Journal de Théorie des Nombres de Bordeaux

F. Halter-Koch

Władysław Narkiewicz

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Journal de Théorie des Nombres de Bordeaux, tome 4, n° 2 (1992), p. 187-198

<http://www.numdam.org/item?id=JTNB_1992__4_2_187_0>

Séminaire de Théorie des Nombres,
Bordeaux 4 *(1992), 187-198*

Polynomial mappings defined by forms with a common factor

par F. HALTER-KOCH AND W. NARKIEWICZ

1. Introduction and Preliminaries

For a field K, we denote by $\widehat{K}$ a fixed algebraic closure of K. For a point $\mathbf{x} = (x_1, \dots, x_n) \in \hat{K}^n$, we set

$$d_K(\mathbf{x}) = \max \left\{[K(x_i) : K] \,\middle|\, i = 1, \dots, n\right\}.$$

A subset $V \subset \hat{K}^n$ is called *K-homogeneous* if there exist homogeneous polynomials $H_1, \dots, H_r \in K[X_1, \dots, X_n]$ such that

$$V = \{\mathbf{x} \in \hat{K}^n \mid H_1(\mathbf{x}) = \dots = H_r(\mathbf{x}) = 0\}.$$

For m, $n \in \mathbb{N}$ and polynomials $F_1, \dots, F_m \in K[X_1, \dots, X_n]$, we define the polynomial mapping

$$F = (F_1, \dots, F_m) : \hat{K}^n \to \hat{K}^m \text{ by } F(\mathbf{x}) = (F_1(\mathbf{x}), \dots F_m(\mathbf{x})),$$

and we set

$$\deg_*(F) = \min \{\deg (F_j) \mid j = 1, \dots, m\},$$
$$\deg^*(F) = \max \{\deg (F_j) \mid j = 1, \dots, m\}.$$

We consider the following finiteness property, which a field K may have or not:

Manuscrit reçu le 29 novembre 1991, révisé le 3 décembre 1992.

(P0)

Given m, $n \in \mathbb{N}$, a K-homogeneous set $V \subset \hat{K}^n$, a subset $\mathfrak{X} \subset V$ and polynomial mappings

$$F = (F_1, \ldots, F_m),\ G = (G_1, \ldots, G_m) : \hat{K}^n \to \hat{K}^m$$

defined by homogeneous polynomials F_j, $G_j \in K[X_1, \ldots, X_n]$, with the following properties:

(1) There exist homogeneous polynomials $F_0, F_1', \ldots, F_m'$ in $K[X_1, \ldots, X_n]$ such that $F_j = F_0 F_j'$ for all $j \in \{1, \ldots, m\}$, and $F_1', \ldots, F_m'$ do not have a common non-trivial zero in V;

(2) $\deg_*(F) > \deg^*(G) + 2\deg(F_0)$;

(3) $\{d_K(\mathbf{x}) \mid \mathbf{x} \in \mathfrak{X}\}$ is bounded;

(4) $F(\mathfrak{X}) \supset G(\mathfrak{X})$;

(5) $G|\mathfrak{X}$ is injective.

Then $\mathfrak{X}$ is a finite set.

An algebraically closed or real closed field clearly never has the property (**P0**). As the main result of this paper, we shall prove that every finitely generated field has the property (**P0**) (Theorem 3). This is well known if $F_0 = 1$ and additionally either $m = n = 1$ or K is a global field, cf. [5], [6].

In [1] we also dealt with the case $F_0 = 1$; there we considered a property which is somewhat weaker than (**P0**) and proved it for a larger class of fields. We also discussed the various hypotheses stated in (**P0**), we outlined the history of the problem, and we gave several further references.

The case $F_0 \neq 1$ was up to now only considered in a rather special case, cf. [7]. The following example (due to the referee) shows that the inequality in (2) cannot be weakened in a trivial way: let K be an infinite field, $m = n = 3$, $V = \hat{K}^3$, $F = (X_1^2 X_3, X_2^2 X_3, X_1 X_2^2 X_3)$ (hence $F_0 = X_3$), $G = (X_1, X_2, X_3)$ and $\mathfrak{X} = \{(x, y, z) \mid x, z \in K^\times\} \subset V$; then (1), (3) and (5) are obviously satisfied, and (4) holds since $(x, y, z) = F(x^{-1}z, x^{-1}z, x^3 z^{-2})$; however, $\deg_*(F) = \deg^*(G) + 2\deg(F_0)$, and $\mathfrak{X}$ is infinite.

The main tools in proving that a field has the property (**P0**) are the theory of height functions (to be explained in § 2), the theory of places of algebraic function fields (§ 3) and the following set-theoretical Lemma, already proved in [1].

MAIN LEMMA. *Let $\mathfrak{X}$, $\mathfrak{Y}$ be sets and $F, G : \mathfrak{X} \to \mathfrak{Y}$ mappings such that $F(\mathfrak{X}) \supset G(\mathfrak{X})$. Let $f : \mathfrak{X} \to \mathbb{R}$ be a mapping such that $f(\mathfrak{X}) \subset \mathbb{R}$ is discrete, and let $C > 0$ be a real constant with the property that x, $y \in \mathfrak{X}$, $f(x) \geq C$ and $F(x) = G(y)$ implies $f(y) > f(x)$. If*

$$\mathfrak{X}_C = \{x \in \mathfrak{X} \mid f(x) < C\},$$

then $F(\mathfrak{X}_C) \supset G(\mathfrak{X}_C)$*. If moreover* $\mathfrak{X}_C$ *is finite and* G *is injective, then* $\mathfrak{X} = \mathfrak{X}_C$.

2. Heights

In this section, we collate the necessary facts from the theory of height functions, following [8].

Let K be a field, equipped with a family of absolute values

$$\mathbf{M} = (|\cdot|_v)_{v \in M_K}$$

satisfying the product formula. We regard M_K as a set of places of K such that, for every $v \in M_K$, $|\cdot|_v$ is an absolute value defining v; we denote by K_v the completion of K at v.

For a finite extension field L of K, let M_L be the set of all places of L extending those of M_K. If $v \in M_K$, $w \in M_L$ and $w \mid v$, then

$$L \underset{K}{\otimes} K_v = \prod_{w|v} \widetilde{L_w},$$

where $\widetilde{L_w}$ is a finite-dimensional local K_v-algebra with residue field L_w. We define an absolute value $|\cdot|_w$ of L by

$$|x|_w = \left|\mathcal{N}_{L_w/K_v}(x)\right|_v^{n_w}, \quad \text{where} \quad n_w = \frac{[\widetilde{L_w} : K_v]}{[L : K] \cdot [L_w : K_v]}.$$

Then $(|\cdot|_w)_{w \in M_L}$ satisfies the product formula.

The family $\mathbf{M} = (|\cdot|_v)_{v \in M_K}$ gives rise to a height function $\mathbf{H} : \hat{K}^n \to \mathbb{R}$, defined for every $n \in \mathbb{N}$, as follows: for $\mathbf{x} = (x_1, \ldots, x_n) \in \hat{K}^n$, let L be a finite extension of K such that $\mathbf{x} \in L^n$, and set

$$\mathbf{H}(\mathbf{x}) = \mathbf{H}(x_1, \ldots, x_n) = \prod_{w \in M_L} \max\{1, |x_1|_w, \ldots, |x_n|_w\}.$$

Thanks to the normalization of $|\cdot|_w$, this definition does not depend on the choice of the field L. We call $\mathbf{H} = \mathbf{H}_{\mathbf{M}}$ the *height function associated with* $\mathbf{M}$.

For a monic polynomial $f = X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0 \in \hat{K}[X]$ of degree $d \in \mathbb{N}$, we set

$$\mathbf{H}(f) = \mathbf{H}(a_0, a_1, \ldots, a_{d-1}).$$

LEMMA 1. *Let K be a field, equipped with a family of absolute values satisfying the product formula, and let* $\mathbf{H}$ *be the associated height function. If* $n \in \mathbb{N}$, $\mathbf{x} \in \hat{K}^n$ *and* $z \in \hat{K}$, *then we have*

$$\mathbf{H}(z)\,\mathbf{H}(z\mathbf{x}) \geq \mathbf{H}(\mathbf{x})\,.$$

Proof. We set $\mathbf{x} = (x_1, \ldots, x_n)$ and $L = K(x_1, \ldots, x_n, z)$. For $w \in M_L$, we choose $i_w \in \{1, \ldots, n\}$ such that $|x_{i_w}|_w = \max\{|x_1|_w, \ldots, |x_n|_w\}$; then the assertion is equivalent with

$$\prod_{w \in M_L} \max\{1, |z|_w\} \cdot \prod_{w \in M_L} \max\{1, |x_{i_w} z|_w\} \geq \prod_{w \in M_L} \max\{1, |x_{i_w}|_w\}\,,$$

and hence also with

$$\prod_{\substack{w \in M_L \\ |z|_w > 1}} |z|_w \cdot \prod_{\substack{w \in M_L \\ |z|_w < 1}} \frac{\max\{1, |x_{i_w} z|_w\}}{\max\{1, |x_{i_w}|_w\}} \cdot \prod_{\substack{w \in M_L \\ |z|_w > 1}} \frac{\max\{1, |x_{i_w} z|_w\}}{\max\{1, |x_{i_w}|_w\}} \geq 1\,.$$

Since obviously

$$\prod_{\substack{w \in M_L \\ |z|_w > 1}} \frac{\max\{1, |x_{i_w} z|_w\}}{\max\{1, |x_{i_w}|_w\}} \geq 1\,,$$

and (by the product formula)

$$\prod_{\substack{w \in M_L \\ |z|_w > 1}} |z|_w = \prod_{\substack{w \in M_L \\ |z|_w < 1}} |z|_w^{-1}\,,$$

it is sufficient to observe that $0 < a < 1$ and $b > 0$ implies $\max\{1, ab\} \geq a \max\{1, b\}$. $\square$

The next Lemma gives information about the behaviour of heights under polynomial mappings.

LEMMA 2. *Let K be a field, equipped with a family of absolute values satisfying the product formula, and let* $\mathbf{H}$ *be the associated height function.*

Let $F = (F_1, \ldots, F_m) : \hat{K}^n \to \hat{K}^m$ *be a polynomial mapping, defined by homogeneous polynomials* $F_1, \ldots, F_m \in K[X_1, \ldots, X_n]$, $d_* = \deg_*(F)$ *and* $d^* = \deg^*(F)$.

i) *There exists a constant $C_1 \in \mathbb{R}_{>0}$ such that, for all $\mathbf{x} \in \hat{K}^n$,*

$$\mathbf{H}(F(\mathbf{x})) \leq C_1 \mathbf{H}(\mathbf{x})^{d^*} .$$

ii) *Let $V \subset \hat{K}^n$ be a K-homogeneous set such that $F_1, \ldots, F_m$ do not have a common non-trivial zero in V. Then there exists a constant $C_2 \in \mathbb{R}_{>0}$ such that, for all $\mathbf{x} \in V$,*

$$\mathbf{H}(F(\mathbf{x})) \geq C_2 \mathbf{H}(\mathbf{x})^{d_*} .$$

Proof.

i) follows almost immediately from the definition of the height function, cf. [8, 2.3].

ii) is proved in [5, Proposition 2]. □

PROPOSITION 1. *Let K be a field, equipped with a family of absolute values satisfying the product formula, and let $\mathbf{H}$ be the associated height function.*

For $m, n \in \mathbb{N}$, let $F = (F_1, \ldots, F_m)$, $G = (G_1, \ldots, G_m) : \hat{K}^n \to \hat{K}^m$ be polynomial mappings defined by homogeneous polynomials $F_j, G_j \in K[X_1, \ldots, X_n]$. Let $F_0 \in K[X_1, \ldots, X_n]$ be a common factor of $F_1, \ldots, F_m$, say $F_j = F_0 F_j'$ where $F_j' \in K[X_1, \ldots, X_n]$, and suppose that $\deg_*(F) > \deg^*(G) + 2\deg(F_0)$. Let $V \subset \hat{K}^n$ be a K-homogeneous set such that $F_1', \ldots, F_m'$ do not have a common non-trivial zero in V.

i) *There exists a constant $C_0 \in \mathbb{R}_{>0}$ such that $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{H}(\mathbf{x}) \geq C_0$ and $F(\mathbf{x}) = G(\mathbf{y})$ implies $\mathbf{H}(\mathbf{y}) > \mathbf{H}(\mathbf{x})$.*

ii) *Let C_0 be a constant satisfying* i), *$\mathfrak{X} \subset V$ a subset such that $F(\mathfrak{X}) \supset G(\mathfrak{X})$, $G|\mathfrak{X}$ is injective, and $\mathfrak{X}_0 = \{\mathbf{x} \in \mathfrak{X} \mid \mathbf{H}(\mathbf{x}) \leq C_0\}$ is finite. Then $\mathfrak{X}$ is finite.*

Proof.

i) We set $d_0 = \deg(F_0)$, $d^* = \deg^*(G)$, $d_* = \deg_*(F)$, and $F' = (F_1', \ldots, F_m') : \hat{K}^n \to \hat{K}^m$. Then we have $\deg_*(F') = d_* - d_0$, $d_* - 2d_0 > d^*$, and Lemma 1 implies

$$\mathbf{H}(F(\mathbf{x})) = \mathbf{H}(F_0(\mathbf{x})F'(\mathbf{x})) \geq \frac{\mathbf{H}(F'(\mathbf{x}))}{\mathbf{H}(F_0(\mathbf{x}))}$$

for all $\mathbf{x} \in \hat{K}^n$. By Lemma 2, there exist constants $C_1, C_2 \in \mathbb{R}_{>0}$ such that $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$ and $F(\mathbf{x}) = G(\mathbf{y})$ implies

$$C_1\mathbf{H}(\mathbf{y})^{d^*} \geq \mathbf{H}(G(\mathbf{y})) = \mathbf{H}(F(\mathbf{x})) \geq \frac{C_2\mathbf{H}(\mathbf{x})^{d_*-d_0}}{C_1\mathbf{H}(\mathbf{x})^{d_0}} = C_2C_1^{-1}\mathbf{H}(\mathbf{x})^{d_*-2d_0},$$

and hence $\mathbf{H}(\mathbf{y}) > \mathbf{H}(\mathbf{x})$, provided that $\mathbf{H}(\mathbf{x})$ is large enough.

ii) follows immediately from **i)** and the Main Lemma, applied for $F, G : \mathfrak{X} \to \hat{K}^m$ and $f = \mathbf{H} : \mathfrak{X} \to \mathbb{R}$. □

THEOREM 1. *Let K be a field, equipped with a family of absolute values satisfying the product formula, and let $\mathbf{H}$ be the associated height function. Suppose that for every $C \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$, the set*

$$\{\mathbf{x} \in \hat{K}^n \mid \mathbf{H}(\mathbf{x}) + d_K(\mathbf{x}) \leq C\}$$

is finite. Then K has the property (**P0**). *In particular, every global field has the property* (**P0**).

Proof. The first assertion follows immediately from Proposition 1. If K is a global field (i. e., either an algebraic number field or an algebraic function field in one variable over a finite field), then we use the finiteness results from [8, 2.5]. □

We close this section with two further Lemmas concerning heights to be used in § 3. For simplicity, we restrict ourselves to the case of non-archimedian absolute values.

LEMMA 3. *Let K be a field, equipped with a family of non-archimedean absolute values satisfying the product formula, and let $\mathbf{H}$ be the associated height function.*

i) *For monic polynomials $f, g \in K[X]$, we have $\mathbf{H}(fg) \leq \mathbf{H}(f)\mathbf{H}(g)$.*

ii) *If $n \in \mathbb{N}$ and $\mathbf{x}, \mathbf{y} \in K^n$, then $\mathbf{H}(\mathbf{x} - \mathbf{y}) \leq \mathbf{H}(\mathbf{x})\mathbf{H}(\mathbf{y})$.*

Proof.

i) is obvious; we even have equality, cf. [2, ch. 3, Prop. 2.4].

ii) Set $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$, and let L be a finite extension of K such that $\mathbf{x}, \mathbf{y} \in L^n$. Then we have

$$\begin{aligned}
\mathbf{H}(\mathbf{x} - \mathbf{y}) &= \prod_{w \in M_L} \max\{1, |x_1 - y_1|_w, \dots, |x_n - y_n|_w\} \\
&\leq \prod_{w \in M_L} \max\{1, |x_1|_w, |y_1|_w, \dots, |x_n|_w, |y_n|_w\} \\
&\leq \prod_{w \in M_L} \max\{1, |x_1|_w, \dots, |x_n|_w\} \cdot \prod_{w \in M_L} \max\{1, |y_1,|_w, \dots, |y_n|_w\} \\
&= \mathbf{H}(\mathbf{x})\mathbf{H}(\mathbf{y}) . \quad \square
\end{aligned}$$

LEMMA 4. *Let K be a field, equipped with a family of non-archimedean absolute values satisfying the product formula, and let $\mathbf{H}$ be the associated height function.*

Let $d, n \in \mathbb{N}$, and let $\mathbf{x} = (x_1, \dots, x_n) \in \hat{K}^n$ be such that $d_K(\mathbf{x}) \leq d$. For $i \in \{1, \dots, n\}$, let $f_i \in K[X]$ be the minimal polynomial of x_i, and $f = f_1 \cdot \ldots \cdot f_n \in K[X]$. Then we have

$$\mathbf{H}(f) \leq \mathbf{H}(\mathbf{x})^{nd} .$$

Proof. For $i \in \{1, \dots, n\}$, we have

$$f_i = \prod_{\nu=1}^{d_i} (X - x_i^{(\nu)}) ,$$

where $x_i^{(\nu)} \in \hat{K}$ are the conjugates of x_i (counted with multiplicity in the inseparable case) and $d_i \leq d$. Since $\mathbf{H}(X - x_i^{(\nu)}) = \mathbf{H}(x_i^{(\nu)}) = \mathbf{H}(x_i)$, we obtain, using Lemma 3,

$$\mathbf{H}(f) = \mathbf{H}\Big(\prod_{i=1}^{n} \prod_{\nu=1}^{d_i} (X - x_i^{(\nu)})\Big) \leq \prod_{i=1}^{n} \mathbf{H}(x_i)^{d_i} \leq \prod_{i=1}^{n} \mathbf{H}(\mathbf{x})^{d_i} \leq \mathbf{H}(\mathbf{x})^{nd} . \quad \square$$

3. Algebraic function fields

We start with some preliminaries on height functions and places of algebraic function fields; for the general theory of places cf. [3, ch. I].

Let $K_1 = K_0(t)$ be a rational function field over an arbitrary field K_0, and let $\mathcal{P}$ be the set of all monic irreducible polynomials in $K_0[t]$. For $p \in \mathcal{P}$, we define an absolute value $|\cdot|_p$ as follows: If

$$x = c \prod_{p \in \mathcal{P}} p^{\nu_p},$$

where $c \in K_0^\times$, $\nu_p \in \mathbb{Z}$ and $\nu_p = 0$ for almost all $p \in \mathcal{P}$, then

$$|x|_p = e^{-\nu_p \deg(p)};$$

for $x = \alpha\beta^{-1}$, where $\alpha, \beta \in K_0[t]$, we set

$$|x|_\infty = e^{\deg(\alpha) - \deg(\beta)}.$$

Then $(|\cdot|_v)_{v \in \mathcal{P} \cup \{\infty\}}$ is a family of non-archimedean absolute values satisfying the product formula. The corresponding height function will be denoted by $\mathbf{H}_\partial$; we call it the *canonical height function* for $K_0(t)$.

If $\mathbf{x} \in K_1^n$, then $\mathbf{x} = (\alpha_1\beta^{-1}, \dots, \alpha_n\beta^{-1})$, where $\alpha_1, \dots, \alpha_n, \beta \in K_0[t]$, β is monic and $\gcd(\alpha_1, \dots, \alpha_n, \beta) = 1$; this implies

$$\max\{|\alpha_1|_p, \dots, |\alpha_n|_p, |\beta|_p\} = 1, \ \forall p \in \mathcal{P}$$

and hence

$$\mathbf{H}_\partial(x) = e^{\partial(\mathbf{x})},$$

where

$$\partial(\mathbf{x}) = \max\{\deg(\alpha_1), \dots, \deg(\alpha_n), \deg(\beta)\}.$$

Every $\tau \in \hat{K}_0$ defines a place $\phi_\tau : K_1 \to \hat{K}_0 \cup \{\infty\}$ by $\phi_\tau(h) = h(\tau)$. A place $\phi : \hat{K}_1 \to \hat{K}_0 \cup \{\infty\}$ is called a K_0-*place* if $\phi | K_1 = \phi_\tau$ for some $\tau \in \hat{K}_0$. Obviously, every ϕ_τ ($\tau \in \hat{K}_0$) extends to a K_0-place of $\hat{K}_1$. For a K_0-place ϕ of $\hat{K}_1$, we denote by $R_\phi = \phi^{-1}(\hat{K}_0) \subset \hat{K}_1$ the local ring associated with ϕ. We extend the ring homomorphism $\phi : R_\phi \to \hat{K}_0$ to homomorphisms of the polynomial rings $\phi : R_\phi[X_1, \dots, X_n] \to \hat{K}_0[X_1, \dots, X_n]$ (by acting on the coefficients) and to mappings $\phi : R_\phi^n \to \hat{K}_0^n$ (by acting on the components). If $C \in R_\phi[X_1, \dots, X_n]$ and $\mathbf{x} \in R_\phi^n$, then $(\phi C)(\phi(\mathbf{x})) = \phi(C(\mathbf{x}))$.

PROPOSITION 2. *Let K_0 be a field which is neither algebraically closed nor real closed, let $K_1 = K_0(t)$ be a rational function field over K_0 and $\mathbf{H}_\partial$ the canonical height function.*

For every $M \in \mathbb{N}$, there exists a K_0-place $\phi : \hat{K}_1 \to \hat{K}_0 \cup \{\infty\}$ with the following property:

If $m \in \mathbb{N}$ and $\mathcal{Z} \subset \hat{K}^m$ is a subset satisfying

$$\mathbf{H}_\partial(\mathbf{z}) + d_K(\mathbf{z}) + m \leq M \quad \text{for all} \quad \mathbf{z} \in \mathcal{Z}\,,$$

then we have $\mathcal{Z} \subset R_\phi^m$, and $\phi|\mathcal{Z} : \mathcal{Z} \to \hat{K}_0^m$ is injective.

Proof. Let $\tau \in \hat{K}_0$ be an element satisfying

$$N = [K_0(\tau) : K_0] > 2M^3 \log M\,,$$

and let $\phi : \hat{K}_1 \to \hat{K}_0 \cup \{\infty\}$ a place extending ϕ_τ. We assert that ϕ has the required property.

Let $m \in \mathbb{N}$ and $\mathcal{Z} \subset \hat{K}^m$ be a subset such that $\mathbf{H}_\partial(\mathbf{z}) + d_K(\mathbf{z}) + m \leq M$ for all $\mathbf{z} \in \mathcal{Z}$. Putting

$$\mathcal{Z}^* = \mathcal{Z} \cup \{\mathbf{z} - \mathbf{z}' \mid \mathbf{z}\,, \mathbf{z}' \in \mathcal{Z}\}\,,$$

it is sufficient to prove that $\mathcal{Z}^* \subset R_\phi^m$, and $\phi(\mathbf{x}) \neq 0$ for all $0 \neq \mathbf{x} \in \mathcal{Z}^*$.

For $\mathbf{x} \in \mathcal{Z}^*$, we have $d_K(\mathbf{x}) \leq M^2$, and $\mathbf{H}_\partial(\mathbf{x}) \leq M^2$ by Lemma 3. For $\mathbf{x} = (x_1, \ldots, x_m)$, we denote by $f_i \in K[X]$ the minimal polynomial of x_i, and we set

$$f = f_1 \cdot \ldots \cdot f_m = X^d + \frac{\alpha_{d-1}}{\beta} X^{d-1} + \cdots + \frac{\alpha_1}{\beta} X + \frac{\alpha_0}{\beta} \in K_1[X]\,,$$

where $\alpha_0, \ldots, \alpha_{d-1}, \beta \in K_0[t]$, β is monic and $\gcd\{\alpha_0, \ldots, \alpha_{d-1}, \beta\} = 1$. Lemma 4 implies

$$\mathbf{H}_\partial(f) \leq \mathbf{H}_\partial(\mathbf{x})^{mM^2} \leq M^{2M^3}\,,$$

and therefore

$$\begin{aligned} \max\{\deg(\alpha_0), \ldots, \deg(\alpha_{d-1}), \deg(\beta)\} &= \partial(\alpha_0\beta^{-1}, \ldots, \alpha_{d-1}\beta^{-1}) \\ &= \log \mathbf{H}_\partial(f) \leq 2M^3 \log M < N\,. \end{aligned}$$

Hence we have $\beta(\tau) \neq 0$, and $\alpha_i \neq 0$ implies $\alpha_i(\tau) \neq 0$ for all $i \in \{0, 1, \ldots, d-1\}$.

Since $\phi(\beta) = \phi_\tau(\beta) = \beta(\tau) \neq 0$, we infer $f \in R_\phi[X]$, $x_1, \ldots, x_m$ are integral over R_ϕ, and since R_ϕ is integrally closed, we obtain $\mathbf{x} \in R_\phi^m$.

If $\mathbf{x} \neq 0$, then we have $x_j \neq 0$ for some $j \in \{1, \ldots, m\}$ and $\alpha_i \neq 0$ for some $i \in \{0, \ldots, d-1\}$. Let $l \in \{0, \ldots, d-1\}$ be minimal such that $\alpha_l \neq 0$; then we obtain

$$0 = f(x_j) = x_j^l \Big(x_j^{d-l} + \frac{\alpha_{l-1}}{\beta} x_j^{d-l-1} + \cdots + \frac{\alpha_l}{\beta} \Big) .$$

Dividing by x_j^l and multiplying with β yields an equation

$$\alpha_l = x_j \gamma , \quad \text{where} \quad \gamma \in R_\phi .$$

Since $\phi(\alpha_l) = \phi_\tau(\alpha_l) = \alpha_l(\tau) \neq 0$, we obtain $\phi(x_j) \neq 0$ and hence $\phi(\mathbf{x}) \neq \mathbf{0}$. $\square$

THEOREM 2. *Let K_0 be a field, and suppose that every finite extension field of K_0 has the property* (**P0**). *Then every finitely generated extension field of K_0 has the property* (**P0**).

Proof. It is sufficient to prove that every finitely generated extension field of transcendence degree 1 over K_0 has the property (**P0**); the general case follows by an obvious induction on the transcendence degree.

Let $K_1 = K_0(t)$ be a rational function field over K_0, and let K be a finite extension of K_1. We shall prove that K has the property (**P0**); obviously, we have $\hat{K} = \hat{K}_1$. Let $\mathbf{H}_\partial$ be the canonical height function associated with $K_0(t)$.

Let $F = (F_1, \ldots, F_m)$, $G = (G_1, \ldots, G_m) : \hat{K}^n \to \hat{K}^m$ be polynomial mappings defined by homogeneous polynomials $F_j, G_j \in K[X_1, \ldots, X_n]$. Let $F_0 \in K[X_1, \ldots, X_n]$ be a common factor of $F_1, \ldots, F_m$, say $F_j = F_0 F_j'$ where $F_j' \in K[X_1, \ldots, X_n]$, and suppose that $\deg_*(F) > \deg^*(G) + 2\deg(F_0)$. Let $V \subset \hat{K}^n$ be a K-homogeneous set such that $F_1', \ldots, F_m'$ do not have a common non-trivial zero in V, and let $\mathfrak{X} \subset V$ be a subset such that $F(\mathfrak{X}) \supset G(\mathfrak{X})$ and $G|\mathfrak{X}$ is injective. By Proposition 1, there exists a constant $C_0 \in \mathbb{R}_{>0}$ such that $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$, $\mathbf{H}_\partial(\mathbf{x}) \geq C_0$ and $F(\mathbf{x}) = G(\mathbf{y})$ implies $\mathbf{H}_\partial(\mathbf{y}) > \mathbf{H}_\partial(\mathbf{x})$. Putting

$$\mathfrak{X}_0 = \{\mathbf{x} \in \mathfrak{X} \mid \mathbf{H}_\partial(\mathbf{x}) \leq C_0\} ,$$

the Main Lemma yields $F(\mathfrak{X}_0) \supset G(\mathfrak{X}_0)$, and according to Proposition 1 it is sufficient to prove that $\mathfrak{X}_0$ is finite. By Lemma 2, there exists a constant $C_1 \in \mathbb{R}_{>0}$ such that

$$\mathbf{H}_\partial(G(\mathbf{x})) \leq C_1 H_\partial(\mathbf{x})^{\deg^*(G)} \leq C_1 C_0^{\deg^*(G)}$$

for all $\mathbf{x} \in \mathfrak{X}_0$.

Let $H_1, \ldots, H_r \in K[X_1, \ldots, X_n]$ be homogeneous polynomials such that $V = \{\mathbf{x} \in \hat{K}^n \mid H_1(\mathbf{x}) = \cdots = H_r(\mathbf{x}) = 0\}$. Since $F_1', \ldots, F_m', H_1, \ldots, H_r$ do not have a common non-trivial zero in $\hat{K}^n$, it follows from Hilbert's Nullstellensatz [9, §130] that there exist polynomials $F_{j\nu}$, $H_{j\nu} \in K[X_1, \ldots, X_n]$ such that

$$X_\nu^q = \sum_{j=1}^m F_{j\nu} F_j' + \sum_{j=1}^r H_{j\nu} H_j$$

for some exponent $q \in \mathbb{N}$ and all $\nu \in \{1, \ldots, n\}$.

Let D be the (finite) set of all coefficients of the polynomials F_0, F_j', G_j, H_j, $F_{j\nu}$ and $H_{j\nu}$. Since K_0 has the property (**P0**), it is neither algebraically closed nor real closed, and Proposition 2 applies: there exists a K_0-place $\phi : \hat{K} \to \hat{K}_0 \cup \{\infty\}$ such that $D \subset R_\phi$, $\mathfrak{X}_0 \subset R_\phi^n$, $\phi|D \cup \{0\}$ is injective and $\phi|G(\mathfrak{X}_0)$ is injective.

There exist elements $y_1, \ldots, y_k \in R_\phi$ such that $K = K_0(t, y_1, \ldots, y_k)$, and consequently $\phi(K) = L \cup \{\infty\}$, where $L = K_0(\phi(t), \phi(y_1), \ldots, \phi(y_k))$. L is a finite extension of K_0, thus it has the property (**P0**). We use the place ϕ to shift the whole situation from K to L. We have $\hat{L} = \hat{K}_0$, the polynomials F_0, F_j', G_j, H_j, $F_{j\nu}$ and $H_{j\nu}$ have coefficients in $K \cap R_\phi$, and consequently their ϕ-images have coefficients in L.

We consider the polynomial mappings $\phi(F) = (\phi(F_1), \ldots, \phi(F_m))$ and $\phi(G) = (\phi(G_1), \ldots, \phi(G_m)) : \hat{K}_0^n \to \hat{K}_0^m$; by construction one has: $\phi(F_j) = \phi(F_0)\phi(F_j')$, $\forall j \in \{1, \ldots, m\}$, and $\deg_*(\phi(F)) > \deg^*(\phi(G)) + 2\deg(\phi(F_0))$. We set

$$V_\phi = \left\{\mathbf{z} \in \hat{K}_0^n \mid \phi(H_1)(\mathbf{z}) = \cdots = \phi(H_r)(\mathbf{z}) = 0\right\};$$

then $V_\phi \subset \hat{K}_0^n$ is a K-homogeneous set, and since

$$X_\nu^q = \sum_{j=1}^m \phi(F_{j\nu})\phi(F_j') + \sum_{j=1}^r \phi(H_{j\nu})\phi(H_j),$$

the polynomials $\phi(F_1'), \ldots, \phi(F_m')$ do not have a common non-trivial zero in V_ϕ.

Obviously, we have $\phi(\mathfrak{X}_0) \subset V_\phi$, $\phi(F)(\phi(\mathfrak{X}_0)) \supset \phi(G)(\phi(\mathfrak{X}))$, and we contend that

$$\phi(G) \circ \phi \mid \mathfrak{X}_0 : \mathfrak{X}_0 \to \hat{K}_0^m$$

is injective, i. e., $\phi|\mathfrak{X}_0$ and $\phi(G)|\phi(\mathfrak{X}_0)$ are both injective. Indeed, if $\mathbf{x}, \mathbf{y} \in \mathfrak{X}_0$ and $\phi(G) \circ \phi(\mathbf{x}) = \phi(G) \circ \phi(\mathbf{y})$, then $\phi(G(\mathbf{x})) = \phi(G(\mathbf{y}))$ and hence $\mathbf{x} = \mathbf{y}$, since $\phi|G(\mathfrak{X}_0)$ and $G|\mathfrak{X}_0$ are both injective.

Since L has the property (**P0**), we conclude that $\phi(\mathfrak{X}_0)$ is finite, and hence $\mathfrak{X}_0$ is also finite. □

THEOREM 3. *Every finitely generated field has the property* (**P0**).

Proof. By Theorem 2, it suffices to prove that every finite extension of a prime field has the property (**P0**). For finite fields, this is obvious; for algebraic number fields, it follows from Theorem 1. □

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Franz Halter-Koch
Institut für Mathematik
Karl-Franzens-Universität
Heinrichstraße 36/IV
A-8010 Graz, Österreich

Władysław Narkiewicz
Instytut Matematyczny
Uniwersytet Wrocławski
Plac Grunwaldzki 2/4
PL-50-384 Wrocław, Polen