbf{1}_{F^\times}$ and $\sigma = \nu^{-1/2}\rho$. Finally, we consider the irreducible constituents π of $\chi_1 \times \chi_2 \rtimes \sigma$. In this case, we have $\alpha = \beta,\ \gamma = \delta$ and $\alpha \neq \gamma$.

Case VIa. Suppose that $\pi = \tau(S, \nu^{-\frac{1}{2}}\rho)$. Then the semisimplification of π_U is

$$(\chi_1\chi_2 \otimes \chi_1 \otimes \sigma)^{\oplus 2} + (\chi_2 \otimes \chi_1 \otimes \sigma).$$

Hence the actions of t_i, $i = 1, 2$ on each component are

$$(\gamma, \gamma^2), (\gamma, \gamma^2), (\gamma, \gamma\alpha)$$

respectively.

Case VIb. If $\pi = \tau(T, \nu^{-\frac{1}{2}}\rho)$, then we have $\dim \pi^{K_P} = \dim \pi^I = 1$ by Table A. 15 in [16], and hence $\pi^I = \pi^{K_P}$. We also get $\pi_U = \chi_2 \otimes \chi_1 \otimes \sigma$, and the elements t_1, t_2 act on π_U by $(\gamma, \gamma\alpha)$. Therefore any non-zero K_P-spherical vector is a p-stabilized vector with the eigenvalues $(\delta_B^{-\frac{1}{2}}(t_1)\gamma, \delta_B^{-\frac{1}{2}}(t_2)\gamma\alpha)$.

Case VIc. Suppose that $\pi = L(\nu^{\frac{1}{2}}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-\frac{1}{2}}\rho)$. Then we get $\pi_U = \chi_2 \otimes \chi_1^{-1} \otimes \chi_1\sigma$, and t_i, $i = 1, 2$ acts on it by $(\alpha, \gamma\alpha)$. By Table A. 15 in [16], the space π^{K_Q} is one-dimensional. Therefore a non-zero K_Q-fixed vector is a p-stabilized vector with the eigenvalues $(\delta_B^{-\frac{1}{2}}(t_1)\alpha, \delta_B^{-\frac{1}{2}}(t_2)\gamma\alpha)$.

Case VId. If $\pi = L(\nu, \mathbf{1}_{F^\times} \rtimes \nu^{-1/2}\rho)$, then π has a non-zero K-fixed vector. The semisimplification of π_U is

$$(\chi_1^{-1} \otimes \chi_2 \otimes \chi_1\sigma)^{\oplus 2} + (\chi_2 \otimes \chi_1^{-1} \otimes \chi_1\sigma).$$

Hence the actions of $t_i,\ i = 1, 2$ on each component are just

$$(\alpha, \alpha^2), (\alpha, \alpha^2), (\alpha, \gamma\alpha)$$

respectively.

In Table 7.1, we list the simultaneous eigenvalues for t_1, t_2 on π_U, where π is a representation in groups I-V. For a representation in group VI, we always have $\alpha = \beta,\ \gamma = \delta$ and $\alpha \neq \gamma$. So for such representations, we list the multiplicity of the simultaneous eigenvalues for t_1, t_2 on π_U in Table 7.2.

7.2. Siegel parahoric case. In this subsection, we compute the eigenvalues of the U_p-operator $U^P_{\varpi,1}$, where P denotes the Siegel parabolic subgroup of $\mathrm{GSp}_4(F)$. The strategy is as follows:

(i) We use the classification of the irreducible smooth representations (π, V) of $\mathrm{GSp}_4(F)$ admitting K_P-fixed vectors in Table A.15 of [16]. As in the previous subsection, we realize π as an irreducible constituent of some unramified principal series representation $\chi_1 \times \chi_2 \rtimes \sigma$. We use the same notation as in subsection 7.1.

(ii) We compute the set S' of eigenvalues of t_1 on $(V_{N_P})^{M_P \cap K_P}$ as follows: The semisimplification $\pi^{\mathrm{ss}}_{N_P}$ of the Jacquet module π_{N_P} of π associated to P is given in Table A.3 of [16]. We denote by $\pi_{N_P, M_P \cap K_P}$ the M_P-submodule of $\pi^{\mathrm{ss}}_{N_P}$ spanned by the $M_P \cap K_P$-fixed vectors in $\pi^{\mathrm{ss}}_{N_P}$. Note that for any irreducible admissible representation τ of $M_P \simeq \mathrm{GL}_2(F) \times F^\times$, we have $\dim \tau^{M_P \cap K_P} \leq 1$. Thus the length of $\pi_{N_P, M_P \cap K_P}$ is equal to $\dim(V_{N_P})^{M_P \cap K_P}$. Since the element t_1 lies in the center of M_P, the eigenvalues of t_1 on $(V_{N_P})^{M_P \cap K_P}$ is just those of t_1 on $\pi_{N_P, M_P \cap K_P}$. So we can easily compute the eigenvalues of t_1 on $(V_{N_P})^{M_P \cap K_P}$ because t_1 acts on each irreducible component of $\pi_{N_P, M_P \cap K_P}$ by the central character. It will turns out that S' is contained in $S = \{\alpha, \beta, \gamma, \delta\}$, where $\alpha, \beta, \gamma, \delta$ are the Satake parameters of $\chi_1 \times \chi_2 \rtimes \sigma$ defined in (7.1).

(iii) We assume that S' contains just $\dim(\pi_{N_P})^{K_P \cap M_P}$-elements. Then $(\pi_{N_P})^{K_P \cap M_P}$ is a semisimple and multiplicity-free Z_{M_P}-module because t_1 generates Z_{M_P} modulo $Z_{M_P}(\mathfrak{o})$. We further assume that π has a non-zero K-fixed vector ϕ_K. Then, given an element s in S', the vector

$$\phi_s := \prod_{\substack{s' \in S' \\ s' \neq s}} (\delta_P^{\frac{1}{2}}(t_1) U^P_{\varpi,1} - s')\phi_K$$

rep.	$(\alpha,\alpha\beta)$	$(\alpha,\alpha\gamma)$	$(\beta,\beta\alpha)$	$(\beta,\beta\delta)$	$(\gamma,\gamma\alpha)$	$(\gamma,\gamma\delta)$	$(\delta,\delta\beta)$	$(\delta,\delta\gamma)$	$\dim\pi^I$	$\dim\pi^K$
I	○	○	○	○	○	○	○	○	8	1
IIa	-	○	-	-	○	○	-	○	4	0
IIb	○	-	○	○	-	-	○	-	4	1
IIIa	-	-	○	○	-	-	○	○	4	0
IIIb	○	○	-	-	○	○	-	-	4	1
IVa	-	-	-	-	-	-	-	○	1	0
IVb	-	-	○	○	-	-	○	-	3	0
IVc	-	○	-	-	○	○	-	-	3	0
IVd	○	-	-	-	-	-	-	-	1	1
Va	-	-	-	-	-	○	-	○	2	0
Vb	-	○	-	-	○	-	-	-	2	0
Vc	-	-	-	○	-	-	○	-	2	0
Vd	○	-	○	-	-	-	-	-	2	1

TABLE 7.1

representation	(α,α^2)	$(\alpha,\alpha\gamma)$	$(\gamma,\gamma\alpha)$	(γ,γ^2)	$\dim\pi^I$	$\dim\pi^K$
VIa	0	0	1	2	3	0
VIb	0	0	1	0	1	0
VIc	0	1	0	0	1	0
VId	2	1	0	0	3	1

TABLE 7.2

is a p-stabilized vector with respect to $\widehat{P} = Q$ with the eigenvalue $\delta_P^{-\frac{1}{2}}(t_1)s$ because of Proposition 2.3 and the results in Section 4.

7.2.1. ***Case I.*** In this case, $\pi_{N_P, M_P \cap K_P}$ is

$$(\chi_1 \times \chi_2) \otimes \sigma + (\chi_1^{-1} \times \chi_2^{-1}) \otimes \chi_1\chi_2\sigma + (\chi_1 \times \chi_2^{-1}) \otimes \chi_2\sigma + (\chi_2 \times \chi_1^{-1}) \otimes \chi_1\sigma.$$

The element t_1 acts on each component of $\pi_{N_P, M_P \cap K_P}$ by

$$\delta, \alpha, \gamma, \beta$$

respectively.

7.2.2. ***Case II.*** **Case IIa**. In this case, we have $\pi_{N_P, M_P \cap K_P} = (\chi_1 \times \chi_2^{-1}) \otimes \chi_2\sigma$. The element t_1 acts on it by γ. Since $\dim V^{K_P} = 1$, any non-zero K_P-fixed vector is a p-stabilized vector with respect to $\widehat{P}$.

Case IIb. In this case, we have

$$\pi_{N_P, M_P \cap K_P} = \chi \mathbf{1}_{\mathrm{GL}(2)} \otimes \sigma + \chi^{-1} \mathbf{1}_{\mathrm{GL}(2)} \otimes \chi^2\sigma + (\chi_2 \times \chi_1^{-1}) \otimes \chi_1\sigma,$$

and t_1 acts on each component by δ, α, β respectively.

7.2.3. ***Case III.*** **Case IIIa**. In this case, we obtain

$$\pi_{N_P, M_P \cap K_P} = (\chi_1 \times \chi_2) \otimes \sigma + (\chi_2 \times \chi_1^{-1}) \otimes \chi_1\sigma,$$

and t_1 acts on each component by δ and β respectively.

Case IIIb. We get

$$\pi_{N_P, M_P \cap K_P} = (\chi_2^{-1} \times \chi_1^{-1}) \otimes \chi_1\chi_2\sigma + (\chi_1 \times \chi_2^{-1}) \otimes \chi_2\sigma.$$

The element t_1 acts on each component by α and γ respectively.

7.2.4. ***Case IV.*** **Case IVa**. The representations in case IVa have no K_P-fixed vectors.

Case IVb. We obtain

$$\pi_{N_P, M_P \cap K_P} = \nu^{3/2} \mathbf{1}_{\mathrm{GL}(2)} \otimes \sigma + (\chi_2 \times \chi_1^{-1}) \otimes \chi_1\sigma.$$

The element t_1 acts on each component by δ and β respectively.

Case IVc. In this case, t_1 acts on $\pi_{N_P, M_P \cap K_P} = (\chi_1 \times \chi_2^{-1}) \otimes \chi_2\sigma$ by γ.

Case IVd. The element t_1 acts on $\pi_{N_P, M_P \cap K_P} = \nu^{-3/2} \mathbf{1}_{\mathrm{GL}(2)} \otimes \chi_1\chi_2\sigma$ by α.

7.2.5. ***Case V.*** **Case Va**. In this case, π admits no K_P-fixed vectors.

Case Vb. We have $\pi_{N_P,M_P\cap K_P} = \nu^{1/2}\xi\mathbf{1}_{\mathrm{GL}(2)} \otimes \chi_2\sigma$, and t_1 acts on it by γ.

Case Vc. The element t_1 acts on $\pi_{N_P,M_P\cap K_P} = \nu^{1/2}\xi\mathbf{1}_{\mathrm{GL}(2)} \otimes \sigma$ by δ.

Case Vd. In this case, we obtain

$$\pi_{N_P,M_P\cap K_P} = \nu^{-1/2}\xi\mathbf{1}_{\mathrm{GL}(2)} \otimes \chi_1\sigma \oplus \nu^{-1/2}\xi\mathbf{1}_{\mathrm{GL}(2)} \otimes \chi_1\chi_2\sigma.$$

So t_1 acts on each component by β and α respectively.

7.2.6. ***Case VI.*** **Case VIa**. In this case, the element t_1 acts on the space $\pi_{N_P,M_P\cap K_P} = \nu^{1/2}\mathbf{1}_{\mathrm{GL}(2)} \otimes \sigma$ by γ.

Case VIb. We have $\pi_{N_P,M_P\cap K_P} = \nu^{1/2}\mathbf{1}_{\mathrm{GL}(2)} \otimes \sigma$. The element t_1 acts on it by γ.

Case VIc. In this case, π has no K_P-fixed vectors.

Case VId. We get $\pi_{N_P,M_P\cap K_P} = (\nu^{-1/2}\mathbf{1}_{\mathrm{GL}(2)} \otimes \chi_1\sigma)^{\oplus 2}$ and t_1 acts on each component by α.

In Table 7.3, we list the eigenvalues for t_1 on $(\pi_{N_P})^{M_P\cap K_P}$, where π is a representation in groups I-V. For representations in group VI, we list the multiplicity of the eigenvalues for t_1 on $(\pi_{N_P})^{M_P\cap K_P}$ in Table 7.4.

representation	α	β	γ	δ	$\dim \pi^{K_P}$	$\dim \pi^K$
I	○	○	○	○	4	1
IIa	-	-	○	-	1	0
IIb	○	○	-	○	3	1
IIIa	-	○	-	○	2	0
IIIb	○	-	○	-	2	1
IVa	-	-	-	-	0	0
IVb	-	○	-	○	2	0
IVc	-	-	○	-	1	0
IVd	○	-	-	-	1	1
Va	-	-	-	-	0	0
Vb	-	-	○	-	1	0
Vc	-	-	-	○	1	0
Vd	○	○	-	-	2	1

TABLE 7.3

representation	α	γ	$\dim \pi^{K_P}$	$\dim \pi^K$
VIa	0	1	1	0
VIb	0	1	1	0
VIc	0	0	0	0
VId	2	0	2	1

TABLE 7.4

7.3. Klingen parahoric case. In this subsection, we compute the eigenvalues of the U_p-operator $U^Q_{\varpi,1} = [K_Q t_2 K_Q]$. The strategy is exactly same to that for the Siegel parahoric case. So we shall be brief here.

(i) According to the classification in Table A.15 of [16], we realize the representations (π, V) of $\mathrm{GSp}_4(F)$ which admit K_Q-fixed vectors as an irreducible constituent of some unramified principal series representation $\chi_1 \times \chi_2 \rtimes \sigma$.

(ii) We compute the set S' of eigenvalues of t_2 on $(V_{N_Q})^{M_Q \cap K_Q}$ as follows: The semisimplification $\pi^{\mathrm{ss}}_{N_Q}$ of π_{N_Q} is given in Table A.3 of [16]. We denote by $\pi_{N_Q, M_Q \cap K_Q}$ the M_Q-submodule of $\pi^{\mathrm{ss}}_{N_Q}$ generated by the $M_Q \cap K_Q$-fixed vectors. Note that for any irreducible admissible representation τ of $M_Q \simeq F^\times \times \mathrm{GSp}_2(F)$, we have $\dim \tau^{M_Q \cap K_Q} \leq 1$. So the length of $\pi_{N_Q, M_Q \cap K_Q}$ is equal to $\dim(V_{N_Q})^{M_Q \cap K_Q}$. Since $t_2 \in Z_{M_Q}$, the eigenvalues of t_2 on $(V_{N_Q})^{M_Q \cap K_Q}$ is just those of t_2 on $\pi_{N_Q, M_Q \cap K_Q}$. So we can compute the eigenvalues of t_2 on $(V_{N_Q})^{M_Q \cap K_Q}$ because t_2 acts on each irreducible component of $\pi_{N_Q, M_Q \cap K_Q}$ by the central character. It will turns out that S' is contained in $S = \{\alpha\beta, \alpha\gamma, \delta\beta, \delta\gamma\}$, where $\alpha, \beta, \gamma, \delta$ are the Satake parameters of $\chi_1 \times \chi_2 \rtimes \sigma$ defined in (7.1).

(iii) If S' contains just $\dim(\pi_{N_Q})^{K_Q \cap M_Q}$-elements, then $(\pi_{N_Q})^{K_Q \cap M_Q}$ is a semisimple and multiplicity-free Z_{M_Q}-module. We further assume that π has a non-zero K-fixed vector ϕ_K. Then, given an element s in S', the vector

$$\phi_s := \prod_{\substack{s' \in S' \\ s' \neq s}} (\delta_Q^{\frac{1}{2}}(t_2) U^Q_{\varpi,1} - s')\phi_K$$

is a p-stabilized vector with respect to $\widehat{Q} = P$ with the eigenvalue $\delta_Q^{-\frac{1}{2}}(t_2)s$ because of Proposition 2.3 and the result in Section 4.

7.3.1. ***Case I.*** In this case, we have

$$\begin{aligned}\pi_{N_Q, M_Q \cap K_Q} =& \chi_1 \otimes (\chi_2 \rtimes \sigma) + \chi_2 \otimes (\chi_1 \rtimes \sigma) + \chi_2^{-1} \otimes (\chi_1 \rtimes \chi_2\sigma) \\ &+ \chi_1^{-1} \otimes (\chi_2 \rtimes \chi_1\sigma).\end{aligned}$$

The element t_2 acts on each irreducible component of $\pi_{N_Q,M_Q\cap K_Q}$ by

$$\delta\gamma, \beta\delta, \alpha\gamma, \alpha\beta$$

respectively.

7.3.2. ***Case II.*** **Case IIa**. In this case, we get $\pi_{N_Q,M_Q\cap K_Q} = \chi_1 \otimes (\chi_2 \rtimes \sigma) + \chi_2^{-1} \otimes (\chi_1 \rtimes \chi_2\sigma)$. The element t_2 acts on each component by $\delta\gamma$ and $\alpha\gamma$ respectively.

Case IIb. The element t_2 acts on each component of $\pi_{N_Q,M_Q\cap K_Q} = \chi_2 \otimes (\chi_1 \rtimes \sigma) + \chi_1^{-1} \otimes (\chi_2 \rtimes \chi_1\sigma)$ by $\beta\delta$ and $\alpha\beta$ respectively.

7.3.3. ***Case III.*** **Case IIIa**. In this case, t_2 acts on $\pi_{N_Q,M_Q\cap K_Q} = \chi_2 \otimes (\chi_1 \rtimes \sigma)$ by $\beta\delta$.

Case IIIb. We have $\pi_{N_Q,M_Q\cap K_Q} = \chi \otimes \rho\mathbf{1}_{\mathrm{GSp}(2)} + \chi^{-1} \otimes \chi\rho\mathbf{1}_{\mathrm{GSp}(2)} + \chi_2^{-1} \otimes (\chi_1 \rtimes \chi_2\sigma)$, and t_2 acts on each component of it by $\delta\gamma, \alpha\beta$ and $\alpha\gamma$ respectively.

7.3.4. ***Case IV.*** **Case IVa**. In this case, π has no K_Q-fixed vectors.

Case IVb. The element t_2 acts on $\pi_{N_Q,M_Q\cap K_Q} = \chi_2 \otimes (\chi_1 \rtimes \sigma)$ by $\beta\delta$.

Case IVc. We have $\pi_{N_Q,M_Q\cap K_Q} = \nu^2 \otimes \nu^{-1}\rho\mathbf{1}_{\mathrm{GSp}(2)} + \chi_2^{-1} \otimes (\chi_1 \rtimes \chi_2\sigma)$, and t_2 acts on each component by $\delta\gamma$ and $\alpha\gamma$ respectively.

Case IVd. The element t_2 acts on $\pi_{N_Q,M_Q\cap K_Q} = \nu^{-2} \otimes \nu\rho\mathbf{1}_{\mathrm{GSp}(2)}$ by $\alpha\beta$.

7.3.5. ***Case V.*** **Case Va**. In this case, t_2 acts on $\pi_{N_Q,M_Q\cap K_Q} = \chi_1 \otimes (\chi_2 \rtimes \sigma)$ by $\delta\gamma$.

Case Vb. In this case, t_2 acts on $\pi_{N_Q,M_Q\cap K_Q} = \chi_2^{-1} \otimes (\chi_1 \rtimes \chi_2\sigma)$ by $\alpha\gamma$.

Case Vc. The element t_2 acts on $\pi_{N_Q,M_Q\cap K_Q} = \chi_2 \otimes (\chi_1 \rtimes \sigma)$ by $\beta\delta$.

Case Vd. In this case, t_2 acts on $\pi_{N_Q,M_Q\cap K_Q} = \nu^{-1/2}\xi \otimes (\xi \rtimes \nu^{1/2}\rho)$ by $\alpha\beta$.

7.3.6. ***Case VI.*** **Case VIa**. We have $\pi_{N_Q,M_Q\cap K_Q} = \chi_1 \otimes (\chi_2 \rtimes \sigma)$, and hence t_2 acts on it by γ^2.

Case VIb. In this case, π has no K_Q-fixed vectors.

Case VIc. The element t_2 acts on $\pi_{N_Q,M_Q\cap K_Q} = \mathbf{1}_{F^\times} \otimes \rho\mathbf{1}_{\mathrm{GSp}(2)}$ by $\alpha\gamma$.

Case VId. In this case, t_2 acts on each component of $\pi_{N_Q,M_Q\cap K_Q} = \mathbf{1}_{F^\times} \otimes \rho\mathbf{1}_{\mathrm{GSp}(2)} + \chi_1^{-1} \otimes (\chi_2 \rtimes \chi_1\sigma)$ by $\alpha\gamma$ and α^2 respectively.

In Table 7.5, we list the eigenvalues for t_2 on $(\pi_{N_Q})^{M_Q \cap K_Q}$, where π is a representation in groups I-V. For representations in group VI, we list the multiplicity of the eigenvalues for t_2 on $(\pi_{N_Q})^{M_Q \cap K_Q}$ in Table 7.6.

representation	$\alpha\beta$	$\alpha\gamma$	$\delta\beta$	$\delta\gamma$	$\dim \pi^{K_Q}$	$\dim \pi^{K}$
I	◯	◯	◯	◯	4	1
IIa	-	◯	-	◯	2	0
IIb	◯	-	◯	-	2	1
IIIa	-	-	◯	-	1	0
IIIb	◯	◯	-	◯	3	1
IVa	-	-	-	-	0	0
IVb	-	-	◯	-	1	0
IVc	-	◯	-	◯	2	0
IVd	◯	-	-	-	1	1
Va	-	-	-	◯	1	0
Vb	-	◯	-	-	1	0
Vc	-	-	◯	-	1	0
Vd	◯	-	-	-	1	1

Table 7.5

representation	α^2	$\alpha\gamma$	γ^2	$\dim \pi^{K_Q}$	$\dim \pi^{K}$
VIa	0	0	1	1	0
VIb	0	0	0	0	0
VIc	0	1	0	1	0
VId	1	1	0	2	1

Table 7.6

8. A relation to global objects

In this section, we will be concerned with global objects and give an answer why we consider the actions of the positive elements in local settings. Basic references of this section are [15] and [9] (see also Section 5 of II in [18]).

Let G be a non-compact connected semisimple algebraic group over $\mathbb{Q}$ and K the maximal compact subgroup of $G(\mathbb{R})$. Let $K_{\mathbb{C}}$ be the complexification of K. Put $D = G(\mathbb{R})/K$. Let $\lambda : K_{\mathbb{C}} \longrightarrow \mathrm{Aut}(E_\lambda)$ be an algebraic representation on a complex vector space E_λ. Then we obtain a homogeneous vector bundle $\mathbb{E}_\lambda = G(\mathbb{R}) \times_{K,\lambda|_K} E_\lambda = \mathrm{Ind}_K^{G(\mathbb{R})} \lambda|_K$ on D. Since D is simply connected, there is a (smooth) trivialization

$$\mathbb{E}_\lambda \xrightarrow{\sim} D \times E_\lambda.$$

With respect to this trivialization, the action of $\gamma \in G(\mathbb{R})$ on $D \times E_\lambda$ is given by

$$\gamma(z,v) = (\gamma z, J_\lambda(\gamma,z)v)$$

for some smooth mapping $J_\lambda : G \times D \longrightarrow \mathrm{Aut}(E_\lambda)$ which is so called an automorphic factor. The action of $G(\mathbb{R})$ induces the cocycle condition $J_\lambda(\gamma_1\gamma_2, z) = J_\lambda(\gamma_1, \gamma_2 z)J_\lambda(\gamma_2, z)$ for all $\gamma_1, \gamma_2 \in G(\mathbb{R})$.

For a holomorphic E_λ-valued function f on D, the action of $g \in G(\mathbb{R})$ is defined by

$$(g^{-1}f)(z) = J_\lambda^{-1}(g,z)f(gz).$$

For a discrete subgroup Γ of $G(\mathbb{Q})$ and a finite character $\chi : \Gamma \longrightarrow \mathbb{C}^\times$, we say that a holomorphic function $f : D \longrightarrow E_\lambda$ is an automorphic form of weight λ and level Γ with the character χ if it satisfies $\gamma f = \chi(\gamma)f$ for any $\gamma \in \Gamma$. We denote by $M_\lambda(\Gamma,\chi)$ the space of automorphic forms of weight λ and level Γ with the character χ.

Next we construct adelic forms from classical automorphic forms. Let G be a connected reductive group over $\mathbb{Q}$ and Z_G the center of G. Then its derived group G^{der} is a semisimple connected algebraic group. Consider the following exact sequence

$$1 \longrightarrow G^{\mathrm{der}} \longrightarrow G \xrightarrow{\nu} T := G/G^{\mathrm{der}} \longrightarrow 1$$

where T is a torus (cf. p.303 of [15]) . If $G = GL_2$ (respectively GSp_{2n}), then $T = \mathbb{G}_m$ and ν is the determinant map (respectively the similitude character). Let $G(\mathbb{R})^+$ be the connected component of the identity element in $G(\mathbb{R})$ with respect to the real topology. For simplicity, in what follows we assume that $G(\mathbb{R})^+ = Z_G(\mathbb{R})^+G^{\mathrm{der}}(\mathbb{R})^+$ (for example, $G = GSp_{2n}, GL_2, U(p,q), p,q > 0$ satisfy this condition).

Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and $\mathbb{A}_f$ the finite adele of $\mathbb{Q}$. Let K be a compact open subgroup of $G(\mathbb{A}_f)$. Assume that $\nu(K) \supset T(\hat{\mathbb{Z}})$. Then it follows from the strong approximation theorem for G^{der} that

$$G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+K = G(\mathbb{Q})Z_G(\mathbb{R})^+G^{\mathrm{der}}(\mathbb{R})^+K. \tag{8.1}$$

Set $K_\infty = Z(\mathbb{R})^+K_\infty^{(1)}$, where $K_\infty^{(1)}$ is the maximal compact subgroup of G^{der}. Then $D := G(\mathbb{R})^+/K_\infty = G^{\mathrm{der}}(\mathbb{R})/K_\infty^{(1)}$ is the bounded symmetric domain endowed with an involution ι. Let $I \in D$ be the fixed point of ι. From the description of D as above, $Z(\mathbb{R})^+$ acts on I trivially. Put $\Gamma = G^{\mathrm{der}}(\mathbb{Q}) \cap K$. For an automorphic form $f \in M_\lambda(\Gamma,\chi)$, we define the function $F_f : G(\mathbb{A}) \longrightarrow E_\lambda$ as follows. By (8.1), it is possible to write a given $g \in G(\mathbb{A})$ as $g = az_\infty g_\infty k$ with $a \in G(\mathbb{Q})$, $z_\infty \in Z_G(\mathbb{R})^+$, $g_\infty \in G^{\mathrm{der}}(\mathbb{R})^+$, and $k \in K$. Then we put

$$F_f(g) = J_\lambda^{-1}(g_\infty, I)f(g_\infty I).$$

We can check that F_f is an automorphic form on $G(\mathbb{A})$ in the sense of [4]. Let π_f be the maximal irreducible subquotient of the representation of $G(\mathbb{A})$ generated by F_f.

We now consider Hecke operators and then compare them with those in local settings. For $\alpha \in G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$, consider the double coset

$$T(\alpha) := \Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i.$$

We write $\alpha_i = \alpha_{i,0}\alpha_{i,1}$, where $\alpha_{i,0} \in Z(\mathbb{R})^+$ and $\alpha_{i,1} \in G^{\mathrm{der}}(\mathbb{R})^+$. Then we define the action of $T(\alpha)$ on f by

$$T(\alpha)f(z) = n(\alpha)_\lambda \sum_i (\alpha_{i,1}f)(z)$$

where $n(\alpha)_\lambda \in \mathbb{Q}^\times$ is a normalized factor depending on λ (and also on G). Assume that G is unramified at a rational p and the p-component K_p of K is a compact open subgroup of $G(\mathbb{Z}_p)$ which contains an Iwahori subgroup. There exists a compact open subgroup K^p of $G(\mathbb{A}_f^{(p)})$ such that $K = K^p \times K_p$. Note that $F_{f,p}$ is an Iwahori fixed vector of $\pi_{f,p}$. For $\alpha \in G(\mathbb{Q}) \cap T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$, thus we obtain

$$\begin{aligned}
T(\alpha)F_f(g) = T(\alpha)F_f(g_\infty) &= T(\alpha)J_\lambda^{-1}(g_\infty, I)F(g_\infty I) \\
&= n(\alpha)_\lambda \sum_i J_\lambda^{-1}(g_\infty, I)J_\lambda^{-1}(\alpha_{i,1}, g_\infty I)F(\alpha_{i,1}g_\infty I) \\
&= n(\alpha)_\lambda \sum_i J_\lambda^{-1}(\alpha_{i,1}g_\infty, I)F(\alpha_{i,1}g_\infty I) \\
&= n(\alpha)_\lambda \sum_i F_f(\alpha_{i,1}g_\infty).
\end{aligned}$$

Note that

$$\begin{aligned}
G(\mathbb{Q})Z_G(\mathbb{R})^+\alpha_{i,1}g_\infty K &= G(\mathbb{Q})Z_G(\mathbb{R})^+\alpha_i g_\infty K \\
&= G(\mathbb{Q})(g_\infty Z_G(\mathbb{R})^+ \times \alpha_i^{-1}K).
\end{aligned}$$

Hence we have

$$\begin{aligned}
T(\alpha)F_f(g) &= n(\alpha)_\lambda \sum_i F_f(g_\infty\alpha_i^{-1}) = n(\alpha)_\lambda \sum_{h\in K\alpha^{-1}K/K} F_f(g_\infty h) \\
&= n(\alpha)_\lambda \int_K F_f(g_\infty h g_f)dg_f = n(\alpha)_\lambda \int_{K\alpha^{-1}K} F_f(g_\infty g_f)dg_f \\
&= n(\alpha)_\lambda \int_{G(\mathbb{A}_f)} T(\alpha^{-1})_K F_f(g_\infty g_f)dg_f = n(\alpha)_\lambda [K_p\alpha^{-1}K_p]F_{f,p}
\end{aligned}$$

where dg_f is the Haar measure on $G(\mathbb{A}_f)$ normalized so that $\mathrm{vol}(K) = 1$ and $T(\alpha^{-1})_K := [K_p\alpha^{-1}K_p] \otimes 1_{K^p}$.

We can naturally define the notion of positive or negative elements in the global setting. In the global setting, we usually consider Hecke operators represented by negative elements. By the arguments above, the computation of such global Hecke operators is reduced to that of Hecke operators associated to positive elements in the local setting. In the next section, we will give examples of p-stabilized forms in the global setting.

9. p-stabilized forms

9.1. GL_2-case. In this subsection we will refer [19] as a basic reference. Put $G = GL_2$. In this case, we have $G^{\mathrm{der}} = SL_2,\ K = SO(2)(\mathbb{R}),\ K_{\mathbb{C}} \simeq \mathbb{C}^\times$ and the corresponding Hermitian symmetric space is the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The automorphic factor is given by $J(\gamma, z) = cz + d$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and for an integer $k \geq 1$ we define the algebraic representation $\sigma_k : K_{\mathbb{C}} \longrightarrow \mathbb{C}^\times, z \mapsto z^k$. For an integer $N \geq 1$, we define the congruence subgroup $\Gamma_0(N)$ (respectively $\Gamma_1(N)$) to be the group consisting of the elements $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $c \equiv 0 \bmod N$ (respectively $a - 1 \equiv c \equiv 0 \bmod N$). For an integer $N \geq 1$ and a Dirichlet character $\chi : \Gamma_0(N) \longrightarrow \mathbb{C}^\times$ so that $\chi|_{\Gamma_1(N)} = 1$, we define $M_k(\Gamma_0(N), \chi) := M_{\sigma_k}(\Gamma_0(N), \chi)$. For a prime $p \not| N$, we define the action of Hecke operator $T_p = [\Gamma_0(N)\mathrm{diag}(1,p)\Gamma_0(N)]$ on $M_k(\Gamma_0(N), \chi)$ by

$$\begin{aligned} T_p f(z) := &\ p^{\frac{k}{2}-1} \sum_{\alpha \in \Gamma_0(N)\backslash\Gamma_0(N)\mathrm{diag}(1,p)\Gamma_0(N)} ((p^{-\frac{1}{2}}\alpha)^{-1} f)(z) \\ = &\ p^{\frac{k}{2}-1} \sum_{\alpha \in \Gamma_0(N)\backslash\Gamma_0(N)\mathrm{diag}(1,p)\Gamma_0(N)} j(p^{-\frac{1}{2}}\alpha, z)^{-k} f(p^{-\frac{1}{2}}\alpha z). \end{aligned}$$

Note that $p^{-\frac{1}{2}}\alpha \in SL_2(\mathbb{R})$. If $p \not| N$, then we define the action of $U_p = [\Gamma_0(pN)\mathrm{diag}(1,p)\Gamma_0(pN)]$ on $M_k(\Gamma_0(pN), \chi)$ by

$$U_p f(z) := p^{\frac{k}{2}-1} \sum_{\alpha \in \Gamma_0(pN)\backslash\Gamma_0(pN)\mathrm{diag}(1,p)\Gamma_0(pN)} ((p^{-\frac{1}{2}}\alpha)^{-1} f)(z).$$

Let $f \in M_k(\Gamma_0(N), \chi)$ be a normalized cusp form which is an eigenform for all $T_p,\ p \not| N$ with the eigenvalue $a_p(f)$. Let α_p, β_p be the Satake parameters at p so that $a_p(f) = \alpha_p + \beta_p$ and $\alpha_p\beta_p = \chi(p)p^{k-1}$. Let π_f be the automorphic representation associated to f and $\pi_{f,p}$ the local component at p. Since $\pi_{f,p}$ is a principal series representation, there exist characters $\chi_i : \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times,\ i = 1, 2$ such that $\pi_{f,p} \simeq \pi(\chi_1, \chi_2)$. We may put $\chi_1(p^{-1})p^{\frac{k-1}{2}} = \alpha_p,\ \chi_2(p^{-1})p^{\frac{k-1}{2}} = \beta_p$. We define p-stabilized forms as follows:

$$f_{\alpha_p}(z) := f(z) - \beta_p f(pz),\ f_{\beta_p}(z) := f(z) - \alpha_p f(pz).$$

Then f_{α_p} and f_{β_p} belong to $M_k(\Gamma_0(pN), \chi)$, and these are Hecke eigenforms for $T_\ell,\ \ell \not| pN$ and U_p-eigenforms with the eigenvalues α_p, β_p respectively.

This can be checked via local computation as follows. By using the strong approximation theorem, it is easy to see that $F_{f(pz)}(g) = F_f(g \cdot \mathrm{diag}(1,p))$. Hence the local component of $F_{f_{\alpha_p}}$ (respectively $F_{f_{\beta_p}}$) at p corresponds to $\chi_1(p)f_1$ (respectively $\chi_2(p)f_2$) of Section 5. By the computation of Section 5 again, we have

$$U_p f_{\alpha_p}(z) = p^{\frac{k}{2}-1}p^{\frac{1}{2}}\chi_2^{-1}(p)f_{\alpha_p}(z) = \alpha_p f_{\alpha_p}(z)$$

and

$$U_p f_{\beta_p}(z) = p^{\frac{k}{2}-1}p^{\frac{1}{2}}\chi_1^{-1}(p)f_{\beta_p}(z) = \beta_p f_{\beta_p}(z).$$

We say the above f is p-ordinary if $\mathrm{ord}_p(a_p(f)) = 0$. If so is f, then we may assume that $\mathrm{ord}_p(\alpha_p) = 0$ and $\mathrm{ord}_p(\beta_p) > 0$. The above computations show us that f_{α_p} can be embedded into a Hida family if f is p-ordinary. Note that f_{β_p} can be embedded into a Coleman family (cf. [7]).

9.2. $\boldsymbol{GSp_4}$-case. Let $\nu : G = GSp_4 \longrightarrow GL_1$ be the similitude character. Put $Sp_4 := \mathrm{Ker}\nu$. In this case, we have $G^{\mathrm{der}} = Sp_4$, $K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{R}) \right\} \simeq \mathrm{U}(2)(\mathbb{R}), K_{\mathbb{C}} \simeq \mathrm{GL}_2(\mathbb{C})$ and the corresponding Hermitian symmetric space is the Siegel upper half-plane $\mathcal{H}_2 = \{Z \in M_2(\mathbb{C}) |\ {}^tZ = Z,\ \mathrm{Im}(Z) > 0\}$. For a pair of positive integers $\underline{k} = (k_1, k_2)$ such that $k_1 \geq k_2$, we define the algebraic representation $\lambda_{\underline{k}}$ of $\mathrm{GL}_2(\mathbb{C})$ by

$$\lambda_{\underline{k}} = \mathrm{Sym}^{k_1-k_2}\mathrm{St}_2 \otimes \det{}^{k_2}\mathrm{St}_2$$

where St_2 is the standard representation of dimension 2 over $\mathbb{C}$. Then the corresponding automorphic factor is defined by

$$J_{\underline{k}}(\gamma, Z) = \lambda_{\underline{k}}(CZ + D)$$

for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{R})$ and $Z \in \mathcal{H}_2$. For an integer $N \geq 1$, we define a principal congruence subgroup $\Gamma(N)$ to be the group consisting of the elements $g \in \mathrm{Sp}_4(\mathbb{Z})$ such that $g \equiv 1 \bmod N$. We also define the level of Γ to be the minimal N satisfying (i) $g\Gamma g^{-1}$ contains $\Gamma(N)$ for some $g \in \mathrm{GSp}_4(\mathbb{Q})$ and (ii) all divisors of the denominator or the numerator of the entries of g divide N. If Γ is of level N, then the closure $\overline{\Gamma}$ in $\mathrm{Sp}_4(\mathbb{A}_f)$ satisfies that the p-component of $\overline{\Gamma}$ is $\mathrm{Sp}_4(\mathbb{Z}_p)$ for all $p \not| N$. For a parabolic subgroup P, let

$I_P = \Gamma_P(p)$ be the group consisting of the elements $g \in \mathrm{Sp}_4(\mathbb{Z})$ such that $(g \bmod p) \in P(\mathbb{F}_p)$. For a discrete subgroup Γ of level N, put $\Gamma_{I_P} := \Gamma \cap I_P$.

For a discrete subgroup Γ of level N and a Dirichlet character $\chi : \Gamma \longrightarrow \mathbb{C}^\times$, we define $M_{\underline{k}}(\Gamma, \chi) := M_{\lambda_{\underline{k}}}(\Gamma, \chi)$. A function of this space is called a Siegel modular form of weight $\underline{k}$ and level Γ with a character χ. If $k = k_1 = k_2$, put $M_k(\Gamma, \chi) := M_{(k,k)}(\Gamma, \chi)$ for short. For a prime $p \not| N$ and $t'_1 = \mathrm{diag}(1,1,p,p)$, $t'_2 = \mathrm{diag}(1,p,p,p^2)$, we define the action of two Hecke operators $T_{p,i} = [\Gamma t'_1 \Gamma]$, $i = 1,2$ on $M_k := \bigcup_{(\Gamma,\chi)} M_k(\Gamma, \chi)$ by

$$\begin{aligned} T_{p,i} f(Z) := &\ p^{i(\frac{k_1+k_2}{2}-3)} \sum_{\alpha \in \Gamma \backslash \Gamma t'_i \Gamma} ((\nu(t'_i)^{-\frac{1}{2}} \alpha)^{-1} f)(Z) \\ = &\ p^{i(\frac{k_1+k_2}{2}-3)} \sum_{\alpha \in \Gamma \backslash \Gamma t'_i \Gamma} J_{\underline{k}}(\nu(t'_i)^{-\frac{1}{2}} \alpha, Z)^{-1} f(\nu(t'_i)^{-\frac{1}{2}} \alpha Z). \end{aligned}$$

Then $T_{p,i} f \in M_k$, but in general, these operators do not preserve $M_{\underline{k}}(\Gamma, \chi)$ (see Lemma 3.1 of [17]). Note that $\nu(t'_i)^{-\frac{1}{2}} \alpha \in \mathrm{Sp}_4(\mathbb{R})$. If $p \not| N$, then we define the action of $U^{P,\mathrm{global}}_{p,i} = [\Gamma_{I_P} t'_i \Gamma_{I_P}]$, $i = 1,2$ on $M_k(\Gamma_{I_P}, \chi)$ by

$$U^{P,\mathrm{global}}_{p,i} f(Z) := p^{i(\frac{k_1+k_2}{2}-3)} \sum_{\alpha \in \Gamma_{I,P} \backslash \Gamma_{I_P} t'_i \Gamma_{I_P}} ((\nu(t'_i)^{-\frac{1}{2}} \alpha)^{-1} f)(Z). \tag{9.1}$$

In what follows, we will discuss about a p-stabilized form of Saito-Kurokawa lift. Let $\Gamma_0(N)$ be the subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ consisting of the elements $g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}$ such that $C_g \equiv 0_2 \bmod N$. Fix a finite character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$. Then we define (and denote it by χ again) the character on $\Gamma_0(N)$ associated to χ by $\chi(g) := \chi(\det(D_g))$ for $g \in \Gamma_0(N)$. For a normalized elliptic newform f of weight $2k-2 \geq 2$ and level N with the character χ^2, there exists a cusp form $F = SK(f)$ in $S_{k-1}(\Gamma_0(N), \chi)$ by [11] so that F is an eigenform for all $T_{p,i}$, $p \not| N$ with the eigenvalue $a_{p,i}(F)$ and these eigenvalues are written as

$$a_{p,1}(F) = \chi(p)(p^{k-1} + p^{k-2}) + a_p(f),$$
$$a_{p,2}(F) = a_p(f)\chi(p)(p^{k-2} + p^{k-3}) + 2\chi^2(p)p^{2k-4} - (p^2+1)\chi^2(p)p^{2k-6}$$

where $a_p(f)$ is the eigenvalue of f for T_p in Section 9.1. Let π_f (respectively Π_F) be the automorphic representation associated to f (respectively $F = SK(f)$) and $\pi_{f,p}$ (respectively $\Pi_{F,p}$) the local component at p. Let $\chi' : \mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ be the character corresponding to χ and denote by χ'_p its local component at p. Since $\pi_{f,p}$ is a principal series representation, there exist characters $\chi_i : \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$, $i = 1,2$ such that $\pi_{f,p} \simeq \pi(\chi_1, \chi_2)$ with the central character $\chi_1\chi_2 = \chi'^{-2}_p$. Hence we have $\chi_1\chi_2(p) = \chi'^2_p(p^{-1})$. Then by [16], we have $\Pi_{F,p} \simeq \chi_1\chi' 1_{\mathrm{GL}_2} \rtimes \chi_1^{-1}\chi_p'^{-2}$ which is the case IIb in Section 7.1.2. Then the eigenvalues of $U^B_{p,i}, i = 1,2$ in the local setting are

$$(p^{\frac{3}{2}}\delta, p^2\delta\beta), (p^{\frac{3}{2}}\alpha, p^2\alpha\beta), (p^{\frac{3}{2}}\beta, p^2\beta\delta), (p^{\frac{3}{2}}\beta, p^2\beta\alpha)$$

where $\alpha = \chi_1(p^{-1}),\ \beta = p^{\frac{1}{2}}\chi'(p),\ \gamma = p^{-\frac{1}{2}}\chi'(p)$ and $\delta = \chi_2(p^{-1})$. We may put $\chi_1(p^{-1})p^{\frac{2k-3}{2}} = \alpha_p(f),\ \chi_2(p^{-1})p^{\frac{2k-3}{2}} = \beta_p(f)$ where $\alpha_p(f), \beta_p(f)$ are eigenvalues of T_p for f. From (9.1) and the observation in Section 7 (7.1-(iii) and 7.1.2 Case IIb),

$$(\beta_p - U^{B,\mathrm{global}}_{p,1})(\chi'(p)p^{k-1} - U^{B,\mathrm{global}}_{p,1})F \tag{9.2}$$

is a p-stabilized form with the eigenvalues $(\alpha_p, p^{k-2}\alpha_p\chi'(p))$ for $U^{B,\mathrm{global}}_{p,i}$, $i = 1,2$. We take the Fourier expansion

$$F = \sum_{T\in \mathrm{Sym}^2(\mathbb{Z})_{>0}} a(T)e^{2\pi\sqrt{-1}\mathrm{tr}(TZ)}$$

where $\mathrm{Sym}^2(\mathbb{Z})_{>0}$ is the subset of $M_2(\mathbb{Q})$ consisting of all symmetric matrices which are positive and semi-integral. Then we have

$$(\beta_p - U^{B,\mathrm{global}}_{p,1})(\chi'(p)p^{k-1} - U^{B,\mathrm{global}}_{p,1})F =$$

(9.3)
$$\sum_{T>0}\Big(p^{k-1}\chi'(p)\beta_p a(T) - (\chi'(p)p^{k-1} + \beta_p)a(pT) + a(p^2T)\Big)e^{2\pi\sqrt{-1}\mathrm{tr}(TZ)},$$

since we can choose a complete system of representatives for $\Gamma_{I,B}\backslash\Gamma_{I_B}t_1'\Gamma_{I_B}$ to be the same as in the case Siegel parabolic and in that case we can compute the action easily (cf. [2]).

Similarly, $(\beta_p - U^{P,\mathrm{global}}_{p,1})(\chi'(p)p^{k-1} - U^{P,\mathrm{global}}_{p,1})F$ is also a p-stabilized form with respect to $\widehat{P} = Q$ with the eigenvalues α_p for $U^{P,\mathrm{global}}_{p,1}$ (see 7.2-(iii) and 7.2.2 Case IIb). As explained above, we have the same expansion. Therefore we have

$$\begin{aligned}&(\beta_p - U^{B,\mathrm{global}}_{p,1})(\chi'(p)p^{k-1} - U^{B,\mathrm{global}}_{p,1})F\\ =&(\beta_p - U^{P,\mathrm{global}}_{p,1})(\chi'(p)p^{k-1} - U^{P,\mathrm{global}}_{p,1})F\end{aligned}$$

for t_1'. If $k \geq 2$, F has p-integral (algebraic) coefficients, then so does $(\beta_p - U^{B,\mathrm{global}}_{p,1})(\chi'(p)p^{k-1} - U^{B,\mathrm{global}}_{p,1})F$. We further assume that $\mathrm{ord}_p(\alpha_p) = 0$ (this implies $\mathrm{ord}_p(\beta_p) > 0$). Then by using the relation of Hecke operators (cf. p.228, Example 4.2.10 of [1]), we see that $\mathrm{ord}_p(a(p^2T_0)) = 0$ if $\mathrm{ord}_p(a(T_0)) = 0$ for some $T_0 \in \mathrm{Sym}^2(\mathbb{Z})_{>0}$. Hence under this assumption (the existence of such T_0), $(\beta_p - U^{B,\mathrm{global}}_{p,1})(\chi'(p)p^{k-1} - U^{B,\mathrm{global}}_{p,1})F$ is not only preserving p-integrality, but also non-vanishing modulo p.

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