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## The imaginary abelian number fields with class numbers equal to their genus class numbers

par KU-YOUNG CHANG et SOUN-HI KWON

RÉSUMÉ. Titre français : Sur les corps abéliens dont le nombre de classes est égal au nombre de genres.

Nous savons qu'il n'existe qu'un nombre fini de corps abéliens imaginaires pour lesquels le nombre de classes est égal au nombre de genres. Ceux de ces corps qui sont cycliques et non quadratiques ont été classés dans [Lou2,4] et dans [CK]. Dans cet article, nous déterminons tous les corps abéliens non cycliques dont le nombre de classes est égal au nombre de genres. Cela achève la classification des corps abéliens possédant une classe par genre, sauf dans le cas des corps quadratiques imaginaires.

ABSTRACT. We know that there exist only finitely many imaginary abelian number fields with class numbers equal to their genus class numbers. Such non-quadratic cyclic number fields are completely determined in [Lou2,4] and [CK]. In this paper we determine all non-cyclic abelian number fields with class numbers equal to their genus class numbers, thus the one class in each genus problem is solved, except for the imaginary quadratic number fields.

### 1. Introduction

For an abelian number field  $N$  the narrow genus field  $G_N$  of  $N$  is the maximal abelian number field containing  $N$  such that the extension  $G_N/N$  is unramified at all finite places. The degree  $[G_N : N]$  is called the genus class number of  $N$  and is denoted by  $g_N$ . The genus class number of  $N$  is easy to determine, namely,  $g_N = \frac{1}{[N:\mathbb{Q}]} \prod_p e_p$ , where  $e_p$  is the ramification index of  $p$  in  $N$ . In particular, when  $N$  is imaginary, the genus field of  $N$  is contained in the Hilbert class field of  $N$ , whence  $g_N$  divides the class number of  $N$ . In [Lou1], Louboutin has proved that there exist only finitely many imaginary abelian number fields with class numbers equal to their genus class numbers. Imaginary non-quadratic cyclic number fields

with class numbers equal to their genus numbers have been recently classified ([Lou2, 4], [CK]). It is known that under a suitable generalized Riemann Hypothesis, there are exactly 65 imaginary quadratic number fields with class numbers equal to their genus numbers ([We] and [Lou5]). These fields are listed in Table I. The aim of this paper is to prove the following result:

**Theorem 1.** *There are exactly 424 imaginary non-quadratic abelian number fields with class numbers equal to their genus class numbers: 77 out of them are cyclic and 347 are non-cyclic. Their degrees are less than or equal to 24, their class numbers are 1, 2 or 4. The conductors of these fields are less than or equal to 65689. These fields are listed in tables at the end of this paper.*

We note that in [M] under the assumption of the Generalized Riemann Hypothesis, Miyada has determined all imaginary abelian number fields  $N$  with class numbers equal to their genus class numbers such that the Galois groups are elementary 2-groups. (There seems to be one misprint in [M, Table 2]: the field  $K = \mathbb{Q}(\sqrt{-2}, \sqrt{-37})$  appears twice, i.e. for  $g_K = h_K = 1$  and for  $g_K = h_K = 2$ . In fact,  $g_K = h_K = 2$ .) However, we do not need any assumption for the proof of Theorem 1. This paper is organized in the following way. Section 2 presents some well-known facts on imaginary abelian number fields. In Section 3 we illustrate our computations. Throughout this paper the following notation will be used. For an imaginary abelian number field  $K$ , let  $K^+$ ,  $f_K$ ,  $h_K$ ,  $\omega_K$ ,  $G_K$ ,  $g_K$ ,  $h_K^-$ ,  $Q_K$  be the maximal real subfield, the conductor, the class number, the number of roots of unity in  $K$ , the genus field, the genus class number, the relative class number and the Hasse unit index of  $K$ , respectively. For an odd prime  $p$  let  $\chi_p$  be an odd Dirichlet character of conductor  $p$ , of degree  $p - 1$ . When  $\rho \geq 2$ , let  $\psi_{p^\rho}$  be an even primitive Dirichlet character of conductor  $p^\rho$ , of order  $p^{\rho-1}$  with  $\psi_{p^\rho}^p = \psi_{p^{\rho-1}}$ . For the prime  $p = 2$ , let  $\chi_4$  be the odd Dirichlet quadratic character of conductor 4. When  $\rho \geq 3$ , let  $\psi_{2^\rho}$  be an even primitive Dirichlet character of conductor  $2^\rho$ , of order  $2^{\rho-2}$  with  $\psi_{2^\rho}^2 = \psi_{2^{\rho-1}}$ .

## 2. Preliminaries

In this section we review some well-known facts concerning imaginary abelian number fields.

**Proposition 1.** (1) *Let  $F$  be an abelian number field. If  $h_F = g_F$ , then  $h_M = g_M$  for any subfield  $M$  of  $F$ .*

(2) *Let  $F$  be an imaginary abelian number field,  $\chi_F$  the group of primitive Dirichlet characters associated to  $F$  and  $\chi_F^-$  the subset of  $\chi \in \chi_F$  such that  $\chi(-1) = -1$ . For a  $\chi \in \chi_F$  let us denote the conductor of  $\chi$  by*

$f_\chi$ . We have

$$h_{\bar{F}} = Q_F \omega_F \prod_{\chi \in \chi_{\bar{F}}} \left( -\frac{1}{2} B_{1,\chi} \right),$$

where  $B_{1,\chi} = \frac{1}{f_\chi} \sum_{a=1}^{f_\chi-1} \chi(a)a$ .

- (3) If  $F$  is an imaginary cyclic number field, then  $Q_F = 1$ . Let  $F \subset K$  be the two CM-fields. If  $[K : F]$  is odd, then  $Q_K = Q_F$ .
- (4) Let  $F$  be an imaginary cyclic number field of degree  $2m$ . If  $h_F = g_F$ , then  $1 \leq m \leq 10$ .

*Proof.* (1) See Lemma 1 in [M]. (2) See Theorem 4.17 in [W]. (3) For the first statement see Satz 24 in [H]. For the second statement see Lemma 2 in [HY1]. (4) See [Lou2 and 4] and [CK]. □

### 3. Proof of Theorem 1

Let  $N$  be an imaginary abelian number field with Galois group  $G$ . When  $G = \prod \mathbb{Z}/p_i^{n_i} \mathbb{Z}$ , we say  $N$  is of type  $(p_1^{n_1}, \dots, p_i^{n_i}, \dots)$ . In addition we put  $*$  when the corresponding subfield is imaginary. For example, if  $N$  is of type  $(4^*, 2)$ , then  $N$  is the compositum of an imaginary cyclic quartic field  $M_1$  and a real quadratic field  $M_2$  with  $M_1 \cap M_2 = \mathbb{Q}$ . Then the maximal real subfield  $N^+$  is of type  $(2, 2)$ . If  $N$  is of type  $(4^*, 2^*)$ , then  $N$  is the compositum of an imaginary cyclic quartic field  $M_1$  and an imaginary quadratic field  $M_2$ . In this case,  $N^+$  is of type  $(4)$ . A field  $N$  of type  $(2^*, 2, 2)$  can be either of type  $(2^*, 2^*, 2)$  or  $(2^*, 2^*, 2^*)$ . However we prefer to say that  $N$  is of type  $(2^*, 2^*, 2^*)$  with as many  $*$  as possible for a technical reason to be seen later.

Let  $N$  be an imaginary abelian number field of type  $(2^{m_1^*}, \dots, 2^{m_s^*}, n_1, \dots, n_r)$  with  $m_1 \geq \dots \geq m_s$ . We may assume that  $n_i \leq 2^{m_s}$  for  $n_i$  which is a 2-power. In order to describe  $N$  we give the associated group of Dirichlet characters  $\langle \tau_1, \dots, \tau_s, \varphi_1, \dots, \varphi_r \rangle$ . Here  $\tau_1, \dots, \tau_s$  are odd primitive Dirichlet characters of order  $2^{m_i}$  for  $1 \leq i \leq s$  and  $\varphi_1, \dots, \varphi_r$  are the even primitive Dirichlet characters of order  $n_j$  for  $1 \leq j \leq r$ . For each prime  $p$  let  $G^{(p)}$  be the  $p$ -Sylow subgroup of  $G$ . Let  $N^{(2)}$  be the maximal subfield of  $N$  of 2-power degree. The scheme of our proof of Theorem 1 is as follows. In 3.1 we determine all imaginary abelian number fields  $N$  with  $h_N = g_N$  such that  $G^{(2)}$  is an elementary 2-group. According to Theorem 2 below, if  $G^{(2)}$  is an elementary 2-group, then the 2-rank of  $G^{(2)}$  is less than or equal to 3. In 3.2 we determine all imaginary abelian number fields  $N$  with  $h_N = g_N$  such that  $N^{(2)}$  is of type  $(2^{m_1^*}, 2^{m_2})$  with  $m_1 > m_2 \geq 1$ . Moreover we prove that if  $N$  is an imaginary abelian number field with  $h_N = g_N$ , then 2-rank of  $G^{(2)}$  is less than or equal to 3. If it is equal to 3, then  $G^{(2)}$  is either  $(2^*, 2^*, 2^*)$  or  $(4^*, 2^*, 2^*)$ . In 3.3 we determine all imaginary abelian

number fields  $N$  with  $h_N = g_N$  such that  $N^{(2)}$  is of type  $(2^{m_1^*}, 2^{m_2^*})$  with  $m_1 \geq 2$  and  $m_1 \geq m_2$ . Finally it remains to determine all fields  $N$  with  $h_N = g_N$  such that  $N^{(2)} = (4^*, 2^*, 2^*)$ . These fields are treated in 3.4. Our determinations rely on the computations of the relative class numbers  $h_{N^-}$  of  $N$ . The class numbers  $h_{N^+}$  of  $N^+$  which we need are known in [G], [L], [Ma] and [Mä]. In the remaining part of this paper we say briefly “character of order  $m$ ” instead of “primitive Dirichlet character of order  $m$ ”.

**3.1.  $N^{(2)}$  is of type  $(2^*, \dots, 2^*)$ .**

**Proposition 2.** *Let  $N$  be an imaginary abelian number field,  $G$  the Galois group of  $N/\mathbb{Q}$ . Assume that  $G$  is not cyclic and that the 2-Sylow subgroup of  $G$  is cyclic. If  $h_N = g_N$ , then  $N$  is of type  $(2^*, 3, 3)$  and  $N$  is associated with  $\langle \chi_3, \psi_9, \chi_7^2 \rangle$ .*

*Proof.* First, we claim that if  $h_N = g_N$ , then  $N$  is of type  $(2^*, 3, \dots, 3)$ . According to [CK] there are 4 imaginary cyclic number fields of degree 10 with class numbers equal to their genus class numbers. By Proposition 1.(1) we cannot have a field  $N$  with  $h_N = g_N$  such that  $N$  is of type  $(2^*, 5, 5)$ . By the same argument if  $N$  is of type  $(2^*, 7, 7)$ ,  $(2^*, 9, 9)$ ,  $(4^*, 3, 3)$  or  $(4^*, 5, 5)$ , then  $h_N \neq g_N$ . According to Proposition 1.(1) it follows that if  $h_N = g_N$ , then  $N$  is of type  $(2^*, 3, \dots, 3)$ . Let us now consider with the fields of type  $(2^*, 3, 3)$ . Let  $\chi$  be an odd quadratic character,  $\varphi_1, \varphi_2$  two cubic characters,  $N$  the field associated with  $\langle \chi, \varphi_1, \varphi_2 \rangle$ . Let  $M_1, M_2, M_3$  and  $M_4$  be the subfields associated with  $\langle \chi, \varphi_1 \rangle$ ,  $\langle \chi, \varphi_1 \varphi_2 \rangle$ ,  $\langle \chi, \varphi_1 \varphi_2^2 \rangle$ , and  $\langle \chi, \varphi_2 \rangle$ , respectively. The fact that  $h_N = g_N$  implies that  $h_{M_i} = g_{M_i}$  for  $1 \leq i \leq 4$ . According to [Lou4], there is only one  $N$  such that  $N = M_1 M_2 M_3 M_4$  with  $h_{M_i} = g_{M_i}$  for  $1 \leq i \leq 4$ . That is,  $N = M_1 M_2 M_3 M_4 = M_1 M_2$ , where  $M_1, M_2, M_3$  and  $M_4$  are associated with  $\langle \chi_3, \psi_9 \rangle$ ,  $\langle \chi_3, \chi_7^2 \rangle$ ,  $\langle \chi_3, \chi_7^4 \psi_9 \rangle$ , and  $\langle \chi_3, \chi_7^2 \rangle$  respectively. We verify that this field has  $h_N = g_N = 1$ . Since there is only one field of type  $(2^*, 3, 3)$  with class number equal to its genus class number, there is no field of type  $(2^*, 3, 3 \dots 3)$ , of degree  $\geq 54$  with class number equal to its genus class number. □

**Theorem 2.** *Let  $N$  be an imaginary abelian number field such that the Galois group  $G$  is an elementary 2-group. Assume that  $g_N = h_N$ . Then we have*

- (1)  $[N : \mathbb{Q}] \leq 2^3$ .
- (2) If  $G = (2, 2)$ , then  $h_N = g_N = 1, 2$  or  $4$ .
- (3) If  $G = (2, 2, 2)$ , then  $h_N = g_N = 1$  or  $2$ .

*Proof.* See Theorem 1 in [M]. □

**Proposition 3.** (1) *Let  $N$  be an imaginary bicyclic biquadratic number field and let  $k_1$  and  $k_2$  be two imaginary quadratic subfields. If  $h_N = g_N$ , then  $h_{k_1}$  and  $h_{k_2}$  are 1, 2 or 4.*

- (2) There are exactly 219 bicyclic biquadratic number fields with class numbers equal to their genus class numbers. These fields are listed in Table III.
- (3) There exist exactly 42 imaginary abelian number fields  $N$  of degree  $> 4$  with class numbers equal to their genus numbers such that  $N^{(2)}$  is of type  $(2^*, 2^*)$ . These fields are listed in Table IV.

*Proof.* (1) Suppose that  $h_N = g_N$  and  $h_{k_1} \geq 8$ . Since  $h_N^- = \frac{Q_N}{2} h_{k_1} h_{k_2}$ , we must have  $h_N^- = 4$ ,  $Q_N = 1$ ,  $h_{k_1} = 8$ ,  $h_{k_2} = 1$  and  $h_{N_+} = 1$ . By [Corollary 1, M], there exist at most three rational primes ramified in  $N$ . The fact that  $h_N = g_N = 4 = \frac{1}{4} \prod e_p$  implies that there exist exactly three ramified primes including 2 with  $e_2 = 4$ . This leads a contradiction. Since the fact that  $e_2 = 4$ ,  $h_{k_2} = 1$  and  $h_{N_+} = 1$  implies  $k_2 \in \{\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2})\}$  and  $N_+ \in \{\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{p}), \mathbb{Q}(\sqrt{2p})\}$  with  $p \equiv 1 \pmod{4}$ , there exist at most two ramified primes in  $N = k_2 N_+$ .

- (2) There are 81 imaginary quadratic number fields with class numbers 1, 2 or 4 ([S1,2] and [A]). Among these 81 fields there are 51 fields with class numbers equal to their genus class numbers. In order to get all bicyclic biquadratic number fields with class numbers equal to their genus class numbers it is sufficient to examine the composita of two of those 51 quadratic fields. Note that we do not assume the Generalized Riemann Hypothesis.
- (3) Let  $N$  be an imaginary abelian number fields of degree  $> 4$ . Assume that  $h_N = g_N$  and  $N^{(2)}$  is of type  $(2^*, 2^*)$ . Bearing in mind the imaginary cyclic fields with class numbers equal to their genus class numbers and Proposition 2 it remains to consider the fields  $N$  of type  $(2^*, 2^*, 3)$  or  $(2^*, 2^*, 5)$ .

i) Let  $\tau_1, \tau_2$  be two odd quadratic characters,  $\varphi$  a cubic characters,  $N$  the field associated with  $\langle \tau_1, \tau_2, \varphi \rangle$ . Let  $M_1$  and  $M_2$  be the imaginary sextic subfields associated with  $\langle \tau_1, \varphi \rangle$  and  $\langle \tau_2, \varphi \rangle$ , respectively. Let  $E$  be the imaginary bicyclic biquadratic subfield associated with  $\langle \tau_1, \tau_2 \rangle$ ,  $k_1$  and  $k_2$  the imaginary quadratic subfields associated with  $\tau_1$  and  $\tau_2$ , respectively. Using [Lou4], we make a list of  $(M_1, M_2)$ 's such that  $h_{M_1} = g_{M_1}$ ,  $h_{M_2} = g_{M_2}$  and  $h_E = g_E$ . There are 40 pairs of  $(M_1, M_2)$  satisfying  $h_{M_1} = g_{M_1}$ ,  $h_{M_2} = g_{M_2}$  and  $h_E = g_E$ . For these 40 fields we determine  $h_N^-$  using the fact that  $h_N^- = Q_N \omega_N \frac{h_{M_1}^- h_{M_2}^-}{\omega_{M_1} \omega_{M_2}}$  and  $Q_N = Q_E$ . The class numbers of real sextic number fields are known by [Mä]. There are 39 fields  $N$  of type  $(2^*, 2^*, 3)$  with  $h_N = g_N$ . (The remaining field is associated with  $\langle \chi_4 \psi_8, \chi_{11}^5, \chi_7^2 \rangle$ . This field has class number 3 and genus class number one.)

- ii) By the same method as i) we verify that there are three imaginary abelian number fields  $N$  of type  $(2^*, 2^*, 5)$  with  $h_N = g_N$ . □

**Proposition 4.** *There are exactly 19 imaginary abelian number fields  $N$  with class numbers equal to their genus class numbers such that the subfield  $N^{(2)}$  is type of  $(2^*, 2^*, 2^*)$ . There are exactly 17 fields of type  $(2^*, 2^*, 2^*)$ , and two fields of type  $(2^*, 2^*, 2^*, 3)$ . These fields are listed in Table V.*

*Proof.* Let  $N$  be an imaginary abelian number field  $N$  with class number equal to its genus class number such that the subfield  $N^{(2)}$  is type of  $(2^*, 2^*, 2^*)$ . According to [Lou4] and [CK], if  $p \geq 11$ , then  $G^{(p)}$  is trivial. Let us begin with the fields of type  $(2^*, 2^*, 2^*)$ .

If  $N$  is of type  $(2^*, 2^*, 2^*)$  and if  $h_N = g_N$ , then  $h_N = g_N = 1$  or  $2$ . Let  $k_1, k_2, k_3$  and  $k_4$  be the four imaginary quadratic subfields of  $N$ . Since  $h_N^- = Q_N \omega_N \prod_{i=1}^4 \frac{h_i}{\omega_i}$ , where  $h_i = h_{k_i}$  and  $\omega_i$  is the number of roots of unity in  $k_i$  for  $i = 1, 2, 3$  and  $4$ , there are at least three imaginary quadratic subfields  $k_i$  with  $h_{k_i} \leq 4$  among the four imaginary quadratic subfields of  $N$ . It remains us to consider the composita  $k_1 k_2 k_3$  of three imaginary quadratic subfields  $k_i$  with class numbers  $1, 2$  or  $4$ ,  $i = 1, 2$  and  $3$ . There are 18 composita  $N = k_1 k_2 k_3$  satisfying the following:

- i)  $h_{k_i} = 1, 2$  or  $4$  for  $i = 1, 2, 3$ .
- ii)  $h_{k_i} = g_{k_i}$  for  $i = 1, 2, 3$ .
- iii) For the six imaginary bicyclic biquadratic subfields  $M$  of  $N$  we have  $h_M = g_M$ .

Computing the unit indices  $Q_N$  by [HY3] and  $h_{N^+}$  by [Satz 5, K] for these 18 fields  $N$ , we verify that there are exactly 17 octic fields  $N$  with  $h_N = g_N$ .

Consider now the fields of type  $(2^*, 2^*, 2^*, 3)$ . Let  $\tau_1, \tau_2, \tau_3$  be three odd quadratic characters,  $\varphi$  a cubic character,  $N$  the field associated with  $\langle \tau_1, \tau_2, \tau_3, \varphi \rangle$ . Let  $M_1, M_2, M_3$  and  $M_4$  be the four imaginary sextic subfields of  $N$  associated with  $\langle \tau_1, \varphi \rangle, \langle \tau_2, \varphi \rangle, \langle \tau_3, \varphi \rangle$ , and  $\langle \tau_1 \tau_2 \tau_3, \varphi \rangle$ , respectively. Let  $E$  be the subfield associated with  $\langle \tau_1, \tau_2, \tau_3 \rangle$ . According to [Lou4], there are only two fields  $N$  such that  $h_{M_i} = g_{M_i}$  for  $1 \leq i \leq 4$  and  $h_E = g_E$ . For these two fields we compute  $h_N^-$  and  $h_{N^+}$ . The results are summarized in Table V.

Finally, it is sufficient to notice that there is no field  $N$  of type  $(2^*, 2^*, 2^*, 5)$  with  $h_N = g_N$ . □

### 3.2. $N^{(2)}$ is of type $(2^{m_1}, 2^{m_2})$ with $m_1 > m_2 \geq 1$ .

**Proposition 5.** (1) *There are 15 imaginary abelian number fields of type  $(4^*, 2)$  with class numbers equal to their genus class numbers. These fields are given in Table VIII*

- (2) Let  $m$  be an odd integer with  $m \geq 3$ . There is no imaginary abelian number field of degree  $8m$  with class number equal to its genus class number such that the subfield  $N^{(2)}$  is of type  $(4^*, 2)$ .
- (3) There is no field of type  $(8^*, 2)$  with class number equal to its genus class number. There is no field of type  $(16^*, 2)$  with class number equal to its genus class number. Consequently, if  $N$  is an imaginary abelian number field with  $h_N = g_N$  such that the subfield  $N^{(2)}$  is of type  $(2^{m_1^*}, 2^{m_2^*})$  with  $m_1 > m_2 \geq 1$ , then  $N$  is of type  $(4^*, 2)$ .
- (4) There is no imaginary abelian number field of type  $(4^*, 2, 2)$  with class number equal to its genus class number.
- (5) If  $N$  is an imaginary abelian number field with  $h_N = g_N$ , then the 2-rank of  $G^{(2)}$  is less than or equal to 3. Moreover, if the 2-rank of  $G^{(2)}$  is equal to 3, then  $G^{(2)} = (2^*, 2^*, 2^*)$  or  $(4^*, 2^*, 2^*)$ .

*Proof.* (1) Let  $\varphi$  be an odd primitive character of order 4,  $\chi$  an even primitive quadratic character,  $N$  the associated field with  $\langle \varphi, \chi \rangle$ . Let  $M_1$  and  $M_2$  be two imaginary cyclic quartic subfields of  $N$  associated with  $\langle \varphi \rangle$  and  $\langle \varphi\chi \rangle$ , respectively. The real quadratic subfields of  $M_1$  and  $M_2$  coincide. In order to obtain  $N$  with  $h_N = g_N$  it is sufficient to consider the composita  $M_1M_2$  such that  $h_{M_1} = g_{M_1}$  and  $h_{M_2} = g_{M_2}$ . There are 35 fields  $N$  such that  $N = M_1M_2$ ,  $h_{M_1} = g_{M_1}$  and  $h_{M_2} = g_{M_2}$ . The Hasse unit indices  $Q_N$  are easily obtained from Sätze 15 and 22 of [H]. The relative class number  $h_N^-$  can be expressed as

$$h_N^- = Q_N \frac{\omega_N}{\omega_{M_1}\omega_{M_2}} h_{M_1}^- h_{M_2}^-.$$

The class number  $h_{N^+}$  can be computed following [K]. It remains 15 fields  $N$  such that  $h_N = g_N$ . Similarly we verify (3).

- (2) Let  $N$  be an imaginary abelian number field of degree  $8m$ , containing a subfield of type  $(4^*, 2)$ . Suppose  $h_N = g_N$ . According to [CK], we have  $m = 3$ . Let  $\varphi, \chi$  and  $\omega$  be an odd character of order 4, an even quadratic character and a cubic character, respectively. Assume that  $N$  is associated with  $\langle \varphi, \chi, \omega \rangle$ . Let  $M_1$  and  $M_2$  be the subfields associated with  $\langle \varphi\omega \rangle$  and  $\langle \varphi\chi\omega \rangle$ , respectively. Using Table II in [CK], we verify that there is no pair of  $(M_1, M_2)$  such that  $h_{M_1} = g_{M_1}$  and  $h_{M_2} = g_{M_2}$ .
- (4) Let  $\varphi$  be an odd primitive character of order 4,  $\chi_1$  and  $\chi_2$  two even primitive quadratic character,  $N$  the associated field with  $\langle \varphi, \chi_1, \chi_2 \rangle$ . Let  $M_1, M_2, M_3$  and  $M_4$  be four imaginary cyclic quartic subfields of  $N$  associated with  $\langle \varphi \rangle, \langle \varphi\chi_1 \rangle, \langle \varphi\chi_2 \rangle$  and  $\langle \varphi\chi_1\chi_2 \rangle$ , respectively. The fields  $M_i, 1 \leq i \leq 4$ , have the same real quadratic subfield. Suppose that  $h_N = g_N$ . Then we have  $h_{M_i} = g_{M_i}$ , for  $1 \leq i \leq 4, h_{M_1M_2} = g_{M_1M_2}, h_{M_1M_3} = g_{M_1M_3}, h_{M_1M_4} = g_{M_1M_4}, h_{M_2M_3} = g_{M_2M_3}, h_{M_2M_4} =$



$g_{M_2M_4}$  and  $h_{M_3M_4} = g_{M_3M_4}$ . We verify that there is no such quadruple of  $(M_1, M_2, M_3, M_4)$ .

- (5) By (3) and (4) it is sufficient to verify that there is no field  $N$  of type  $(4^*, 4^*, 2^*)$  with  $h_N = g_N$ . Suppose that there is a field  $N$  of type  $(4^*, 4^*, 2^*)$  with  $h_N = g_N$ . Let  $\varphi_1$  and  $\varphi_2$  be two odd primitive characters of order 4,  $\chi$  the odd quadratic character such that  $N$  is associated to the group  $\langle \varphi_1, \varphi_2, \chi \rangle$ . Then we would have three fields of type  $(4^*, 4^*)$  with class numbers equal to their genus class numbers, i.e. the fields associated with  $\langle \varphi_1, \varphi_2 \rangle, \langle \varphi_1, \varphi_1\varphi_2\chi \rangle$  and  $\langle \varphi_2, \varphi_1\varphi_2\chi \rangle$ . This contradicts Proposition 8.(1) below. □

**3.3.  $N^{(2)}$  is of type  $(2^{m_1^*}, 2^{m_2^*})$  with  $m_1 \geq 2$  and  $m_1 \geq m_2$ .**

Bearing in mind Proposition 5.(3) if  $h_N = g_N$  and  $N^{(2)}$  is of type  $(2^{m_1^*}, 2^{m_2^*})$  with  $m_2 \geq 2$ , then  $m_1 = m_2 = 2$ . We need only consider  $N$  such that  $N^{(2)}$  is of type  $(4^*, 2^*), (8^*, 2^*), (16^*, 2^*)$  or  $(4^*, 4^*)$ .

**Proposition 6.** *There exist 39 imaginary abelian number fields  $N$  with  $h_N = g_N$  such that the subfield  $N^{(2)}$  is of type  $(4^*, 2^*)$ : 36 of them are of type  $(4^*, 2^*)$  and 3 of them are of type  $(4^*, 2^*, 3)$ . These fields are listed in Table VI.*

*Proof.* According to [CK], if  $N$  is an imaginary abelian number field with  $h_N = g_N$  such that the subfield  $N^{(2)}$  is of type  $(4^*, 2^*)$ , then  $[N : \mathbb{Q}] \leq 40$ . Our first step is to determine all fields of type  $(4^*, 2^*)$ . Let  $\varphi$  be an odd character of order 4,  $\chi$  an odd quadratic character,  $N$  the field associated with  $\langle \varphi, \chi \rangle$ . Let  $M$  be the field associated with  $\langle \varphi \rangle$  and let  $k_1$  and  $k_2$  be the two imaginary quadratic subfields associated with  $\langle \chi \rangle$  and  $\langle \varphi^2\chi \rangle$ , respectively. Using [Lou2] we make a list of  $N$  such that  $h_M = g_M$  and  $h_E = g_E$ , where  $E$  is the field associated  $\langle \varphi^2, \chi \rangle$ . We use the formula

$$h_N^- = Q_N \omega_N \prod_{\chi \in X_N^-} \left( -\frac{1}{2} B_{1,\chi} \right) = Q_N \omega_N \frac{h_M^- h_{k_1} h_{k_2}}{\omega_M \omega_{k_1} \omega_{k_2}}.$$

To determine  $Q_N$  we use the tables in [H], [YH1] and [YH2] for the fields of conductors  $\leq 200$ , and Sätze 15 and 22 in [H] for the fields of conductor  $> 200$ . For example let us consider  $Q_N$  for the field  $N$  which is associated with  $\langle \chi_5\psi_8, \chi_7^3 \rangle$ . We have  $N_+ = \mathbb{Q} \left( \sqrt{7(5 + \sqrt{5})} \right)$  and  $N = N_+ (\sqrt{-7})$ .

In order to determine  $Q_N$  it is sufficient to know whether the prime ideal of  $N$  lying above 7 is principal or not. Using the function `IdealIsPrincipal` of KASH([KT]) we know that this prime ideal is principal. Hence we conclude that  $Q_N = 2$ . On the other hand, the class number of real quartic fields associated with  $\langle \varphi\chi \rangle$  are known by [G]. Using the above results we can

easily determine all fields of type  $(4^*, 2^*, 3)$  and verify that there is no field of type  $(4^*, 2^*, 5)$  with class number equal to its genus class number. The computational results are compiled in Table VI.  $\square$

**Proposition 7.** (1) *Let  $N$  be an imaginary abelian number field with  $h_N = g_N$ . If the subfield  $N^{(2)}$  is of type  $(8^*, 2^*)$ , then  $N$  is of type  $(8^*, 2^*)$ . There exist three imaginary abelian number fields of type  $(8^*, 2^*)$  with class numbers equal to their genus class numbers. These fields are given in Table VII.*

(2) *There is no imaginary number field of type  $(16^*, 2^*)$  with class number equal to its genus class number.*

*Proof.* (1) According to [CK] we know that for an odd integer  $m \geq 3$  there is no imaginary cyclic number field of degree  $8m$  with class number equal to its genus class number. Let  $\varphi$  be an odd character of order 8,  $\chi$  an odd quadratic character,  $N$  the field associated with  $\langle \varphi, \chi \rangle$ . Let  $M, L, F, E, k_1$ , and  $k_2$  be the subfields associated with  $\langle \varphi \rangle, \langle \varphi^2 \chi \rangle, \langle \varphi^2, \chi \rangle, \langle \varphi^4, \chi \rangle, \langle \chi \rangle$  and  $\langle \varphi^4 \chi \rangle$ , respectively. By [Lou2], Propositions 3 and 6 we have three fields  $N$  such that  $h_M = g_M, h_L = g_L, h_F = g_F$  and  $h_E = g_E$ . These fields are listed in Table VII. Using the formula

$$h_N^- = Q_N \frac{\omega_N}{\omega_M \omega_L \omega_{k_1} \omega_{k_2}} h_M^- h_L^- h_{k_1} h_{k_2}$$

we obtain immediately  $h_N^-$ . The class numbers of real cyclic fields of conductor  $\leq 100$  are known by [Ma]. The class number of the real abelian number fields of conductor 160 in Table VII is equal to 2 according to Theorem 3 of [L].

(2) It is clear from the fact that there is only one imaginary cyclic number field of degree 16 whose class number is equal to its genus class number ([Lou2]).  $\square$

**Proposition 8.** (1) *There are two imaginary abelian number fields of type  $(4^*, 4^*)$  with class numbers equal to their genus class number. These fields are given in Table IX.*

(2) *Let  $m$  be an odd integer with  $m \geq 3$ . There is no imaginary abelian number field of degree  $16m$  with class number equal to its genus class number such that the subfield  $N^{(2)}$  is of type  $(4^*, 4^*)$ .*

*Proof.* (1) Let  $\varphi_1$  and  $\varphi_2$  be two odd quartic characters,  $N$  the field associated with  $\langle \varphi_1, \varphi_2 \rangle$ . Let  $M_1, M_2, M_3$  and  $M_4$  be the fields associated with  $\langle \varphi_1 \rangle, \langle \varphi_2 \rangle, \langle \varphi_1 \varphi_2^2 \rangle$  and  $\langle \varphi_1^2 \varphi_2 \rangle$ , respectively. We have only two fields  $N$  such that  $h_{M_i} = g_{M_i}$  for  $1 \leq i \leq 4$ ,  $h_{M_1 M_3} = g_{M_1 M_3}$  and  $h_{M_2 M_4} = g_{M_2 M_4}$ . These two fields have class number one ([Y]).

(2) It follows immediately from Proposition 5.(2). □

**3.4.  $N^{(2)}$  is of type  $(4^*, 2^*, 2^*)$ .**

**Proposition 9.** (1) *There are 7 imaginary abelian number fields of type  $(4^*, 2^*, 2^*)$  with class numbers equal to their genus class numbers. These fields are given in Table X.*

(2) *Let  $m$  be an odd integer  $\geq 3$ . There is no imaginary number field of degree  $16m$  with class number equal to its genus class number such that the subfield  $N^{(2)}$  is of type  $(4^*, 2^*, 2^*)$ .*

*Proof.* (1) Let  $\varphi$  be an odd character of order 4,  $\chi_1$  and  $\chi_2$  two odd quadratic characters,  $N$  the field associated with  $\langle \varphi, \chi_1, \chi_2 \rangle$ . Let  $M_1, M_2, F_1, F_2, L$  and  $E$  be the imaginary subfields associated with  $\langle \varphi \rangle, \langle \varphi \chi_1 \chi_2 \rangle, \langle \varphi, \chi_1 \rangle, \langle \varphi, \chi_2 \rangle, \langle \varphi, \chi_1 \chi_2 \rangle$  and  $\langle \varphi^2, \chi_1, \chi_2 \rangle$ , respectively. Using [Lou2], Propositions 4 and 6 we verify that there are 7 fields  $N$  with  $h_{M_1} = g_{M_1}, h_{M_2} = g_{M_2}, h_{F_1} = g_{F_1}, h_{F_2} = g_{F_2}, h_L = g_L$  and  $h_E = g_E$ . For these 7 fields we compute  $h_N^-$  and  $h_{N^+}$ . Note that

$$h_N^- = \frac{Q_N}{Q_E} \omega_N \frac{h_{M_1}^-}{\omega_{M_1}} \frac{h_{M_2}^-}{\omega_{M_2}} \frac{h_E^-}{\omega_E}.$$

(2) The proof of (2) is similar to that of Proposition 5.(2). □

To conclude, the proof of Theorem 1 is completed by Propositions 2-9. All computations were carried out using PARI-GP[P] and KANT V4[KT]. **Acknowledgements.** The authors wish to express their gratitude to S. Louboutin and K. Yamamura for suggesting the problem and for many stimulating conversations.

**4. Tables**

TABLE I. Cyclic number fields

$$\chi_5(2) = e^{2\pi i/4}, \chi_7(3) = e^{2\pi i/6}, \psi_9(2) = e^{2\pi i/3}, \chi_{13}(2) = e^{2\pi i/12},$$

$$\chi_{17}(3) = e^{2\pi i/16}, \chi_{19}(2) = e^{2\pi i/18}, \chi_{31}(3) = e^{2\pi i/30}$$

Type	$h_N = g_N$	$f_N$
$2^*$	1	3, 4, 7, 8, 11, 19, 43, 67, 163
	2	15, 20, 24, 35, 40, 51, 52, 88, 91, 115, 123, 148, 187, 232 235, 267, 403, 427
4	4	84, 120, 132, 168, 195, 228, 280, 312, 340, 372, 408, 435, 483 520, 532, 555, 595, 627, 708, 715, 760, 795, 1012, 1435
		8
	16	5460

TABLE I continued.

Type	$N$	$f_N$	$h_N^-$	$h_{N^+}$	$g_N$	Type	$N$	$f_N$	$h_N^-$	$h_{N^+}$	$g_N$	
$4^*$	$\chi_5$	5	1	1	1	$(2^*, 3)$	$\chi_7^3, \chi_7^2$	7	1	1	1	
	$\chi_{13}^3$	13	1	1	1		$\chi_3, \psi_9$	9	1	1	1	
	$\chi_4\psi_{16}$	16	1	1	1		$\chi_{19}^9, \chi_{19}^6$	19	1	1	1	
	$\chi_{29}^7$	29	1	1	1		$\chi_3, \chi_7^2$	21	1	1	1	
	$\chi_{37}^9$	37	1	1	1		$\chi_4, \chi_7^2$	28	1	1	1	
	$\chi_5\psi_8$	40	2	1	2		$\chi_5^2\chi_7^3, \chi_7^2$	35	2	1	2	
	$\chi_3\psi_{16}$	48	2	1	2		$\chi_4, \psi_9$	36	1	1	1	
	$\chi_{53}^{13}$	53	1	1	1		$\chi_3, \chi_{13}^4$	39	1	1	1	
	$\chi_3\chi_4\chi_5$	60	4	1	4		$\chi_{43}^{21}, \chi_{43}^{14}$	43	1	1	1	
	$\chi_{61}^{15}$	61	1	1	1		$\chi_3\chi_5^2, \psi_9$	45	2	1	2	
	$\chi_5\chi_{13}^6$	65	2	1	2		$\chi_4\chi_{13}^6, \chi_{13}^4$	52	2	1	2	
	$\chi_5^2\chi_{13}$	65	2	1	2		$\chi_4\psi_8, \chi_7^2$	56	1	1	1	
	$\chi_5^2\chi_4\psi_{16}$	80	2	1	2		$\chi_3, \chi_7^2\psi_9$	63	1	3	3	
	$\chi_5\psi_{16}$	80	2	2	4		$\chi_3, \chi_7^4\psi_9$	63	1	3	3	
	$\chi_5\chi_{17}^8$	85	2	1	2		$\chi_7^3, \psi_9$	63	1	1	1	
	$\chi_5\chi_{17}^{12}$	85	2	2	4		$\chi_7^3, \chi_7^4\psi_9$	63	1	3	3	
	$\psi_8\chi_{13}^3$	104	2	1	2		$\chi_{67}^{33}, \chi_{67}^{22}$	67	1	1	1	
	$\chi_3\chi_5\chi_7^3$	105	4	1	4		$\chi_3\psi_8, \psi_9$	72	2	1	2	
	$\chi_7^3\chi_{17}^4$	119	2	1	2		$\chi_4, \chi_{19}^6$	76	1	1	1	
	$\chi_3\chi_4\psi_8\chi_5$	120	4	1	4		$\chi_{11}^5, \chi_7^2$	77	1	1	1	
	$\chi_4\chi_5\chi_7^3$	140	4	1	4		$\chi_3\chi_4\chi_7^3, \chi_7^2$	84	4	1	4	
	$\chi_3\chi_5\chi_{13}^9$	195	4	2	8		$\chi_7^3, \chi_{13}^4$	91	1	1	1	
	$\chi_3\chi_5^2\chi_{17}^4$	255	4	1	4		$\chi_7^3, \chi_7^4, \chi_{13}^4$	91	1	3	3	
	$(4^*, 3)$	$\chi_{13}^3, \chi_{13}^4$	13	1	1		1	$\chi_7^3\chi_{13}^6, \chi_7^2\chi_{13}^4$	91	2	3	6
		$\chi_5, \chi_7^2$	35	1	1		1	$\chi_7^3\chi_{13}^6, \chi_7^4\chi_{13}^4$	91	2	3	6
$\chi_{37}^9, \chi_{37}^{12}$		37	1	1	1	$\chi_3, \chi_{31}^{10}$	93	1	1	1		
$\chi_5, \psi_9$		45	1	1	1	$\chi_4\psi_8, \chi_{13}^4$	104	1	1	1		
$\chi_{61}^{15}, \chi_{61}^{20}$		61	1	1	1	$\chi_3\chi_5^2, \chi_7^2$	105	2	1	2		
$\chi_{13}^3, \chi_7^2\chi_{13}^4$		91	1	3	3	$\chi_3, \psi_9\chi_{13}^8$	117	1	3	3		
						$\chi_3, \chi_{43}^{14}$	129	1	1	1		
$(2^*, 7)$	$\chi_{43}^{21}, \chi_{43}^6$	43	1	1	1	$\chi_7^3, \chi_7^4, \chi_{19}^6$	133	1	3	3		
	$\chi_7^3, \psi_{49}$	49	1	1	1	$\chi_{19}^9, \psi_9\chi_{19}^{12}$	171	1	3	3		
$16^*$	$\chi_{17}$	17	1	1	1	$\chi_7^3, \chi_7^4\chi_{31}^{10}$	217	1	3	3		
$(2^*, 9)$	$\chi_3, \psi_{27}$	27	1	1	1	$\chi_{19}^9, \chi_{13}^8\chi_{19}^6$	247	1	3	3		
	$\chi_{19}^9, \chi_{19}^2$	19	1	1	1							
$(4^*, 5)$	$\chi_5, \psi_{25}$	25	1	1	1	$8^*$	$\chi_4\psi_{32}$	32	1	1	1	
$(2^*, 5)$	$\chi_{11}^5, \chi_{11}^2$	11	1	1	1		$\chi_{41}^5$	41	1	1	1	
	$\chi_3, \chi_{11}^2$	33	1	1	1		$\chi_3\chi_{17}^2$	51	2	1	2	
	$\chi_4, \chi_{11}^2$	44	1	1	1		$\chi_5\chi_{17}^2$	85	2	2	4	
	$\chi_3\chi_5^2, \psi_{25}$	75	1	2	2							

TABLE II.  $(2^*, 3, 3)$

$N$	$N^+$	$f_N$	$Q_N$	$h_N^-$	$h_{N^+}$	$g_N$
$\chi_3, \chi_7^2, \psi_9$	$\chi_7^2, \psi_9$	63	1	1	1	1

TABLE III. ( $2^*$ ,  $2^*$ )

$$h_N = g_N = 1$$

$f_{k_1}$	$f_{k_2}$
3	4, 7, 8, 11, 15, 19, 24, 43, 51, 67, 123, 163, 267
4	7, 8, 11, 19, 20, 43, 52, 67, 148, 163
7	8, 11, 19, 35, 43, 91, 163, 427
8	11, 19, 40, 43, 67, 232
11	19, 67, 88, 163, 187
19	67, 163
43	67, 163
67	163

$$h_N = g_N = 2$$

$f_{k_1}$	$f_{k_2}$
3	20, 35, 40, 84, 88, 115, 132, 168, 187, 228, 232, 235, 372, 483, 627, 708
4	15, 24, 35, 40, 84, 88, 91, 115, 132, 228, 232, 372, 403, 532, 708, 1012
7	15, 20, 40, 51, 52, 84, 115, 123, 168, 187, 235, 267, 403, 483, 532
8	15, 20, 24, 35, 52, 88, 91, 115, 148, 168, 235, 403, 427
11	24, 51, 52, 91, 123, 132, 232, 403, 427, 627, 1012
15	20, 35, 40, 43, 67, 115, 163, 235
19	24, 52, 88, 91, 123, 148, 228, 232, 403, 532, 627
20	35, 40, 43, 67, 115, 163, 235
24	43, 67, 88, 163
35	40, 43, 67, 115, 163, 235
40	43, 67, 115, 163, 235
43	88, 115, 148, 232, 235, 267, 427
51	163, 187
52	67, 91, 163, 403
67	88, 123, 235, 403
88	163
91	163, 403
115	163, 235
148	163
163	187, 232, 235, 267, 403

$$h_N = g_N = 4$$

$f_{k_1}$	$f_{k_2}$
4	120, 168, 280, 312, 760
8	84, 120, 280, 312, 372, 532, 760
20	88, 120, 280, 760
24	52, 84, 132, 148, 168, 228, 372, 708
40	88, 120, 280, 760
52	312
88	132, 148, 1012

TABLE IV.

(2\*, 2\*, 3)

$\tau_1$	$\tau_2$	$\varphi$	$N^+$	$f_N$	$h_{M_1}^-$	$h_{M_2}^-$	$Q_N$	$h_N^-$	$h_{N^+}$	$g_N$	nb.	
$\chi_7^3$	$\chi_3$	$\chi_7^2$	$\chi_3\chi_7^3, \chi_7^2$	21	1	1	2	1	1	1	1	
	$\chi_4$	$\chi_7^2$	$\chi_4\chi_7^3, \chi_7^2$	28	1	1	2	1	1	1	2	
	$\chi_5^2\chi_7^3$	$\chi_7^2$	$\chi_5^2, \chi_7^2$	35	1	2	1	1	1	1	3	
	$\chi_4\psi_8$	$\chi_7^2$	$\chi_4\chi_7^3\psi_8, \chi_7^2$	56	1	1	2	1	1	1	4	
	$\chi_{11}^5$	$\chi_7^2$	$\chi_7^3\chi_{11}^5, \chi_7^2$	77	1	1	2	1	1	1	5	
	$\chi_3\chi_4\chi_7^3$	$\chi_7^2$	$\chi_3\chi_4, \chi_7^2$	84	1	4	1	2	1	2	6	
	$\chi_3\chi_5^2$	$\chi_7^2$	$\chi_3\chi_5^2\chi_7^3, \chi_7^2$	105	1	2	1	1	2	2	7	
$\chi_3$	$\chi_4$	$\chi_7^2$	$\chi_3\chi_4, \chi_7^2$	84	1	1	2	1	1	1	8	
	$\chi_5^2\chi_7^3$	$\chi_7^2$	$\chi_3\chi_5^2\chi_7^3, \chi_7^2$	105	1	2	1	1	2	2	9	
	$\chi_4\psi_8$	$\chi_7^2$	$\chi_3\chi_4\psi_8, \chi_7^2$	168	1	1	2	1	1	1	10	
	$\chi_{11}^5$	$\chi_7^2$	$\chi_3\chi_{11}^5, \chi_7^2$	231	1	1	2	1	1	1	11	
	$\chi_3\chi_4\chi_7^3$	$\chi_7^2$	$\chi_4\chi_7^3, \chi_7^2$	84	1	4	1	2	1	2	12	
	$\chi_3\chi_5^2$	$\chi_7^2$	$\chi_5^2, \chi_7^2$	105	1	2	1	1	1	1	13	
	$\chi_5^2\chi_7^3$	$\chi_7^2$	$\chi_4\chi_5^2\chi_7^3, \chi_7^2$	140	1	2	1	1	2	2	14	
$\chi_4$	$\chi_4\psi_8$	$\chi_7^2$	$\psi_8, \chi_7^2$	56	1	1	1	1	1	1	15	
	$\chi_{11}^5$	$\chi_7^2$	$\chi_4\chi_{11}^5, \chi_7^2$	308	1	1	2	1	1	1	16	
	$\chi_3\chi_4\chi_7^3$	$\chi_7^2$	$\chi_3\chi_7^3, \chi_7^2$	84	1	4	1	2	1	2	17	
	$\chi_3\chi_5^2$	$\chi_7^2$	$\chi_3\chi_4\chi_5^2, \chi_7^2$	420	1	2	1	1	2	2	18	
	$\chi_5^2\chi_7^3$	$\chi_7^2$	$\chi_4\psi_8\chi_5^2\chi_7^3, \chi_7^2$	280	2	1	1	1	2	2	19	
	$\chi_3\chi_5^2$	$\chi_7^2$	$\chi_3\chi_7^3, \chi_7^2$	105	2	2	1	2	1	2	20	
	$\chi_4\psi_8$	$\chi_3\chi_4\chi_7^3$	$\chi_7^2$	$\chi_3\chi_7^3\psi_8, \chi_7^2$	168	1	4	1	2	2	4	21
$\chi_3$	$\chi_3\chi_5^2$	$\chi_7^2$	$\chi_3\chi_4\psi_8\chi_5^2, \chi_7^2$	840	1	2	1	1	2	2	22	
	$\chi_4$	$\psi_9$	$\chi_3\chi_4, \psi_9$	36	1	1	2	1	1	1	23	
	$\chi_3\chi_5^2$	$\psi_9$	$\chi_5^2, \psi_9$	45	1	2	1	1	1	1	24	
	$\chi_7^3$	$\psi_9$	$\chi_3\chi_7^3, \psi_9$	63	1	1	2	1	1	1	25	
	$\chi_3\psi_8$	$\psi_9$	$\psi_8, \psi_9$	72	1	2	1	1	1	1	26	
	$\chi_4$	$\chi_3\chi_5^2$	$\psi_9$	$\chi_3\chi_4\chi_5^2, \psi_9$	180	1	2	1	1	2	2	27
		$\chi_7^3$	$\psi_9$	$\chi_4\chi_7^3, \psi_9$	252	1	1	2	1	1	1	28
$\chi_3\psi_8$		$\psi_9$	$\chi_3\chi_4\psi_8, \psi_9$	72	1	2	2	2	1	2	29	
$\chi_3\chi_5^2$	$\chi_7^3$	$\psi_9$	$\chi_3\chi_5^2\chi_7^3, \psi_9$	315	2	1	1	1	2	2	30	
$\chi_3$	$\chi_7^3$	$\chi_{13}^4$	$\chi_3\chi_7^3, \chi_{13}^4$	273	1	1	2	1	1	1	31	
	$\chi_4\psi_8$	$\chi_{13}^4$	$\chi_3\chi_4\psi_8, \chi_{13}^4$	312	1	1	2	1	1	1	32	
$\chi_4\chi_{13}^6$	$\chi_7^3$	$\chi_{13}^4$	$\chi_4\chi_7^3\chi_{13}^6, \chi_{13}^4$	364	2	1	1	1	2	2	33	
	$\chi_4\psi_8$	$\chi_{13}^4$	$\psi_8\chi_{13}^6, \chi_{13}^4$	208	2	1	1	1	2	2	34	
$\chi_7^3$	$\chi_4\psi_8$	$\chi_{13}^4$	$\chi_4\psi_8\chi_7^3, \chi_{13}^4$	728	1	1	2	1	1	1	35	
$\chi_4$	$\chi_{19}^9$	$\chi_{13}^4$	$\chi_4\chi_{19}^9, \chi_{13}^4$	76	1	1	2	1	1	1	36	
$\chi_3$	$\chi_{43}^{14}$	$\chi_{13}^4$	$\chi_3\chi_{43}^{14}, \chi_{13}^4$	129	1	1	2	1	1	1	37	
$\chi_3$	$\chi_7^3$	$\chi_7^4\psi_9$	$\chi_3\chi_7^3, \chi_7^4\psi_9$	63	1	1	2	1	3	3	38	
$\chi_7^3$	$\chi_7^3\chi_{13}^6$	$\chi_7^4\chi_{13}^4$	$\chi_{13}^6, \chi_7^4\chi_{13}^4$	91	1	2	1	1	3	3	39	

$$\chi_7(3) = e^{2\pi i/6}, \chi_{13}(2) = e^{2\pi i/12}$$

(2\*, 2\*, 5)

$N$	$N^+$	$f_N$	$Q_N$	$h_N^-$	$h_{N^+}$	$g_N$
$\chi_3, \chi_{11}^5, \chi_{11}^2$	$\chi_3\chi_{11}^5, \chi_{11}^2$	33	2	1	1	1
$\chi_3, \chi_4, \chi_{11}^2$	$\chi_3\chi_4, \chi_{11}^2$	44	2	1	1	1
$\chi_4, \chi_{11}^5, \chi_{11}^2$	$\chi_4\chi_{11}^5, \chi_{11}^2$	132	2	1	1	1

TABLE V.

(2\*, 2\*, 2\*)

$h_N = g_N$	$(f_{k_1}, f_{k_2}, f_{k_3})$
1	(3, 4, 7), (3, 4, 8), (3, 4, 11), (3, 4, 15), (3, 4, 19), (3, 7, 8), (3, 7, 15) (3, 8, 15), (3, 11, 19), (3, 11, 24), (3, 11, 51), (4, 7, 19), (4, 7, 20), (4, 7, 52), (4, 8, 11), (4, 8, 20), (7, 8, 35)

(2\*, 2\*, 2\*, 3)

$N$	$N^+$	$f_N$	$h_{M_1}^-$	$h_{M_2}^-$	$h_{M_3}^-$	$h_{M_4}^-$	$Q_N$	$h_N^-$	$h_{N^+}$	$g_N$
$\chi_3, \chi_4, \chi_7^3, \chi_7^2$	$\chi_7^2, \chi_3\chi_4, \chi_3\chi_7^3$	84	1	1	1	4	2	1	1	1
$\chi_3, \chi_7^3, \chi_3\chi_5^2, \chi_7^2$	$\chi_7^2, \chi_5^2, \chi_3\chi_7^3$	105	1	1	2	2	2	1	1	1

TABLE VI.

(4\*, 2\*)

$\varphi$	$\chi$	$N^+$	$f_N$	$h_M^-$	$h_{k_1}$	$h_{k_2}$	$Q_N$	$h_N^-$	$h_{N^+}$	$g_N$	nb.
$\chi_5$	$\chi_3$	$\chi_3\chi_5$	15	1	1	2	2	1	1	1	1
	$\chi_4$	$\chi_4\chi_5$	20	1	1	2	2	1	1	1	2
	$\chi_7^3$	$\chi_5\chi_7^3$	35	1	1	2	2	1	1	1	3
	$\chi_4\psi_8$	$\chi_4\psi_8\chi_5$	40	1	1	2	2	1	1	1	4
$\chi_{13}^3$	$\chi_4$	$\chi_4\chi_{13}^3$	52	1	1	2	2	1	1	1	5
	$\chi_7^2$	$\chi_7^2\chi_{13}^3$	91	1	1	2	2	1	1	1	6
$\chi_4\psi_{16}$	$\chi_3$	$\chi_3\chi_4\psi_{16}$	48	1	1	2	2	1	1	1	7
	$\chi_4$	$\psi_{16}$	16	1	1	1	1	1	1	1	8
	$\chi_{11}^5$	$\chi_4\psi_{16}\chi_{11}^5$	176	1	1	2	2	1	1	1	9
	$\chi_4\chi_5^2$	$\chi_5^2\psi_{16}$	80	1	2	2	1	1	2	2	10
$\chi_{37}^9$	$\chi_4$	$\chi_4\chi_{37}^9$	148	1	1	2	2	1	1	1	11
$\chi_5\psi_8$	$\chi_4$	$\chi_4\psi_8\chi_5$	40	2	1	2	2	2	1	2	12
	$\chi_4\psi_8$	$\chi_4\chi_5$	40	2	1	2	2	2	1	2	13
$\chi_{29}^7$	$\chi_4\psi_8$	$\chi_4\psi_8\chi_{29}^7$	232	1	1	2	2	1	1	1	14
$\chi_3\chi_4\chi_5$	$\chi_3$	$\chi_4\chi_5$	60	4	1	2	1	2	1	2	15
	$\chi_4$	$\chi_3\chi_5$	60	4	1	2	1	2	1	2	16
	$\chi_4\psi_8$	$\chi_3\chi_5\psi_8$	120	4	1	2	1	2	2	4	17
$\chi_4\psi_{16}\chi_5^2$	$\chi_4$	$\chi_5^2\psi_{16}$	80	2	1	1	1	1	2	2	18
	$\chi_4\chi_5^2$	$\psi_{16}$	80	2	2	2	1	2	1	2	19
$\chi_5\psi_{16}$	$\chi_4$	$\chi_4\psi_{16}\chi_5$	80	2	1	2	2	2	2	4	20
	$\chi_4\psi_8$	$\chi_4\psi_{16}\chi_5$	80	2	1	2	2	2	2	4	21
$\chi_3\psi_{16}$	$\chi_3$	$\psi_{16}$	48	2	1	2	1	1	1	1	22
	$\chi_4$	$\chi_3\chi_4\psi_{16}$	48	2	1	1	2	2	1	2	23
	$\chi_{11}^5$	$\chi_3\chi_{11}^5\psi_{16}$	528	2	1	2	1	1	2	2	24
	$\chi_4\chi_5^2$	$\chi_3\chi_4\psi_{16}\chi_5^2$	240	2	2	2	1	2	2	4	25
$\chi_3\chi_5\chi_7^3$	$\chi_3$	$\chi_5\chi_7^3$	105	4	1	2	1	2	1	2	26
	$\chi_7^2$	$\chi_3\chi_5$	105	4	1	2	1	2	1	2	27
$\chi_7^3\chi_{17}^4$	$\chi_3$	$\chi_3\chi_7^3\chi_{17}^4$	357	2	1	2	1	1	2	2	28
	$\chi_{11}^5$	$\chi_7^3\chi_{11}^5\chi_{17}^4$	1309	2	1	2	1	1	2	2	29
$\psi_8\chi_{13}^3$	$\chi_4$	$\chi_4\psi_8\chi_{13}^3$	104	2	1	2	2	2	1	2	30
$\chi_3\chi_4\psi_8\chi_5$	$\chi_3$	$\chi_4\psi_8\chi_5$	120	4	1	2	1	2	1	2	31
	$\chi_4$	$\chi_3\chi_5\psi_8$	120	4	1	2	1	2	2	4	32
	$\chi_4\psi_8$	$\chi_3\chi_5$	120	4	1	2	1	2	1	2	33
$\chi_4\chi_5\chi_7^3$	$\chi_4$	$\chi_5\chi_7^3$	140	4	1	2	1	2	1	2	34
	$\chi_7^2$	$\chi_4\chi_5$	140	4	1	2	1	2	1	2	35
	$\chi_4\psi_8$	$\chi_5\chi_7^3\psi_8$	280	4	1	2	1	2	2	4	36

TABLE VI continued.

(4\*, 2\*, 3)

$N$	$N^+$	$f_N$	$Q_N$	$h_N^-$	$h_{N^+}$	$g_N$
$\chi_5, \chi_7^3, \chi_7^2$	$\chi_5\chi_7^3, \chi_7^2$	35	2	1	1	1
$\chi_3, \chi_5, \psi_9$	$\chi_3\chi_5, \psi_9$	45	2	1	1	1
$\chi_3, \chi_5, \chi_7^2$	$\chi_3\chi_5, \chi_7^2$	105	2	1	1	1

TABLE VII. (8\*, 2\*)

$N$	$N^+$	$f_N$	$h_M^-$	$h_L^-$	$h_E^-$	$Q_N$	$h_N^-$	$h_{N^+}$	$g_N$
$\chi_4\psi_{32}, \chi_4$	$\psi_{32}$	32	1	1	1	1	1	1	1
$\chi_4\psi_{32}, \chi_3$	$\chi_3\chi_4\psi_{32}$	96	1	2	1	2	1	1	1
$\chi_4\psi_{32}, \chi_4\chi_5^2$	$\chi_5^2\psi_{32}$	160	1	2	2	1	1	2	2

TABLE VIII. (4\*, 2)

$\varphi$	$\chi$	$N^+$	$f_N$	$h_{M_1}^-$	$h_{M_2}^-$	$Q_N$	$h_N^-$	$h_{N^+}$	$g_N$	nb.
$\chi_5$	$\psi_8$	$\chi_5^2, \psi_8$	40	1	2	1	1	1	1	1
	$\chi_3\chi_4$	$\chi_5^2, \chi_3\chi_4$	60	1	4	1	2	1	2	2
	$\chi_{13}$	$\chi_5^2, \chi_{13}$	65	1	2	1	1	1	1	3
	$\chi_{17}$	$\chi_5^2, \chi_{17}$	85	1	2	1	1	1	1	4
	$\chi_3\chi_7^3$	$\chi_5^2, \chi_3\chi_7^3$	105	1	4	1	2	1	2	5
	$\chi_3\chi_4\psi_8$	$\chi_5^2, \chi_3\chi_4\psi_8$	120	1	4	1	2	1	2	6
	$\chi_4\chi_7^3$	$\chi_5^2, \chi_4\chi_7^3$	140	1	4	1	2	1	2	7
$\chi_{13}^3$	$\chi_5^2$	$\chi_{13}^3, \chi_5^2$	65	1	2	1	1	1	1	8
	$\psi_8$	$\chi_{13}^3, \psi_8$	104	1	2	1	1	1	1	9
$\chi_4\psi_{16}$	$\chi_3\chi_4$	$\psi_8, \chi_3\chi_4$	48	1	2	2	2	1	2	10
	$\chi_5^2$	$\psi_8, \chi_5^2$	80	1	2	1	1	1	1	11
$\chi_5\psi_8$	$\chi_3\chi_4\psi_8$	$\chi_5^2, \chi_3\chi_4\psi_8$	120	2	4	1	4	1	4	12
	$\chi_3\chi_4$	$\chi_5^2, \chi_3\chi_4$	120	2	4	1	4	1	4	13
	$\chi_4\psi_8\chi_7^3$	$\chi_5^2, \chi_4\psi_8\chi_7^3$	280	2	4	1	4	1	4	14
$\chi_3\psi_{16}$	$\chi_3\chi_4\chi_5^2$	$\psi_8, \chi_3\chi_4\chi_5^2$	240	2	2	1	2	2	4	15

TABLE IX. (4\*, 4\*)

$N$	$N^+$	$f_N$	$h_{M_1}^-$	$h_{M_2}^-$	$h_{M_3}^-$	$h_{M_4}^-$	$Q_N$	$h_N^-$	$h_{N^+}$	$g_N$
$\chi_5, \chi_{13}^3$	$\chi_5^2, \chi_5\chi_{13}^3$	65	1	1	2	2	2	1	1	1
$\chi_5, \chi_4\psi_{16}$	$\chi_5^2, \chi_4\psi_{16}\chi_5$	80	1	1	2	2	2	1	1	1

TABLE X. (4\*, 2\*, 2\*)

$N$	$f_N$	$h_{M_1}^-$	$h_{M_2}^-$	$h_E^-$	$Q_E$	$Q_N$	$h_N^-$	$h_{N^+}$	$g_N$
$\chi_5, \chi_4, \chi_4\psi_8$	40	1	2	1	1	2	1	1	1
$\chi_5, \chi_3, \chi_4$	60	1	4	1	2	2	1	1	1
$\chi_5, \chi_3, \chi_7^3$	105	1	4	1	2	2	1	1	1
$\chi_5, \chi_3, \chi_4\psi_8$	120	1	4	1	2	2	1	1	1
$\chi_5, \chi_4, \chi_7^3$	140	1	4	1	2	2	1	1	1
$\chi_4\psi_{16}, \chi_3, \chi_4$	48	1	2	1	2	2	1	1	1
$\chi_4\psi_{16}, \chi_4, \chi_4\chi_5^2$	80	1	2	1	1	1	1	1	1



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