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par KOHJI MATSUMOTO et YOSHIO TANIGAWA

RÉSUMÉ. Dans cet article, nous considérons que certaines séries de Dirichlet multiples, dont nous montrons le prolongement analytique en utilisant la formule intégrale de Mellin-Barnes. Des majorations de ces séries sont également obtenues.

ABSTRACT. Multiple Dirichlet series of several complex variables are considered. Using the Mellin-Barnes integral formula, we prove the analytic continuation and an upper bound estimate.

1. Introduction and statement of results

Let $s = \sigma + it$ be a complex variable, and

$$(1.1) \qquad \varphi_k(s) = \sum_{n=1}^{\infty} \frac{a_k(n)}{n^s}$$

be a Dirichlet series with complex coefficients $a_k(n)$ $(1 \leq k \leq r)$. We assume

(I) $\varphi_k(s)$ is convergent absolutely for $\sigma > \alpha_k (> 0)$;

(II) $\varphi_k(s)$ can be continued meromorphically to the whole complex plane $\mathbb{C}$, and holomorphic except for a possible pole at $s = \alpha_k$ of order at most 1, whose residue we denote by R_k;

(III) in any fixed strip $\sigma_1 \leq \sigma \leq \sigma_2$, the order estimate $\varphi_k(\sigma + it) = O(|t|^A)$ holds as $|t| \to \infty$, where $A = A(\sigma_1, \sigma_2)$ be a non-negative constant.

In the present paper we introduce the multiple Dirichlet series

$$(1.2) \quad \Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r))$$

$$= \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{a_1(m_1)}{m_1^{s_1}} \frac{a_2(m_2)}{(m_1 + m_2)^{s_2}} \cdots \frac{a_r(m_r)}{(m_1 + \cdots + m_r)^{s_r}}$$

associated with $\varphi_1, \ldots, \varphi_r$, where $s_k = \sigma_k + it_k$ $(1 \leq k \leq r)$ are complex variables, and prove several basic properties. It is clear that the multiple

series (1.2) converges absolutely if $\sigma_k > \alpha_k$ $(1 \leq k \leq r)$. Let $\mathbb{N}, \mathbb{N}_0$ be the sets of positive integers and non-negative integers, respectively. We first prove the following

Theorem 1. *The multiple series* $\Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r))$ *can be continued meromorphically to the whole* $\mathbb{C}^r$ *space, and its possible singularities are located only on the subsets of* $\mathbb{C}^r$ *each of which is defined by one of the following equations:*

$$(1.3) \qquad s_j + \cdots + s_r = \alpha_j + \delta_{j+1}\alpha_{j+1} + \cdots + \delta_r\alpha_r - n,$$

where $1 \leq j \leq r$, $n \in \mathbb{N}_0$, *and* $\delta_k = 0$ *or* 1 $(2 \leq k \leq r)$. *When* $j = r$, *(1.3) is to be read as* $s_r = \alpha_r - n$. *Moreover,*

  (i) *in the case* $j = r \geq 2$, *if* $\alpha_r \in \mathbb{N}$, *then the possible values of* n *are* $0, 1, 2, \ldots, \alpha_r - 1$;

  (ii) *in the case* $2 \leq j \leq r - 1$, *if* $\alpha_j \in \mathbb{N}$ *and* $\delta_{j+1} = \cdots = \delta_r = 1$, *then the possible values of* n *are* $0, 1, 2, \ldots, \alpha_j - 1$;

  (iii) *in the case* $j = 1$, *if* $r = 1$ *or if* $\delta_2 = \cdots = \delta_r = 1$, *then* $n = 0$.

Theorem 2. *If* $\varphi_k(s)$ *is entire for* $1 \leq k \leq r$, *then*

$$\Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r))$$

is also entire.

Our $\Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r))$ includes various interesting special cases. The function $\varphi_k(s)$ can be the Riemann zeta-function $\zeta(s)$, the Dirichlet L-function $L(s, \chi)$, the automorphic L-function $L(s, f)$ attached to a certain cusp form, etc. When $\varphi_k(s) = \zeta(s)$ $(1 \leq k \leq r)$, our Φ_r is nothing but the well-known Euler-Zagier sum, whose analytic continuation was recently established by various methods (Arakawa and Kaneko [3], Zhao [13], Akiyama, Egami and Tanigawa [1], Matsumoto [10]; see also Goncharov [6]). Our proof of Theorems 1 and 2 is a generalization of the method given in Matsumoto [10].

When $\varphi_k(s) = L(s, \chi_k)$ $(1 \leq k \leq r)$, where $\chi_1, \ldots, \chi_r$ are Dirichlet characters, the corresponding Φ_r may be called the multiple Dirichlet L-function. The special values at positive integer arguments of these kinds of multiple L-functions have been studied by Goncharov, Arakawa and Kaneko [4] and others. It is also to be noted that a different type of multiple Dirichlet L-functions of the form

$$\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\chi_1(m_1)}{m_1^{s_1}} \frac{\chi_2(m_1 + m_2)}{(m_1 + m_2)^{s_2}} \cdots \frac{\chi_r(m_1 + \cdots + m_r)}{(m_1 + \cdots + m_r)^{s_r}}$$

has been studied by Goncharov [5], Akiyama and Ishikawa [2], and Ishikawa [8, 9].

By Theorem 2, the multiple Dirichlet L-function

$$\Phi_r((s_1,\ldots,s_r);(L(\cdot,\chi_1),\ldots,L(\cdot,\chi_r)))$$

is entire if $\chi_1,\ldots,\chi_r$ are non-principal. Also, the multiple automorphic L-function

$$\Phi_r((s_1,\ldots,s_r);(L(\cdot,f_1),\ldots,L(\cdot,f_r)))$$

is entire, where $f_1,\ldots,f_r$ are cusp forms. On the other hand, in the present paper we assume that the order of the pole at $s=\alpha_k$ is at most 1, but this is only for simplicity. We may consider the case when $\varphi_k(s)$ has the pole of higher order by the same method.

A natural next problem is to study the order estimate of Φ_r. In this paper we treat the simplest case Φ_2. By the assumption (III) there exists a non-negative constant $\theta_k(\sigma)$ for any σ such that

(1.4) $$\varphi_k(\sigma+it)=O(|t|^{\theta_k(\sigma)})$$

holds as $|t|\to\infty$. We shall prove

Theorem 3. *Let η be a small positive number. The estimate*

$$\Phi_2((s_1,s_2);(\varphi_1,\varphi_2)) \ll (1+|t_2|)^{-\alpha_2} (1+|t_1+t_2|)^{\theta_1(\sigma_1+\sigma_2-\alpha_2)} R_2$$

$$+ (1+|t_2|)^{\theta_2(\eta)+\frac{1}{2}+\max\{0,\frac{1}{2}-\sigma_2+\eta\}} \max\{1+|t_2|,1+|t_1+t_2|\}^{\theta_1(\sigma_1+\sigma_2-\eta)}$$

holds in the region $\{(s_1,s_2)|\ \sigma_1+\sigma_2>\alpha_1+\eta,\sigma_2>\eta\}$, *except the points near the set of singularities.*

Therefore, if some non-trivial estimate of $\varphi_k(s)$ is known, then Theorem 3 gives a non-trivial estimate of Φ_2. For instance, when $\varphi_k(s)=\zeta(s),L(s,\chi)$, or the automorphic L-function $L(s,f)$ (see Good [7]), sharp values of $\theta_k(\sigma)$ are known.

It is clearly an interesting problem to generalize Theorem 2 to the case of $\Phi_r, r\geq 3$.

2. Proof of Theorems 1 and 2

We prove the theorems by induction. The argument is similar to that developed in the last section of [10], or in [11], [12].

The case $r=1$ is clear from the assumption (II). Consider the case Φ_r, assuming that the theorems are true for Φ_{r-1}. First assume $\sigma_k>\alpha_k$ ($1\leq k\leq r$). Recall the classical Mellin-Barnes formula

(2.1) $$\Gamma(s)(1+\lambda)^{-s}=\frac{1}{2\pi i}\int_{(c)}\Gamma(s+z)\Gamma(-z)\lambda^z dz,$$

where $\Re s > 0, |\arg \lambda| < \pi, \lambda \neq 0, -\Re s < c < 0$, and the path of integration is the vertical line $\Re z = c$. Put $s = s_r$ and

$$\lambda = \frac{m_r}{m_1 + \cdots + m_{r-1}}$$

in (2.1); we may assume $-\sigma_r < c < -\alpha_r$. Then multiply the both sides by

$$a_1(m_1) \cdots a_r(m_r) m_1^{-s_1} \cdots (m_1 + \cdots + m_{r-2})^{-s_{r-2}} (m_1 + \cdots + m_{r-1})^{-s_{r-1}-s_r}$$

and sum up with respect to $m_1, \ldots, m_r$. We obtain

$$(2.2) \quad \Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r)) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)}$$
$$\times \Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z); (\varphi_1, \ldots, \varphi_{r-1}))\varphi_r(-z)dz.$$

Next we shall shift the path to $\Re z = M - \eta$, where M is a non-negative integer and η is a small positive number. The function $\varphi_r(-z)$ is of polynomial order with respect to $\Im z$ by the assumption (III), while

$$\Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z); (\varphi_1, \ldots, \varphi_{r-1})) = O(1)$$

in the region $c \leq \Re z \leq M - \eta$ because of the expression (1.2). Hence the integrand on the right-hand side of (2.2) tends to zero when $|\Im z| \to \infty$, so this shifting is possible. Counting the residues of the poles at $s = -\alpha_r, 0, 1, 2, \ldots, M - 1$, we obtain

$$(2.3) \qquad \qquad \Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r))$$

$$= \frac{\Gamma(s_r - \alpha_r)\Gamma(\alpha_r)}{\Gamma(s_r)} \Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r - \alpha_r); (\varphi_1, \ldots, \varphi_{r-1}))R_r$$

$$+ \sum_{l=0}^{M-1} \binom{-s_r}{l} \Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + l); (\varphi_1, \ldots, \varphi_{r-1}))\varphi_r(-l)$$

$$+ \frac{1}{2\pi i} \int_{(M-\eta)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)}$$
$$\times \Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z); (\varphi_1, \ldots, \varphi_{r-1}))\varphi_r(-z)dz.$$

The first two terms on the right-hand side are meromorphic in the whole $\mathbb{C}^r$ space by the induction assumption, while we can see that the last integral term is holomorphic in the region

$$\left\{ (s_1, \ldots, s_r) \middle| \begin{array}{c} \sigma_k + \cdots + \sigma_r > \alpha_k + \cdots + \alpha_{r-1} - M + \eta \quad (1 \leq k \leq r-1) \\ \sigma_r > -M + \eta \end{array} \right\}$$

by using the induction assumption on the location of singularities of Φ_{r-1}. Since M is arbitrary, this implies the meromorphic continuation of Φ_r to the whole $\mathbb{C}^r$ space.

The location of singularities of Φ_r can also be seen from (2.3). In fact, the factor $\Gamma(s_r - \alpha_r)$ is singular on $s_r = \alpha_r - n \ (n \in \mathbb{N}_0)$. If $\alpha_r \in \mathbb{N}$,

then this singularity is cancelled by the factor $\Gamma(s_r)$ when $n \geq \alpha_r$. By the induction assumption, the factor

$$\Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r - \alpha_r); (\varphi_1, \ldots, \varphi_{r-1}))$$

is singular on

$$s_j + \cdots + s_r - \alpha_r = \alpha_j + \delta_{j+1}\alpha_{j+1} + \cdots + \delta_{r-1}\alpha_{r-1} - n,$$

where $1 \leq j \leq r - 1$ and $n \in \mathbb{N}_0$, but

(i) if $j \geq 2$, $\alpha_j \in \mathbb{N}$ and $\delta_{j+1} = \cdots = \delta_{r-1} = 1$ then $0 \leq n \leq \alpha_j - 1$, and

(ii) if $j = 1$, and if $r = 2$ or $\delta_2 = \cdots = \delta_{r-1} = 1$ then $n = 0$.

Similarly the factor

$$\Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + l); (\varphi_1, \ldots, \varphi_{r-1})) \quad (l \in \mathbb{N}_0)$$

is singular on

$$s_j + \cdots + s_r + l = \alpha_j + \delta_{j+1}\alpha_{j+1} + \cdots + \delta_{r-1}\alpha_{r-1} - n$$

$(1 \leq j \leq r - 1, n \in \mathbb{N}_0)$, with the same restrictions as above. Collecting the above information we obtain the assertion on the location of singularities of Φ_r.

Lastly, if $\varphi_r(s)$ is entire then $R_r = 0$, hence the first term on the right-hand side of (2.3) vanishes. Therefore inductively we can show that Φ_r is entire. All the assertions of Theorems 1 and 2 are now established.

3. Estimation of certain integrals

In this section, as a preparation for the proof of Theorem 3, we consider the integral of the form

$$(3.1) \quad J(u, v) = \int_{-\infty}^{\infty} (1 + |u + y|)^p (1 + |v + y|)^q (1 + |y|)^r$$
$$\times \exp\left(A|u + y| + B|y|\right) dy,$$

where u, v, p, q, r, A, B are real numbers with $A + B < 0$. To evaluate this integral, we use the estimate

$$(3.2) \quad \int_{-\infty}^{\infty} (1 + |y|)^\rho \exp(A|u + y| + B|y|) dy$$
$$= O\left((1 + |u|)^{\rho+\delta} e^{B|u|} + e^{A|u|}\right),$$

where u, ρ, A, B are real numbers with $A + B < 0, \delta = 1$ or 0 according as $A = B$ or $A \neq B$, and the implied constant depends only on ρ, A and B. This estimate is a special case of Lemma 3 in Matsumoto [11].

Temporarily we assume $|u| \geq |v|$. Divide the integral (3.1) as

$$\int_{|y|>2|u|} + \int_{2|u| \geq |y| > 2|v|} + \int_{|y| \leq 2|v|},$$

and estimate the factor $(1+|u+y|)^p (1+|v+y|)^q (1+|y|)^r$ in the integrand as

$$\begin{cases} O\big((1+|y|)^{p+q+r}\big) & \text{if } |y| > 2|u|, \\[2mm] O\big(\max\{1, U^p\}(1+|y|)^{q+r}\big) & \text{if } 2|u| \geq |y| > 2|v|, \\[2mm] O\big(\max\{1, U^p\}\max\{1, V^q\}(1+|y|)^r\big) & \text{if } |y| \leq 2|v|, \end{cases}$$

where $U = 1+|u|$ and $V = 1+|v|$. Then replace the intervals of all integrals by $(-\infty, \infty)$, and apply (3.2). The result is that

$$\begin{aligned} J(u,v) \ll & U^{p+q+r+\delta}e^{B|u|} + e^{A|u|} \\ & + \max\{1, U^p\}\{U^{q+r+\delta}e^{B|u|} + e^{A|u|}\} \\ & + \max\{1, U^p\}\max\{1, V^q\}\{U^{r+\delta}e^{B|u|} + e^{A|u|}\}, \end{aligned}$$

hence

Lemma 1. *Using the notations as above, we have*

$$(3.3) \quad J(u,v) \ll (1 + U^p)(1 + U^q + V^q)U^{r+\delta}e^{B|u|}$$
$$+ (1 + U^p)(1 + V^q)e^{A|u|}.$$

This is proved under the assumption $|u| \geq |v|$, but the case $|u| < |v|$ can be treated similarly, and the same conclusion (3.3) holds.

In particular, if $A = B = -\frac{\pi}{2}$ then $\delta = 1$, then (3.3) implies the following

Lemma 2. *We have*

$$\int_{-\infty}^{\infty} (1+|u+y|)^p(1+|v+y|)^q(1+|y|)^r \exp\left(-\frac{\pi}{2}|u+y| - \frac{\pi}{2}|y|\right) dy$$
$$= O\left((1 + U^p)(1 + U^q + V^q)(1 + U^{r+1})\exp\left(-\frac{\pi}{2}|u|\right)\right),$$

with the implied constant depending only on p, q and r.

4. Proof of Theorem 3

Putting $r = 2$ and $M = 0$ in (2.3), we have

$$\Phi_2((s_1, s_2); (\varphi_1, \varphi_2)) = \frac{\Gamma(s_2 - \alpha_2)\Gamma(\alpha_2)}{\Gamma(s_2)} \varphi_1(s_1 + s_2 - \alpha_2)R_2$$
$$+ \frac{1}{2\pi i} \int_{(-\eta)} \frac{\Gamma(s_2 + z)\Gamma(-z)}{\Gamma(s_2)} \varphi_1(s_1 + s_2 + z)\varphi_2(-z)dz,$$

and the last integral is holomorphic in the region

$$\{(s_1, s_2) \mid \sigma_1 + \sigma_2 > \alpha_1 + \eta, \quad \sigma_2 > \eta\}.$$

Hence in this region, by using Stirling's formula and (1.4), we have

$$\Phi_2((s_1, s_2); (\varphi_1, \varphi_2)) \ll (1 + |t_2|)^{-\alpha_2} (1 + |t_1 + t_2|)^{\theta_1(\sigma_1 + \sigma_2 - \alpha_2)} R_2$$

$$+ (1 + |t_2|)^{-\sigma_2 + \frac{1}{2}} e^{\frac{\pi}{2}|t_2|} \int_{-\infty}^{\infty} (1 + |t_2 + y|)^{\sigma_2 - \eta - \frac{1}{2}} (1 + |t_1 + t_2 + y|)^{\theta_1(\sigma_1 + \sigma_2 - \eta)}$$

$$\times (1 + |y|)^{\theta_2(\eta) + \eta - \frac{1}{2}} \exp\left(-\frac{\pi}{2}|t_2 + y| - \frac{\pi}{2}|y|\right) dy.$$

We apply Lemma 2 with $u = t_2, v = t_1 + t_2, p = \sigma_2 - \eta - \frac{1}{2}, q = \theta_1(\sigma_1 + \sigma_2 - \eta)$, and $r = \theta_2(\eta) + \eta - \frac{1}{2}$. Since $q \geq 0$ and $r + 1 \geq 0$, the estimate of Lemma 2 can be written as

$$O\left((1 + U^p)U^{r+1} \max\{U, V\}^q \exp\left(-\frac{\pi}{2}|u|\right)\right).$$

Hence

$$\Phi_2((s_1, s_2); (\varphi_1, \varphi_2))$$

$$\ll (1 + |t_2|)^{-\alpha_2} (1 + |t_1 + t_2|)^{\theta_1(\sigma_1 + \sigma_2 - \alpha_2)} R_2 + (1 + |t_2|)^{\theta_2(\eta) - \sigma_2 + \eta + 1}$$

$$\times \left\{1 + (1 + |t_2|)^{\sigma_2 - \eta - \frac{1}{2}}\right\} \max\left\{1 + |t_2|, 1 + |t_1 + t_2|\right\}^{\theta_1(\sigma_1 + \sigma_2 - \eta)},$$

which implies the assertion of Theorem 3.

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