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Mathieu DUTOUR SIKIRIĆ et Konstantin RYBNIKOV

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Delaunay polytopes derived from the Leech lattice

par MATHIEU DUTOUR SIKIRIĆ et KONSTANTIN RYBNIKOV

RÉSUMÉ. Un polytope de Delaunay dans un réseau L est parfait si les transformations affines qui préservent sa propriété d'être Delaunay sont des compositions d'homothéties et d'isométries. Les polytopes de Delaunay parfaits sont rares en petite dimension et ici nous considérons ceux qui apparaissent dans des sections du réseau de Leech.

Par ce moyen, nous trouvons des réseaux ayant plusieurs orbites de polytopes de Delaunay parfaits. Nous exhibons aussi des polytopes de Delaunay parfait qui restent Delaunay dans des super-réseaux. Aussi nous trouvons des polytopes de Delaunay parfait ayant des groupes d'automorphismes relativement petit par rapport à leurs réseaux. Nous prouvons aussi que certains polytopes de Delaunay parfaits ont un nombre de lamination égal à 5 alors que les polytopes de Delaunay précédemment connus ont un nombre de lamination égal à 3.

Une construction bien connu de polytopes de Delaunay centralement symétriques utilise des polytope de Delaunay parfait antisymétriques. Nous classifions complètement les types de polytopes de Delaunay parfaits qui peuvent apparaître dans cette construction.

Enfin, nous prouvons une borne supérieure sur le rayon de recouvrement du réseau $\Lambda_{24}(v)^*$ qui généralise la borne de Smith. Nous prouvons que cette borne est atteinte seulement pour Λ_{23}^* qui est le meilleur recouvrement de \mathbb{R}^{23} connu.

ABSTRACT. A Delaunay polytope in a lattice L is perfect if any affine transformation that preserve its Delaunay property is a composite of an homothety and an isometry. Perfect Delaunay polytopes are rare in low dimension and here we consider the ones that one can get in lattice that are sections of the Leech lattice.

By doing so we are able to find lattices with several orbits of perfect Delaunay polytopes. Also we exhibit Delaunay polytopes which remain Delaunay in some superlattices. We found perfect Delaunay polytopes with small automorphism group relative to

the automorphism group of the lattice. And we prove that some perfect Delaunay polytopes have lamination number 5, which is higher than previously known 3.

A well known construction of centrally symmetric perfect Delaunay polytopes uses a laminated construction from an antisymmetric perfect Delaunay polytope. We fully classify the types of perfect Delaunay polytopes that can occur.

Finally, we derived an upper bound for the covering radius of $\Lambda_{24}(v)^*$, which generalizes the Smith bound and we prove that this bound is met only by Λ_{23}^* , the best known lattice covering in \mathbb{R}^{23} .

1. Introduction

Given an n -lattice $L \subset \mathbb{R}^n$, a sphere $S = S(c, r)$ of center $c \in \mathbb{R}^n$ and radius r is called an *empty sphere* for L if there is no $v \in L$ such that $\|v - c\| < r$. A polytope $D = D_L(c)$ in \mathbb{R}^n (not necessarily of full dimension) is called a *Delaunay polytope* in L if the set of its vertices $\text{vert } D$ is $S \cap L$ where S is an *empty sphere* for L centered at c . An n -dimensional Delaunay polytope D in L is *perfect* with respect to L (or, extreme, as in [10]) if every linear bijective transformation ϕ of \mathbb{R}^n that maps D onto a Delaunay polytope in $\phi(L)$ is a composition of a homothety and an isometry. Perfect Delaunay polytopes were introduced in [10, 9, 20, 21] and their theory is closely related to the second Voronoi decomposition [35].

Up to similarity the unit interval in \mathbb{Z}^1 and the Gosset polytope 2_{21} in the root lattice E_6 are the only perfect Delaunay polytopes in dimension $n \leq 6$ [8]. In an upcoming work, we have classified the 7-dimensional perfect Delaunay polytopes and found 2 polytopes [17]: the polytope 3_{21} in the root lattice E_7 and a 35 vertex perfect Delaunay polytope [22]. Classification in dimension 8 seems much harder to obtain, a list of 27 8-dimensional perfect Delaunay polytopes is given in [15] and it is likely that the list is complete. The enumeration is obtained by application of the method of [18]; the same method partially applied in dimension 9 gives 100000 perfect Delaunay polytopes. In [24, 25, 11] several infinite series are built that proves the existence of perfect Delaunay polytopes for any dimension $n \geq 6$.

It is expected that very few general structural properties of perfect Delaunay polytopes are true in general and therefore we searched for specific examples. The problem is that the dimensions where we can do full or partial enumeration are too low for interesting phenomena to occur. Therefore, following [9], we try another route by looking for the polytopes obtained by sections of the Leech lattice Λ_{24} . More precisely, given $v \in \Lambda_{24}$ we define $\Lambda_{24}(v)$ to be the lattice of vectors of Λ_{24} orthogonal to v . In Theorem 3, we prove an upper bound on the covering radius of $\Lambda_{24}(v)^*$ and we prove that this upper bound is met only for vectors v of norm 4, which gives the

covering density of Λ_{23}^* , the lattice holding the covering density record in dimension 23 [32, Table 3].

Then, in Section 4 we define the notion of main Delaunay polytopes for lattices L with $\Lambda_{24}(v) \subset L \subset \Lambda_{24}(v)^*$. For 16 vectors of smallest norm in the Leech lattice, Table 4.1 gives the corresponding 23-dimensional perfect main Delaunay polytopes. We found 13 lattices having several orbits of perfect Delaunay polytopes. Furthermore the table shows 2 perfect Delaunay polytopes which admit extensions to a superlattice with the same vertex set and 4 perfect Delaunay polytopes which admit extension with twice the vertex set (see Theorem 4.2 for details). This phenomenon has a direct analog in the theory of perfect lattice packings [28]. The analogy between perfect lattice packings and perfect Delaunay polytopes was first considered in [16].

Due to its rigidity property, a perfect Delaunay polytope uniquely defines the lattice in which it is contained. In Theorem 4.1 we give a perfect Delaunay polytope D of Λ_{22} for which the size of the automorphism group is one sixth of the size of the automorphism group of the lattice. The value of the quotient in size of the previously known example was always 1 or 2.

The lamination number of a n -dimensional polytope D is the minimum number of $(n - 1)$ -dimensional lattice layer that needs to be used in order to cover the vertex-set of D . In Theorem 4.3 we give two perfect Delaunay polytopes with lamination number 5, while all the previously known ones had lamination number 3, i.e. the minimum possible for perfect Delaunay polytopes [15, Theorem 10]. It is conjectured [1] that a n -dimensional polytope, whose vertices belong to a lattice L and is free of lattice point in its interior, has lamination number at most $n + 1$. [26] inquired about the possible lamination number of Delaunay polytopes and conjectured that they cannot have large lamination number. We expect that there exist Delaunay polytopes with arbitrarily high lamination number.

From an antisymmetric n -dimensional perfect Delaunay polytope D embedded in \mathbb{R}^{n+1} , it is always possible to get a centrally symmetric Delaunay polytope by stacking D and $v - D$ for a suitably chosen vector $v \notin L \otimes \mathbb{R}$. An interesting question is whether or not by varying the distance between $L \otimes \mathbb{R}$ and $v + L \otimes \mathbb{R}$ we can get a perfect Delaunay polytope. This question was solved in [10, Lemma 15.3.7] by assuming that there is an additional vertex in the interior of the cylinder defined by D and $v - D$. In Theorem 5.1 we classify the two possible scenario that can occur, i.e. empty cylinder or perfect Delaunay polytope. In Theorem 5.2 the structure of the cylinder obtained in the first case is elucidated. Corollary 5.1 gives some sufficient conditions for being in first or second case and this allows to find empty cylinder, which were unknown before.

There are many open questions on perfect Delaunay polytopes. One of them concerns the property of being a t -design. Very few Delaunay polytopes with $t \geq 4$ are known and we do not have any technique for doing their enumeration. Table 4.1 gives 6 such polytopes. A n -dimensional Delaunay polytope D in a lattice L is called basic if there are $n + 1$ vertices v_0, v_1, \dots, v_n of D such that for every vertex v of D there exists $\lambda_i \in \mathbb{Z}$ such that $v = \sum_{i=0}^n \lambda_i v_i$ and $1 = \sum_{i=0}^n \lambda_i$. Non-basic Delaunay polytopes exist [13] but it is unknown whether non-basic perfect Delaunay polytopes exist.

The paper is organized as follows. Section 2 introduces basic notions on lattices, Delaunay polytopes in lattices, Leech lattice and spherical designs. In Section 3 we define the lattices $\Lambda_{24}(v)$ and $\Lambda_{24}(v, d)$. Then we prove an upper bound on the covering radius of $\Lambda_{24}(v)^*$ and determine the covering radius of Λ_{23}^* . In Section 4 we define the main Delaunay polytopes of the lattices $\Lambda_{24}(v, d)$ and prove the announced existence results. In Section 5 we fully determine the Delaunay polytopes that one can obtain by lamination over an antisymmetric Delaunay polytope.

2. Definitions

2.1. Lattices. A lattice L is a subgroup of the vector space \mathbb{R}^n of the form $\mathbb{Z}v_1 + \dots + \mathbb{Z}v_k$, where v_1, \dots, v_k are independent vectors. The determinant $\det L$ is defined as the k -dimensional volume of the parallelepiped

$$\{x_1 v_1 + \dots + x_k v_k : 0 \leq x_i \leq 1 \text{ for all } i\}.$$

For $\mathbb{K} = \mathbb{R}$ or \mathbb{Q} we denote by $L \otimes \mathbb{K}$ the set $\mathbb{K}v_1 + \dots + \mathbb{K}v_k$. A lattice of rank k is called a k -dimensional lattice or simply a k -lattice. If L is a sublattice of L' , then L' is called a *superlattice* of L . The pair $L \subset L'$, where both lattices are of the same rank, is called a *centering* of L .

If $L \subset \mathbb{R}^n$ is a lattice then the dual lattice L^* is defined as follows:

$$L^* = \{x \in L \otimes \mathbb{R} \text{ such that for all } y \in L \text{ we have } \langle x, y \rangle \in \mathbb{Z}\}.$$

A lattice is called *integral* if $\langle x, y \rangle \in \mathbb{Z}$ for $x, y \in L$, i.e. if $L \subset L^*$. A lattice is called unimodular if $\det L = 1$. A lattice is *self-dual*, that is $L = L^*$, if and only if it is integral and unimodular.

If B^n is the n -dimensional unit ball centered at the origin and L is an n -dimensional lattice, then the *covering radius* also called inhomogeneous minimum $\text{cov } L$ is defined as follows:

$$\text{cov } L = \min\{\mu : L + \mu B^n \text{ covers } \mathbb{R}^n\}.$$

It is easy to see (see, e.g. [5, Section 2.1.3]) that $\text{cov } L$ is equal to the maximum circumradius of the Delaunay polytopes of L .

The group $O_n(\mathbb{R})$ is the group of isometries of \mathbb{R}^n preserving the origin. The automorphism group $\text{Aut } L$ of a lattice $L \subset \mathbb{R}^n$ is defined as

$$\text{Aut } L = \{f \in O_n(\mathbb{R}) \text{ s.t. } f(L) = L\}.$$

In [30] a practical algorithm for computing $\text{Aut } L$ is given. The full affine isometry group of a lattice L is denoted by $\text{Iso } L$.

A n -dimensional lattice admits $2^n - 1$ sublattices of index 2. For the dimensions that we are considering the most practical method is to enumerate directly all $2^n - 1$ possibilities and then use $\text{Aut } L$ for reducing by isomorphism. Note that we can also get the index 2 superlattices by this method.

A $(n-1)$ -dimensional lattice L' of L is called *primitive* if $(L' \otimes \mathbb{R}) \cap L = L'$. A *lamination* of an n -dimensional lattice L is a partition of L of the form $\cup_{k \in \mathbb{Z}} (L' + kv)$, where L' is a primitive $(n-1)$ -dimensional sublattice $L' \subset L$ and v is a vector in $L \setminus L'$. Set $\Lambda_0 = \mathbb{Z}^0$. A laminated n -lattice Λ_n is defined, up to similarity, as the densest n -lattice that has a lamination $\cup_{k \in \mathbb{Z}} (L' + kv)$, with L' isometric to Λ_{n-1} . For $n \leq 8$, the Λ_n are root lattices and are the best lattice packings in their dimensions. The laminated lattices are uniquely defined for $n \in \{9, 10, 15, \dots, 24\}$. See [5, Chapter 6] for more details on their construction and other properties.

2.2. Delaunay Polytopes in Lattices. Given a lattice $L \subset \mathbb{R}^n$ a point $x \in \mathbb{R}^n$ defines a (not necessarily full dimensional) *Delaunay polytope* $D_L(x)$ by

$$D_L(x) = \text{conv} \left\{ v \in L : \|x - v\| = \min_{w \in L} \|x - w\| \right\}.$$

The polytope $D_L(x)$ can be computed easily from x and L by using implementations of the *Closest Vector Problem*, which is generally feasible in the dimensions considered here. Given a full dimensional Delaunay polytope D denote by $c(D)$ its center and by $L(D)$ the lattice it affinely generates, i.e. the lattice generated by the difference between vertices of D .

Let us define $E_2(n)$ to be the space of polynomial functions of degree at most 2 on \mathbb{R}^n . If D is a Delaunay polytope with empty sphere $S(c, r)$ for $L = \mathbb{Z}v_1 + \dots + \mathbb{Z}v_n$ (rank $L = n$) then the function

$$\begin{aligned} f_D : \mathbb{Z}^n &\rightarrow \mathbb{R} \\ x = (x_1, \dots, x_n) &\mapsto \left\| \sum_{i=1}^n x_i v_i - c \right\|^2 - r^2 \end{aligned}$$

is a polynomial of degree 2 such that $f_D(x) \geq 0$ for all $x \in \mathbb{Z}^n$ and $f_D(x) = 0$ if and only if $\sum_{i=1}^n x_i v_i$ is a vertex of D .

Definition. Let D be a n -dimensional Delaunay polytope in a lattice $L = \mathbb{Z}v_1 + \dots + \mathbb{Z}v_n$.

(i) We define the cone of quadratic function

$$C_D = \left\{ \begin{array}{l} f \in E_2(n) : f(x) \geq 0 \text{ for all } x \in \mathbb{Z}^n, \text{ and } f(x) = 0 \\ \text{if and only if } \sum x_i v_i \text{ is a vertex of } D \end{array} \right\}$$

(ii) The dimension of C_D is called the perfection rank and denoted by $\text{perfrank } D$.

(iii) D is perfect if and only if $\text{perfrank } D = 1$.

The perfection rank of D can be computed easily by linear algebra computations. Let $(w^j)_{1 \leq j \leq M}$ the coordinates of the vertex set of D in the basis $(v_i)_{1 \leq i \leq n}$. Then $\text{perfrank } D$ is the dimension of the solution space of the vector space

$$\{f \in E_2(n) \text{ s.t. } f(w^1) = 0, \dots, f(w^M) = 0\}.$$

As a consequence a n -dimensional perfect Delaunay polytopes has at least $\frac{(n+1)(n+2)}{2} - 1$ vertices. The above notion of perfection is computable but is not immediately equivalent to the geometric notion given at the beginning of this paper. For a proof of the equivalence, see for example [21] or for a different viewpoint [10].

The isometry group of a Delaunay polytope $D = D_L(c)$ is denoted by $\text{Iso}_L D$. The subgroup $\text{Aut}_L D$ is the group of isometries of L preserving D . It can happen that $\text{Aut}_L D \neq \text{Iso}_L D$ but if $L(D) = L$ then we have equality.

Definition. Let D be a n -dimensional Delaunay polytope in a lattice L .

(i) For a $(n-1)$ -dimensional primitive sublattice L' of L , the non-empty sections of $\text{vert } D$ by hyperplanes $L' + kv$ are called laminae.

(ii) The lamination number of D is the minimum number of laminae over all primitive $(n-1)$ -dimensional sublattices L' of L .

We do not know any systematic method for computing the lamination number of a polytope. However, it is easy to enumeration the 2-laminations if any exist: one takes $n+1$ independent vertices $S = \{v_0, \dots, v_n\}$, iterate over all $2^n - 1$ 2-laminations of S and keeps the ones that extends to 2-laminations of D .

Definition. Given a vector $v \in L \otimes \mathbb{Q}$, denote by $\text{den}(v)$ the least common denominator of its coordinate, i.e. the smallest integer $d > 0$ such that $dv \in L$.

A Delaunay polytope D is either centrally-symmetric with respect to its circumcenter c , or *antisymmetric*, in which case for any $v \in \text{vert } D$ we have $2c - v \notin \text{vert } D$. Note that $\text{den}(c(D)) = 2$ if and only if D is centrally symmetric.

2.3. The Leech lattice and related lattices. The lattice Λ_{24} is known as *the Leech lattice*. The Leech lattice is integral and unimodular and therefore self-dual. Every vector $v \in \Lambda_{24}$ has even norm and the norm of non-zero vectors is at least 4. There are 196560 such minimal vectors, which is the maximal possible number in dimension 24 [5, Chapter 14]. It is known that the Leech lattice is the best lattice packing in dimension 24 [3].

The symmetry group $\text{Aut } \Lambda_{24}$ is the Conway group Co_0 , which is a double cover of the simple group Co_1 . The list of all 307 orbits of Delaunay polytopes of Λ_{24} is known and among them 23 orbits have the largest possible circumradius, i.e. $\sqrt{2}$. Those Delaunay polytopes are described by affine root systems; full details are in [4], [5, Chapter 25]. It is expected that the Leech lattice is the best lattice covering in dimension 24.

The lattices Λ_{23} , Λ_{23}^* have 93150, respectively 4600, minimal vectors of norm 4, respectively 3. The 4600 minimal vectors of Λ_{23}^* generate an index 2 sublattice called O_{23} . The self-dual lattice O_{23} is known as the *shorter Leech lattice*, as it was constructed by John Leech from the 23-dimensional Golay code (after O'Connor and Pall's work [29]). The lattice Λ_{23} is an index 2 sublattice of O_{23} ; thus, Λ_{23} is an index 4 sublattice of Λ_{23}^* . More precisely, $\Lambda_{23}^*/\Lambda_{23} = \mathbb{Z}/4\mathbb{Z}$. The automorphism group of these three lattices is $\mathbb{Z}_2 \times Co_2$ and the corresponding three integral representations exhaust the list of 23-dimensional integral representations of Co_2 enumerated in [31].

2.4. Spherical designs. We say that a finite, nonempty subset X in \mathbb{R}^n carries a *spherical t -design* (see [7] for details on spherical designs) if there is a similarity transformation mapping X to points on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ so that for the spherical measure $d\omega$ on S^{n-1} and for all polynomials $f \in \mathbb{R}[x_1, \dots, x_n]$ up to degree t we have

$$\frac{1}{|Y|} \sum_{y \in Y} f(y) = \frac{1}{\omega(S^{n-1})} \int_{S^{n-1}} f(y) d\omega(y).$$

The maximal possible t is called the *strength* of the design. A Delaunay polytope of center c and radius r defines a spherical t -design if and only if for all $k \leq t$ we have the following identity

$$\frac{1}{|\text{vert } D|} \sum_{x, y \in \text{vert } D} \langle x - c, y - c \rangle^k = \begin{cases} 0, & \text{for all odd } k \\ \frac{1 \cdot 3 \cdots (k-1)}{n(n+2) \cdots (n+k-2)} r^{2k} & \text{for all even } k. \end{cases}$$

This characterization is computationally easier and the proof of equivalence is done in [7], where the notion of spherical design was introduced and their harmonic analysis developed. A Delaunay polytope defines a 0-design on its empty sphere and it is a 1-design if and only if its circumcenter is equal to its isobarycenter. In [16] it is proved that if a Delaunay polytope D defines a 4-design, then D is perfect. This is the counterpart of a similar

theorem for perfect forms [34]. Almost nothing is known on the enumeration of perfect Delaunay polytope with defines t designs of strength $t \geq 4$.

All the computation of this paper were realized using the **polyhedral** package [37].

3. The lattices $\Lambda_{24}(v, d)$ and the covering density of Λ_{23}^*

Let v be a primitive vector of the Leech lattice Λ_{24} , i.e. one such that $\frac{v}{d} \notin \Lambda_{24}$ for all natural number $d > 1$. Then, we define

$$\Lambda_{24}(v) = \{x \in \Lambda_{24} \text{ such that } \langle x, v \rangle = 0\}.$$

By the appendix of [28] we have

$$\Lambda_{24}(v)^*/\Lambda_{24}(v) \simeq \mathbb{Z}/\|v\|^2\mathbb{Z}.$$

Any $d \in \mathbb{N}$ dividing $\|v\|^2$ determines uniquely a lattice $\Lambda_{24}(v, d)$ with

$$\Lambda_{24}(v) \subset \Lambda_{24}(v, d) \subset \Lambda_{24}(v)^* \quad \text{and} \quad [\Lambda_{24}(v, d) : \Lambda_{24}(v)] = d.$$

We have $\Lambda_{24}(v, 1) = \Lambda_{24}(v)$ and $\Lambda_{24}(v, \|v\|^2) = \Lambda_{24}(v)^*$.

Theorem 3.1. *Let v be a primitive vector of the Leech lattice and define*

$$r(v) = \sqrt{2 - \frac{1}{4\|v\|^2}}.$$

- (i) *The covering radius of the lattice $\Lambda_{24}(v)^*$ is at most $r(v)$;*
- (ii) *the only lattice $\Lambda_{24}(v)^*$ with covering radius $r(v)$ is $\Lambda_{23}^* = \Lambda_{24}(v)^*$ for v a vector of norm 4.*

The Delaunay polytopes of maximal circumradius of Λ_{23}^ belong to a single orbit of Delaunay polytopes, whose representatives have 64 vertices and 2688 symmetries.*

Proof. Denote by p_v the orthogonal projection operator of Λ_{24} onto $\Lambda_{24}(v) \otimes \mathbb{R}$. The dual lattice $\Lambda_{24}(v)^*$ is equal to the projection $p_v(\Lambda_{24})$ of Λ_{24} . We suppose that the covering radius of $\Lambda_{24}(v)^*$ is strictly greater than $r(v)$, that is that there exists a vector $w \in \Lambda_{24}(v) \otimes \mathbb{R}$ such that for every $x \in \Lambda_{24}(v)^*$ we have $\|x - w\| > r(v)$. Define $v' = \frac{1}{\|v\|^2}v$ and $w' = w + \frac{1}{2}v'$. For every $y \in \Lambda_{24}$ set

$$h_y = y - w' = p_v(y - w) + \alpha v' = (p_v(y) - w) + \alpha v',$$

where

$$\alpha = \langle v, h_y \rangle = -\frac{1}{2} + \langle v, y \rangle \in \frac{1}{2} + \mathbb{Z}.$$

Thus we get

$$(3.1) \quad \begin{aligned} \|h_y\|^2 &= \|p_v(y) - w\|^2 + \alpha^2 \|v'\|^2 \\ &> r(v)^2 + \alpha^2 \|v'\|^2 \geq 2. \end{aligned}$$

The inequality $\|h_y\|^2 > 2$ contradicts the fact that the covering radius of Λ_{24} is $\sqrt{2}$.

Suppose now that D is a Delaunay polytope of $\Lambda_{24}(v)^*$ of center c and circumradius $r(v)$. Define $c' = c + \frac{1}{2}v'$. By an argument similar to (i) we get that $\|y - c'\|^2 \geq 2$ for every $y \in \Lambda_{24}$. Thus c' is the center of a Delaunay polytope D' of Λ_{24} of circumradius $\sqrt{2}$. Define $f(x) = \langle x, v \rangle$. We have $f(c') = \frac{1}{2}$ and $f(v) \in \mathbb{Z}$ for $v \in \text{vert } D' \subset \Lambda_{24}$.

If $v \in \text{vert } D'$ then we have

$$\|p_v(v) - c\|^2 = 2 - \frac{1}{\|v\|^2} \left(\frac{1}{2} - f(v) \right)^2.$$

Thus in order for D to be a Delaunay polytope of circumradius $r(v)$, it is necessary that $f(v) = 0$ or 1 for $v \in \text{vert } D'$. So D' has lamination number 2 and the vector v is defined up to a scalar multiple by the corresponding 2-lamination. For a n -dimensional polytope P a 2-lamination in two layers L_0, L_1 corresponds to a partition of $\text{vert } P$ in two subsets P_0 and P_1 . If $\mathcal{S} = \{v_1, \dots, v_{n+1}\}$ is a set of $n + 1$ independent vertices of P then the possible partitions $\{P_0, P_1\}$ are determined by the intersections $\mathcal{S} \cap P_0$. Thus there are at most $2^{n+1} - 2$ 2-laminations on P and they can be enumerated by considering all subsets of an independent set \mathcal{S} of $\text{vert } P$ and checking if they correspond to a partition $\{P_0, P_1\}$. By $D_{\Lambda_{24}}(c_1), \dots, D_{\Lambda_{24}}(c_{23})$ we denote representatives of the 23 orbits of Delaunay polytopes of circumradius $\sqrt{2}$. Given a polytope $D_{\Lambda_{24}}(c_i)$ we enumerate its 2-laminations; determine the possible vectors v ; keep the ones such that the projection $p_v(c_i)$ determines a Delaunay polytope of $\Lambda_{24}(v)^*$ of circumradius $r(v)$. It turns out that, up to equivalence, only one such vector v satisfies the required conditions. This vector is of norm 4 and so Λ_{23}^* is the only lattice meeting the bound. \square

The upper bound of the above theorem was proved in [33] for the lattice Λ_{23}^* , which holds the record lattice covering in dimension 23. The Delaunay polytope of Λ_{24} that determines the Delaunay polytope of Λ_{23}^* of maximal circumradius is named A_3^8 [5, Chapter 23]. If one takes the Gram matrix of Λ_{23} from [36] and apply a matrix inversion then the center of an empty sphere of maximum radius is

$$\frac{1}{4} (3, 1, 0, 2, 1, 1, 2, 0, 3, 2, 3, 3, 2, 1, 2, 2, 3, 3, 0, 2, 1, 3, 3).$$

The covering density of Λ_{23}^* is 15.3217885165555.

The only general method for computing the covering radius of a lattice is to compute the full Delaunay tessellation. For Λ_{23} , respectively O_{23} , there are 709, respectively 5, orbits of Delaunay polytopes [12]. For Λ_{23}^* the same program yields several hundred thousands of orbits before the computation could terminate.

4. The main Delaunay polytopes of $\Lambda_{24}(v, d)$

Let us say that a vector $v \in \Lambda_{24}$ has type n if it is of norm $2n$ and has type $n_{a,b}$ if it is also the sum of two vectors of types a and b . $\text{Aut } \Lambda_{24}$ is transitive on the vectors of following types:

(4.1)

$$2, 3, 4, 5, 6_{2,2}, 6_{3,2}, 7, 8_{2,2}, 8_{3,2}, 8_{4,2}, 9_{3,3}, 9_{4,2}, 10_{3,3}, 10_{4,2}, 10_{5,2}, 11_{4,3}, 11_{5,2}$$

and this exhausts the list of vectors of norm at most 22 (see [5, Section 10.3.3]). By v_n , respectively $v_{n;a,b}$, we denote a vector of type n , respectively $n_{a,b}$. Note that $2v_2$ is of type $8_{2,2}$. The lattice $\Lambda_{24}(v_2)$ is Λ_{23} . The lattice $\Lambda_{24}(v_2, 2)$ is the shorter Leech lattice O_{23} .

Denote by $\text{Stab}(v)$ the stabilizer of $v \in \Lambda_{24}$ under $\text{Aut } \Lambda_{24}$. $\text{Stab}(v)$ defines a subgroup of $\text{Aut } \Lambda_{24}(v, d)$. The stabilizers of the 17 vectors $v_n, v_{n;a,b}$ are given in [5, Table 10.4] and many of them involve sporadic simple groups.

Definition. For a given vector $v \in \Lambda_{24}$ and $\alpha \in \mathbb{Z}$, define:

(i) $c(v, \alpha)$ to be the projection of 0 on the hyperplane

$$\{x \in \mathbb{R}^n \text{ s.t. } \langle v, x \rangle = \alpha\}.$$

(ii) The Delaunay polytope

$$D(v, d, \alpha) = D_{\Lambda_{24}(v,d)}(c(v, \alpha)).$$

(iii) The set of vectors

$$\text{Min}_{v,\alpha} \Lambda_{24} = \{x \in \text{Min } \Lambda_{24} : \langle x, v \rangle = \alpha\}.$$

The Delaunay polytopes $D(v, d, \alpha)$ contain $\text{Stab}(v)$ in their stabilizer.

Proposition 4.1. *Let us take a non-zero vector $v \in \Lambda_{24}$. We have:*

(i) *If $\text{Min}_{0,v} \Lambda_{24} \neq \emptyset$ then $\text{Min } \Lambda_{24}(v) = \text{Min}_{0,v} \Lambda_{24}$.*

(ii) *If $\alpha \neq 0$ and $\text{Min}_{\alpha,v} \Lambda_{24}$ is 23-dimensional then*

$$D(v, d, \alpha) = \text{Min}_{v,\alpha} \Lambda_{24}$$

and its covering radius is $d(v, \alpha) = \sqrt{4 - \frac{\alpha^2}{\|v\|^2}}$.

Proof. By construction a non-zero vector $v \in \Lambda_{24}(v)$ has norm at least 4. So, if there is one such vector then the minimum is 4 and (i) holds.

Since $c(v, \alpha)$ is orthogonal to the hyperplane $\{x \in \mathbb{R}^n \text{ s.t. } \langle x, v \rangle = \alpha\}$ we obtain that all elements of $\text{Min}_{\alpha,v} \Lambda_{24}$ are at equal distance from $c(v, \alpha)$. Furthermore, if there were a vector at closer distance then it would define a non-zero vector of norm smaller than 4, which is impossible. \square

We call a Delaunay polytopes $D(v, d, \alpha)$ with covering radius $d(v, \alpha)$ *main*. In the following, we will use the main Delaunay polytopes $D(v, d, \alpha)$ as a source of example of polytopes with specific properties. We will be chiefly interested in the case $d = 1$ and see how the Delaunay polytopes

Type of v	Main Delaunay polytopes	
2	$D(v, 1, 1); (47104, 4, 7, 1), D(v, 2, 1); (94208, 2, 7, 1)$ $D(v, 1, 2); (4600, 2, 7, 1)$	
3	$D(v, 1, 1); (48600, 6, 5, 1)$ $D(v, 1, 2); (11178, 3, 5, 1), D(v, 2, 2); (11178, 3, 5, 2)$ $D(v, 1, 3); (552, 2, 5, 1), D(v, 3, 3); (552, 2, 5, 3)$	
4	$D(v, 1, 1); (47104, 8, 3, 1)$ $D(v, 1, 2); (16192, 4, 3, 1), D(v, 2, 2); (32384, 2, 3, 1)$	
5	$D(v, 1, 1); (45100, 10, 1, 1)$	$D(v, 1, 2); (19450, 5, 1, 1)$
6 _{2,2}	$D(v, 1, 2); (22518, 6, 1, 1)$	
6 _{3,2}	$D(v, 1, 1); (43056, 12, 2, 1)$ $D(v, 1, 2); (21528, 6, 2, 1)$ $D(v, 1, 3); (6072, 4, 2, 1), D(v, 2, 3); (12144, 2, 3, 1)$	
7	$D(v, 1, 1); (41152, 14, 0, 1)$ $D(v, 1, 3); (7900, 14, 0, 1)$	$D(v, 1, 2); (22825, 7, 0, 1)$
8 _{3,2}	$D(v, 1, 2); (24576, 8, 0, 1), D(v, 2, 2); (47104, 4, 7, 1),$ $D(v, 4, 2); (94208, 2, 7, 1)$	
8 _{4,2}	$D(v, 1, 1); (39424, 16, 1, 1)$ $D(v, 1, 2); (23608, 8, 1, 1)$ $D(v, 1, 3); (9472, 16, 1, 1)$ $D(v, 1, 4); (2268, 4, 1, 1), D(v, 2, 4); (4536, 2, 1, 1)$	
9 _{3,3}	$D(v, 1, 1); (37908, 18, 1, 1)$ $D(v, 1, 3); (10758, 6, 1, 1)$	$D(v, 1, 2); (24057, 9, 1, 1)$ $D(v, 1, 4); (3159, 9, 1, 1)$
9 _{4,2}	$D(v, 1, 1); (37743, 18, 0, 1)$ $D(v, 1, 2); (24035, 9, 0, 1)$ $D(v, 1, 3); (10879, 6, 0, 1), D(v, 3, 3); (32384, 2, 3, 1)$	
10 _{3,3}	$D(v, 1, 2); (25300, 10, 1, 1)$	$D(v, 1, 4); (4325, 5, 1, 1)$
10 _{4,2}	$D(v, 1, 2); (25036, 10, 0, 1)$	$D(v, 1, 4); (3489, 5, 0, 1)$
10 _{5,2}	$D(v, 1, 1); (36454, 20, 0, 1)$ $D(v, 1, 3); (11882, 20, 0, 1)$	$D(v, 1, 2); (24266, 10, 0, 1)$ $D(v, 1, 4); (3993, 5, 0, 1)$
11 _{4,3}	$D(v, 1, 1); (35200, 22, 0, 1)$ $D(v, 1, 3); (12760, 22, 0, 1)$	$D(v, 1, 2); (24332, 11, 0, 1)$ $D(v, 1, 4); (4832, 11, 0, 1)$
11 _{5,2}	$D(v, 1, 1); (34782, 22, 0, 1)$ $D(v, 1, 3); (13122, 22, 0, 1)$	$D(v, 1, 2); (24200, 11, 0, 1)$

TABLE 4.1. First column gives the 17 types of vector of Λ_{24} of norm at most 22 except 8_{2,2}. The entries “ $D(v, d, \alpha): (N, den, s, ind)$ ” in second column correspond to the 23-dimensional main Delaunay polytope $D(v, d, \alpha)$ of $\Lambda_{24}(v, d)$ with N vertices, denominator of circumcenter $den = den(c(D))$, strength s of t -design and index ind of $L(D)$ in $\Lambda_{24}(v, d)$

is extended for larger d . Since $\text{Min } \Lambda_{24}$ has a finite number of elements, there is a finite number of such main perfect Delaunay polytopes but we are not able at this point to determine the complete list. Therefore we limit ourselves to v from the first 17 types.

In Table 4.1 we give informations about the main perfect Delaunay polytopes associated to the vectors of norm at most 22, according to their types. We omit 8_{22} which is covered by type 2. For every perfect main Delaunay D we give the number N of vertices, the denominator $den = den(c(D))$ of the circumcenter $c(D)$, the strength s of the spherical t -design and the index ind of $L(D)$ in $\Lambda_{24}(v, d)$ by the symbol “ $D(v, d, \alpha); (N, den, s, ind)$ ”. Two additional 22-dimensional perfect Delaunay polytope occur as $\text{Min}_{v_4,4} \Lambda_{24}$ and $\text{Min}_{v_6;2,2,4} \Lambda_{24}$. Their invariants are $(275, 5, 4, 1)$ and $(891, 3, 5, 1)$.

The remarkable centrally symmetric perfect Delaunay $D(v_3, 1, 3)$ with invariants $(552, 2, 5, 1)$ was first identified in [9], it defines 276 equiangular lines [27], it is universally optimal [2] and it gives the facet of maximal incidence of the contact polytope of Λ_{24} [19]. It was noted in [10] that a 22-dimensional antisymmetric perfect Delaunay with 275 vertices is included in $D(v_3, 1, 3)$. This polytope is $\text{Min}_{v_4,4} \Lambda_{24}$. The polytopes $D(v_5, 1, 4)$ and $D(v_3, 1, 3)$ define spherical t -design of strength 4, respectively 5 just like Gosset’s 2_{21} and 3_{21} , which are perfect Delaunay polytopes in E_6 and E_7 . The set $\text{Min } \Lambda_{23}(v_3)^*$ is equivalent to $D(v_3, 1, 3)$ and the set $\text{Min } L_{22}^*$ is equivalent to $D(v_5, 1, 3) \cup (2c(D(v_5, 1, 3)) - D(v_5, 1, 3))$. Similarly 3_{21} is equivalent to $\text{Min } E_7^*$ and $\text{Min } E_6^*$ is equivalent to $2_{21} \cup (2c(2_{21}) - 2_{21})$.

Many lattices $\Lambda_{24}(v)$ have several orbits of perfect Delaunay polytopes. No such example is known in dimension $n \leq 9$. It turns out that for a given vector v of the 17 cases the strength of the spherical t -design is always the same for all main full dimensional Delaunay polytopes. In particular, for vectors of type 2, 3, 4, and $6_{3,2}$ the main Delaunay polytopes P define spherical t -designs for $t = 7, 5, 3$ and 2. Our proof was obtained by direct computation and it would be interesting to have a less computational proof, for example, using modular forms in the spirit of the theory of strongly perfect lattices explained in [34].

Theorem 4.1. *Let D be the 22-dimensional Delaunay cell $D(v_6;2,2, 1, 4)$ with 891 vertices.*

- (i) D affinely generates the 22-dimensional lattice Λ_{22} .
- (ii) We have $|\text{Aut } \Lambda_{22}| = 6|\text{Aut } D|$.

Proof. The lattice $L(D)$ is computed by simple linear algebra computation. Then the isomorphism with Λ_{22} is established by using the program ISOM of Plesken and Souvignier [30].

The automorphism group of $\text{Aut } \Lambda_{22}$ is computed by the program AUTO from the same package. The automorphism group of D is computed by the algorithms explained in [12]. \square

For all other known perfect Delaunay polytopes D in a lattice L we have $|\text{Aut } L| = |\text{Aut } D|$ if D is centrally symmetric and $|\text{Aut } L| = 2|\text{Aut } D|$ if D is antisymmetric.

Theorem 4.2. (i) *The perfect main Delaunay polytopes $D(v_3, 1, 2)$ and $D(v_3, 1, 3)$ are still Delaunay polytopes in the superlattices $\Lambda_{24}(v_3, 2)$ and $\Lambda_{24}(v_3, 3)$ of index 1 and 3.*

(ii) *The polytopes $D(v_2, 1, 1)$, $D(v_4, 1, 2)$, $D(v_{6;3,2}, 1, 3)$ and $D(v_{8;4,2}, 1, 4)$ are antisymmetric, have $\text{den}(c(D)) = 4$ and admit a centrally symmetric extension in a superlattice of index 2.*

Proof. For each given example in (i) and (ii), we consider the corresponding lattices and compute the closest vector problem. \square

Note that in [14] we obtained Delaunay polytopes with the same property by a different method. Above property (i), respectively (ii), is a direct analog of the relation between the set of minimal vectors of the pair of perfect lattices $A_9 \subset A_9^2$ with $\text{Min } A_9 = \text{Min } A_9^2$ [6] and $D_8 \subset E_8$ with $\text{Min } D_8 \subset \text{Min } E_8$

Theorem 4.3. *The polytopes $D(v_3, 1, 1)$ and $D(v_3, 1, 2)$ have lamination number 5.*

Proof. Let us assume that $l(D) \leq 4$ for $D = D(v_3, 1, 1)$ or $D(v_3, 1, 2)$. This means that we can find a 22-dimensional sublattice L' and four vectors w_1, w_2, w_3, w_4 such that the layers $L_i = w_i + L'$ cover $\text{vert } D$.

We checked with a computer that $L(D) = \Lambda_{24}(v_3)$, i.e., the difference vectors of $\text{vert } D$ generate $\Lambda_{24}(v_3)$. There exists a linear function f on $\Lambda_{24}(v_3)$ such that $L' = \ker f$ and $f(\Lambda_{24}(v_3)) = \mathbb{Z}$. We define an index 2 sublattice

$$L'_2 = \{w \in \Lambda_{24}(v_3) \text{ such that } f(w) \in 2\mathbb{Z}\}$$

of $\Lambda_{24}(v_3)$ and take $w \in \Lambda_{24}(v_3)$ such that $f(w) = 1$. It is not possible for L'_2 or $w + L'_2$ to contain all four layers $w_i + L'$ since if it were so, then D would not be generating. So, $L'_2 \cap D$ or $(w + L'_2) \cap D$ contains at most 2 layers. If one of them contains just one layer, then it is of dimension at most 22. By enumerating all index 2 sublattices of $\Lambda_{24}(v_3)$ we found that $L'_2 \cap D$ and $(w + L'_2) \cap D$ are always 23-dimensional. So, $L'_2 \cap D$ and $(w + L'_2) \cap D$ are contained in two layers and thus have lamination number 2. We enumerated their 2-laminations by using the same method as in Theorem 3.1 and found that each 2-lamination of $L'_2 \cap D$, respectively $(w + L'_2) \cap D$, induces a lamination of $(w + L'_2) \cap D$, respectively $L'_2 \cap D$ with at least three layers. So, the lamination number of D is 5. \square

5. Construction of Perfect Delaunay Polytopes by Lamination

In [10, Lemma 15.3.7], [25], [15], [23] a construction of centrally symmetric Delaunay $(n + 1)$ -polytopes from antisymmetric Delaunay n -polytopes

is considered. Here we reconsider this construction and completely classify the types of Delaunay polytopes that one can obtain.

Definition. Let D be a n -dimensional Delaunay polytope of circumcenter c in a lattice $L \subset \mathbb{R}^{n+1}$.

Define $e_{n+1} \in \mathbb{R}^{n+1}$ s.t. $c - (e_{n+1} - c)$ is orthogonal to $L \otimes \mathbb{R}$ and δ is the distance between $L \otimes \mathbb{R}$ and $e_{n+1} + L \otimes \mathbb{R}$.

Define $L(\delta) = L + \mathbb{Z}e_{n+1}$.

We choose $c - (e_{n+1} - c)$ orthogonal to $L \otimes \mathbb{R}$ in order for $\text{vert } D$ and $e_{n+1} - \text{vert } D$ to lie on a common sphere. For $\delta > 0$ there are in fact two possible choices for e_{n+1} but they are actually isometrically equivalent.

Theorem 5.1. *Let D be a Delaunay polytope in a n -dimensional lattice L of center c . For $i \in \mathbb{Z}$, define $D_i = D_L((1 - 2i)c)$ and denote by r_i the common distance between $(1 - 2i)c$ and vertices of D_i . Either:*

- (i) *For all i , $r_i \geq r_0$. Then $L(0)$ is an index 2 superlattice of L such that $D' = D_{L(0)}(c)$ is a centrally symmetric Delaunay n -polytope containing $D \cup (2c - D)$ with $\text{perfrank } D' \leq \text{perfrank } D$.*
- (ii) *Or there exists i such that $r_i < r_0$. Then there exists $\delta_s > 0$ such that $D' = D_{L(\delta_s)}(c')$ with $c' = \frac{1}{2}e_{n+1}$ is a centrally symmetric Delaunay $(n + 1)$ -polytope containing $D \cup (2c' - D)$ with $\text{perfrank } D' \leq \text{perfrank } D$.*

Proof. Define

$$r_i(\delta) = \sqrt{r_i + \delta \left(i - \frac{1}{2} \right)^2}$$

and $c' = \frac{1}{2}e_{n+1}$. The sphere circumscribing D and $e_{n+1} - D$ is $S(c', r_0(\delta))$ and we have $r_0(\delta) = r_1(\delta)$. For $i \in \mathbb{Z}$, the set of closest points in layer $L + ie_{n+1}$ to c' is

$$S_i(c) = ie_{n+1} + \text{vert } D_L((1 - 2i)c)$$

and the common distance to c' is $r_i(\delta)$. If there exists an index i such that $r_i < r_0$ then there exists $\delta_i > 0$ such that $r_i(\delta_i) = r_0(\delta_i)$ and $S_i(c)$ is outside of $S(c', r_0(\delta))$ if and only if $\delta \geq \delta_i$. If one takes $\delta_s = \max_{i \in \mathbb{Z}} \delta_i$ then $S(c', r_0(\delta))$ is an empty sphere if and only if $\delta \geq \delta_s$ and $D' = S(c', r_0(\delta)) \cap L(\delta)$ has more than two layers if and only if $\delta = \delta_s$. In that case $L(\delta_s)$ is determined by L and thus $\text{perfrank } D' \leq \text{perfrank } D$.

On the other hand, if for all i $r_i \geq r_0$ then $\delta_s = 0$ and $L(0)$ is actually an n -dimensional superlattice of L . We have $\text{perfrank } D' \leq \text{perfrank } D$ since D' has more vertices than D . \square

If a Delaunay polytope falls into case (i) then we say that this Delaunay polytope is of the *first type* and otherwise it is of the *second type*.

Theorem 5.2. *Let D be a Delaunay polytope of the first type and define for $\delta > 0$ the polyhedron*

$$D_{cyl} = \text{conv } V_{cyl} \text{ with } V_{cyl} = \bigcup_{i \in \mathbb{Z} \text{ s.t. } r_i = r_0} (ie_{n+1} + \text{vert } D_i),$$

The following holds:

- (i) $V_{cyl} \subset L(\delta)$,
- (ii) V_{cyl} is contained in a cylinder \mathcal{C} ,
- (iii) V_{cyl} is arithmetically equivalent to the product $\text{vert } D' \times \mathbb{Z}$ where D' is the n -dimensional Delaunay polytope of case (i) in Theorem 5.1.

Proof. (i) and (ii) are clear. Let us write $L(\delta) = \mathbb{Z}v_1 + \dots + \mathbb{Z}v_{n+1}$ and take a function $f \in E_2(n+1)$ such that $f(x) = 0$ if and only if $\sum_i x_i v_i \in \mathcal{C}$ and $f(x) > 0$ for $x \notin L(\delta) \cap \mathcal{C}$. The quadratic form of f is positive semidefinite and has a kernel of rank 1. By [21, Corollary 2.5] the set of $x \in \mathbb{Z}^{n+1}$ such that $f(x) = 0$ is of the form $R + L$ with L a lattice of dimension 1 and R affinely equivalent to a n -dimensional Delaunay polytope. So, V_{cyl} is arithmetically equivalent to $\text{vert } D_2 \times \mathbb{Z}$ with D_2 a Delaunay polytope. By projecting D_2 on \mathbb{R}^n we get actually D' which proves (iii). \square

Corollary 5.1. *Take D a Delaunay polytope of a lattice L of center c .*

- (i) *If $\text{den}(c) = 2$ or 4 then D is of first type.*
- (ii) *If $\text{den}(c)$ is odd then D is of second type.*

Proof. If $\text{den}(c) = 2$ then D is centrally symmetric and thus of the first type. If $\text{den}(c) = 4$, then D is antisymmetric and therefore $-D$ is also a Delaunay polytope. Thus when $\text{den}(c) = 4$ there is a Delaunay polytope centered at $3c$; $r_i = r_0$ for all $i \in \mathbb{Z}$ and by Theorem 5.1 D is of first type. If $\text{den}(c)$ is odd then there exists an index i such that $(1 - 2i)c \in L$ and thus $r_i = 0$. So, by Theorem 5.1 D is of second type. \square

Table 4.1 gives many perfect Delaunay polytopes D with $\text{den}(c(D))$ odd, and which are thus of second type. It was an interesting open question whether there exist Delaunay polytopes of first type, which are anti-symmetric. Theorem 4.2.(ii) gives four perfect Delaunay polytope D with $\text{den}(c(D)) = 4$ which are thus of first type and antisymmetric.

Using the method of [18], we obtained 100000 perfect Delaunay polytopes in dimension 9. All the ones of first type were centrally symmetric. All the centrally symmetric ones were obtained by the construction of Theorem 5.1 but we think that this is not the case in a large enough dimension.

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Mathieu DUTOUR SIKIRIĆ
Rudjer Bosković Institute
54 ulica Bijenicka
1000 Zagreb, Croatia
E-mail: Mathieu.Dutour@gmail.com
URL: <http://drobilica.irb.hr/~mathieu/>

Konstantin RYBNIKOV
Department of Mathematical Sciences
University of Massachusetts at Lowell
MA 01854, Lowell, USA
E-mail: Konstantin_Rybnikov@uml.edu