JOURNAL
de Théorie des Nombres
de BORDEAUX
*anciennement Séminaire de Théorie des Nombres de Bordeaux*

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Tome 26, nº 1 (2014), p. 17-24.

<http://jtnb.cedram.org/item?id=JTNB_2014__26_1_17_0>

*Journal de Théorie des Nombres*
*de Bordeaux* **26** (2014), 17–24

# On a conjecture of Dekking : The sum of digits of even numbers

par Iurie BOREICO, Daniel EL-BAZ et Thomas STOLL

Résumé. *A propos d'une conjecture de Dekking : la somme des chiffres des nombres pairs*
Soient $q \geq 2$ et $s_q$ la fonction somme des chiffres en base $q$. Pour $j = 0, 1, \ldots, q-1$ on considère

$$\#\{0 \leq n < N : \quad s_q(2n) \equiv j \pmod q\}.$$

En 1983, F. M. Dekking a conjecturé que cette quantité est strictement supérieure à $N/q$ et, respectivement, strictement inférieure à $N/q$ pour une infinité de $N$, affirmant ce faisant l'absence d'un phénomène de dérive (ou phénomène de Newman). Dans cet article, nous démontrons sa conjecture.

Abstract. Let $q \geq 2$ and denote by $s_q$ the sum-of-digits function in base $q$. For $j = 0, 1, \ldots, q-1$ consider

$$\#\{0 \leq n < N : \quad s_q(2n) \equiv j \pmod q\}.$$

In 1983, F. M. Dekking conjectured that this quantity is greater than $N/q$ and, respectively, less than $N/q$ for infinitely many $N$, thereby claiming an absence of a drift (or Newman) phenomenon. In this paper we prove his conjecture.

## 1. Introduction

Let $q \geq 2$ and denote by $s_q : \mathbb{N} \to \mathbb{N}$ the sum-of-digits function in the $q$-ary digital representation of integers. In his influential paper from 1968, Gelfond [5] proved the following result.[1]

**Theorem 1.1.** *Let $q, d, m \geq 2$ and $a, j$ be integers with $0 \leq a < d$ and $0 \leq j < m$. If $(m, q-1) = 1$ then*

(1.1)
$$\#\{\ 0 \leq n < N : \quad n \equiv a \pmod d, \quad s_q(n) \equiv j \pmod m\} = \frac{N}{dm} + g(N),$$

*where $g(N) = O_q(N^\lambda)$ with $\lambda = \frac{1}{2\log q} \log \frac{q\ \sin(\pi/2m)}{\sin(\pi/2mq)} < 1$.*

Manuscrit reçu le 18 novembre 2012, accepté le 19 février 2013.

[1] As usual, we write $f(N) = O(1)$ if $|f(N)| < C$ for some absolute constant $C$, and $f(N) = O_q(1)$ if the implied constant depends on $q$.

Shevelev [8, 9] recently determined the optimal exponent $\lambda$ in the error term in Gelfond's asymptotic formula when $q = m = 2$, and Shparlinski [10] showed that in this case it can be arbitrarily small for sufficiently large primes $d$.

The oscillatory behaviour of the error term $g(N)$ in (1.1) is still not completely understood. The story can be said to have originated with the observation by Moser in the 1960s that for the quintuple of parameters

$$(q, a, d, j, m) \equiv (2, 0, 3, 0, 2) \tag{1.2}$$

the error term seems to have *constant* positive sign, *i.e.*, $g(N) > 0$ for all $N \geq 1$. In 1969, Newman [7] (with a much more precise result by Coquet [2]) proved this observation and there is at present a large number of articles which establish so-called *Newman phenomena*, *Newman-like phenomena* or *drifting phenomena* for general classes of quintuples $(q, a, d, j, m)$ extending (1.2). The two main techniques come from a direct inspection of the recurrence relations using the $q$-additivity of the sum-of-digits function, and from the determination of the maximal and minimal value of a related fractal function which is continuous but nowhere differentiable [6, 2, 11]. We refer the reader to the monograph of Allouche and Shallit [1] and the article of Drmota and Stoll [4] for a list of references. Characterizing all $(q, a, d, j, m)$ for which one has a Newman-like phenomenon is still wide open.

The aim of the present article is to prove a related conjecture Dekking (see [3, "Final Remark", p. 32-11]) made in 1983 at the Séminaire de Théorie des Nombres de Bordeaux concerning a *non-drifting phenomenon*, that is, a situation where the error $g(N)$ is *oscillating in sign* (as $N \to \infty$). To our knowledge, this conjecture has not yet been addressed in the literature, and we will provide a self-contained proof here.

**Conjecture (Dekking, 1983):** Let $q \geq 2$ and $0 \leq j < q$ and set

$$(q, a, d, j, m) \equiv (q, 0, 2, j, q).$$

Then $g(N) < 0$ and $g(N) > 0$ infinitely often.

Dekking was mostly interested in finding the optimal error term in (1.1) (or, as he puts it, the *typical exponent* of the error term) and obtained various results for the cases $q = 2$, $d$ arbitrary, and $d = 2$, $q$ arbitrary. As for the conjecture, he proved the case of $q = 3$, $j = 0, 1, 2$ via an argument with a geometrical flavour.

Our main result is as follows.

**Theorem 1.2.** *Let $q \geq 2$, $0 \leq j < q$ and set*

$$(q, a, d, j, m) \equiv (q, a, d, j, q).$$

(i) *If $d \mid q$, then $g(N) = O(1)$ and $g(N)$ changes signs infinitely often as $N \to \infty$.*
(ii) *If $d \mid q-1$, then $g(N)$ can take arbitrarily large positive values as well as arbitrarily large negative values as $N \to \infty$.*

In the case of $d = 2$ this proves Dekking's conjecture and covers all bases $q \geq 2$.

## 2. Proof of Theorem 1.2

For an integer $n \geq 0$, we write

$$n = (\varepsilon_k, \varepsilon_{k-1}, \ldots, \varepsilon_0)_q$$

to refer to its $q$-ary digital expansion $n = \sum_{i=0}^{k} \varepsilon_i q^i$. Let $U(n) = \{z \in \mathbb{C} \mid z^n = 1\}$ denote the set of the $n$th roots of unity. We will make use of the following well-known formula from discrete Fourier analysis.

**Proposition 2.1.** *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathbb{C}[x]$, $n \geq 1$, $l \geq 0$ and set $\omega_n = e^{2\pi i/n}$. Then*

$$\sum_{k \equiv l \ (\bmod n)} a_k x^k = \frac{1}{n} \sum_{s=0}^{n-1} \omega_n^{-ls} f(\omega_n^s x).$$

*Proof.* The coefficient of $x^j$ in $\frac{1}{n}\sum_{s=0}^{n-1} \omega_n^{-ls} f(\omega_n^s x)$ is $\frac{1}{n}\sum_{s=0}^{n-1} a_j \omega_n^{s(j-l)}$, that is $a_j$ if $j \equiv l \pmod{n}$ and 0 otherwise. □

We deal with (i) $d \mid q$ and (ii) $d \mid q-1$ in Theorem 1.2 separately in the two subsequent sections.

**2.1. The case $d \mid q$.** For $d = 2$, $q$ even, Dekking remarked and left to the readers of his article (see [3, Remark before Proposition 5, p.32-08]) that the typical exponent $\lambda$ equals 0, *i.e.*, $g(N) = O(1)$. This is due to the fact that when $q$ is even then the parity of an integer is completely encoded in the last digit of its base $q$ expansion. A similar situation applies when $d \mid q$. In order to find the oscillatory behaviour of $g(N)$, we calculate $g(N)$ explicitly.

Define

$$f_j(n) = c_j(n) - \frac{1}{q},$$

where

$$c_j(n) = \begin{cases} 1 & \text{if } s_q(n) \equiv j \pmod{q}; \\ 0 & \text{otherwise.} \end{cases}$$

Consider

$$D_j(N) = \sum_{\substack{0 \leq n < N \\ n \equiv a \pmod{d}}} f_j(n), \tag{2.1}$$

thus

$$g(N) = D_j(N) - \frac{N}{dq} + \frac{1}{q}\left\lceil \frac{N-a}{d} \right\rceil. \tag{2.2}$$

We want to find infinitely many values of $N$ such that $g(N) > 0$, respectively, $g(N) < 0$. Since an integer in base $q$ (with $q$ divisible by $d$) gives remainder $a$ mod $d$ if and only if its last digit in base $q$ gives remainder $a$ mod $d$, we get for $N = (\varepsilon_k, \dots, \varepsilon_0)_q$,

$$\begin{aligned} D_j(N) = \sum_{r=2}^{k} \sum_{\delta=0}^{\varepsilon_r - 1} \sum_{\substack{0 \le i_0, i_1, \dots, i_{r-1} \le q-1 \\ i_0 \equiv a \pmod d}} & f_j((\varepsilon_k, \dots, \varepsilon_{r+1}, \delta, i_{r-1}, \dots, i_0)_q) \\ + \sum_{\delta=0}^{\varepsilon_1 - 1} \sum_{\substack{i_0=0 \\ i_0 \equiv a \pmod d}}^{q-1} & f_j((\varepsilon_k, \dots, \varepsilon_2, \delta, i_0)_q) \\ + \sum_{\substack{\delta=0 \\ \delta \equiv a \pmod d}}^{\varepsilon_0 - 1} & f_j((\varepsilon_k, \dots, \varepsilon_1, \delta)_q). \end{aligned}$$

For $r \ge 2$ we get

$$\begin{aligned} \sum_{\substack{0 \le i_0, i_1, \dots, i_{r-1} \le q-1 \\ i_0 \equiv a \pmod d}} & f_j((\varepsilon_k, \dots, \varepsilon_{r+1}, \delta, i_{r-1}, \dots, i_0)_q) \\ &= D_{j-\varepsilon_k - \dots - \varepsilon_{r+1} - \delta}(q^r) = 0. \end{aligned}$$

Set $\alpha = j - s_q(N) + \varepsilon_1 + \varepsilon_0$ and $\beta = j - s_q(N) + \varepsilon_0$. For the other two terms we then get by a direct calculation,

$$\sum_{\delta=0}^{\varepsilon_1 - 1} \sum_{\substack{i_0=0 \\ i_0 \equiv a \pmod d}}^{q-1} f_j((\varepsilon_k, \dots, \varepsilon_2, \delta, i_0)_q) = -\frac{\varepsilon_1}{d} + \sum_{\delta=0}^{\varepsilon_1 - 1} \sum_{\substack{0 \le i_0 < q \\ i_0 \equiv a \pmod d \\ i_0 \equiv \alpha - \delta \pmod q}} 1 \tag{2.3}$$

and

$$\sum_{\substack{\delta=0 \\ \delta \equiv a \pmod d}}^{\varepsilon_0 - 1} f_j((\varepsilon_k, \dots, \varepsilon_1, \delta)_q) = -\frac{1}{q}\left\lceil \frac{\varepsilon_0 - a}{d} \right\rceil + \sum_{\substack{0 \le \delta < \varepsilon_0 \\ \delta \equiv a \pmod d \\ \delta \equiv \beta \pmod q}} 1. \tag{2.4}$$

From (2.2), (2.3) and (2.4) it is straightforward to find sequences of positive integers $N$ with $g(N) > 0$, respectively $g(N) < 0$. In fact, if $a \neq 0$ we can take all $N$ with $\varepsilon_1 = 0$, $\varepsilon_0 = a$ to get $g(N) = -\frac{a}{qd} < 0$. For $a = 0$ we take all $N$ with $\varepsilon_1 = 1$, $\varepsilon_0 = a$ and $s_q(N) \not\equiv j+1 \pmod d$ to get $g(N) = -1/d < 0$. On the other hand, if $a + 1 < q$ we may take all $N$ with $\varepsilon_1 = 0$, $\varepsilon_0 = a+1$ to find $g(N) = 1 + \frac{1}{d} - \frac{a+1}{qd} - \frac{1}{q} > 0$. If $a + 1 = q$ (which again implies

$d = q$) we take all $N$ with $\varepsilon_1 = 1$, $\varepsilon_0 = 0$ and $s_q(N) \equiv j + 2 \pmod q$ to get $g(N) = -\frac{1}{d} + 1 > 0$. This completes the proof in this case.

**2.2. The case $d \mid q - 1$.** In what follows, set

$$E_{a,j}(k) = \#\{\, 0 \le n < q^k \;:\; n \equiv a \pmod d,\; s_q(n) \equiv j \pmod q\},$$

where $a, j$ are fixed integers with $0 \le a < d$, $0 \le j < q$ and $k \ge 1$. Consider the generating polynomial in two variables

$$P(x, y) = \prod_{i=0}^{k-1} (1 + xy^{q^i} + x^2 y^{2q^i} + \cdots + x^{q-1} y^{(q-1)q^i}),$$

which encodes the digits of integers less than $q^k$ in base $q$. Denote by $[x^u y^v]P(x, y)$ the coefficient of $x^u y^v$ in the expansion of $P(x, y)$. By Proposition 2.1,

$$E_{a,j}(k) = \sum_{\substack{u \equiv j \,(\mathrm{mod}\ q) \\ v \equiv a \,(\mathrm{mod}\ d)}} [x^u y^v] P(x, y) = \frac{1}{dq} \sum_{\substack{\omega \in U(q) \\ \varepsilon \in U(d)}} \omega^{-j} \varepsilon^{-a} P(\omega, \varepsilon).$$

For $\varepsilon \in U(d)$ with $d \mid q - 1$ we have $\varepsilon^{lq^i} = \varepsilon^l$ for $0 \le l \le q - 1$ and thus

$$P(\omega, \varepsilon) = (1 + \omega\varepsilon + \omega^2 \varepsilon^2 + \ldots + \omega^{q-1} \varepsilon^{q-1})^k.$$

Since $\omega\varepsilon = 1$ if and only if $\omega = \varepsilon = 1$ ($d$ and $q$ are coprime) and $\omega^q \varepsilon^q = \varepsilon$ we get

$$E_{a,j}(k) - \frac{q^{k-1}}{d} = \frac{1}{dq} \sum_{\substack{\omega \in U(q) \\ \varepsilon \in U(d) \\ \omega\varepsilon \ne 1}} \omega^{-j} \varepsilon^{-a} \left( \frac{1 - \varepsilon}{1 - \omega\varepsilon} \right)^k. \tag{2.5}$$

We now take a closer look at the dominant term on the right hand side in (2.5). Note that for $\omega \in U(q), \varepsilon \in U(d)$ with $\omega\varepsilon \ne 1$, we have

$$\frac{1}{\pi} \arg \left( \frac{1 - \varepsilon}{1 - \omega\varepsilon} \right) \in \mathbb{Q}.$$

We claim that the numbers $\dfrac{1 - \varepsilon}{1 - \omega\varepsilon}$ are all pairwise distinct. Indeed, for any point on the unit circle $z \ne 1$, it can easily be seen (geometrically or otherwise) that $\arg((1 - z)^2) = \arg(z) + \pi$. It follows that

$$\arg \left( \left( \frac{1 - \varepsilon}{1 - \omega\varepsilon} \right)^2 \right) = -\arg(\omega).$$

Therefore, if

$$\frac{1 - \varepsilon}{1 - \omega\varepsilon} = \frac{1 - \varepsilon'}{1 - \omega'\varepsilon'}$$

then we conclude that $\omega$ and $\omega'$ have the same argument so $\omega = \omega'$, and then $\varepsilon = \varepsilon'$. This means that there are no cancellations in (2.5).

Write

$$R = \max\left\{\left|\frac{1-\varepsilon}{1-\omega\varepsilon}\right| : \quad \omega \in U(q),\ \varepsilon \in U(d),\ \omega\varepsilon \neq 1\right\}$$

and let $r_1, r_2, \ldots, r_h$ be all of the numbers $(1-\varepsilon)/(1-\omega\varepsilon)$ whose absolute value equals $R$.

The set $U(d)$ divides the unit circle into $d \geq 2$ equal parts, so it always contains an element $\varepsilon_0$ in the open half-plane $\mathrm{Re}(\varepsilon) < 0$. Similarly, $U(q)$ must contain an element $\omega_0$ in the closed half-plane $\mathrm{Re}(\varepsilon_0\omega) \geq 0$. Then $|1-\varepsilon_0| > \sqrt{2}$ while $|1-\omega_0\varepsilon_0| \leq \sqrt{2}$, thus

$$\left|\frac{1-\varepsilon_0}{1-\omega_0\varepsilon_0}\right| > 1.$$

Note also that $\omega_0\varepsilon_0 \neq 1$ as $(d,q) = 1$ and $\varepsilon_0 \neq 1$.

It follows that $R > 1$, which in particular implies that the value 1 is not among these $r_i$. Then, as $k \to \infty$,

$$\sum_{\substack{\omega\in U(q)\\ \varepsilon\in U(d)\\ \omega\varepsilon\neq 1}} \omega^{-j}\varepsilon^{-a}\left(\frac{1-\varepsilon}{1-\omega\varepsilon}\right)^k \quad \sim \quad R^k \sum_{i=1}^{h} c_i \left(\frac{r_i}{R}\right)^k,$$

for certain $c_i \in \mathbb{C}$ which are not all zero. As the $r_i$ all have arguments equal to rational multiples of $\pi$, the $r_i/R$, $i = 1, \ldots, h$, are roots of unity. Therefore there exists an integer $M \geq 1$ such that $(r_i/R)^M = 1$ for all $i$. Write

$$c'(k) = \sum_{i=1}^{h} c_i \left(\frac{r_i}{R}\right)^k.$$

Since $E_{a,j}(k)$ is real and $c'(k+M) = c'(k)$ for all $k$ we must have that $c'(k) \in \mathbb{R}$ for all $k$. Moreover,

$$\sum_{k=0}^{M-1} c'(k) = \sum_{i=1}^{h} c_i \sum_{k=0}^{M-1} \left(\frac{r_i}{R}\right)^k = 0,$$

since $r_i$ is not real for all $i$. Thus, among all the $c'(k)$ there is at least one positive and at least one negative value. Let $-c_1' = c'(k_1) < 0$ be the smallest negative value and $c_2 = c'(k_2) > 0$ be the largest positive value among them. Then, as $k \to \infty$,

$$E_{a,j}(k) - \frac{q^{k-1}}{d} \sim -\frac{c_1'}{dq} R^k < 0, \qquad \text{for} \quad k \equiv k_1 \pmod{M}$$

and

$$E_{a,j}(k) - \frac{q^{k-1}}{d} \sim \frac{c_2'}{dq} R^k > 0, \qquad \text{for} \quad k \equiv k_2 \pmod{M}.$$

This completes the proof.

## 3. Acknowledgment

We would like to express our gratitude to J.-P. Allouche, F. M. Dekking and J. Shallit for related correspondence on this problem. T. Stoll was supported by the Agence Nationale de la Recherche, grant ANR-10-BLAN 0103 MUNUM.

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