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On the spectrum of the Thue-Morse quasicrystal and the rarefaction phenomenon

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In honor of the* 60*-th birthday of Henri Cohen...

Résumé. On explore le spectre d'un peigne de Dirac pondéré supporté par le quasicristal de Thue-Morse au moyen de la Conjecture de Bombieri-Taylor, pour les pics de Bragg, et d'une nouvelle conjecture que l'on appelle Conjecture de Aubry-Godrèche-Luck, pour la composante singulière continue. La décomposition de la transformée de Fourier du peigne de Dirac pondéré est obtenue dans le cadre de la théorie des distributions tempérées. Nous montrons que l'asymptotique de l'arithmétique des sommes p-raréfiées de Thue-Morse (Dumont ; Goldstein, Kelly and Speer ; Grabner ; Drmota and Skalba,...), précisément les fonctions fractales des sommes de chiffres, jouent un rôle fondamental dans la description de la composante singulière continue du spectre, combinées à des résultats classiques sur les produits de Riesz de Peyrière et de M. Queffélec. Les lois d'échelle dominantes des suites de mesures approximantes sont contrôlées sur une partie de la composante singulière continue par certaines inégalités dans lesquelles le nombre de classes de diviseurs et le régulateur de corps quadratiques réels interviennent.

Abstract. The spectrum of a weighted Dirac comb on the Thue-Morse quasicrystal is investigated by means of the Bombieri-Taylor conjecture, for Bragg peaks, and of a new conjecture that we call Aubry-Godrèche-Luck conjecture, for the singular continuous component. The decomposition of the Fourier transform of the weighted Dirac comb is obtained in terms of tempered distributions. We show that the asymptotic arithmetics of the p-rarefied sums of the Thue-Morse sequence (Dumont; Goldstein, Kelly and Speer; Grabner; Drmota and Skalba,...), namely the fractality of sum-of-digits functions, play a fundamental role in the description of the singular continous part of the spectrum, combined with some classical results on Riesz products of Peyrière and M.

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Mots clefs. Thue-Morse quasicrystal, spectrum, singular continuous component, rarefied sums, sum-of-digits fractal functions, approximation to distribution.

Queffélec. The dominant scaling of the sequences of approximant measures on a part of the singular component is controlled by certain inequalities in which are involved the class number and the regulator of real quadratic fields.

Contents

1. Introduction

The $\pm$ Prouhet-Thue-Morse sequence $(\eta_n)_{n\in\mathbb{N}}$ is defined by

$$\eta_n = (-1)^{s(n)} \qquad n \geq 0, \tag{1.1}$$

where $s(n)$ is equal to the sum of the 2-digits $n_0 + n_1 + n_2 + \ldots$ in the binary expansion of $n = n_0 + n_1 2 + n_2 2^2 + \ldots$ It can be viewed as a fixed point of the substitution $1 \to 1\ \overline{1}, \overline{1} \to \overline{1}\ 1$ on the two letter alphabet $\{\pm 1\}$, starting with 1 ($\overline{1}$ stands for -1). There exists a large literature on this sequence [AMF] [Q1]. Let a and b be two positive real numbers such that $0 < b < a$. Though there exists an infinite number of ways of constructing a regular aperiodic point set of the line [La1] from the sequence $(\eta_n)_{n\in\mathbb{N}}$, we adopt the following definition, which seems to be fairly canonical. We call the Thue-Morse quasicrystal, denoted by $\Lambda_{a,b}$, or simply by Λ (without mentioning the parametrization with a and b), the point set

$$\Lambda \ := \ \Lambda^+ \cup (-\Lambda^+) \quad \subset \mathbb{R} \tag{1.2}$$

where $\Lambda_{a,b}^{+}$, or simply Λ^{+}, on $\mathbb{R}^{+}$, is equal to

$$(1.3)\quad \{0\} \cup \Big\{ f(n) := \sum_{0 \le m \le n-1} \big(\frac{1}{2}(a+b) + \frac{1}{2}(a-b)\eta_m\big) \mid n = 1, 2, 3, \ldots \Big\}.$$

The function f defined by (1.3) is extended to $\mathbb{Z}$ by symmetry: we put

$$(1.4)\qquad f(0) = 0 \text{ by convention and } f(n) = -f(-n) \text{ for } n \in \mathbb{Z},\ n < 0.$$

For all $n \in \mathbb{Z}$, $|f(n+1) - f(n)|$ is equal either to a or b so that the closed (generic) intervals of respective lengths a and b are the two prototiles of the aperiodic tiling of the line $\mathbb{R}$ whose $(f(n))_{n\in\mathbb{Z}}$ is the set of vertices. The point set $\Lambda_{a,b}$ is a Delone set since it is relatively dense and uniformly discrete, as set of vertices of a tiling [GVG] [M]. Recall that a point set $X \subset \mathbb{R}$ which is a Delone set and such that it exists a finite set F_X such that $X - X \subset X + F_X$ is called a Meyer set. It is straightforward to prove that

$$(a+b)\,\mathbb{Z} \ \subset\ \Lambda_{a,b}$$

and moreover that

$$\Lambda_{a,b} \ -\ \Lambda_{a,b} \ \subset\ \Lambda_{a,b} + F,$$

where $F = \{\pm a, \pm b, \pm 2a, \pm 2b, \pm a \pm b\}$, so that $\Lambda_{a,b}$ is a Meyer set [La1] [M] [VG]. The Thue-Morse quasicrystal, and any weighted Dirac comb on it, is considered as a somehow myterious point set, intermediate between chaotic, or random, and periodic [AGL] [AT] [B] [CSM] [GL1] [GL2] [KIR] [Lu] [PCA] [WWVG], and the interest for such systems in physics is obvious from many viewpoints.

In this note we study the spectrum of a weighted Dirac comb μ on the point set $\Lambda_{a,b}$ by using arithmetic methods, more precisely by involving sum-of-digits fractal functions associated with the rarefied sums of the Thue-Morse sequence (Coquet [Ct], Dumont [D], Gelfond [Gd], Grabner [Gr1], Newman [N], Goldstein, Kelly and Speer [GKS], Drmota and Skalba [DS1] [DS2], ...). For this, we hold for true two conjectures which are expressed in terms of scaling laws of approximant measures: the Bombieri-Taylor Conjecture and a new conjecture that we call Aubry-Godrèche-Luck Conjecture (Subsection 3.2). In the language of physics, the spectrum measures the extent to which the intensity diffracted by μ is concentrated at a real number k (wave vector). It can be observed through the square modulus of the Fourier transform of μ at $\{k\}$ [Cy] [G], or, possibly, of its autocorrelation [H] [La2].

On one hand the spectrum of the symbolic dynamical system associated with the Prouhet-Thue-Morse sequence is known to be singular continuous:

if

$$\widehat{\eta_n}(k) := \sum_{j=0}^{2^n-1} \eta_j \exp(-2i\pi jk) \tag{1.5}$$

denotes its Fourier transform, then

$$|\widehat{\eta_n}(k)|^2 = 2^n \prod_{j=0}^{n-1} (1 - \cos(2\pi 2^j k)) \;=\; 2^{2n} \prod_{j=0}^{n-1} \sin^2(\pi 2^j k) \tag{1.6}$$

is a Riesz product constructed on the sequence $(2^j)_{j\geq 0}$ which has the property that the sequence of measures

$$\left\{2^{-n}|\widehat{\eta_n}(k)|^2 dk\right\}_{n\geq 0} \tag{1.7}$$

has a unique accumulation point, its limit, for the vague topology, is a singular continuous measure (Peyriere [P] §4.1, Allouche, Mendès-France [AMF] Appendix I, p. 337). On the other hand, Queffélec ([Q2] §6.3.2.1) has shown that replacing the alphabet $\{\pm 1\}$ by $\{0, 1\}$ leads to a new component of the measure, which is discrete and exactly localized at the elements of the group $\mathbb{D}_2$ of 2-adic rational numbers in the one-dimensional torus $\mathbb{R}/2\pi\mathbb{Z}$, explicitly

$$\mathbb{D}_2 := \{2\pi \frac{m}{2^n} \mid 0 \leq m \leq 2^n, n = 0, 1, 2, \ldots\}. \tag{1.8}$$

Therefore the spectrum of $\Lambda_{a,b}$, and more generally the Fourier transform of a weighted Dirac comb μ supported by $\Lambda_{a,b}$, is expected to be the sum of a discrete part and a singular continuous part, each of them being a function of a and b. In Subsection 3.1, Theorem 3.6 (the proof of which is similar to the proof of Theorem 4.1 in [GVG] for sets $\mathbb{Z}_\beta$ of β-integers, with β a quadratic unitary Pisot number) shows that it is the case in the context of tempered distributions. The weights $(\omega(n))$ appearing in μ should be more than bounded and should obey mild further assumptions so that it is possible to fully exploit the resources of the theory of tempered distributions. These constraints on $(\omega(n))$ are made precise by Hypothesis **(H)** in Section 3.1.

We explore the particular Dirac comb on $\Lambda_{a,b}$ for which the weights are all equal to 1 on the real positive line. In Subsection 3.3 we deduce the Bragg component of its spectrum by classical results on Riesz products of Peyrière [P] and Queffélec [Q2] and by the Bombieri-Taylor conjecture (Theorem 3.9). In the context of this paper, we use the Bombieri-Taylor argument, for the pure point part of the spectrum, and the Aubry-Godrèche-Luck argument, for the singular part of the spectrum, as effective conjectures. In Subsection 3.4 we show the deep relation between the p-rarefied sums of the Thue-Morse sequence and the singular continuous component of its spectrum. We use the sum-of-digits fractal functions of the rarefaction phenomenon, recalled in Section 4, in agreement with the Aubry-Godrèche-Luck argument (Subsection 3.2) to deduce, in Subsection 5.1, the singular part $\mathbb{S}$ of the spectrum, up to a zero measure subset (Theorem 5.1). In the subsequent Subsections we explore the extinction properties and the questions of visibility of the singular component of the spectrum as a function of the sequence of finite approximants. This leads us to characterize prime numbers p for which the sequences of approximant measures at the wave vectors $k = \frac{4\pi}{a+b}\frac{t}{2^h p}$ have the property of being "size-increasing" at k; more generally, for some classes of prime numbers p the Cohen-Lenstra heuristics of real quadratic fields is required [CL1] [CL2] [CM]. In Section 6 we indicate how to deal with other weighted Dirac combs on the Thue-Morse quasicrystal $\Lambda_{a,b}$ and prove that the diffraction process remains somehow invariant under the Marcinkiewicz equivalence relation, for the Bragg peaks, by the Bombieri-Taylor argument.

In addition to the mathematical interest which links diffraction spectra to arithmetics as mentioned in the title, the present results represent the solutions of many experimental questions of physicists on Thue-Morse quasicrystals which remained unexplained until now.

2. Averaging sequences of finite approximants

Definition 2.1. An averaging sequence $(U_l)_{l\geq 0}$ of finite approximants of $\Lambda_{a,b}$ is given by a closed interval J whose interior contains the origin and a strictly increasing sequence $(\rho_l)_{l\geq 0}$ of positive numbers such that

$$U_l \ = \ \rho_l J \cap \Lambda_{a,b}, \qquad l = 0, 1, 2, \ldots$$

A natural averaging sequence of such approximants for the Thue-Morse quasicrystal is yielded by the sequence $((-B_N) \cup B_N)_{N\geq 0}$, where

$$B_N \ = \ \{x \in \Lambda_{a,b} \mid 0 \leq x \leq f(N)\} \qquad N = 0, 1, 2, \ldots \tag{2.9}$$

Obviously $\operatorname{Card}(B_N) = 2N + 1$ for $N = 0, 1, 2, \ldots$

Similarly an averaging sequence can be defined for a Radon measure $\mu = \sum_{n\in\mathbb{Z}} \omega(n)\delta_{f(n)}$ supported by $\Lambda_{a,b}$, where $\omega(x)$ is a bounded complex-valued function, called weight, and where $\delta_{f(n)}$ denotes the normalized Dirac measure supported by the singleton $\{f(n)\}$. The support of the weight function ω is denoted by

$$\operatorname{supp}(\omega) := \{n \in \mathbb{Z} \mid \omega(n) \neq 0\}.$$

Introducing

$$\Lambda_{a,b}^{\operatorname{supp}(\omega)} \ = \ \bigcup_{n\in \operatorname{supp}(\omega)} \{f(n)\},$$

then an averaging sequence $(U_l)_{l\geq 0}$ of finite approximants of $\Lambda_{a,b}^{\mathrm{supp}(\omega)}$ is given by a closed interval J whose interior contains the origin and a strictly increasing sequence $(\rho_l)_{l\geq 0}$ of positive numbers such that

$$U_l \;=\; \rho_l J \cap \Lambda_{a,b}^{\mathrm{supp}(\omega)}, \qquad l = 0, 1, 2, \ldots.$$

Let us denote by $\mathcal{A}_{a,b}$, resp. $\mathcal{A}_{a,b}^{\mathrm{supp}(\omega)}$, the set of the averaging sequences of finite approximants of $\Lambda_{a,b}$, resp. $\Lambda_{a,b}^{\mathrm{supp}(\omega)}$. If the affine hull of $\mathrm{supp}(\omega)$ is $\mathbb{R}$, then $\mathcal{A}_{a,b} = \mathcal{A}_{a,b}^{\mathrm{supp}(\omega)}$.

3. Diffraction spectra

3.1. Fourier transform of a weighted Dirac comb on $\boldsymbol{\Lambda_{a,b}}$. Let δ_{x_0} be the Dirac measure supported by x_0. Let us consider the following Radon measure supported by the Thue-Morse quasicrystal $\Lambda_{a,b}$:

$$\mu = \sum_{n\in\mathbb{Z}} \omega(n)\delta_{f(n)}. \tag{3.10}$$

where $\omega(x)$ is a bounded complex-valued function (weight function). The measure (3.10) is translation bounded, i.e. for all compact $K \subset \mathbb{R}$ there exists α_K such that $\sup_{\alpha\in\mathbb{R}} |\mu|(K+\alpha) \leq \alpha_K$, and so is a tempered distribution. Its Fourier transform $\widehat{\mu}$ is also a tempered distribution defined by

$$\widehat{\mu}(k) = \mu(e^{-ikx}) = \sum_{n\in\mathbb{Z}} \omega(n) e^{-ikf(n)} \tag{3.11}$$

It may or may not be a measure.

Let us now transform (3.11) using the general form of $f(n)$, $n \in \mathbb{Z}$, given by (1.2) and (1.4). For a real number $x \in \mathbb{R}$, the integer part of x is denoted by $\lfloor x \rfloor$ and its fractional part by $\{x\}$.

Lemma 3.1. *If $n \in \mathbb{Z}$ is even, then $f(n) = n(a+b)/2$. If $n \geq 1$ is odd, then*

$$f(n) \;=\; \frac{n}{2}(a+b) + \frac{a-b}{2}\eta_{n-1}. \tag{3.12}$$

Proof. Obvious since, for every even integer $m \geq 0$, $\eta_m + \eta_{m+1} = 0$. ☐

We deduce that $f(n)$ has the general form $f(n) = \alpha_1 n + \alpha_0 + \alpha_2\{x(n)\}$ for all integers $n \geq 1$, where

$$\alpha_1 = (a+b)/2, \quad \alpha_0 = -(a-b)/2, \quad \alpha_2 = 2(a-b), \quad x(n) = (1+\eta_{n-1})/4.$$

Indeed, $x(n) \in [0,1)$ is equal to its fractional part. Now, for any $x \in \mathbb{R}$, the "fractional part" function $x \to \{x\}$ is periodic of period 1 and so is the piecewise continuous function $e^{-ik\alpha_2\{x\}}$. Let

$$c_m(k) = \int_0^1 e^{-ik\alpha_2\{x\}} e^{-2\pi imx} dx \tag{3.13}$$

$$= i\frac{e^{-ik\alpha_2} - 1}{2\pi m + k\alpha_2} \;=\; (-1)^m e^{-ik\frac{\alpha_2}{2}} \operatorname{sinc}(\frac{\alpha_2}{2}k + m\pi) \tag{3.14}$$

be the coefficients of the expansion of $e^{-ik\alpha_2\{x\}}$ in Fourier series. In (3.14), sinc denotes the cardinal sine function $\operatorname{sinc}(x) = \sin x/x$. Thus

$$e^{-ik\alpha_2\{x\}} \;=\; \sum_{-\infty}^{+\infty} c_m(k) e^{2i\pi mx} \tag{3.15}$$

where the convergence is punctual in the usual sense of Fourier series for piecewise continuous functions. Thus

$$\sum_{n\geq 1} \omega(n) e^{-ikf(n)} = \sum_{n\geq 1} \omega(n) e^{-ik\left[\frac{n}{2}(a+b)+\frac{1}{2}(a-b)\eta_{n-1}\right]}$$

$$= \sum_{n\geq 1} \omega(n) e^{-ik\left[\frac{n}{2}(a+b)-\frac{a-b}{2}\right]} \times \left(\sum_{m\in\mathbb{Z}} c_m(k) e^{2i\pi m\frac{1+\eta_{n-1}}{4}}\right) \tag{3.16}$$

$$= \sum_{m\in\mathbb{Z}} \sum_{n\geq 1} c_m(k)\omega(n) e^{-ik\left[\frac{n}{2}(a+b)-\frac{a-b}{2}\right]} \times e^{2i\pi m\frac{1+\eta_{n-1}}{4}}. \tag{3.17}$$

At the moment, we have carried out the interchange of summation in (3.16) and (3.17). The hypothesis **(H)** below allows to justify these formal manipulations, within the theory of tempered distributions.

Lemma 3.2. *For every integer $n \geq 1$,*

$$s(n) \;=\; \frac{(-1)^{s(n)} - 1}{2} \qquad (\mathrm{mod}\ 2). \tag{3.18}$$

Proof. Obvious. □

We deduce

$$\sum_{n\geq 1} \omega(n) e^{-ikf(n)} = \sum_{m\in\mathbb{Z}} \sum_{n\geq 1} c_m(k)\omega(n) e^{-ik\left[\frac{n}{2}(a+b)-\frac{a-b}{2}\right]} \times e^{i\pi m(1+s(n-1))},$$

and then, since $e^{i\pi} = -1$,

$$= \sum_{m\in\mathbb{Z}} \sum_{n\geq 1} \left(c_m(k)\,(-\eta_{n-1})^m\,\omega(n) e^{-i\pi k\left(n-\frac{a-b}{a+b}\right)\frac{(a+b)}{2\pi}}\right).$$

Therefore

$$\widehat{\mu}(k) \;=\; \sum_{n\geq 1} \omega(-n) e^{+ikf(n)} + \omega(0) + \sum_{n\geq 1} \omega(n) e^{-ikf(n)}$$

$$= \omega(0) + \sum_{m\in\mathbb{Z}}\sum_{n\geq 1}\left(c_m(k)\,(-\eta_{n-1})^m\,\omega(n)e^{-i\pi\left(n-\frac{a-b}{a+b}\right)k\frac{(a+b)}{2\pi}}\right)$$
$$+ \sum_{m\in\mathbb{Z}}\sum_{n\geq 1}\left(c_m(-k)\,(-\eta_{n-1})^m\,\omega(-n)e^{+i\pi\left(n-\frac{a-b}{a+b}\right)k\frac{(a+b)}{2\pi}}\right).$$

Let us observe that $(-\eta_{n-1})^m = 1$ if m is even and that $(-\eta_{n-1})^m = -\eta_{n-1}$ if m is odd, for all $n \geq 1$. Then

$$\widehat{\mu}(k) = \omega(0) + \sum_{m\in\mathbb{Z}} c_m(k)\left[\frac{1+(-1)^m}{2}\left(\sum_{n\geq 1}\omega(n)e^{-i\pi\left(n-\frac{a-b}{a+b}\right)k\frac{(a+b)}{2\pi}}\right)\right.$$
$$\left.-\frac{1-(-1)^m}{2}\left(\sum_{n\geq 1}\omega(n)\eta_{n-1}e^{-i\pi\left(n-\frac{a-b}{a+b}\right)k\frac{(a+b)}{2\pi}}\right)\right]$$
$$(3.19)\qquad + \sum_{m\in\mathbb{Z}} c_m(-k)\left[\frac{1+(-1)^m}{2}\left(\sum_{n\geq 1}\omega(-n)e^{+i\pi\left(n-\frac{a-b}{a+b}\right)k\frac{(a+b)}{2\pi}}\right)\right.$$
$$\left.-\frac{1-(-1)^m}{2}\left(\sum_{n\geq 1}\omega(-n)\eta_{n-1}e^{+i\pi\left(n-\frac{a-b}{a+b}\right)k\frac{(a+b)}{2\pi}}\right)\right].$$

Suppose

$$(3.20)\qquad (a-b)/(a+b) \in \mathbb{Q} \qquad (\Longleftrightarrow\ \frac{a}{b} \in \mathbb{Q})$$

and define

$$g_{a,b} := \gcd(a-b, a+b) \geq 1.$$

Lemma 3.3. *If $a, b \in \mathbb{N}\setminus\{0\}$ are such that $a/b \in \mathbb{Q}$ is an irreducible fraction, then $g_{a,b} = 2$ if a and b are both odd, and $g_{a,b} = 1$ if a is odd and b is even, or if a is even and b is odd.*

Proof. Obvious. □

Remark 3.4. The hypothesis (3.20) is fundamental since the functions

$$\sum_{n\geq 1}\left(\omega(\pm n)e^{\mp 2i\pi(n-\frac{a-b}{a+b})x\frac{a+b}{g_{a,b}}}\right)$$

and

$$\sum_{n\geq 1}\left(\omega(\pm n)\eta_{n-1}e^{\mp 2i\pi(n-\frac{a-b}{a+b})x\frac{a+b}{g_{a,b}}}\right)$$

are then periodic (in x) of period 1 and that they can be considered, under some assumptions (see below), as the derivative in the distributional sense of a function of period 1. The existence of these four new functions of period 1 would be impossible if (3.20) would not hold. Then if $(a-b)/(a+b) \notin \mathbb{Q}$ the decomposition into a discrete part and a continuous part of the Fourier transform, which is deduced from them (see below), would be impossible except if $\widehat{\mu}(k)$ is equal to 0. In other terms we can say that the condition $(a-b)/(a+b) \notin \mathbb{Q}$ cancels the diffraction produced by the weighted Dirac comb (3.10), supported by the Thue-Morse quasicrystal $\Lambda_{a,b}$; this one does not occur whatever the weight function $\omega(x)$ is (no concentration of intensity at any wave vector k).

Remark 3.5. If $a = b > 0$ and $\omega(n) = 1$ for $n \geq 0$ and $\omega(n) = 0$ for $n \leq -1$, then the tempered distribution $\widehat{\mu}(k)$ is an infinite sum and is reminiscent of the Fourier transform (1.5) of the ± 1 Prouet-Thue-Morse sequence in the usual way. Let us observe that the sections of this tempered distribution (finite sums obtained by truncating its "tail") provide the Riesz products (1.6) when particular averaging sequences of length a power of 2 are used. We will consider this dependance on the choice of the averaging sequence through the rarefaction phenomenon (Section 4).

Let us assume (3.20), i.e. $a/b \in \mathbb{Q}$. Then we make the following hypothesis:
Hypothesis **(H)**. We suppose that the Fourier series

$$\sum_{n\geq 1}\Big(\frac{\omega(n)\, e^{-2i\pi(n-\frac{a-b}{a+b})x\frac{a+b}{g_{a,b}}}}{\frac{-i\pi(a+b)}{g_{a,b}}\big(n-\frac{a-b}{a+b}\big)}\Big), \quad \text{resp.} \ \sum_{n\geq 1}\Big(\frac{\omega(n)\eta_{n-1}\, e^{-2i\pi(n-\frac{a-b}{a+b})x\frac{a+b}{g_{a,b}}}}{\frac{-i\pi(a+b)}{g_{a,b}}\big(n-\frac{a-b}{a+b}\big)}\Big),$$

resp.

$$\sum_{n\geq 1}\Big(\frac{\omega(-n)\, e^{+2i\pi(n-\frac{a-b}{a+b})x\frac{a+b}{g_{a,b}}}}{\frac{+i\pi(a+b)}{g_{a,b}}\big(n-\frac{a-b}{a+b}\big)}\Big), \quad \text{resp.} \ \sum_{n\geq 1}\Big(\frac{\omega(-n)\eta_{n-1}\, e^{+2i\pi(n-\frac{a-b}{a+b})x\frac{a+b}{g_{a,b}}}}{\frac{+i\pi(a+b)}{g_{a,b}}\big(n-\frac{a-b}{a+b}\big)}\Big),$$

converges in the punctual sense to a periodic piecewise continuous function

$$f_{\omega+}(x), \text{ resp. } f_{\omega\eta+}(x), \ f_{\omega-}(x), \ f_{\omega\eta-}(x),$$

of period 1, which has a derivative

$$f'_{\omega+}(x), \text{ resp. } f'_{\omega\eta+}(x), \ f'_{\omega-}(x), \ f'_{\omega\eta-}(x),$$

continuous bounded on the open set

$$\mathbb{R} - \cup_p\{a_{p,+}\}, \text{ resp. } \mathbb{R} - \cup_p\{a_{p,\eta+}\}, \ \mathbb{R} - \cup_p\{a_{p,-}\}, \ \mathbb{R} - \cup_p\{a_{p,\eta-}\},$$

where

$$a_{p,+}, \text{resp. } a_{p,\eta+}, \ a_{p,-}, \ a_{p,\eta-},$$

are the respective discontinuity points of

$$f_{\omega+}(x), \text{resp. } f_{\omega\eta+}(x), \ f_{\omega-}(x), \ f_{\omega\eta-}(x).$$

Let us insist on the importance of this hypothesis **(H)**: it constrains the weight function in such a way that the interchange of summation in (3.16) and (3.17) is made possible. In any case, the nature of the weight function obeying **(H)** deserves more investigation.

Let

$$\sigma_{p+} = f_{\omega+}(a_{p,+}+0) - f_{\omega+}(a_{p,+}-0),$$

$$\sigma_{p-} = f_{\omega-}(a_{p,-}+0) - f_{\omega-}(a_{p,-}-0),$$

$$\sigma_{p\eta+} = f_{\omega\eta+}(a_{p,\eta+}+0) - f_{\omega\eta+}(a_{p,\eta+}-0),$$

$$\sigma_{p\eta-} = f_{\omega\eta-}(a_{p,\eta-}+0) - f_{\omega\eta-}(a_{p,\eta-}-0),$$

be the respective jumps at $a_{p,+}$, resp. $a_{p,-},\ a_{p,\eta+},\ a_{p,\eta-}$, of

$$f_{\omega+}(x),\ \text{resp. } f_{\omega-}(x),\ f_{\omega\eta+}(x),\ f_{\omega\eta-}(x).$$

Then, by a classical result ([Sz], Chap. II, §2), we have the equality

$$\sum_{n\geq 1}\Big(\omega(n)e^{-2i\pi(n-\frac{a-b}{a+b})\left(\frac{g_{a,b}}{4\pi}k\right)\frac{a+b}{g_{a,b}}}\Big)$$

$$= f'_{\omega+}\Big(\frac{g_{a,b}}{4\pi}k\Big) + \sum_{p\in\mathbb{Z}}\sigma_{p+}\delta\Big(\frac{g_{a,b}}{4\pi}k - a_{p,+}\Big) \tag{3.21}$$

where $f'_{\omega+}$ has to be taken in a distributional sense. We have similar equalities for the three other summations. Then we are led to the following (formal) formula for the Fourier transform of μ:

$$\begin{aligned}\widehat{\mu}(k) = \omega(0) &\\ + \sum_{m\in\mathbb{Z}} c_m(k)&\Bigg[\frac{1+(-1)^m}{2}\Bigg(f'_{\omega+}\Big(\frac{g_{a,b}}{4\pi}k\Big) + \sum_{p\in\mathbb{Z}}\sigma_{p+}\delta\Big(\frac{g_{a,b}}{4\pi}k - a_{p,+}\Big)\Bigg)\\ &-\frac{1-(-1)^m}{2}\Bigg(f'_{\omega\eta+}\Big(\frac{g_{a,b}}{4\pi}k\Big) + \sum_{p\in\mathbb{Z}}\sigma_{p\eta+}\delta\Big(\frac{g_{a,b}}{4\pi}k - a_{p,\eta+}\Big)\Bigg)\Bigg]\\ + \sum_{m\in\mathbb{Z}} c_m(-k)&\Bigg[\frac{1+(-1)^m}{2}\Bigg(f'_{\omega-}\Big(\frac{g_{a,b}}{4\pi}k\Big) + \sum_{p\in\mathbb{Z}}\sigma_{p-}\delta\Big(\frac{g_{a,b}}{4\pi}k - a_{p,-}\Big)\Bigg)\\ &-\frac{1-(-1)^m}{2}\Bigg(f'_{\omega\eta-}\Big(\frac{g_{a,b}}{4\pi}k\Big) + \sum_{p\in\mathbb{Z}}\sigma_{p\eta-}\delta\Big(\frac{g_{a,b}}{4\pi}k - a_{p,\eta-}\Big)\Bigg)\Bigg]\end{aligned} \tag{3.22}$$

within the framework of the distribution theory.

We now consider the convergence of the part of $\widehat{\mu}(k)$ which does not contain the Dirac measures (with the jumps).

Given $T > 0$ and the set of Fourier coefficients $\{c_m(k) \mid m \in \mathbb{Z}\}$ in (3.14), let J_T be the space of 4-tuples $\{g_1(k), g_2(k), g_3(k), g_4(k)\}$, where each $g_i(k)$ is a complex valued function, such that the series

$$\sum_{m\in\mathbb{Z}} \Big(c_m(k)\Big[\frac{1+(-1)^m}{2}\,(g_1(k/T)) - \frac{1-(-1)^m}{2}\,(g_2(k/T))\Big]$$

$$+c_m(-k)\Big[\frac{1+(-1)^m}{2}\,(g_3(k/T)) - \frac{1-(-1)^m}{2}\,(g_4(k/T))\Big]\Big) \tag{3.23}$$

converges (in the punctual sense) to a function $G(k)$ which is slowly increasing and locally integrable in k: there exists $A > 0$ and $\nu > 0$ such that $|G(k)| < A|k|^\nu$ for $k \to +\infty$. As a matter of fact, for any $T > 0$, all 4-tuples of Fourier exponentials $e^{i\lambda k}$ are in J_T for all $\lambda \in \mathbb{R}$, by (3.15).

If we assume that $\{f'_{\omega+}, f'_{\omega-}, f'_{\omega\eta+}, f'_{\omega\eta-}\} \in J_T$ with $T = 4\pi/g_{a,b}$ then, assuming true the hypothesis **(H)**, the first part of $\widehat{\mu}(k)$ which does not contain the Dirac measures defines a regular tempered distribution, say $\widehat{\mu}_r(k)$. From (3.22), we infer that if the Fourier transform of the measure μ can be interpreted as a measure too, then the first term $\widehat{\mu}_r(k)$ may be interpreted as a "continuous part" of that measure.

Summarizing we have proved the following result.

Theorem 3.6. *Assume $a/b \in \mathbb{Q}$ and let $g_{a,b} = \gcd(a-b, a+b)$. Suppose that the hypothesis* **(H)** *is true and that $\{f'_{\omega+}, f'_{\omega-}, f'_{\omega\eta+}, f'_{\omega\eta-}\}$ belongs to J_T with $T = 4\pi/g_{a,b}$. Then*

(i) *the Fourier transfom $\widehat{\mu}$ of μ is the sum of a pure point tempered distribution $\widehat{\mu}_{pp}$ and a regular tempered distribution $\widehat{\mu}_r$ as follows:*

$$\widehat{\mu} \;=\; \widehat{\mu}_{pp} + \widehat{\mu}_r$$

with $\widehat{\mu}_r(k) \;=\; \omega(0)$

$$+\sum_{m\in\mathbb{Z}} \Big(c_m(k)\Big[\frac{1+(-1)^m}{2}\,(f'_{\omega+}(k/T)) - \frac{1-(-1)^m}{2}\,\Big(f'_{\omega\eta+}(k/T)\Big)\Big]$$

$$+c_m(-k)\Big[\frac{1+(-1)^m}{2}\,\Big(f'_{\omega\eta+}(k/T)\Big) - \frac{1-(-1)^m}{2}\,\Big(f'_{\omega\eta-}(k/T)\Big)\Big]\Big), \tag{3.24}$$

and with $\widehat{\mu}_{pp}(k) \;=$

$$\sum_{m,p\in\mathbb{Z}} \Big(c_m(k)\Big[\frac{1+(-1)^m}{2}\sigma_{p+}\delta\Big(\frac{k}{T} - a_{p,+}\Big) - \frac{1-(-1)^m}{2}\sigma_{p-}\delta\Big(\frac{k}{T} - a_{p,-}\Big)\Big]$$

(3.25)

$$+c_m(-k)\Big[\frac{1+(-1)^m}{2}\sigma_{p\eta+}\delta\Big(\frac{k}{T} - a_{p,\eta+}\Big) - \frac{1-(-1)^m}{2}\sigma_{p\eta-}\delta\Big(\frac{k}{T} - a_{p,\eta-}\Big)\Big]\Big),$$

(ii) *the pure point part* $\widehat{\mu}_{pp}$ *is supported by the union*

$$T \times \bigcup_{p\in\mathbb{Z}} \{a_{p,+}, a_{p,-}, a_{p,\eta+}, a_{p,\eta-}\}.$$

3.2. Scaling behaviour of approximant measures. It has becoming traditional for the last two decades to define the Bragg spectrum, i.e. the more or less bright spots which are observed in a diffraction experiment on a long-range order material (say defined by χ, as a weighted sum of Dirac measures localized at the atomic sites), as the pure-point component of the diffraction measure, defined by the Fourier transform of the so-called autocorrelation γ of the measure χ (for the definition of γ in the context of aperiodic point sets, see [H]).

This calls for some assumptions: on the point set of diffractive sites (whether it is a model set, a Meyer set or a Delone set with additional properties, etc), on the uniqueness of the autocorrelation γ (Theorem 3.4 in [H]), on the existence of a limit in the sense of Bohr-Besicovich for the averaged Fourier transform of finite approximants of the measure χ (Theorems 5.1 and 5.3 in [H]). Under these assumptions, the Bragg component of the spectrum can be computed using the so-called Bombieri-Taylor argument.

Let us make precise this context in the case of a weighted Dirac comb $\mu = \sum_{n\in\mathbb{Z}} \omega(n)\delta_{f(n)}$ on $\Lambda_{a,b}$ as (3.10). In the general context of diffraction physics, the diffracted intensity at k, per diffracting site, (with $a/b \in \mathbb{Q}$; see Remark 3.4) is [Cy] [G] [H] [La2]:

$$I_\omega(k) \;=\; \limsup_{l\to+\infty} \left| \frac{1}{\mathrm{card}(U_l)} \sum_{f(n)\in U_l} \omega(n) e^{-ikf(n)} \right|^2 , \tag{3.26}$$

for any averaging sequence $(U_l)_{l\geq 0} \in \mathcal{A}_{a,b}$.

The Bombieri-Taylor argument asserts that when (3.26) takes a nonzero value, for an averaging sequence $(U_l)_{l\geq 0}$, then we have the equality

$$I_\omega(k) \;=\; \widehat{\gamma}_{pp}(\{k\})$$

where $\widehat{\gamma}_{pp}(\{k\})$ is the value at $\{k\}$ of the pure-point part of the Fourier transform $\widehat{\gamma}$ of the autocorrelation γ.

The Bombieri-Taylor argument has appeared in [BT1] and [BT2] without proof. Since then it has become a conjecture (see Hof [H]) since its proof seems fairly difficult to establish in general for Delone sets or Meyer sets [H]. It is only proved for model sets (Hof [H]), for some Meyer sets (Strungaru [Su]), and in particular contexts (de Oliveira [Oa], Lenz [Lz], ...). We do not intend to prove it but to reformulate it in terms of the scaling properties of a family of approximant measures that we define below in a way which seems consistent with approaches followed by many authors (Cheng, Savit and Merlin [CSM], Aubry, Godrèche and Luck [AGL] [Lu], Kolar, Iochum and Raymond [KIR]). These measures will be "approximant" to the Fourier transform of the autocorrelation γ but we need not to have a precise definition of γ, or to know whether γ is unique and well-defined, etc. This is a simplification of the problem. Furthermore, this reformulation is more appropriate for expressing the other argument, maned by us (AGL) Aubry-Godrèche-Luck argument. The AGL argument is known to allow the computation of the singular continous component of the spectrum.

The following definition is natural, see (1.7).

Definition 3.7. Given $\mu = \sum_{n\in\mathbb{Z}} \omega(n)\delta_{f(n)}$ on $\Lambda_{a,b}$ and an averaging sequence $\mathcal{U} = (U_l)_{l\geq 0} \in \mathcal{A}_{a,b}^{supp(\omega)}$ of finite approximants of $\Lambda_{a,b}^{supp(\omega)}$, we define the l-th approximant measure $\nu_{\mathcal{U},l}(k)dk$ as

$$\nu_{\mathcal{U},l}(k)dk \;:=\; \frac{1}{\mathrm{card}(U_l)} \left| \sum_{f(n)\in U_l} \omega(n)e^{-ikf(n)} \right|^2 dk \tag{3.27}$$

Definition 3.8. Let $k \in \mathbb{R}$. Then

(i) **(BT) Bombieri-Taylor argument**: k belongs to the set of Bragg peaks of the spectrum of $\Lambda_{a,b}$ if and only if

$$0 < \liminf_{l\to+\infty} \frac{\nu_{\mathcal{U},l}(k)dk}{\mathrm{card}(U_l)dk} \leq \limsup_{l\to+\infty} \frac{\nu_{\mathcal{U},l}(k)dk}{\mathrm{card}(U_l)dk} < +\infty. \tag{3.28}$$

for at least one sequence of finite approximants $\mathcal{U} = (U_l)_{l\geq 0} \in \mathcal{A}_{a,b}^{supp(\omega)}$,

(ii) **(AGL) Aubry-Godrèche-Luck argument**: k belongs to the singular continuous part of the spectrum of $\Lambda_{a,b}$ if and only if there exists $\alpha \in (-1,1)$, which depends only upon k (not upon $\mathcal{U}$), such that

$$0 < \liminf_{l\to+\infty} \frac{\nu_{\mathcal{U},l}(k)dk}{\mathrm{card}(U_l)^\alpha dk} \leq \limsup_{l\to+\infty} \frac{\nu_{\mathcal{U},l}(k)dk}{\mathrm{card}(U_l)^\alpha dk} < +\infty \tag{3.29}$$

for all $\mathcal{U} \;=\; (U_l)_{l\geq 0} \in \mathcal{A}_{a,b}^{supp(\omega)}$ such that

$$0 \neq \liminf_{l\to+\infty} \frac{\nu_{\mathcal{U},l}(k)dk}{\mathrm{card}(U_l)^\alpha dk},$$

or such that

$$0 < \limsup_{l\to+\infty} \frac{\nu_{\mathcal{U},l}(k)dk}{\mathrm{card}(U_l)^\alpha dk} < +\infty \tag{3.30}$$

for all $\mathcal{U} \;=\; (U_l)_{l\geq 0} \in \mathcal{A}_{a,b}^{supp(\omega)}$ such that

$$0 \;=\; \liminf_{l\to+\infty} \frac{\nu_{\mathcal{U},l}(k)dk}{\mathrm{card}(U_l)^\alpha dk}.$$

The fact that the exponent α is < 1 in the (AGL)-argument, compared to the Bombieri-Taylor Conjecture (BT), means that the concentration of intensity in the diffraction process is "less infinite" along the singular continuous component than in the case of the Bragg peaks.

3.3. The Bragg component. In the rest of the paper, we assume $a/b \in \mathbb{Q}$ so that the diffraction phenomenon from any weighted Dirac comb on $\Lambda_{a,b}$ can occur, by Remark 3.4. Then without restriction of generality we can assume $a, b \in \mathbb{Q}$ (with still $0 < b < a$).

Let us start with the Thue-Morse quasicrystal defined by

$$\omega(n) = 0 \quad \text{if } n \leq 0 \text{ and } \omega(n) = 1 \quad \text{if } n \geq 1,$$

up till Section 5, and postpone the study of other weighted Dirac combs on $\Lambda_{a,b}$ to Section 6. Hence $\mathrm{supp}(\omega) = \mathbb{N} \setminus \{0\}$. Let

$$U_l := [0, f(l)] \cap \left(\Lambda_{a,b} \setminus \{0\}\right) \qquad \text{for } l \geq 1, \tag{3.31}$$

so that $(U_l)_{l\geq 1} \in \mathcal{A}_{a,b}^{\mathrm{supp}(\omega)}$ and $\mathrm{card}(U_l) = l$. We call this sequence the canonical sequence, any other sequence of finite approximants in $\mathcal{A}_{a,b}^{\mathrm{supp}(\omega)}$ being a subsequence of it.

Let $k \in \mathbb{R}$ and

$$\kappa(k) := \sum_{m\in 2\mathbb{Z}} c_m(k) \qquad \text{and} \quad \kappa_\eta(k) := \sum_{m\in 2\mathbb{Z}+1} c_m(k).$$

By (3.19), we have

(3.32)
$$\widehat{\mu}(k) = \kappa(k)\big(\sum_{n\geq 1} e^{-\frac{i}{2}((a+b)n-(a-b))k}\big) - \kappa_\eta(k)\big(\sum_{n\geq 1} \eta_{n-1} e^{-\frac{i}{2}((a+b)n-(a-b))k}\big).$$

Theorem 3.9. *The Bragg component, denoted $\mathcal{B}_{a,b}^{(\mu)}$, of the spectrum of the Thue-Morse quasicrystal given by $\mu = \sum_{n\geq 1} \delta_{f(n)}$ on $\Lambda_{a,b}$ is exactly the periodized normalized group of 2-adic rational numbers*

(3.33)
$$\frac{4\pi}{a+b}\mathbb{Z} + \frac{2}{a+b}\mathbb{D}_2 \;=\; \left\{ \frac{4\pi}{a+b}\big(r + \frac{m}{2^h}\big) \mid r \in \mathbb{Z} \;\; \text{and} \;\; \begin{matrix} 0 \leq m \leq 2^h, \\ h = 0, 1, 2, \ldots \end{matrix} \right\}.$$

Proof. First we write $\widehat{\mu}(k)$ in (3.32) as

$$\widehat{\mu}(k) = (\kappa(k) + \kappa_\eta(k))\big(\sum_{n\geq 1} e^{-\frac{i}{2}((a+b)n-(a-b))k}\big)$$
$$- 2\kappa_\eta(k)\big(\sum_{n\geq 1} \frac{1+\eta_{n-1}}{2} e^{-\frac{i}{2}((a+b)n-(a-b))k}\big)$$

and we take the l-th approximant measures for $l = 2^h$ with $h = 0, 1, 2, \ldots$ for $k = \frac{4\pi}{a+b} q$. We have

$$\nu_{\mathcal{U},l}(k)dk = \frac{1}{2^h} \Bigg| \Big(\kappa(\frac{4\pi}{a+b}q) + \kappa_\eta(\frac{4\pi}{a+b}q)\Big)\Big(\sum_{n=1}^{2^h} e^{-2i\pi(n-\frac{a-b}{a+b})q}\Big)$$

$$-2\kappa_\eta(\frac{4\pi}{a+b}q)\Big(\sum_{n=1}^{2^h} \frac{1+\eta_{n-1}}{2} e^{-2i\pi(n-\frac{a-b}{a+b})q}\Big)\Bigg|^2 \frac{4\pi}{a+b}dq. \tag{3.34}$$

We recognize the first summation as the Fourier transform of a lattice and the second summation as the Fourier transform of the Thue-Morse sequence on the alphabet $\{0, 1\}$ studied by Queffélec [Q2] §6.3.2.1 for which the spectrum is composed of a Bragg component and a singular continuous component (see (1.5) to (1.8) in the Introduction). We now use the (BT)-argument (3.28) to these approximant measures, allowing h to go to $+\infty$, to claim that the Bragg peaks arise from either the first summation or the second summation. The corresponding intensities can be computed from $\kappa(\frac{4\pi}{a+b}q)$ and $\kappa_\eta(\frac{4\pi}{a+b}q)$, using (3.34), passing to the limit $h \to +\infty$, and may take a zero value (possible extinctions). $\square$

3.4. Rarefied sums of the Thue-Morse sequence and singular continuous component of the spectrum. From the expression (3.34) of the approximant measure with $l = 2^h$, from [Q2] §6.3.2.1 and the (AGL)-argument (3.29) (3.30), we infer that, apart from the Bragg component of the spectrum of the Thue-Morse quasicrystal given by $\mu = \sum_{n\geq 1} \delta_{f(n)}$ on $\Lambda_{a,b}$, the continuous component exists (i.e. is not trivial) and is only singular.

Assume that $k = (4\pi q)/(a + b)$ does not belong to the Bragg component (3.33) and that $\mathcal{U}$ is the canonical sequence of finite approximants (3.31) in the following. Then we see, when k runs over the singular continuous part of the spectrum, using the (AGL)-argument, that the scaling behaviour of (3.34) with a power of $l = 2^h$ is dictated and dominated by the scaling behaviour of the second summation $\sum_{n=1}^{2^h} \frac{1+\eta_{n-1}}{2} e^{-2i\pi(n-\frac{a-b}{a+b})q}$ with a power of 2^h.

More generally, when k lies in the singular continuous part of the spectrum, one easily asserts that the scaling exponent α introduced in (3.29) (3.30), for the scaling behaviour of $\nu_{\mathcal{U},l}(k)dk$, takes only into account the scaling behaviour of the dominant term

$$\sum_{n=1}^{l} \eta_{n-1} e^{-2i\pi(n-\frac{a-b}{a+b})q},$$

with a (dominant) power of l, if it exists, l going to infinity, and if β denotes this exponent, one has

$$\alpha \;=\; 2\beta - 1.$$

It amounts to understand the scaling behaviour of the sums

$$\sum_{n=1}^{l} \eta_{n-1} e^{-2i\pi nq} \tag{3.35}$$

with a power of l, when l goes to infinity. We now make precise these notions.

Definition 3.10. Let $p \in \mathbb{N} \setminus \{0,1,2\}$. The subsequences $\eta_i,\ \eta_{i+p},\ \eta_{i+2p},\ \eta_{i+3p},\ldots$ are referred to as p-rarefied Thue-Morse sequences for $i = 0, 1, \ldots, p-1$ (p is not necessarily a prime number). Their partial sums, called p-rarefied sums of the Thue-Morse sequence, are denoted

$$S_{p,i}(n) \;=\; \sum_{\substack{0 \le j < n \\ j \equiv i \pmod p}} \eta_j \qquad n = 1,2,\ldots \tag{3.36}$$

The following Lemma is obvious.

Lemma 3.11. *Let $q \in \mathbb{R}$ and denote $W_q := \{e^{-2i\pi nq} \mid n \in \mathbb{Z}\}$ the corresponding countable subset of the unit circle $|z| = 1$. Then,*

$$\#\{n \in \mathbb{Z} \mid e^{-2i\pi nq} = v\} \;=\; \begin{cases} 0 & \textit{if and only if} \quad v \notin W_q, \\ 1 & \textit{if and only if} \quad v \in W_q \ \textit{and}\ q \notin \mathbb{Q}, \\ \infty & \textit{if and only if} \quad v \in W_q \ \textit{and}\ q \in \mathbb{Q}. \end{cases}$$

If $q \in \mathbb{Q}, v \in W_q$, and if we write $q = \frac{t}{p}$, with $t \in \mathbb{Z}, p \in \mathbb{N} \setminus \{0\}$, and $gcd(t,p) = 1$, then there exists an integer $r \in \{0,1,2,\ldots,p-1\}$ such that

$$\{n \in \mathbb{Z} \mid e^{-2i\pi nq} = v\} = r + p\mathbb{Z}.$$

First, let us define the following function $\alpha_l(k)$ of $k \in \mathbb{R}$ and $l \geq 1$, with $\mathrm{card}(U_l) = l$, as follows

$$\mathrm{card}(U_l)^{\alpha_l(k)} := \frac{1}{\mathrm{card}(U_l)} \left| \sum_{j=0}^{l-1} \eta_j e^{-2i\pi jk} \right|^2 . \tag{3.37}$$

Proposition 3.12. *For almost all $k \in \mathbb{R}$*

$$\lim_{n \to +\infty} \alpha_{2^n}(k) \;=\; -1. \tag{3.38}$$

Proof. This is fairly classical. We have

$$\left| \sum_{j=0}^{2^n-1} \eta_j e^{-2i\pi jk} \right|^2 \;=\; 2^{2n} \prod_{j=0}^{n-1} \sin^2(\pi 2^j k) \;=\; 2^{2n} \prod_{j=0}^{n-1} \sin^2(\pi \varphi_j)$$

where $\varphi_j = 2^j \varphi_0, j \geq 0$, with $\varphi_0 = k$. We deduce

$$\alpha_{2^n}(k) = 1 + \frac{2}{n \log 2} \sum_{j=0}^{n-1} \log\left|\sin(\pi\varphi_j)\right|.$$

By a theorem of Raikov [R] [K], for almost every initial value $\varphi_0 = k$, we can change the summation into an integral and obtain the claim

$$\begin{aligned}\lim_{n\to+\infty} \alpha_{2^n}(k) &= 1 + \lim_{n\to+\infty} \frac{2}{n \log 2} \sum_{j=0}^{n-1} \log\left|\sin(\pi\varphi_n)\right| \\ &= 1 + \frac{2}{\log 2} \int_0^1 \log\left|\sin(\pi x)\right| dx = -1.\end{aligned}$$

□

Corollary 3.13. *The singular continuous component of the spectrum of the Thue-Morse quasicrystal given by $\mu = \sum_{n\geq 1} \delta_{f(n)}$ on $\Lambda_{a,b}$ is a subset of measure zero of $\mathbb{R}$.*

Proof. Assume that k belongs to the singular component. Then, by the (AGL)-argument, there exists $\alpha \in (-1, 1)$ such that

$$0 \leq \liminf_{l\to+\infty} \frac{\nu_{\mathcal{U},l}(k)dk}{\mathrm{card}(U_l)^\alpha dk} \leq \limsup_{l\to+\infty} \frac{\nu_{\mathcal{U},l}(k)dk}{\mathrm{card}(U_l)^\alpha dk} < +\infty \tag{3.39}$$

for all $\mathcal{U} \ = \ (U_l)_{l\geq 0} \in \mathcal{A}_{a,b}^{supp(\omega)}$ such that

$$0 \neq \limsup_{l\to+\infty} \frac{\nu_{\mathcal{U},l}(k)dk}{\mathrm{card}(U_l)^\alpha dk}.$$

Here, with the particular subsequence of finite approximants (U_{2^n}), by (3.34) and Proposition 3.12, we have

$$\limsup_{n\to+\infty} \frac{\nu_{\mathcal{U},2^n}(k)dk}{\mathrm{card}(U_{2^n})^\alpha dk} \ = \ \left|\kappa_\eta(k)\right|^2 \times \limsup_{n\to+\infty} \frac{\mathrm{card}(U_{2^n})^{\alpha_{2^n}(k)}}{\mathrm{card}(U_{2^n})^\alpha dk} dk \ = \ 0$$

for almost all k, since $\lim_{n\to\infty}(-\alpha + \alpha_{2^n}(k)) = -\alpha - 1 < 0$. By the block structure of the Prouhet-Thue-Morse sequence we deduce that

$$\limsup_{l\to+\infty} \frac{\nu_{\mathcal{U},l}(k)dk}{\mathrm{card}(U_l)^\alpha dk} \ = \ 0$$

holds for all $\mathcal{U} \ = \ (U_l)_{l\geq 0} \in \mathcal{A}_{a,b}^{supp(\omega)}$ in a similar way. This is impossible, except possibly for the ks which lie in a set of measure zero for which Proposition 3.12 does not hold. □

Using the (AGL)-argument in a careful study of $\frac{4\pi}{a+b}\mathbb{Q}$, and invoking Corollary 3.13, we will deduce in Section 5 that this singular continuous component is mostly

$$\frac{4\pi}{a+b}\mathbb{Q},$$

perhaps up to a subset of $\mathbb{R} \setminus \frac{4\pi}{a+b}\mathbb{Q}$ of measure zero.

Now let $q = \frac{t}{p} \in \mathbb{Q}$, with $t \in \mathbb{Z}, p \in \mathbb{N} \setminus \{0\}$, and $\gcd(t,p) = 1$. Assume that

$$\frac{4\pi}{a+b}q \notin \mathcal{B}_{a,b}^{(\mu)}.$$

This implies that $p \neq 1$ and that p is not a power of 2. For these values of p, by Lemma 3.11, we have, for any integer $N \geq 1$,

$$\frac{1}{Np+1}\left|\sum_{n=1}^{Np+1} \eta_{n-1} e^{-2i\pi n \frac{t}{p}}\right|^2 = \frac{1}{Np+1}\left|\sum_{j=0}^{p-1} S_{p,j}(Np) e^{-2i\pi \frac{jt}{p}}\right|^2. \tag{3.40}$$

The equality (3.40) is the key relation for reducing the problem to the asymptotic behaviour of the p-rarefied sums $S_{p,j}(Np)$ when N goes to infinity. We will continue in Section 5 the characterization of the singular component of the spectrum, but before we need to recall some basic facts about the p-rarefied sums of the Thue-Morse sequence (in Section 4) when p is a prime number, and develop the non-prime case as in Grabner [Gr1] [Gr2].

4. The rarefaction phenomenon

We will follow Goldstein, Kelly and Speer [GKS] (which originates in Dumont [D]) for the fractal structure of the p-rarefied sums of the Thue-Morse sequence. In their paper p is always an odd prime number whereas we need to consider the case in which p is an integer ≥ 3 which is not necessarily prime (by (3.40)), and which is not a power of 2.

In 1969 Newman [N] proved a remarkable conjecture of Moser which asserts that

$$S_{3,0}(n) > 0 \qquad \text{for any } n \geq 1,$$

namely that the 3-rarefied sequence of the Thue-Morse sequence contains a dominant proportion of ones. More precisely Newman proved, for all $n \geq 1$,

$$\frac{3^{-\beta}}{20} < \frac{S_{3,0}(n)}{n^\beta} < 5\,.\,3^{-\beta} \qquad \text{with } \beta = \frac{\log 3}{\log 4}, \tag{4.41}$$

and that the limit $\lim_{n\to+\infty} S_{3,0}(n)/n^\beta$ does not exist.

Then Coquet [Ct] improved (4.41) by the more precise statement

$$S_{3,0}(n) = n^{\frac{\log 3}{\log 4}}\,.\,\psi_{3,0}\left(\frac{\log n}{\log 4}\right) + \frac{\epsilon_{3,0}(n)}{3} \tag{4.42}$$

where $\epsilon_{3,0}(n) \in \{0, \pm 1\}$ and where $\psi_{3,0}(x)$ is a continuous nowhere differentiable function of period 1 which assumes all its values in the closed interval

$$[\inf_{x\in[0,1]} \psi_{3,0}(x), \sup_{x\in[0,1]} \psi_{3,0}(x)] = \left[\liminf \frac{S_{3,0}(n)}{n^{\frac{\log 3}{\log 4}}}, \limsup \frac{S_{3,0}(n)}{n^{\frac{\log 3}{\log 4}}}\right].$$

The latter is explicitly given by

$$\left[\left(\frac{1}{3}\right)^{\frac{\log 3}{\log 4}} \frac{2\sqrt{3}}{3}, \frac{55}{3}\left(\frac{1}{65}\right)^{\frac{\log 3}{\log 4}}\right].$$

Grabner [Gr1] obtained in 1993 the Newman-type strict inequality for $p = 5$

$$S_{5,0}(n) > 0 \qquad \text{for any } n \geq 1,$$

and a Coquet-type fractal description of $S_{5,0}(n)$ as

$$S_{5,0}(n) = n^{\frac{\log 5}{\log 16}} \Phi_{5,0}\left(\frac{\log n}{\log 16}\right) + \frac{\epsilon_{5,0}(n)}{5} \qquad \text{for all } n \geq 1$$

where $\Phi_{5,0}(x)$ is also a continuous nowhere differentiable function of period 1. We refer to [Gr1] for the details. Similar results were found by Grabner, Herendi and Tichy for $p = 17$ [GHT].

In 1992 Goldstein, Kelly and Speer [GKS] proposed a general matrix approach to deal simultaneously with all the sums $S_{p,j}(n)$, $j \in \{0, 1, 2, \ldots, p-1\}$. They proved that the Coquet-type expressions are very general, at least when p is an odd prime number. They obtained, for all odd prime numbers p and all $n \in \mathbb{N}$, the existence of

- p continuous nowhere differentiable functions $\psi_{p,j}(x)$, $j = 0, 1, 2, \ldots, p-1$, of period 1,
- an exponent $\beta = \frac{\log \lambda_1}{s \log 2}$ which remarkably depends only upon p and not on j,
- p error terms $E_{p,j}(n)$ which are bounded above uniformly, for some constant $C > 0$, for all $n \in \mathbb{N}$ and all $j \in \{0, 1, \ldots, p-1\}$, in the sense

$$|E_{p,j}(n)| \leq C n^{\beta_1} \qquad \text{with } \beta_1 = \begin{cases} \frac{\log \lambda_2}{s \log 2} & \text{for } \lambda_2 > 1 \\ 0 & \text{for } \lambda_2 < 1 \end{cases}$$

such that

$$S_{p,j}(n) = n^{\beta} \psi_{p,j}\left(\frac{\log n}{r\, s\, \log 2}\right) + E_{p,j}(n), \tag{4.43}$$

where

- $r \in \{1, 2, 4\}$ is an integer which remarkably depends upon p and not on j,

- s is the order of 2 in the group $(\mathbb{Z}/p\mathbb{Z})^*$ of the invertible elements of the field $\mathbb{Z}/p\mathbb{Z}$ (therefore depends upon p and not of j),
- the real numbers
$$\lambda_1 > \lambda_2 > \ldots \geq 0$$
are the moduli of the eigenvalues, in decreasing order, of the $p \times p$ matrix
$$M = \left(S_{p,i-j}(2^s)\right)_{0 \leq i,j \leq p-1}$$
with integral coefficients.

Proposition 4.1. *Let $\mathbf{S}(n) := (S_{p,0}(n), S_{p,1}(n), \ldots, S_{p,p-1}(n))^t$ denote the vector of $\mathbb{R}^p$ with integer entries (^{t} for transposition). Then*

$$\mathbf{S}(2^s n) \;=\; M\,\mathbf{S}(n). \tag{4.44}$$

Proof. [GKS] p. 3. □

These authors obtain the fractal functions $\psi_{p,j}(x)$ by vectorial interpolation of (4.44) by constructing a fractal function $\mathbf{F}(x) := (F_0(x), F_1(x), \ldots, F_{p-1}(x))^t$ which obeys the self-similar property

$$\mathbf{F}(2^s x) \;=\; M\,\mathbf{F}(x). \tag{4.45}$$

on $[0, +\infty)$ and is such that $\mathbf{F} = \mathbf{S}$ on $\mathbb{N}$.

Proposition 4.2. *Let $\langle 2\rangle$ denote the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ generated by 2 and let $a\langle 2\rangle$ any of its cosets. Then the eigenvalues of M are*

$$\xi_a = (-2i)^s \prod_{j \in a\langle 2\rangle} \sin(2\pi j/p) \tag{4.46}$$

and satisfy

$$\prod_a \xi_a = p \tag{4.47}$$

Proof. Proposition 3.3 in [GKS]. □

As a corollary of (4.47)

$$\lambda_1 > 1 \qquad\qquad \lambda_1 \leq 2^s \tag{4.48}$$

always hold, the second largest λ_2 among the magnitudes of eigenvalues of M can be < 1 or > 1, and the first and second magnitudes of eigenvalues λ_1 and λ_2 can be explicitly computed for some classes of prime numbers p. These authors proved that the asymptotic growth of the summation $S_{p,j}(n)$ is of the order of n^β in the sense that

$$-\infty \;<\; \liminf_{n\to\infty} \frac{S_{p,j}(n)}{n^\beta} \;<\; \limsup_{n\to\infty} \frac{S_{p,j}(n)}{n^\beta} \;<\; +\infty \tag{4.49}$$

holds, for all odd prime number p and all $j = 0, 1, \ldots, p-1$. Let us observe that the p error terms $E_{p,j}(n)$ become negligible at large n in (4.43) since

$\beta_1 < \beta$, and therefore have no influence on the computation of the lower and upper bounds in (4.49).

Several authors have investigated the bounds liminf and limsup in (4.49) and how the region between these bounds was attained by the continous fractal functions $\psi_{p,j}(x)$: Coquet [Ct], Dumont [D], Drmota and Skalba [DS1] [DS2], Grabner [Gr1], Grabner, Herendi and Tichy [GHT]. It appears that it is a non-trivial problem to decide whether the continuous function $\psi_{p,j}(x)$ has a zero or not. The only known examples where $\psi_{p,0}(x)$ has no zero are $p = 3^k 5^l$ ([Gr1]) and $p = 17$ ([GHT]). Dumont [D] has shown that $\psi_{3,0}(x)$ and $\psi_{3,1}(x)$ have no zero, but that $\psi_{3,2}(x)$ has a zero. By the same method he proved that for all prime numbers $p \equiv 3 \pmod 4$ and $j \in \{0, 1, \ldots, p-1\}$ such that the order of 2 in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$ is $(p-1)/2$, then

$$\psi_{p,j}(x) \text{ has a zero.}$$

It was observed in [DS2] that the assertion that $\psi_{p,j}(x)$ has no zero is more or less equivalent to the Newman-type inequality

$$S_{p,j}(n) > 0 \quad \text{for almost all } n \text{ (or } S_{p,j}(n) < 0 \quad \text{for almost all } n)$$

where "almost all" means "all but finitely many". In [DS2] Drmota and Skalba prove (here p denotes an arbitrary integer ≥ 3 and $N \geq 1$ an integer)

$$p \text{ divisible by 3 or } p = 4^N + 1 \implies S_{p,j}(n) > 0 \quad \text{for almost all } n.$$

They show that the only prime numbers $p \leq 1000$ which satisfy $S_{p,j}(n) > 0$ for almost all n are $p = 3, 5, 17, 43, 257, 683$. They also give an asymptotic estimate for large prime numbers p for which this property holds.

We now mention some results on the asymptotics of the sums $S_{p,i}(n)$ when $p \geq 3$ is not a prime number.

Proposition 4.3. *Let $1 \leq r_1, r_2$ be integers. The asymptotic behaviour of the sum $S_{p,0}(n)$ with $p = 3^{r_1} 5^{r_2}$ is given by*

(4.50)
$$S_{p,0}(p\,N) \;=\; \frac{1}{5^{r_2}} S_{3^{r_1},0}(p\,N) + \frac{1}{3^{r_1}} S_{5^{r_2},0}(p\,N) \;+\; O(\log N)\,, \qquad N \geq 1,$$

where

$$S_{3^{r_1},0}(p\,N) = \frac{1}{3^{r_1}-1}\,(p\,N)^{\alpha}\,F_0\big(\log_4(p\,N/3)\big) + \left(\frac{p\,N}{3^{r_1}}\right)^{\alpha/3} F_1\Big(\frac{1}{3}\log_4\Big(\frac{p\,N}{3^{r_1}}\Big)\Big)$$

$$+ \ldots + \left(\frac{p\,N}{3^{r_1}}\right)^{\alpha/(3^{r_1-1})} F_{r_1-1}\Big(\frac{1}{3^{r_1-1}}\log_4\Big(\frac{p\,N}{3^{r_1}}\Big)\Big) + \frac{\epsilon_{3^{r_1}}\,(p\,N/3^{r_1})}{3^{r_1}}, \tag{4.51}$$

$$\begin{aligned} S_{5^{r_2},0}(p\,N) = & \frac{1}{5^{r_2}-1}\,(p\,N)^{\beta}\,G_0(\log_{16}(p\,N/5)) \\ & + \left(\frac{p\,N}{5^{r_2}}\right)^{\beta/5} G_1\left(\frac{1}{5}\log_{16}\left(\frac{p\,N}{5^{r_2}}\right)\right) + \dots \\ & + \left(\frac{p\,N}{5^{r_2}}\right)^{\beta/(5^{r_2-1})} G_{r_2-1}\left(\frac{1}{5^{r_2-1}}\log_{16}\left(\frac{p\,N}{5^{r_2}}\right)\right) + \frac{\epsilon_{5^{r_2}}\,(p\,N/5^{r_2})}{5^{r_2}}, \end{aligned} \tag{4.52}$$

where $F_0, F_1, \dots, F_{r_1-1}, G_0, G_1, \dots, G_{r_2-1}$ are continuous nowhere differentiable functions of period 1*, with $\epsilon_{3^{r_1}}\,(p\,N/3^{r_1})\,, \epsilon_{5^{r_2}}\,(p\,N/5^{r_2}) \in \{0, \pm 1\}$, and*

$$\alpha = \frac{\log 3}{2\log 2}, \qquad \beta = \frac{\log 5}{4\log 2}.$$

Proof. [Gr1] pp. 40–41. □

Remark 4.4. Since $\alpha = \frac{\log 3}{2\log 2} > \beta = \frac{\log 5}{4\log 2}$ and that $\frac{\log 3}{2\log 2}$ is the scaling exponent relative to $S_{3,0}(3\,N)$, Proposition 4.3 shows that the scaling behaviour of $S_{p,0}(p\,N)$ with $p = 3^{r_1}5^{r_2}$ is dominated by that of $S_{3,0}(3\,N)$, which is the rarefied sum of the Thue-Morse sequence relative to the prime number 3; in this respect this prime number 3 can be qualified as "dominant" in the prime factor decomposition of $p = 3^{r_1}5^{r_2}$. Hence Proposition 4.3 tends to show in general that the computation of the dominant scaling exponent of $S_{p,i}(p\,N)$, with $p = p_1^{i_1}p_2^{i_2}\dots p_m^{i_m}$, its prime factor decomposition (with $p_1 \neq 2$), $0 \le i \le p-1$, amounts to computing the scaling exponent relative to only one sum $S_{p_j,i}(n)$ where p_j is the "dominant" prime number in the decomposition $p = p_1^{i_1}p_2^{i_2}\dots p_m^{i_m}$.

5. The singular continous component

5.1. Main Theorem.

Theorem 5.1. *The singular continous component of the spectrum of the Thue-Morse quasicrystal given by $\mu = \sum_{n\ge 1}\delta_{f(n)}$ on $\Lambda_{a,b}$, is the set*

$$\mathbb{S} := \Big\{\frac{4\pi}{a+b}\frac{t}{2^h\,p} \mid t \in \mathbb{Z},\ p \textit{ odd integer } \ge 3,\ h \ge 0,\ gcd(t, 2^h p) = 1, \\ \kappa_\eta\Big(\frac{4\pi}{a+b}\frac{t}{2^h\,p}\Big) \neq 0\Big\}$$

up to a subset of measure zero of $\mathbb{R} \setminus \frac{4\pi}{a+b}\mathbb{Q}$.

Proof. We will only consider that $k \in \frac{4\pi}{a+b}\mathbb{Q}$ and leave apart a possible subset of measure zero in $\mathbb{R} \setminus \frac{4\pi}{a+b}\mathbb{Q}$ by Corollary 3.13. There are two cases: either (i) $k = \frac{4\pi}{a+b}\frac{t}{p}$ with $t \in \mathbb{Z}$, $p \ge 3$ an odd integer, $\gcd(t,p) = 1$, or (ii) $k = \frac{4\pi}{a+b}\frac{t}{2^h p}$ with $t \in \mathbb{Z}$, $p \ge 3$ an odd integer, $h \ge 1$, $\gcd(t, 2^h p) = 1$.

(i) Let us consider the first case. Let us assume first that p is an odd prime number.

Definition 5.2. Let $\xi_k := e^{-2i\pi\frac{t}{p}}$. The compact subset of the complex plane

$$\mathrm{R}(k) := \Big\{ \sum_{j=0}^{p-1} y_j \, \xi_k^j \Big\} \tag{5.53}$$

where y_j runs over the closed interval $\Big[\inf_{x\in[0,1]} \psi_{p,j}(x), \sup_{x\in[0,1]} \psi_{p,j}(x)\Big]$, for $j = 0, 1, \ldots, p-1$, is called the *rarefaction domain* at (the wave vector) k.

The rarefaction domain $\mathrm{R}(k)$ has a facetted boundary and may be a polygon. It may or may not contain the origin, and is convex. It belongs to the linear span of the powers of the root of unity ξ_k and inherits the properties of the fractal functions $\psi_{p,j}(x)$, $j = 0, 1, \ldots, p-1$, which are bounded. It is not reduced to a single point since, for all $j = 0, 1, \ldots, p-1$, the two bounds $\inf_{x\in[0,1]} \psi_{p,j}(x)$ and $\sup_{x\in[0,1]} \psi_{p,j}(x)$ are distinct by construction.

We now show that the (AGL)-argument can be invoked, with suitable values of $\alpha = \alpha(k)$, for all such elements k. For $\mathcal{U}$ the canonical sequence (3.31), recall the expression of the l-th approximant measure, with $l \geq 1$,

$$\begin{aligned} \nu_{\mathcal{U},l}(k)dk = \frac{1}{l} \Bigg| \Big(\kappa\big(\frac{4\pi}{a+b}\frac{t}{p}\big) + \kappa_\eta\big(\frac{4\pi}{a+b}\frac{t}{p}\big)\Big)\Big(\sum_{n=1}^{l} e^{-2i\pi(n-\frac{a-b}{a+b})\frac{t}{p}}\Big) \\ -2\kappa_\eta\big(\frac{4\pi}{a+b}\frac{t}{p}\big)\Big(\sum_{n=1}^{l} \frac{1+\eta_{n-1}}{2} e^{-2i\pi(n-\frac{a-b}{a+b})\frac{t}{p}}\Big)\Bigg|^2 \, dk. \end{aligned} \tag{5.54}$$

The rarefaction domain $\mathrm{R}(k)$ is related to the l-th approximant measure $\nu_{\mathcal{U},l}(k)dk$, for $l = Np+1$ and $N \geq 1$ a positive integer, as follows: by (3.40), (4.43) and (5.53) we have

$$\begin{aligned} \frac{1}{Np+1}\Bigg|\sum_{n=1}^{Np+1} \eta_{n-1} e^{-2i\pi n\frac{t}{p}}\Bigg|^2 &= \frac{1}{Np+1}\Bigg|\sum_{j=0}^{p-1} S_{p,j}(Np) e^{-2i\pi\frac{jt}{p}}\Bigg|^2. \\ &= \frac{1}{Np+1}(Np)^{2\beta}\Big|z + \mathcal{O}\Big((Np)^{\beta_1-\beta}\Big)\Big|^2 \end{aligned} \tag{5.55}$$

where $z \in \mathrm{R}(k)$, $s \geq 1$ is the smallest integer such that $2^s \equiv 1 \,(\mathrm{mod}\ p)$, where λ_1 and λ_2 ($< \lambda_1$) are the first and second magnitudes of eigenvalues of the matrix M (see Section 4), where $\beta = \frac{\log\lambda_1}{s\log 2}$ and where $\beta_1 = \frac{\log\lambda_2}{s\log 2}$ if $\lambda_2 > 1$, and $\beta_1 = 0$ if $\lambda_2 < 1$. These quantities are relative to the p-rarefied sums of the Thue-Morse sequence. Since k does not belong to the

Bragg component $\mathcal{B}_{a,b}^{(\mu)}$ the second summation in (5.54) is dominant in the asymptotic behaviour of $\nu_{\mathcal{U},l}(k)dk$. Therefore

$$0 \le \liminf_{N\to+\infty} \frac{\nu_{\mathcal{U},Np+1}(k)dk}{\mathrm{card}(U_{Np+1})^{\alpha}dk} < \limsup_{N\to+\infty} \frac{\nu_{\mathcal{U},Np+1}(k)dk}{\mathrm{card}(U_{Np+1})^{\alpha}dk} < +\infty \tag{5.56}$$

with

$$\alpha \;=\; 2\beta - 1 \;=\; 2\,\frac{\log\lambda_1}{s\log 2} - 1, \tag{5.57}$$

and

$$\liminf_{N\to+\infty} \frac{\nu_{\mathcal{U},Np+1}(k)dk}{\mathrm{card}(U_{Np+1})^{\alpha}dk} \;=\; \left|\kappa_\eta\Big(\frac{4\pi}{a+b}\frac{t}{p}\Big)\right|^2 \times \inf_{z\in\mathrm{R}(k)} |z|^2\,, \tag{5.58}$$

$$\limsup_{N\to+\infty} \frac{\nu_{\mathcal{U},Np+1}(k)dk}{\mathrm{card}(U_{Np+1})^{\alpha}dk} \;=\; \left|\kappa_\eta\Big(\frac{4\pi}{a+b}\frac{t}{p}\Big)\right|^2 \times \sup_{z\in\mathrm{R}(k)} |z|^2 \;>0. \tag{5.59}$$

It is easy to check that (5.56) holds for any other subsequence of $\mathcal{U}$ with the same exponent $\alpha = 2\beta - 1$. Let us notice that $1 < \lambda_1 < 2^s$ by (4.46), which implies that $-1 < \alpha < 1$. Then we deduce the claim from the (AGL)-argument.

General case: if now $p \ge 3$ is an odd integer, Cheng, Savit and Merlin ([CSM] II.B.4) have shown that a scaling exponent always exists. By Remark 4.4 this scaling exponent seems to be equal, in the rarefaction phenomenon, to the scaling exponent of the Thue-Morse rarefied sequence relative to the “dominant” prime number in the prime number decomposition of p. However, no general result is known to the authors on this subject, except the case $p = 3^{r_1}5^{r_2}$ with $r_1, r_2 \ge 1$ (i.e. Proposition 4.3 due to Grabner [Gr1]). We now proceed as in the previous case.

(ii) the second case can be deduced from the first one (i) as follows, now with $l = 2^n$, $n \ge 1$:

$$\begin{aligned}
\frac{1}{2^n}\left|\sum_{j=1}^{2^n} \eta_{j-1} e^{-2i\pi j\frac{t}{2^h p}}\right|^2 &= 2^n \prod_{j=0}^{n-1} \sin^2(\pi 2^j \frac{t}{2^h p})\\
&= 2^h \prod_{j=0}^{h-1} \sin^2(\pi 2^j \frac{t}{2^h p}) \times 2^{n-h} \prod_{j=0}^{n-h-1} \sin^2(\pi 2^j \frac{t}{p})\\
&= 2^h \prod_{j=0}^{h-1} \sin^2(\pi 2^j \frac{t}{2^h p}) \times \frac{1}{2^{n-h}}\left|\sum_{j=1}^{2^{n-h}} \eta_{j-1} e^{-2i\pi j\frac{t}{p}}\right|^2.
\end{aligned} \tag{5.60}$$

We have $\prod_{j=0}^{h-1} \sin^2(\pi 2^j \frac{t}{2^h p}) \neq 0$. Then, by (5.60), we can apply the asymptotic laws relative to the p-rarefied sums of the Thue-Morse sequence as in (i). Therefore

$$0 \leq \liminf_{n\to+\infty} \frac{\nu_{\mathcal{U},2^n}(k)dk}{\mathrm{card}(U_{2^n})^\alpha dk} < \limsup_{n\to+\infty} \frac{\nu_{\mathcal{U},2^n}(k)dk}{\mathrm{card}(U_{2^n})^\alpha dk} < +\infty \tag{5.61}$$

with

$$\alpha = \alpha(k) = \alpha(2^h k) = \alpha(\frac{4\pi}{a+b}\frac{t}{p}) = 2\beta - 1 = 2\frac{\log\lambda_1}{s\log 2} - 1, \tag{5.62}$$

and
(5.63)

$$\liminf_{n\to+\infty} \frac{\nu_{\mathcal{U},2^n}(k)dk}{\mathrm{card}(U_{2^n})^\alpha dk} = |\kappa_\eta(k)|^2 \times 2^h \prod_{j=0}^{h-1} \sin^2(\pi 2^j \frac{t}{2^h p}) \times \inf_{z\in\mathrm{R}(2^h k)} |z|^2,$$

(5.64)

$$\limsup_{n\to+\infty} \frac{\nu_{\mathcal{U},2^n}(k)dk}{\mathrm{card}(U_{2^n})^\alpha dk} = |\kappa_\eta(k)|^2 \times 2^h \prod_{j=0}^{h-1} \sin^2(\pi 2^j \frac{t}{2^h p}) \times \sup_{z\in\mathrm{R}(2^h k)} |z|^2.$$

The inequalities (5.61) hold for any other subsequence of $\mathcal{U}$ with the same exponent $\alpha = 2\beta - 1$. The (AGL)-argument is now invoked and gives the claim. □

5.2. Extinction properties. We say that a subsequence $\mathcal{V} = (V_i)_{i\geq 0}$ with $V_i = [0, f(N_i)] \cap (\Lambda_{a,b} \setminus \{0\})$, of the canonical sequence $\mathcal{U}$ (3.31), has the extinction property at $k = \frac{4\pi}{a+b}\frac{t}{p} \in \mathbb{S}$ if

$$\lim_{i\to+\infty} \frac{\nu_{\mathcal{V},i}(k)dk}{\mathrm{card}(V_i)^\alpha dk} = 0$$

for the exponent α which corresponds uniquely to k in the (AGL)-argument.

Proposition 5.3. *Let $k = \frac{4\pi}{a+b}\frac{t}{p} \in \mathbb{S}$ with p odd. If the origin 0 is not contained in the rarefaction domain $R(k)$, then there exists no sequence of finite approximants $\mathcal{V}$ in $\mathcal{A}_{a,b}^{supp(\omega)}$ which possesses the extinction property at k.*

Proof. Assume that such a sequence of finite approximants $\mathcal{V} = (V_i)_i$ exists. Then we would have

$$\begin{aligned} \liminf_{i\to+\infty} \frac{\nu_{\mathcal{V},i}(k)dk}{\mathrm{card}(V_i)^\alpha dk} &= \lim_{i\to+\infty} \frac{\nu_{\mathcal{V},i}(k)dk}{\mathrm{card}(V_i)^\alpha dk} \\ &= 0 = \left|\kappa_\eta\Big(\frac{4\pi}{a+b}\frac{t}{p}\Big)\right|^2 \times \inf_{z\in\mathrm{R}(k)} |z|^2. \end{aligned} \tag{5.65}$$

But $\inf_{z\in\mathrm{R}(k)} |z|^2 \neq 0$ and $\kappa_\eta\Big(\frac{4\pi}{a+b}\frac{t}{p}\Big) \neq 0$. Contradiction. □

Now let $k = \frac{4\pi}{a+b}\frac{t}{p} \in \mathbb{S}$ and assume $0 \in \mathrm{R}(k)$. The problem of finding a sequence of finite approximants $\mathcal{V} = (V_i)$, with $V_i = [0, f(N_i\, p)] \cap (\Lambda_{a,b} \backslash \{0\})$, which has the extinction property at k amounts to exhibit explicitly an increasing sequence (N_i) of positive integers which satisfies

$$\lim_{i\to\infty} \sum_{j=0}^{p-1} \psi_{p,j}\left(\frac{\log(N_i\, p)}{rs\log 2}\right) \xi_k^j \;=\; 0,$$

by (4.43), and in particular sequences (N_i) for which

$$\sum_{j=0}^{p-1} \psi_{p,j}\left(\frac{\log(N_i\, p)}{rs\log 2}\right) \xi_k^j \;=\; 0 \qquad \text{for all } i$$

as soon as $i \geq i_0$ for a certain i_0. This problem concerns at the same time all the p functions $\psi_{p,j}(x)$ (whose individual cancellation properties are recalled in Section 4) and seems as difficult as finding particular sequences of integers which realize the min and the max of the fractal functions $\psi_{p,j}(x)$ (see Coquet [Ct] for $p = 3$ and $j = 0$).

Let us notice that most of the sequences of finite approximants have not the extinction property at k.

5.3. Growth regimes of approximant measures and visibility in the spectrum.

Definition 5.4. Let $k = \frac{4\pi}{a+b}\frac{t}{2^h\, p} \in \mathbb{S}$. We say that the sequence of approximant measures $\left(\nu_{\mathcal{U},l}(k)dk\right)_{l\geq 0}$ is

(i) *size-increasing at* k : if and only if $\alpha \in (0,1)$,
(ii) *étale at* k : if and only if $\alpha = 0$,
(iii) *size-decreasing at* k : if and only if $\alpha \in (-1,0)$.

These three regimes correspond exactly, by (5.57) and (5.62), to the three cases of exponents

$$1/2 < \beta < 1,\;\; \beta = 1/2,\;\; 0 < \beta < 1/2$$

relative to the asymptotic laws of the p-rarefied sums of the Thue-Morse sequence.

On the other hand, it is usual in physics [Cy] [G] [Lu] to represent the function "intensity per diffracting site" $I_l(k)$ (see (3.26)) as a function of k, for various values of sizes $l = \mathrm{card}(U_l)$ of finite approximant point sets of μ. In this respect, it is of common use to increase l in order to have a more precise understanding of the spectrum. Here, it is illusory to do so since, on the set of ks at which the sequence of approximant measures is "étale" and "size-decreasing", the functions

$$l \;\mapsto\; I_l(k)$$

tend to zero much more rapidly than on the set of ks at which the sequence of approximant measures is "size-increasing": it suffices to compare (3.26) and (3.27) to observe this. This leads to a unique visibility in the spectrum, when l is large enough, of the singular peaks at the ks for which the sequence of approximant measures is "size-increasing", and of the Bragg peaks characterized by the fact that $\limsup_{l\to+\infty} I_l(k)$ is a nonzero constant, by the Bombieri-Taylor Conjecture.

In the numerical study [PCA] most of the peaks in the spectrum, which are not Bragg peaks, are guessed to be labeled using "$2^h p$" at the denominator of the wave vector with the prime number $p = 3$, not with any other odd prime. Therefore it is important to characterize the set of odd prime numbers p for which the p-rarefied sums of the Thue-Morse sequence have the property that the sequences of approximant measures are "size-increasing" at $k = \frac{4\pi}{a+b}\frac{t}{2^h p} \in \mathbb{S}$. In Subsection 5.4 we ask this general question and solve it for some classes of prime numbers, using [GKS].

5.4. Classes of prime numbers. Denote by $\mathbb{P} = \{3, 5, \ldots\}$ the set of odd prime numbers. In the following, we only consider prime numbers p which are odd, so that 2 is invertible modulo p. The order of 2 in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$, denoted by s, is ≥ 1.

Problem : Characterize the set of odd prime numbers p for which the exponent $\beta = (\log \lambda_1)/(s \log 2)$ is $> 1/2$ (recall that λ_1 is the first magnitude of eigenvalues of the matrix $M = (S_{p,i-j}(2^s))_{0\leq i,j\leq p-1})$.

5.4.1. *The class $\mathcal{P}_1$.* Let $\mathcal{P}_1 := \{p \in \mathbb{P} \mid s = p - 1\}$. We have

$$\mathcal{P}_1 = \{3, 5, 11, 13, 19, 29, 37, 53, 59, 61, 67, 83, 101, 107, 131, 139,$$
$$149, 163, 173, 179, 181, 197, \ldots\}.$$

It is a conjecture of Artin [Le] that $\mathcal{P}_1$ is infinite. The moduli of the eigenvalues of the matrix $M = (S_{p,i-j}(2^{p-1}))_{0\leq i,j\leq p-1}$ are

$$\lambda_1 \;=\; p \text{ (which is } (p-1)\text{-fold degenerated)}, \qquad \lambda_2 \;=\; 0.$$

The integer r is equal to 1, the error terms $E_{p,j}(n)$ are bounded, and (4.43) reads as

$$S_{p,j}(n) \;=\; n^{\beta}\psi_{p,j}\left(\frac{\log n}{(p-1)\,\log 2}\right) - \frac{1}{p}\left(\frac{1-(-1)^n}{2}\right)\eta_n, \tag{5.66}$$

with

$$\beta \;=\; \frac{\log p}{(p-1)\log 2}$$

with all functions $\psi_{p,j}(x)$ continuous, nowhere differentiable, of period 1.

Proposition 5.5. *The prime numbers $p \in \mathcal{P}_1$ for which $\beta > 1/2$ are* 3 *and* 5.

Proof. Indeed, these values of p are given by the inequality:

$$\frac{\log p}{(p-1)\log 2} > \frac{1}{2}\,. \tag{5.67}$$

Since $x \to \frac{\log x}{(x-1)\log 2}$ goes to zero when x tends to infinity, only a finite number of values of p satisfy (5.67). An easy computation gives 3 and 5. □

5.4.2. *The class $\mathcal{P}_{2,1}$.* Let $\mathcal{P}_{2,1} := \{p \in \mathbb{P} \mid s = \frac{p-1}{2},\ p \equiv 1\,(\mathrm{mod}\ 4)\}$. We have

$$\mathcal{P}_{2,1} = \{17, 41, 97, 137, 193, \ldots\}.$$

The moduli of the eigenvalues of the matrix $M = (S_{p,i-j}(2^{\frac{p-1}{2}}))_{0\le i,j\le p-1}$ are

$$\lambda_1 \ = \ \epsilon^h\sqrt{p} \quad (\text{with degeneracy} \ge \frac{p-1}{2}), \qquad \lambda_2 \ = \ \epsilon^{-h}\sqrt{p} \ \in (0,1),$$

where h is equal to the class number of the field $\mathbb{Q}(\sqrt{p})$ and $\epsilon > 1$ the fundamental unit in the real quadratic field $\mathbb{Q}(\sqrt{p})$. The integer r is equal to 1, the error terms $E_{p,j}(n)$ are bounded in modulus by

$$2^{\frac{p-1}{2}}\frac{1}{\sqrt{p}-\epsilon^h},$$

and (4.43) reads as

$$S_{p,j}(n) \ = \ n^{\beta}\psi_{p,j}\left(\frac{2\log n}{(p-1)\,\log 2}\right) + E_{p,j}(n) \tag{5.68}$$

with

$$\beta \ = \ \frac{\log p + 2h\log\epsilon}{(p-1)\log 2}$$

with all functions $\psi_{p,j}(x)$ continuous, nowhere differentiable, of period 1.

Proposition 5.6. *Only finitely many prime numbers p in $\mathcal{P}_{2,1}$ give rise to the inequality $\beta > 1/2$. There is only one such prime number: $p = 17$.*

Proof. Let $L(z,\chi_p)$ the Dirichlet L-function with character χ_p ([BS] p. 380). The discriminant of $\mathbb{Q}(\sqrt{p})$ is equal to p and we have the analytic class number formula ([BS] p. 385)

$$2h\log\epsilon \ = \ \sqrt{p}\,L(1,\chi_p). \tag{5.69}$$

By Hua's inequality ([Ha] Theorem 13.3, p. 328)

$$L(1,\chi_p) \ < \ \frac{\log p}{2} + 1$$

we deduce

$$\beta \ < \ \frac{\log p + \sqrt{p}\,(\frac{\log p}{2} + 1)}{(p-1)\log 2}.$$

Since the function

$$x \to \frac{\log x + \sqrt{x}\,(\frac{\log x}{2} + 1)}{(x-1)\log 2} \tag{5.70}$$

tends to zero when x goes to infinity, we deduce the claim.

In order to deduce the possible values we notice that (5.70) intersects the line $y = 1/2$ between $x = 125$ and 126. The Table below, obtained from [BS] pp. 472–474, and PARI-GP, gives the first couples (p, β) (with $\omega = (1+\sqrt{p})/2$).

p	17	41	97	137	197 ...
h	1	1	1	1	1
ϵ	$3+2\omega$	$27+10\omega$	$5035+1138\omega$	$1595+298\omega$	$1637147+253970\omega$
β	0.6332..	0.4339..	0.3490..	0.2398..	0.2672..

□

A complete investigation of the values of β, for all $p \in \mathcal{P}_{2,1}$, requires the knowledge of the class number h and the regulator $\log \epsilon$ in real quadratic fields, in the Cohen-Lenstra heuristics [CL1] [CL2] [CM].

5.4.3. ***The class*** $\boldsymbol{\mathcal{P}_{2,3}}$**.** Let $\mathcal{P}_{2,3} := \{p \in \mathbb{P} \mid s = \frac{p-1}{2},\ p \equiv 3 \,(\mathrm{mod}\ 4)\}$. We have

$$\mathcal{P}_{2,3} = \{7, 23, 47, 71, 79, 103, 167, 191, 199, \ldots\}.$$

All the eigenvalues of the matrix $M = (S_{p,i-j}(2^{\frac{p-1}{2}}))_{0 \le i,j \le p-1}$ have the same modulus which is equal to

$$\lambda_1 \;=\; \sqrt{p}.$$

The integer r is equal to 4, the error terms $E_{p,j}(n)$ are bounded, and (4.43) reads as

$$S_{p,j}(n) \;=\; n^{\beta}\psi_{p,j}\left(\frac{\log n}{2(p-1)\,\log 2}\right) - \frac{1}{p}\left(\frac{1-(-1)^n}{2}\right)\eta_n \tag{5.71}$$

with

$$\beta \;=\; \frac{\log p}{(p-1)\log 2},$$

with all functions $\psi_{p,j}(x)$ continuous, nowhere differentiable, of period 1.

Proposition 5.7. *There exists no prime number p in $\mathcal{P}_{2,3}$ for which $\beta > 1/2$.*

Proof. Same proof as (5.5). □

5.4.4. ***An inequality.*** For the prime number classes other than $\mathcal{P}_1$, $\mathcal{P}_{2,1}$ or $\mathcal{P}_{2,3}$, the corresponding exponents β which control the rarefaction phenomenon have the following property ([GKS] p. 14):

$$\beta \;=\; \frac{\log \lambda_1}{s \log 2} > \frac{\log p}{(p-1)\log 2}.$$

It is expected that the number of such odd prime numbers for which $\beta > 1/2$ is zero or, possibly, is finite.

6. Other Dirac combs and Marcinkiewicz classes

In the case of the equally weighted sum $\mu = \sum_{n\geq 1} \delta_{f(n)}$ the fractality of the sum-of-digits functions $\psi_{p,j}(x)$ used for describing the singular continuous part of the spectrum of μ arises from the approximation, given by (4.45), of the solution of the matrix equation (4.44). In the general case of a weighted Dirac comb on the Thue-Morse quasicrystal $\Lambda_{a,b}$, the computation of scaling exponents arises from the existence of a matrix equation like (4.44) which correlates the block structure of the Prouhet-Thue-Morse sequence with the powers of 2 and the sequence of the weights $(\omega(n))_{n\in\mathbb{Z}}$; then the computation of solutions by interpolation as in (4.43), with fractal powers of n, becomes possible. The Bragg and singular continuous components can be deduced in a similar way.

Now let us turn to classes of weighted Dirac combs on $\Lambda_{a,b}$. Let $\mathcal{L}$ be the space of complex-valued functions on $\Lambda_{a,b}$ (or equivalently on $\mathbb{Z}$ through $n \to f(n)$). For $w \in \mathcal{L}$, we denote by $\|w\|$ the pseudo-norm ("norm 1") of Marcinkiewicz of w defined as

$$\|w\| \;=\; \limsup_{l\to+\infty} \frac{1}{\mathrm{Card}(U_l)} \sum_{n\in\mathbb{Z}, f(n)\in U_l} |w(n)|$$

where $(U_l)_l$ is an averaging sequence of finite approximants of $\Lambda_{a,b}$. The Marcinkiewicz space $\mathcal{M}$ is the quotient space of the subspace

$$\{g \in \mathcal{L} \mid \|g\| < +\infty\}$$

of $\mathcal{L}$ by the equivalence relation $\mathcal{R}$ defined by

$$h \,\mathcal{R}\, g \iff \|h-g\| = 0 \tag{6.72}$$

(Bertrandias [Bs1] [B-VK], Vo Khac [VK]). The class of w is denoted by $\overline{w}$ in $\mathcal{M}$. Though the definition of $\|\cdot\|$ depends upon the chosen averaging sequence $(U_l)_l$ the space $\mathcal{M}$ obviously does not. This equivalence relation is called Marcinkiewicz equivalence relation. The vector space $\mathcal{M}$ is normed with $\|\overline{g}\| = \|g\|$, and is complete (Bertrandias [Bs1] [B-VK], Vo Khac [VK]). By $\mathcal{L}^\infty$ we will mean the subspace of $\mathcal{L}$ of bounded weights endowed with the $\mathcal{M}$-topology in the sequel.

The following proposition shows that the Bombieri-Taylor (BT) argument is compatible with the Marcinkiewicz equivalence relation $\mathcal{R}$, i.e. that two weighted Dirac combs on $\Lambda_{a,b}$ which are Marcinkiewicz-equivalent present the same Bragg component in the spectrum.

Proposition 6.1. *Let $h, g \in \mathcal{L}$ such that $\overline{h}, \overline{g} \in \mathcal{M}$. Then, for $q \in \mathbb{R}$,*

$$I_h(q) \;\leq\; \|h\|^2, \tag{6.73}$$

$$h \;\mathcal{R}\; g \implies I_h(q) \;=\; I_g(q). \tag{6.74}$$

Proof. Immediate. □

The relation (6.74) means that the set of weighted Dirac combs on the Thue-Morse quasicrystal is classified by the Marcinkiewicz relation. The intensity function I_w (per diffracting site) is a class function on $\mathcal{M}$, when running over the Bragg component of the spectrum.

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