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Wintenberger's functor for abelian extensions

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Résumé. Soit k un corps fini. Wintenberger a utilisé le corps des normes pour donner une équivalence entre une catégorie dont les objets E/F sont des extensions abéliennes de Lie p-adiques totalement ramifiées (où F est un corps local avec corps résiduel k), et une catégorie dont les objets sont des paires (K, A), où $K \cong k((T))$ et A est un sous-groupe abélien de Lie p-adique de $\mathrm{Aut}_k(K)$. Dans ce papier, nous étendons cette équivalence en permettant à $\mathrm{Gal}(E/F)$ et à A d'être des pro-p groupes abéliens arbitraires.

Abstract. Let k be a finite field. Wintenberger used the field of norms to give an equivalence between a category whose objects are totally ramified abelian p-adic Lie extensions E/F, where F is a local field with residue field k, and a category whose objects are pairs (K, A), where $K \cong k((T))$ and A is an abelian p-adic Lie subgroup of $\mathrm{Aut}_k(K)$. In this paper we extend this equivalence to allow $\mathrm{Gal}(E/F)$ and A to be arbitrary abelian pro-p groups.

1. Introduction

Let k be a finite field with $q = p^f$ elements. We define a category $\mathcal{A}$ whose objects are totally ramified abelian extensions E/F, where F is a local field with residue field k, and $[E : F]$ is infinite if F has characteristic 0. An $\mathcal{A}$-morphism from E/F to E'/F' is defined to be a continuous embedding $\rho : E \to E'$ such that

(1) ρ induces the identity on k.
(2) E' is a finite separable extension of $\rho(E)$.
(3) F' is a finite separable extension of $\rho(F)$.

Let $\rho^* : \mathrm{Gal}(E'/F') \to \mathrm{Gal}(E/F)$ be the map induced by ρ. It follows from (2) and (3) that ρ^* has finite kernel and finite cokernel.

For each local field K with residue field k we let $\mathrm{Aut}_k(K)$ denote the group of continuous automorphisms of K which induce the identity map on k. Define a metric on $\mathrm{Aut}_k(K)$ by setting $d(\sigma, \tau) = 2^{-a}$, where $a = v_K(\sigma(\pi_K) - \tau(\pi_K))$ and π_K is any uniformizer of K.

We define a category $\mathcal{B}$ whose objects are pairs (K, A), where K is a local field of characteristic p with residue field k, and A is a closed abelian

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subgroup of $\mathrm{Aut}_k(K)$. A $\mathcal{B}$-morphism from (K, A) to (K', A') is defined to be a continuous embedding $\sigma : K \to K'$ such that

(1) σ induces the identity on k.
(2) K' is a finite separable extension of $\sigma(K)$.
(3) A' stabilizes $\sigma(K)$, and the image of A' in $\mathrm{Aut}_k(\sigma(K)) \cong \mathrm{Aut}_k(K)$ is an open subgroup of A.

Let $\sigma^* : A' \to A$ be the map induced by σ. It follows from (2) and (3) that σ^* has finite kernel and finite cokernel.

Let $X_F(E)$ denote the field of norms of the extension E/F, as defined in [7]. Then $X_F(E) \cong k((T))$ and there is a faithful k-linear action of $\mathrm{Gal}(E/F)$ on $X_F(E)$. It follows from the functorial properties of the field of norms construction that there is a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ defined by

$$\mathcal{F}(E/F) = (X_F(E), \mathrm{Gal}(E/F)). \tag{1.1}$$

We wish to prove the following:

Theorem 1.1. *$\mathcal{F}$ is an equivalence of categories.*

Wintenberger ([5, 6]; see also [2]) has shown that $\mathcal{F}$ induces an equivalence between the full subcategory $\mathcal{A}_{Lie}$ of $\mathcal{A}$ consisting of extensions E/F such that $\mathrm{Gal}(E/F)$ is an abelian p-adic Lie group, and the full subcategory $\mathcal{B}_{Lie}$ of $\mathcal{B}$ consisting of pairs (K, A) such that A is an abelian p-adic Lie group. The proof of Theorem 1.1 is based on reducing to the equivalence between $\mathcal{A}_{Lie}$ and $\mathcal{B}_{Lie}$. Note that, contrary to [2, 6], we consider finite groups to be p-adic Lie groups. The equivalence of categories proved in [2, 5, 6] extends trivially to include the case of finite groups.

The following result, proved by Laubie [3], can also be proved as a consequence of Theorem 1.1:

Corollary 1.2. *Let $(K, A) \in \mathcal{B}$. Then there is $E/F \in \mathcal{A}$ such that A is isomorphic to $G = \mathrm{Gal}(E/F)$ as a filtered group. That is, there exists an isomorphism $i : A \to G$ such that $i(A[x]) = G[x]$ for all $x \geq 0$, where $A[x]$, $G[x]$ denote the ramification subgroups of A, G with respect to the lower numbering.*

The finite field $k \cong \mathbb{F}_q$ is fixed throughout the paper, as is the field $K = k((T))$ of formal Laurent series in one variable over k. We work with complete discretely valued fields F whose residue field is identified with k, and with totally ramified abelian extensions of such fields. The ring of integers of F is denoted by $\mathcal{O}_F$ and the maximal ideal of $\mathcal{O}_F$ is denoted by $\mathcal{M}_F$. We let v_F denote the valuation on the separable closure F^{sep} of F which is normalized so that $v_F(F^\times) = \mathbb{Z}$, and we let v_p denote the p-adic valuation on $\mathbb{Z}$. We say that the profinite group G is finitely generated if there is a finite set $S \subset G$ such that $\langle S \rangle$ is dense in G.

2. Ramification theory and the field of norms

In this section we recall some facts from ramification theory, and summarize the construction of the field of norms for extensions in $\mathcal{A}$.

Let $E/F \in \mathcal{A}$. Then $G = \mathrm{Gal}(E/F)$ has a decreasing filtration by the upper ramification subgroups $G(x)$, defined for nonnegative real x. (See for instance [4, IV].) We say that $u \geq 0$ is an upper ramification break of G if $G(u+\epsilon) \precneqq G(u)$ for every $\epsilon > 0$. Since G is abelian, by the Hasse-Arf Theorem [4, V §7, Th. 1] every upper ramification break of G is an integer. In addition, since F has finite residue field and E/F is a totally ramified abelian extension, it follows from class field theory that E/F is arithmetically profinite (APF) in the sense of [7, §1]. This means that for every $x \geq 0$ the upper ramification subgroup $G(x)$ has finite index in $G = G(0)$. This allows us to define the Hasse-Herbrand functions

$$\psi_{E/F}(x) = \int_0^x |G(0) : G(t)|\, dt \tag{2.1}$$

and $\phi_{E/F}(x) = \psi_{E/F}^{-1}(x)$. The ramification subgroups of G with the lower numbering are defined by $G[x] = G(\phi_{E/F}(x))$ for $x \geq 0$. We say that $l \geq 0$ is a lower ramification break for G if $G[l+\epsilon] \precneqq G[l]$ for every $\epsilon > 0$. It is clear from the definitions that l is a lower ramification break if and only if $\phi_{E/F}(l)$ is an upper ramification break.

When $(K, A) \in \mathcal{B}$ the abelian subgroup A of $\mathrm{Aut}_k(K)$ also has a ramification filtration. The lower ramification subgroups of A are defined by

$$A[x] = \{\sigma \in A : v_K(\sigma(T) - T) \geq x + 1\} \tag{2.2}$$

for $x \geq 0$. Since $A[x]$ has finite index in $A = A[0]$ for every $x \geq 0$, the function

$$\phi_A(x) = \int_0^x \frac{dt}{|A[0] : A[t]|} \tag{2.3}$$

is strictly increasing. We define the ramification subgroups of A with the upper numbering by $A(x) = A[\psi_A(x)]$, where $\psi_A(x) = \phi_A^{-1}(x)$. The upper and lower ramification breaks of A are defined in the same way as the upper and lower ramification breaks of $\mathrm{Gal}(E/F)$. The lower ramification breaks of A are certainly integers, and Laubie's result (Corollary 1.2) together with the Hasse-Arf theorem imply that the upper ramification breaks of A are integers as well.

For $E/F \in \mathcal{A}$ let $i(E/F)$ denote the smallest (upper or lower) ramification break of the extension E/F. The following basic result from ramification theory is presumably well-known (cf. [7, 3.2.5.5]).

Lemma 2.1. *Let $M/F \in \mathcal{A}$ and let F'/F be a finite totally ramified abelian extension which is linearly disjoint from M/F. Assume that $M' = MF'$ has residue field k, so that $M'/F' \in \mathcal{A}$. Then $i(M'/F') \leq \psi_{F'/F}(i(M/F))$,* *with equality if the largest upper ramification break u of F'/F is less than $i(M/F)$.*

Proof. Set $G = \mathrm{Gal}(M'/F)$, $H = \mathrm{Gal}(M'/M)$, and $N = \mathrm{Gal}(M'/F')$. Then $G = HN \cong H \times N$. Let $y = \phi_{F'/F}(i(M'/F'))$. Then

$$N = N(i(M'/F')) = N(\psi_{F'/F}(y)) = G(y) \cap N. \tag{2.4}$$

It follows that $G(y) \supset N$, and hence that $G/H = G(y)H/H = (G/H)(y)$. Therefore $y \leq i(M/F)$, which implies $i(M'/F') \leq \psi_{F'/F}(i(M/F))$.

If $u < i(M/F)$ then the group

$$(G/N)(i(M/F)) = G(i(M/F))N/N \tag{2.5}$$

is trivial. It follows that $G(i(M/F)) \subset N$, and hence that

$$N(\psi_{F'/F}(i(M/F))) = G(i(M/F)) \cap N = G(i(M/F)). \tag{2.6}$$

The restriction map from $\mathrm{Gal}(M'/F') = N$ to $\mathrm{Gal}(M/F) \cong G/H$ carries $G(i(M/F))$ onto

$$G(i(M/F))H/H = (G/H)(i(M/F)) = G/H. \tag{2.7}$$

Thus $N(\psi_{F'/F}(i(M/F))) = N$, so we have $i(M'/F') \geq \psi_{F'/F}(i(M/F))$. Since the opposite inequality holds in general, we conclude that $i(M'/F') = \psi_{F'/F}(i(M/F))$ if $u < i(M/F)$. □

Let $E/F \in \mathcal{A}$. Since E/F is an APF extension, the field of norms of E/F is defined: Let $\mathcal{E}_{E/F}$ denote the set of finite subextensions of E/F, and for $L', L \in \mathcal{E}_{E/F}$ such that $L' \supset L$ let $\mathrm{N}_{L'/L} : L' \to L$ denote the norm map. The field of norms $X_F(E)$ of E/F is defined to be the inverse limit of $L \in \mathcal{E}_{E/F}$ with respect to the norms. We denote an element of $X_F(E)$ by $\alpha_{E/F} = (\alpha_L)_{L \in \mathcal{E}_{E/F}}$. Multiplication in $X_F(E)$ is defined componentwise, and addition is defined by the rule $\alpha_{E/F} + \beta_{E/F} = \gamma_{E/F}$, where

$$\gamma_L = \lim_{L' \in \mathcal{E}_{E/L}} \mathrm{N}_{L'/L}(\alpha_{L'} + \beta_{L'}) \tag{2.8}$$

for $L \in \mathcal{E}_{E/F}$.

We embed k into $X_F(E)$ as follows: Let F_0/F be the maximum tamely ramified subextension of E/F, and for $\zeta \in k$ let $\tilde{\zeta}_{F_0}$ be the Teichmüller lift of ζ in $\mathcal{O}_{F_0}$. Note that for any $L \in \mathcal{E}_{E/F_0}$ the degree of the extension L/F_0 is a power of p. Therefore there is a unique $\tilde{\zeta}_L \in L$ such that $\tilde{\zeta}_L$ is the Teichmüller lift of some element of k and $\tilde{\zeta}_L^{[L:F_0]} = \tilde{\zeta}_{F_0}$. Define $f_{E/F}(\zeta)$ to be the unique element of $X_F(E)$ whose L component is $\tilde{\zeta}_L$ for every $L \in \mathcal{E}_{E/F_0}$. Then the map $f_{E/F} : k \to X_F(E)$ is a field embedding. By choosing a uniformizer for $X_F(E)$ we get a k-isomorphism $X_F(E) \cong k((T))$. If E/F is finite then there is a field isomorphism $\iota : X_F(E) \to E$ given by

$\iota(\alpha_{E/F}) = \alpha_E$. This isomorphism is not k-linear in general, since for $\zeta \in k$ we have $\iota(f_{E/F}(\zeta)) = \zeta^{p^{-a}}$, with $a = v_p([E : F])$.

The ring of integers $\mathcal{O}_{X_F(E)}$ consists of those $\alpha_{E/F} \in X_F(E)$ such that $\alpha_L \in \mathcal{O}_L$ for all $L \in \mathcal{E}_{E/F}$ (or equivalently, for some $L \in \mathcal{E}_{E/F}$). A uniformizer $\pi_{E/F} = (\pi_L)_{L\in\mathcal{E}_{E/F}}$ for $X_F(E)$ consists of a uniformizer π_L for each finite subextension L/F of E/F. Furthermore, for each subextension M/F of E/F such that $M/F \in \mathcal{A}$, $\pi_{E/F}$ gives a uniformizer $\pi_{M/F} = (\pi_L)_{L\in\mathcal{E}_{M/F}}$ for $X_F(M)$. The action of $\mathrm{Gal}(E/F)$ on the fields $L \in \mathcal{E}_{E/F}$ induces a k-linear action of $\mathrm{Gal}(E/F)$ on $X_F(E)$. By identifying $\mathrm{Gal}(E/F)$ with the subgroup of $\mathrm{Aut}_k(X_F(E))$ which it induces, we get the functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ defined in (1.1).

Let E' be a finite separable extension of E. Then there is $M \in \mathcal{E}_{E/F}$ and a finite extension M' of M such that $E' = EM'$ and E, M' are linearly disjoint over M. The extension E'/F need not be in $\mathcal{A}$, but it is an APF extension, so the field of norms $X_F(E')$ can be constructed by a method similar to that described above. We define an embedding $j : X_F(E) \to X_F(E')$ as follows. For $\alpha_{E/F} \in X_F(E)$ set $j(\alpha_{E/F}) = \beta_{E'/F}$, where $\beta_{E'/F}$ is the unique element of $X_F(E')$ such that $\beta_{LM'} = \alpha_L$ for all $L \in \mathcal{E}_{E/M}$ [7, 3.1.1]. The embedding j makes $X_F(E')$ into a finite separable extension of $X_F(E)$ of degree $[E' : E]$; in this setting we denote $X_F(E')$ by $X_{E/F}(E')$. If $E'' \supset E' \supset E$ are finite separable extensions then $X_{E/F}(E')/X_F(E)$ is a subextension of $X_{E/F}(E'')/X_F(E)$. Let D/E be an infinite separable extension. Then $X_{E/F}(D)$ is defined to be the union of $X_{E/F}(E')$ as E' ranges over the finite subextensions of D/E.

Let $E/F \in \mathcal{A}$ and recall that $i(E/F)$ is the smallest ramification break of E/F. Define

$$r(E/F) = \left\lceil \frac{p-1}{p} \cdot i(E/F) \right\rceil . \tag{2.9}$$

The proof of Theorem 1.1 depends on the following two propositions, the first of which was proved by Wintenberger:

Proposition 2.2. *Let $E/F \in \mathcal{A}$, let $L \in \mathcal{E}_{E/F}$, and define*

$$\xi_L : \mathcal{O}_{X_F(E)} \longrightarrow \mathcal{O}_L/\mathcal{M}_L^{r(E/L)} \tag{2.10}$$

by $\xi_L(\alpha_{E/F}) = \alpha_L \pmod{\mathcal{M}_L^{r(E/L)}}$. Then

(a) ξ_L is a surjective ring homomorphism.

(b) If $L \supset F_0$ then ξ_L induces the automorphism $\zeta \mapsto \zeta^{p^{-a}}$ on k, where $a = v_p([L : F])$.

Proof. This follows from Proposition 2.2.1 of [7]. □

Proposition 2.3. *Let $E/F \in \mathcal{A}$ and let F'/F be a finite totally ramified abelian extension which is linearly disjoint from E/F. Assume that $E' = EF'$ has residue field k, so that $E'/F' \in \mathcal{A}$. Then the following diagram commutes, where the bottom horizontal map is induced by the inclusion $\mathcal{O}_F \hookrightarrow \mathcal{O}_{F'}$:*

$$\begin{array}{ccc} \mathcal{O}_{X_F(E)} & \overset{j}{\longrightarrow} & \mathcal{O}_{X_{E/F}(E')} \\ \xi_F \downarrow & & \downarrow \xi_{F'} \\ \mathcal{O}_F/\mathcal{M}_F^{r(E/F)} & \longrightarrow & \mathcal{O}_{F'}/\mathcal{M}_{F'}^{r(E'/F')} \end{array} \tag{2.11}$$

Furthermore, if $F = F_0$ then for all $\zeta \in k$ we have $j \circ f_{E/F}(\zeta) = f_{E'/F}(\zeta^{p^b})$, where $b = v_p([F' : F])$.

Proof. Using Lemma 2.1 we get

$$i(E'/F') \leq \psi_{F'/F}(i(E/F)) \leq [F' : F]i(E/F). \tag{2.12}$$

Thus $r(E'/F') \leq [F' : F]r(E/F)$, so the bottom horizontal map in the diagram is well-defined. Let $\alpha_{E/F} = (\alpha_M)_{M\in\mathcal{E}_{E/F}}$ be an element of $\mathcal{O}_{X_F(E)}$. Then the F'-component of $j(\alpha_{E/F})$ is α_F. It follows that $\xi_F(\alpha_{E/F})$ and $\xi_{F'}(j(\alpha_{E/F}))$ are both congruent to α_F modulo $\mathcal{M}_{F'}^{r(E'/F')}$, which proves the commutativity of (2.11). Now suppose $F = F_0$. Then it follows from Proposition 2.2(b) that ξ_F induces the identity on k, and that $\xi_{F'}$ induces the automorphism $\zeta \mapsto \zeta^{p^{-b}}$ on k. Therefore by the commutativity of (2.11) we see that j induces the automorphism $\zeta \mapsto \zeta^{p^b}$ on k. Hence $j \circ f_{E/F}(\zeta) = f_{E'/F}(\zeta^{p^b})$ for all $\zeta \in k$. □

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. To do this, we must show that the functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is essentially surjective and fully faithful.

We begin by showing that $\mathcal{F}$ is essentially surjective. Let $K = k((T))$ and set $\Gamma = \mathrm{Aut}_k(K)$. Let A be a closed abelian subgroup of Γ. Then A is a p-adic Lie group if and only if A is finitely generated. Since $\mathcal{F}$ induces an equivalence between the categories $\mathcal{A}_{Lie}$ and $\mathcal{B}_{Lie}$, it suffices to prove that (K, A) lies in the essential image of $\mathcal{F}$ in the case where A is *not* finitely generated.

Let $F \cong k((T))$, let E/F be a finite totally ramified abelian extension, and let π be a uniformizer of E. Then for each $\sigma \in \mathrm{Gal}(E/F)$ there is a unique $f_\sigma \in k[[T]]$ such that $\sigma(\pi) = f_\sigma(\pi)$. Let $a = v_p([E : F])$ and define

$$G(E/F, \pi) = \{\gamma \in \Gamma : \gamma(T) = f_\sigma^{p^a}(T) \text{ for some } \sigma \in \mathrm{Gal}(E/F)\}, \tag{3.1}$$

where $f_\sigma^{p^a}(T)$ is the power series obtained from $f_\sigma(T)$ by replacing the coefficients by their p^a powers. Then $G(E/F,\pi)$ is a subgroup of Γ which is isomorphic to $\mathrm{Gal}(E/F)$.

Let $l_0 < l_1 < l_2 < \dots$ denote the positive lower ramification breaks of A. For $n \geq 0$ set $r_n = \lceil \frac{p-1}{p} \cdot l_n \rceil$ and let $\overline{\Gamma}_n$ denote the quotient of Γ by the lower ramification subgroup

$$\Gamma[r_n - 1] = \{\sigma \in \Gamma : \sigma(T) \equiv T \pmod{T^{r_n}}\}. \tag{3.2}$$

For each subgroup H of Γ define $\overline{H}$ to be the image of H in $\overline{\Gamma}_n$. Let $\mathcal{S}_n$ denote the set of pairs (E,π) such that

(1) E/F is a totally ramified abelian subextension of F^{sep}/F such that $\mathrm{Gal}(E/F)[l_n]$ is trivial. (Such an extension is necessarily finite.)

(2) π is a uniformizer of E such that $\overline{G(E/F,\pi)} = \overline{A}$.

We define a metric on $\mathcal{S}_n$ by setting $d((E,\pi),(E',\pi')) = 1$ if $E \neq E'$, and $d((E,\pi),(E,\pi')) = 2^{-v_F(\pi-\pi')}$. Since there are only finitely many extensions E/F satisfying (1), and (2) depends only on the class of π modulo $\mathcal{M}_E^{r_n}$, the metric space $\mathcal{S}_n$ is compact.

Lemma 3.1. *Let $n \geq 1$, let $(E,\pi) \in \mathcal{S}_n$, let $\tilde{E}$ denote the fixed field of* $\mathrm{Gal}(E/F)[l_{n-1}]$*, and set $\tilde{\pi} = N_{E/\tilde{E}}(\pi)$. Then $(\tilde{E},\tilde{\pi}) \in \mathcal{S}_{n-1}$, and the map $\nu_n : \mathcal{S}_n \to \mathcal{S}_{n-1}$ defined by $\nu_n(E,\pi) = (\tilde{E},\tilde{\pi})$ is continuous.*

Proof. It follows from the definitions that $\tilde{E}/F$ is a totally ramified abelian extension and that $\mathrm{Gal}(\tilde{E}/F)[l_{n-1}]$ is trivial. Choose $\sigma \in \mathrm{Gal}(E/F)$ and let $\tilde{\sigma}$ denote the restriction of σ to $\tilde{E}$. By Proposition 2.2(a) the norm $\mathrm{N}_{E/\tilde{E}}$ induces a ring homomorphism from $\mathcal{O}_E$ to $\mathcal{O}_{\tilde{E}}/\mathcal{M}_{\tilde{E}}^{r_{n-1}}$. Therefore

$$\tilde{\sigma}(\tilde{\pi}) = \mathrm{N}_{E/\tilde{E}}(\sigma(\pi)) \tag{3.3}$$

$$= \mathrm{N}_{E/\tilde{E}}(f_\sigma(\pi)) \tag{3.4}$$

$$\equiv f_\sigma^{p^b}(\mathrm{N}_{E/\tilde{E}}(\pi)) \pmod{\mathcal{M}_{\tilde{E}}^{r_{n-1}}}, \tag{3.5}$$

$$\equiv f_\sigma^{p^b}(\tilde{\pi}) \pmod{\mathcal{M}_{\tilde{E}}^{r_{n-1}}}, \tag{3.6}$$

where $b = v_p([E:\tilde{E}])$. Let $\tilde{a} = v_p([\tilde{E}:F])$ and let $f_{\tilde{\sigma}} \in k[[T]]$ be such that $\tilde{\sigma}(\tilde{\pi}) = f_{\tilde{\sigma}}(\tilde{\pi})$. Then by (3.6) we have

$$f_{\tilde{\sigma}}(T) \equiv f_\sigma^{p^b}(T) \pmod{T^{r_{n-1}}} \tag{3.7}$$

$$f_{\tilde{\sigma}}^{p^{\tilde{a}}}(T) \equiv f_\sigma^{p^a}(T) \pmod{T^{r_{n-1}}}, \tag{3.8}$$

where $a = v_p([E:F]) = \tilde{a} + b$. It follows that $G(\tilde{E}/F,\tilde{\pi})$ and $G(E/F,\pi)$ have the same image in $\overline{\Gamma}_{n-1}$, and hence that $G(\tilde{E}/F,\tilde{\pi})$ and A have the same image in $\overline{\Gamma}_{n-1}$. Hence $(\tilde{E},\tilde{\pi}) \in \mathcal{S}_{n-1}$, so $\nu_n(E,\pi) = (\tilde{E},\tilde{\pi})$ defines a map from $\mathcal{S}_n$ to $\mathcal{S}_{n-1}$. The fact that ν_n is continuous follows easily from the definitions. □

Since each $A/A[l_n]$ is finite there is a sequence $A_0 \leq A_1 \leq A_2 \leq \dots$ of finitely generated closed subgroups of A such that $A[l_n]A_n = A$ for all $n \geq 0$. Recall that $\mathcal{F}$ induces an equivalence of categories between $\mathcal{A}_{Lie}$ and $\mathcal{B}_{Lie}$. Since $(K, A_n) \in \mathcal{B}_{Lie}$, for $n \geq 0$ there exists $L_n/F_n \in \mathcal{A}_{Lie}$ such that $\mathcal{F}(L_n/F_n)$ is $\mathcal{B}$-isomorphic to (K, A_n). Since A is abelian, the action of A on K gives a $\mathcal{B}$-action of A on the pair (K, A_n). Since $\mathcal{F}(L_n/F_n) \cong (K, A_n)$ and $\mathcal{F}$ induces an equivalence between $\mathcal{A}_{Lie}$ and $\mathcal{B}_{Lie}$, the action of A on K is induced by a faithful $\mathcal{A}$-action of A on L_n/F_n. Since $\mathrm{Gal}(L_n/F_n) \cong A_n$ is finitely generated, and A is not finitely generated, this implies that $\mathrm{Aut}_k(F_n)$ is not finitely generated. Therefore F_n has characteristic p. Thus we may fix $F \cong k((T))$ and assume $F_n = F$ and $L_n \subset F^{sep}$ for all $n \geq 0$.

For $n \geq 0$ let $i_n : (K, A_n) \to (X_F(L_n), \mathrm{Gal}(L_n/F))$ be a $\mathcal{B}$-isomorphism, and set $\pi_{L_n/F} = i_n(T)$. Then for each $\gamma \in A_n$ there is a unique $\sigma_\gamma \in \mathrm{Gal}(L_n/F)$ such that $i_n(\gamma(T)) = \sigma_\gamma(\pi_{L_n/F})$. Furthermore, the map $\gamma \mapsto \sigma_\gamma$ gives an isomorphism from A_n to $\mathrm{Gal}(L_n/F)$. Let $E_n \subset L_n$ be the fixed field of $\mathrm{Gal}(L_n/F)[l_n]$. Suppose $E_n \subsetneqq L_n$; then $i(L_n/E_n) \geq l_n$ and $r(L_n/E_n) \geq r_n$. Write $\sigma_\gamma(\pi_{E_n}) = f(\pi_{E_n})$ and $\sigma_\gamma(\pi_{L_n/F}) = g(\pi_{L_n/F})$, with $f(T), g(T) \in k[[T]]$. Since $\sigma_\gamma(\xi_{E_n}(\pi_{L_n/F})) = \xi_{E_n}(\sigma_\gamma(\pi_{L_n/F}))$ we get

$$f(\pi_{E_n}) \equiv \xi_{E_n}(g(\pi_{L_n/F})) \pmod{\mathcal{M}_{E_n}^{r(L_n/E_n)}}. \tag{3.9}$$

Hence by Proposition 2.2 we have

$$f(T) \equiv g^{p^{-a}}(T) \pmod{T^{r(L_n/E_n)}} \tag{3.10}$$

$$f^{p^a}(T) \equiv \gamma(T) \qquad \pmod{T^{r(L_n/E_n)}}, \tag{3.11}$$

where $a = v_p([E_n : F])$. Since $r(L_n/E_n) \geq r_n$ this implies $\overline{G(E_n/F, \pi_{E_n})} = \overline{A}_n$. On the other hand, if $E_n = L_n$ then $f^{p^a}(T) = \gamma(T)$ and $G(E_n/F, \pi_{E_n}) = A_n$. Since $l_n \geq r_n$, we get $\overline{G(E_n/F, \pi_{E_n})} = \overline{A}_n = \overline{A}$ in either case. Thus $(E_n, \pi_{E_n}) \in \mathcal{S}_n$, and hence $\mathcal{S}_n \neq \varnothing$.

Recall that Lemma 3.1 gives us a continuous map $\nu_n : \mathcal{S}_n \to \mathcal{S}_{n-1}$ for each $n \geq 1$. Since each $\mathcal{S}_n$ is compact and nonempty, by Tychonoff's theorem there exists a sequence of pairs $(E_n, \pi_{E_n}) \in \mathcal{S}_n$ such that

$$\nu_n(E_n, \pi_{E_n}) = (E_{n-1}, \pi_{E_{n-1}}) \tag{3.12}$$

for $n \geq 1$. By the definition of ν_n we have $F \subset E_0 \subset E_1 \subset E_2 \subset \dots$. Let $E_\infty = \cup_{n \geq 0} E_n$. Then E_∞ is a totally ramified abelian extension of F, and there is a unique uniformizer $\pi_{E_\infty/F}$ for $X_F(E_\infty)$ whose E_n-component is π_{E_n} for all $n \geq 0$. Let τ denote the unique k-isomorphism from $K = k((T))$ to $X_F(E_\infty)$ such that $\tau(T) = \pi_{E_\infty/F}$. It follows from our construction that τ induces a $\mathcal{B}$-isomorphism from (K, A) to

$$\mathcal{F}(E_\infty/F) = (X_F(E_\infty), \mathrm{Gal}(E_\infty/F)). \tag{3.13}$$

Thus (K, A) lies in the essential image of $\mathcal{F}$, so $\mathcal{F}$ is essentially surjective.

We now show that $\mathcal{F}$ is faithful. Let E/F and E'/F' be elements of $\mathcal{A}$, and set $G = \mathrm{Gal}(E/F)$ and $G' = \mathrm{Gal}(E'/F')$. We need to show that the map

$$\Psi : \mathrm{Hom}_{\mathcal{A}}(E/F, E'/F') \longrightarrow \mathrm{Hom}_{\mathcal{B}}((X_F(E), G), (X_{F'}(E'), G')) \tag{3.14}$$

induced by the field of norms functor is one-to-one. Suppose $\rho_1, \rho_2 \in \mathrm{Hom}_{\mathcal{A}}(E/F, E'/F')$ satisfy $\Psi(\rho_1) = \Psi(\rho_2)$. Let $\pi_{E/F} = (\pi_L)_{L\in\mathcal{E}_{E/F}}$ be a uniformizer for $X_F(E)$. Then $\Psi(\rho_1)(\pi_{E/F}) = \Psi(\rho_2)(\pi_{E/F})$, and hence $(\rho_1(\pi_L))_{L\in\mathcal{E}_{E/F}} = (\rho_2(\pi_L))_{L\in\mathcal{E}_{E/F}}$. It follows that $\rho_1(\pi_L) = \rho_2(\pi_L)$ for every $L \in \mathcal{E}_{E/F}$. Since ρ_1 and ρ_2 induce the identity map on the residue field k, this implies that $\rho_1 = \rho_2$.

It remains to show that $\mathcal{F}$ is full, i. e., that Ψ is onto. It follows from the arguments given in the proof of [6, Th. 2.1] that the codomain of Ψ is empty if $\mathrm{char}(F) \neq \mathrm{char}(F')$, and that Ψ is onto if G and G' are finitely generated. Thus Ψ is onto if either $\mathrm{char}(F) = 0$ or $\mathrm{char}(F') = 0$. If one of G, G' is finitely generated and the other is not then the domain and codomain of Ψ are both empty. Hence it suffices to prove that Ψ is onto in the case where $\mathrm{char}(F) = \mathrm{char}(F') = p$ and neither of G, G' is finitely generated.

We first show that every isomorphism lies in the image of Ψ. Let

$$\tau : (X_F(E), G) \longrightarrow (X_{F'}(E'), G') \tag{3.15}$$

be a $\mathcal{B}$-isomorphism. Let $l_0 < l_1 < l_2 < \ldots$ denote the positive lower ramification breaks of G and let $u_0 < u_1 < u_2 < \ldots$ denote the corresponding upper ramification breaks. For $n \geq 0$ let F_n denote the fixed field of $G[l_n] = G(u_n)$. If $\lim_{n\to\infty} l_n/[F_n : F] = \infty$ then an argument similar to that used in [5, §2] shows that τ is induced by an $\mathcal{A}$-isomorphism from E/F to E'/F'. This limit condition holds for instance if $\mathrm{char}(F) = p$ and $\mathrm{Gal}(E/F)$ is finitely generated, but it can fail if $\mathrm{Gal}(E/F)$ is not finitely generated. Therefore we use a different method to prove that τ lies in the image of Ψ, based on a characterization of F_n/F in terms of $(X_F(E), G)$.

Fix $n \geq 1$, let d denote the F_n-valuation of the different of F_n/F, and let c be an integer such that $c > \phi_{F_n/F}(\frac{p}{p-1}(l_{n-1} + d))$. Since $G/G(c)$ is finite there exists a finitely generated closed subgroup H of G such that $G(c)H = G$. Let $M \subset E$ be the fixed field of H and set $M_n = F_nM$. Then F_n/F and M_n/M are finite abelian extensions. On the other hand, since G is not finitely generated, $\mathrm{Gal}(M/F) \cong G/H$ is not finitely generated, and hence M/F is an infinite abelian extension.

Proposition 3.2. *Let $\pi_{E/F}$ be a uniformizer for $X_F(E)$ and recall that $\pi_{E/F}$ determines uniformizers π_F, π_{F_n}, $\pi_{M/F}$, and $\pi_{M_n/F}$ for the fields F, F_n, $X_F(M)$, and $X_{M/F}(M_n)$. There exists a k-isomorphism*

$$\zeta : X_{M/F}(M_n)/X_F(M) \longrightarrow F_n/F \tag{3.16}$$

such that

(1) $\zeta(\pi_{M/F}) = \pi_F$;
(2) $\zeta(\pi_{M_n/F}) \equiv \pi_{F_n} \pmod{\mathcal{M}_{F_n}^{l_{n-1}+1}}$;
(3) $\gamma \cdot \zeta(\pi_{M_n/F}) = \zeta(\gamma \cdot \pi_{M_n/F})$ *for every* $\gamma \in G$.

The proof of this proposition depends on the following lemma (cf. [1, p. 88]).

Lemma 3.3. *Let F be a local field, let $g(T) \in \mathcal{O}_F[T]$ be a separable monic Eisenstein polynomial, and let $\alpha \in F^{sep}$ be a root of $g(T)$. Set $E = F(\alpha)$ and let $d = v_E(g'(\alpha))$ be the E-valuation of the different of the extension E/F. Then for any $\eta \in F^{sep}$ there is a root β of $g(X)$ such that $v_E(\eta-\beta) \geq v_E(g(\eta)) - d$.*

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $g(T)$, and choose $1 \leq j \leq n$ to maximize $w = v_E(\eta - \alpha_j)$. For $1 \leq i \leq n$ we have

$$v_E(\eta - \alpha_i) \geq \min\{w, v_E(\alpha_j - \alpha_i)\}, \tag{3.17}$$

with equality if $w \neq v_E(\alpha_j - \alpha_i)$. Since $w \geq v_E(\eta - \alpha_i)$, this implies that for $i \neq j$ we have $v_E(\eta - \alpha_i) \leq v_E(\alpha_j - \alpha_i)$. Since

$$g(\eta) = (\eta - \alpha_1)(\eta - \alpha_2) \ldots (\eta - \alpha_n), \tag{3.18}$$

we get

$$v_E(g(\eta)) \leq w + \sum_{\substack{1 \leq i \leq n \\ i \neq j}} v_E(\alpha_j - \alpha_i) = w + d. \tag{3.19}$$

Setting $\beta = \alpha_j$ gives $v_E(\eta - \beta) = w \geq v_E(g(\eta)) - d$. □

Proof of Proposition 3.2. Since $G(c)H = G$ and $c > \phi_{F_n/F}(l_{n-1}) = u_{n-1}$ we get $G(u_n)H = G$. It follows that M and F_n are linearly disjoint over F. The equality $G(c)H = G$ also implies that $i(M/F) \geq c > u_{n-1}$. Therefore by Lemma 2.1 we have

$$i(M_n/F_n) = \psi_{F_n/F}(i(M/F)) \geq \psi_{F_n/F}(c). \tag{3.20}$$

It follows that $r(M_n/F_n) \geq s$, where $r(M_n/F_n)$ is defined by (2.9) and $s = \lceil \frac{p-1}{p} \cdot \psi_{F_n/F}(c) \rceil$. Let $g(T)$ be the minimum polynomial for $\pi_{M_n/F}$ over $X_F(M)$, and let $g_F(T) \in \mathcal{O}_F[T]$ be the polynomial obtained by applying the canonical map $\lambda : X_F(M) \to F$ given by $\lambda(\alpha_{M/F}) = \alpha_F$ to the coefficients of $g(T)$. Since $g(\pi_{M_n/F}) = 0$, it follows from Propositions 2.2(a) and 2.3 that $v_{F_n}(g_F(\pi_{F_n})) \geq r(M_n/F_n) \geq s$.

Let $\mu : X_F(M) \to F$ be the unique k-algebra isomorphism such that $\mu(\pi_{M/F}) = \pi_F$. Then by Proposition 2.2 we have

$$\mu(\alpha_{M/F}) \equiv \alpha_F \pmod{\mathcal{M}_F^{r(M/F)}} \tag{3.21}$$

for all $\alpha_{M/F} \in \mathcal{O}_{X_F(M)}$. Let $g^\mu(T) \in \mathcal{O}_F[T]$ be the polynomial obtained by applying μ to the coefficients of $g(T)$. Then

$$g^\mu(T) \equiv g_F(T) \pmod{\mathcal{M}_F^{r(M/F)}}. \tag{3.22}$$

It follows from the inequalities

$$[F_n : F] \cdot i(M/F) \geq [F_n : F] \cdot c \geq \psi_{F_n/F}(c) \tag{3.23}$$

that $[F_n : F]\cdot r(M/F) \geq s$. Since we also have $v_{F_n}(g_F(\pi_{F_n})) \geq s$ this implies that $v_{F_n}(g^\mu(\pi_{F_n})) \geq s > l_{n-1} + d$. It follows from Lemma 3.3 that there is a root β of $g^\mu(T)$ such that $v_{F_n}(\pi_{F_n} - \beta) > l_{n-1}$. Therefore by Krasner's Lemma we have $F(\beta) \supset F(\pi_{F_n})$. Since $[F(\beta) : F] = \deg(g) = [F(\pi_{F_n}) : F]$ we deduce that $F(\beta) = F(\pi_{F_n}) = F_n$. Since $\pi_{M_n/F}$ is a root of $g(T)$, and β is a root of $g^\mu(T)$, the isomorphism μ from $X_F(M)$ to F extends uniquely to an isomorphism ζ from $X_{M/F}(M_n)/X_F(M)$ to F_n/F such that $\zeta(\pi_{M_n/F}) = \beta \equiv \pi_{F_n} \pmod{\mathcal{M}_{F_n}^{l_{n-1}+1}}$.

We now show that ζ is H-equivariant. Let $\gamma \in H$ and define $h_\gamma \in k[[T]]$ by

$$h_\gamma(\pi_{M_n/F}) = \gamma \cdot \pi_{M_n/F} = (\gamma \cdot \pi_L)_{L \in \mathcal{E}_{M_n/F}}, \tag{3.24}$$

where we identify k with a subfield of $X_F(M)$ using the map $f_{M/F}$. It follows from Propositions 2.2 and 2.3 that

$$\gamma \cdot \pi_{F_n} \equiv h_\gamma(\pi_{F_n}) \pmod{\mathcal{M}_{F_n}^{r(M_n/F_n)}}. \tag{3.25}$$

Since $\zeta(\pi_{M_n/F}) \equiv \pi_{F_n} \pmod{\mathcal{M}_{F_n}^{l_{n-1}+1}}$ and $r(M_n/F_n) \geq s \geq l_{n-1} + 1$ this implies

$$\begin{aligned}
\zeta(\gamma \cdot \pi_{M_n/F}) &= \zeta(h_\gamma(\pi_{M_n/F})) && (3.26)\\
&= h_\gamma(\zeta(\pi_{M_n/F})) && (3.27)\\
&\equiv h_\gamma(\pi_{F_n}) \pmod{\mathcal{M}_{F_n}^{l_{n-1}+1}} && (3.28)\\
&\equiv \gamma \cdot \pi_{F_n} \pmod{\mathcal{M}_{F_n}^{l_{n-1}+1}} && (3.29)\\
&\equiv \gamma \cdot \zeta(\pi_{M_n/F}) \pmod{\mathcal{M}_{F_n}^{l_{n-1}+1}}. && (3.30)
\end{aligned}$$

Since $\zeta(\gamma \cdot \pi_{M_n/F})$ and $\gamma \cdot \zeta(\pi_{M_n/F})$ are both roots of $g^\mu(T)$, and any two distinct roots π, π' of $g^\mu(T)$ must satisfy $v_{F_n}(\pi - \pi') \leq l_{n-1}$, we deduce that $\gamma\cdot\zeta(\pi_{M_n/F}) = \zeta(\gamma\cdot\pi_{M_n/F})$. Since ζ and γ are k-linear and continuous, it follows that $\gamma \cdot \zeta(\alpha) = \zeta(\gamma \cdot \alpha)$ for all $\alpha \in X_{M/F}(M_n)$. □

Since τ is a $\mathcal{B}$-isomorphism, $\tau^* : G' \to G$ is a group isomorphism. For $\gamma \in G$ set $\gamma' = (\tau^*)^{-1}(\gamma)$, and for $N \leq G$ set $N' = (\tau^*)^{-1}(N)$. Then τ induces an isomorphism from $(X_F(E), N)$ to $(X_{F'}(E'), N')$. In particular, τ gives an isomorphism from $(X_F(E),H)$ to $(X_{F'}(E'),H')$. Using the isomorphism $X_{X_F(M)}(X_{M/F}(E))\cong X_F(E)$ from [7, 3.4.1] we get an isomorphism

$$\tau_H:(X_{X_F(M)}(X_{M/F}(E)),H)\longrightarrow(X_{X_{F'}(M')}(X_{M'/F'}(E')),H'), \tag{3.31}$$

where $M'\subset E'$ is the fixed field of H'. Since H is an abelian p-adic Lie group, it follows from [2, 5, 6] that τ_H is induced by an $\mathcal{A}$-isomorphism

$$\rho:X_{M/F}(E)/X_F(M)\longrightarrow X_{M'/F'}(E')/X_{F'}(M'). \tag{3.32}$$

By restricting ρ we get an isomorphism

$$\rho_n:X_{M/F}(M_n)/X_F(M)\longrightarrow X_{M'/F'}(M_n')/X_{F'}(M'), \tag{3.33}$$

where $M_n'=(M')_n=F_n'M'$ is the fixed field of $H'[l_n]=H[l_n]'$. Furthermore, for $\gamma\in H$ and $\alpha\in X_{M/F}(M_n)$ we have $\rho_n(\gamma(\alpha))=\gamma'(\rho_n(\alpha))$.

Let $\pi_{E/F}$ be a uniformizer for $X_F(E)$, set $\pi_{E'/F'}=\tau(\pi_{E/F})$, and let

$$\zeta:X_{M/F}(M_n)/X_F(M)\longrightarrow F_n/F \tag{3.34}$$

$$\zeta':X_{M'/F'}(M_n')/X_{F'}(M')\longrightarrow F_n'/F' \tag{3.35}$$

be the isomorphisms given by Proposition 3.2. Then $\omega_n=\zeta'\circ\rho_n\circ\zeta^{-1}$ is a k-linear isomorphism from F_n/F to F_n'/F'. It follows from Proposition 3.2 that for $n\geq 1$ we have

$$\omega_n(\pi_{F_n})\equiv\pi_{F_n'}\pmod{\mathcal{M}_{F_n'}^{l_{n-1}+1}} \tag{3.36}$$

and

$$\omega_n(\gamma(\pi_{F_n}))=\gamma'(\omega_n(\pi_{F_n})) \tag{3.37}$$

for all $\gamma\in H$. Since the restriction map from $H=\mathrm{Gal}(E/M)$ to $\mathrm{Gal}(F_n/F)$ is onto, (3.37) is actually valid for all $\gamma\in G$.

Let $\mathcal{I}_n$ denote the set of k-isomorphisms $\omega_n:F_n/F\to F_n'/F'$, and let $\mathcal{T}_n$ denote the subset of $\mathcal{I}_n$ consisting of those ω_n which satisfy (3.36) and (3.37) for all $\gamma\in G$. Since l_{n-1} is the only ramification break of F_n'/F_{n-1}' we have $\psi_{F_n'/F_{n-1}'}(l_{n-1})=l_{n-1}$. Therefore by (3.36) and [4, V §6, Prop. 8], for any $\omega_n\in\mathcal{T}_n$ we have

$$\mathrm{N}_{F_n'/F_{n-1}'}(\omega_n(\pi_{F_n}))\equiv\mathrm{N}_{F_n'/F_{n-1}'}(\pi_{F_n'})\pmod{\mathcal{M}_{F_{n-1}'}^{l_{n-1}+1}}. \tag{3.38}$$

Suppose $n\geq 2$. Since $\mathrm{N}_{F_n/F_{n-1}}(\pi_{F_n})=\pi_{F_{n-1}}$ and $\mathrm{N}_{F_n'/F_{n-1}'}(\pi_{F_n'})=\pi_{F_{n-1}'}$, it follows from (3.38) and (3.37) that

$$\omega_n(\pi_{F_{n-1}})\equiv\pi_{F_{n-1}'}\pmod{\mathcal{M}_{F_{n-1}'}^{l_{n-1}+1}}. \tag{3.39}$$

Since $l_{n-1}>l_{n-2}$ this implies that the restriction $\omega_n\mapsto\omega_n|_{F_{n-1}}$ gives a map from $\mathcal{T}_n$ to $\mathcal{T}_{n-1}$.

Define a metric on $\mathcal{I}_n$ by setting $d(\omega_n,\tilde{\omega}_n)=2^{-a}$, where

$$a=v_{F_n'}(\omega_n(\pi_{F_n})-\tilde{\omega}_n(\pi_{F_n})). \tag{3.40}$$

Then $\mathcal{I}_n$ is compact, since it can be identified with the set of uniformizers for F_n'. Therefore the closed subset $\mathcal{T}_n$ of $\mathcal{I}_n$ is compact as well. Since each $\mathcal{T}_n$ is nonempty, by Tychonoff's theorem there is a sequence $(\omega_n)_{n\geq 1}$ such that $\omega_n \in \mathcal{T}_n$ and $\omega_{n+1}|_{F_n} = \omega_n$ for all $n \geq 1$. Since $E = \cup_{n\geq 1} F_n$ and $E' = \cup_{n\geq 1} F_n'$, the isomorphisms $\omega_n : F_n/F \to F_n'/F'$ combine to give an $\mathcal{A}$-isomorphism $\Omega : E/F \to E'/F'$. Let $\theta = \Psi(\Omega)$ be the $\mathcal{B}$-isomorphism induced by Ω and let $m_n = \min\{l_{n-1}+1, r(E/F_n)\}$. It follows from (3.36) and Proposition 2.2(a) that

$$\theta(\pi_{E/F}) \equiv \pi_{E'/F'} \pmod{\mathcal{M}_{X_{F'}(E')}^{m_n}} \tag{3.41}$$

for every $n \geq 1$. Since $\lim_{n\to\infty} m_n = \infty$ we get $\theta(\pi_{E/F}) = \pi_{E'/F'} = \tau(\pi_{E/F})$. Hence $\tau = \theta = \Psi(\Omega)$.

Now let σ be an arbitrary element of $\mathrm{Hom}_{\mathcal{B}}((X_F(E), G), (X_{F'}(E'), G'))$. Since $X_{F'}(E')$ is a finite separable extension of $\sigma(X_F(E))$, by [7, 3.2.2] there is a finite separable extension $\tilde{E}/E$ such that σ extends to an isomorphism $\tau : X_{E/F}(\tilde{E}) \to X_{F'}(E')$. It follows that each $\gamma' \in G'$ induces an automorphism $\tilde{\gamma} = \tau^{-1}\circ\gamma'\circ\tau$ of $X_{E/F}(\tilde{E})$ whose restriction to $X_F(E)$ is $\sigma^*(\gamma') \in G$. Since $X_{E/F}(F^{sep})$ is a separable closure of $X_F(E)$ [7, Cor. 3.2.3], $\tilde{\gamma}$ can be extended to an automorphism $\overline{\gamma}$ of $X_{E/F}(F^{sep})$. Since $\overline{\gamma}$ stabilizes $X_F(E)$, and $\overline{\gamma}|_{X_F(E)} = \sigma^*(\gamma')$ is induced by an element of $G = \mathrm{Gal}(E/F)$, it follows from [7, Rem. 3.2.4] that $\overline{\gamma}$ is induced by an element of $\mathrm{Gal}(F^{sep}/F)$, which we also denote by $\overline{\gamma}$. Since $\overline{\gamma}$ stabilizes $X_{E/F}(\tilde{E})$, it stabilizes $\tilde{E}$ as well. Thus $\overline{\gamma}|_{\tilde{E}}$ is a k-automorphism of $\tilde{E}$ which is uniquely determined by γ'. Since $\overline{\gamma}|_{\tilde{E}}$ induces the automorphism $\tilde{\gamma}$ of $X_{E/F}(\tilde{E})$, we denote $\overline{\gamma}|_{\tilde{E}}$ by $\tilde{\gamma}$.

Let $\tilde{F}$ denote the subfield of $\tilde{E}$ which is fixed by the subgroup $\tilde{G} = \{\tilde{\gamma} : \gamma' \in G'\}$ of $\mathrm{Aut}_k(\tilde{E})$. Then $\tilde{F} \supset F$, so $\tilde{E}/\tilde{F}$ is a Galois extension, with $\mathrm{Gal}(\tilde{E}/\tilde{F}) = \tilde{G}$. Since the image of $\tilde{G} \cong G'$ in G is open, $\tilde{F}$ is a finite separable extension of F, and hence $\tilde{F} \cong k((T))$ is a local field with residue field k. Therefore $(X_{\tilde{F}}(\tilde{E}), \tilde{G})$ is an object in $\mathcal{B}$, and τ gives a $\mathcal{B}$-isomorphism from $(X_{\tilde{F}}(\tilde{E}), \tilde{G})$ to $(X_{F'}(E'), G')$. By the arguments given above, τ is induced by an $\mathcal{A}$-isomorphism $\Omega : \tilde{E}/\tilde{F} \to E'/F'$. Since $\tilde{E}/E$ and $\tilde{F}/F$ are finite separable extensions, the embedding $E \hookrightarrow \tilde{E}$ induces an $\mathcal{A}$-morphism $i : E/F \to \tilde{E}/\tilde{F}$. Let

$$\alpha : (X_F(E), G) \longrightarrow (X_{\tilde{F}}(\tilde{E}), \tilde{G}) \tag{3.42}$$

be the $\mathcal{B}$-morphism induced by i. Then $\sigma = \tau \circ \alpha = \Psi(\Omega \circ i)$.

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