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The noncommutative factor theorem for lattices in product groups

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THE NONCOMMUTATIVE FACTOR THEOREM FOR LATTICES IN PRODUCT GROUPS

BY RÉMI BOUTONNET & CYRIL HOUDAYER

ABSTRACT. — We prove a noncommutative Bader-Shalom factor theorem for lattices with dense projections in product groups. As an application of this result and our previous works, we obtain a noncommutative Margulis factor theorem for all irreducible lattices $\Gamma < G$ in higher rank semisimple algebraic groups. Namely, we give a complete description of all intermediate von Neumann subalgebras $L(\Gamma) \subset M \subset L(\Gamma \curvearrowright G/P)$ sitting between the group von Neumann algebra and the group measure space von Neumann algebra associated with the action on the Furstenberg-Poisson boundary.

RÉSUMÉ (Le théorème du facteur non commutatif pour des réseaux dans des groupes produits)

Nous montrons un théorème du facteur de Bader-Shalom non commutatif pour les réseaux avec des projections denses dans les groupes produits. Comme application de ce résultat et de nos travaux précédents, nous obtenons un théorème du facteur de Margulis non commutatif pour tous les réseaux irréductibles $\Gamma < G$ dans les groupes algébriques semi-simples de rang supérieur. En particulier, nous donnons une description complète de toutes les sous-algèbres de von Neumann intermédiaires $L(\Gamma) \subset M \subset L(\Gamma \curvearrowright G/P)$ situées entre l'algèbre de von Neumann de groupe et l'algèbre de von Neumann d'espace de mesure de groupe associée à l'action sur le bord de Furstenberg-Poisson.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let H be a locally compact second countable (lcsc) group. A Borel probability measure $\mu \in \text{Prob}(H)$ is said to be *admissible* if μ is absolutely continuous with respect

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to the Haar measure, $\text{supp}(\mu)$ generates H as a semigroup, and $\text{supp}(\mu)$ contains a neighborhood of the identity element $e \in H$. A standard probability space (X, ν) is said to be a (H, μ) -space if it is endowed with a nonsingular action $H \curvearrowright (X, \nu)$ for which the probability measure ν is μ -stationary, that is, $\nu = \mu * \nu$.

Following [Fur63b, Fur63a], for every admissible Borel probability measure $\mu \in \text{Prob}(H)$, we denote by (B, ν_B) the (H, μ) -Furstenberg-Poisson boundary. Recall that (B, ν_B) is the unique (H, μ) -space for which the H -equivariant Poisson transform

$$L^\infty(B, \nu_B) \longrightarrow \text{Har}^\infty(H, \mu) : f \longmapsto \left(h \mapsto \int_B f(hb) d\nu_B(b) \right)$$

is isometric and surjective. Here, $\text{Har}^\infty(H, \mu)$ denotes the space of bounded (right) μ -harmonic functions on H . A (H, μ) -boundary (C, ν_C) is a H -equivariant measurable factor of (B, ν_B) . In that sense, (B, ν_B) is the maximal (H, μ) -boundary. For every (H, μ) -boundary (C, ν_C) , we regard $L^\infty(C) \subset L^\infty(B)$ as a H -invariant von Neumann subalgebra. Note that the center $\mathcal{Z}(H)$ acts trivially on (B, ν_B) . We refer to [Fur02, BS06] for further details on the Furstenberg-Poisson boundary.

DEFINITION. — We say that the pair (H, μ) satisfies the *boundary freeness condition* if for every nontrivial (H, μ) -boundary (C, ν_C) and every $h \in H \setminus \mathcal{Z}(H)$, we have $\nu_C(\text{Fix}_C(h)) = 0$ where $\text{Fix}_C(h) = \{c \in C \mid hc = c\}$.

EXAMPLE. — Let k be a local field, \mathbf{H} a k -isotropic almost k -simple linear algebraic k -group and $\mu \in \text{Prob}(\mathbf{H}(k))$ an admissible Borel probability measure. A combination of [BS06, Cor. 5.2] and [BBHP22, Lem. 6.2] shows that the pair $(\mathbf{H}(k), \mu)$ satisfies the boundary freeness condition.

Recall that the *quasi-center* $\mathcal{Q}\mathcal{Z}(H)$ is the (not necessarily closed) subgroup of all elements $h \in H$ for which the centralizer $\mathcal{Z}_H(h)$ is open in H . We have $\mathcal{Z}(H) < \mathcal{Q}\mathcal{Z}(H)$.

In order to state our main result, we introduce the following notation. Let $d \geq 2$. For every $i \in \{1, \dots, d\}$, let G_i be a lcsc group and $\mu_i \in \text{Prob}(G_i)$ an admissible Borel probability measure. Denote by (B_i, ν_{B_i}) the (G_i, μ_i) -Furstenberg-Poisson boundary. Set $(G, \mu) = \prod_{i=1}^d (G_i, \mu_i)$ and $(B, \nu_B) = \prod_{i=1}^d (B_i, \nu_{B_i})$. Then (B, ν_B) is the (G, μ) -Furstenberg-Poisson boundary (see [BS06, Cor. 3.2]). For every $i \in \{1, \dots, d\}$, denote by $p_i : G \rightarrow G_i$ and by $\hat{p}_i : G \rightarrow \prod_{j \neq i} G_j$ the canonical homomorphisms.

DEFINITION. — Let $\Gamma < G$ be a lattice, that is, $\Gamma < G$ is a discrete subgroup with finite covolume. We say that $\Gamma < G$ is *embedded with dense projections* if for every $i \in \{1, \dots, d\}$, the restriction $p_i|_\Gamma : \Gamma \rightarrow G_i$ is injective and $\hat{p}_i(\Gamma) < \prod_{j \neq i} G_j$ is dense.

Let $\Gamma < G$ be a lattice embedded with dense projections and assume that its center $\mathcal{Z}(\Gamma)$ is finite. Note that $\mathcal{Z}(\Gamma) < \prod_{i=1}^d \mathcal{Z}(G_i)$ and so $\mathcal{Z}(\Gamma)$ acts trivially on (B, ν_B) . Set $\Lambda = \Gamma / \mathcal{Z}(\Gamma)$. Consider the well-defined ergodic action $\Lambda \curvearrowright (B, \nu_B)$ and denote by $L(\Lambda \curvearrowright B)$ its associated *group measure space* von Neumann algebra. Whenever

(C, ν_C) is a (G, μ) -boundary, we regard $L(\Lambda \curvearrowright C) \subset L(\Lambda \curvearrowright B)$ as a von Neumann subalgebra.

Our main result is the following noncommutative analogue of Bader-Shalom’s factor theorem for lattices in product groups (see [BS06, Th. 1.7]). We call such a result a *Noncommutative Factor Theorem (NFT)*.

THEOREM A (NFT for lattices in products). — *Keep the same notation as above. For every $i \in \{1, \dots, d\}$, assume that $\mathcal{QZ}(G_i) = \mathcal{Z}(G_i)$ and that the pair (G_i, μ_i) satisfies the boundary freeness condition. Let $L(\Lambda) \subset M \subset L(\Lambda \curvearrowright B)$ be an intermediate von Neumann subalgebra.*

Then for every $i \in \{1, \dots, d\}$, there exists a unique (G_i, μ_i) -boundary (C_i, ν_{C_i}) such that with $(C, \nu_C) = \prod_{i=1}^d (C_i, \nu_{C_i})$, we have $M = L(\Lambda \curvearrowright C)$.

We point out that unlike the proof of the recent NFT for lattices in higher rank simple algebraic k -groups obtained in [BBH21] (see also [Hou21]), we cannot rely on the noncommutative Nevo-Zimmer theorem from [BH21, BBH21]. The proof of Theorem A consists of two steps. Firstly, we show that when $L(\Lambda) \subset M \subset L(\Lambda \curvearrowright B)$ and $L(\Lambda) \neq M$, there exist $i \in \{1, \dots, d\}$ and a nontrivial (G_i, μ_i) -boundary (C_i, ν_{C_i}) such that $L(\Lambda \curvearrowright C_i) \subset M$. To do this, we consider the conjugation action $\Lambda \curvearrowright M$, we exploit the assumption that $\mathcal{QZ}(G_i) = \mathcal{Z}(G_i)$ and we use the dichotomy theorem for boundary structures [BBHP22, Th. 5.8] (in lieu of the noncommutative Nevo-Zimmer theorem [BH21, BBH21]). Secondly, we exploit the assumption that the pair (G_i, μ_i) satisfies the boundary freeness condition and we combine Bader-Shalom’s factor theorem [BS06] and Suzuki’s result [Suz20] to show that there exists a unique (G, μ) -boundary (C, ν_C) such that $M = L(\Lambda \curvearrowright C)$.

Next, we apply Theorem A to the setting of higher rank lattices. We introduce the following notation. Let $d \geq 1$. For every $i \in \{1, \dots, d\}$, let k_i be a local field, \mathbf{G}_i a simply connected k_i -isotropic almost k_i -simple linear algebraic k_i -group and set $G_i = \mathbf{G}_i(k_i)$.

DEFINITION. — Set $G = \prod_{i=1}^d G_i$. We say that $\Gamma < G$ is a *higher rank lattice* if the following conditions are satisfied:

- (i) $\Gamma < G$ is a discrete subgroup with finite covolume;
- (ii) If $d \geq 2$, then $\Gamma < G$ is embedded with dense projections;
- (iii) $\sum_{i=1}^d \text{rk}_{k_i}(\mathbf{G}_i) \geq 2$.

Observe that the higher rank assumption $\sum_{i=1}^d \text{rk}_{k_i}(\mathbf{G}_i) \geq 2$ implies that exactly one of the following two situations happens.

- Either $d = 1$ (*simple case*). Then $k_1 = k$ and $\mathbf{G}_1 = \mathbf{G}$ is an almost k -simple algebraic k -group such that $\text{rk}_k(\mathbf{G}) \geq 2$.
- Or $d \geq 2$ (*semisimple or product case*).

Let $\Gamma < G$ be a higher rank lattice. For every $i \in \{1, \dots, d\}$, choose a minimal parabolic k_i -subgroup $\mathbf{P}_i < \mathbf{G}_i$ and set $P_i = \mathbf{P}_i(k_i)$. Set $P = \prod_{i=1}^d P_i$ and endow the homogeneous space G/P with its unique G -invariant measure class. Note that

$\mathcal{L}(\Gamma) < \mathcal{L}(G) < P$. Set $\Lambda = \Gamma/\mathcal{L}(\Gamma)$. Consider the well-defined ergodic action $\Lambda \curvearrowright G/P$ and its associated group measure space von Neumann algebra $L(\Lambda \curvearrowright G/P)$.

As an application of Theorem A and [BBH21, Th.D], we obtain the following noncommutative analogue of Margulis' factor theorem for all higher rank lattices (see [Mar91, Th. IV.2.11]).

THEOREM B (NFT for higher rank lattices). — *Keep the same notation as above. Let $L(\Lambda) \subset M \subset L(\Lambda \curvearrowright G/P)$ be an intermediate von Neumann subalgebra.*

Then for every $i \in \{1, \dots, d\}$, there exists a unique parabolic k_i -subgroup $P_i < Q_i < G_i$ such that with $Q = \prod_{i=1}^d Q_i(k_i)$, we have $M = L(\Lambda \curvearrowright G/Q)$. In particular, the rank $\sum_{i=1}^d \text{rk}_{k_i}(G_i)$ is an invariant of the inclusion $L(\Lambda) \subset L(\Lambda \curvearrowright G/P)$.

We actually prove in Theorem 3.3 a more general version of Theorem B where the algebraic k_i -group G_i may not be simply connected and the lattice $\Gamma < G$ may not be embedded with dense projections. We refer to [BBH21, §6] and [Hou21, §5] for a discussion of the relevance of Theorem B regarding Connes' rigidity conjecture for the group von Neumann algebras of higher rank lattices.

Finally, we apply Theorem A to the setting of lattices in products of trees. We introduce the following notation. Let $d \geq 2$. For every $i \in \{1, \dots, d\}$, let T_i be a bi-regular tree and denote by $\text{Aut}^+(T_i)$ the group of bi-coloring preserving automorphisms of T_i . Let $\Gamma < \prod_{i=1}^d \text{Aut}^+(T_i)$ be a uniform lattice. Denote by G_i the closure of the image of Γ in $\text{Aut}^+(T_i)$ and assume that G_i is 2-transitive on the boundary ∂T_i . Assume that $\Gamma < \prod_{i=1}^d G_i$ is embedded with dense projections. Endow ∂T_i with its unique G_i -invariant measure class and $B = \prod_{i=1}^d \partial T_i$ with the product measure class.

As an application of Theorem A, we obtain the following noncommutative analogue of Burger-Mozes' factor theorem for lattices in product of trees (see [BM00b, Th. 4.6]).

THEOREM C (NFT for lattices in products of trees). — *Keep the same notation as above. Let $L(\Gamma) \subset M \subset L(\Gamma \curvearrowright B)$ be an intermediate von Neumann subalgebra.*

Then there exists a unique subset $J \subset \{1, \dots, d\}$ such that with $B_J = \prod_{j \in J} \partial T_j$, we have $M = L(\Gamma \curvearrowright B_J)$.

2. CONTINUOUS ELEMENTS IN NONCOMMUTATIVE BOUNDARIES

Let H be a lcsc group and M a von Neumann algebra. We say that M is a *H-von Neumann algebra* if it is endowed with a continuous action $H \curvearrowright M$. Let M and N be *H-von Neumann algebras*. We say that a unital normal mapping $\Psi : M \rightarrow N$ is a *H-map* if Ψ is *H-equivariant* with respect to the actions $H \curvearrowright M$ and $H \curvearrowright N$.

For $i \in \{1, 2\}$, let G_i be a lcsc group and $\mu_i \in \text{Prob}(G_i)$ an admissible Borel probability measure. Denote by (B_i, ν_{B_i}) the (G_i, μ_i) -Furstenberg-Poisson boundary. Set $(G, \mu) = (G_1 \times G_2, \mu_1 \otimes \mu_2)$ and $(B, \nu_B) = (B_1 \times B_2, \nu_{B_1} \otimes \nu_{B_2})$. Denote by $p_i : G \rightarrow G_i$ the canonical homomorphism. Let $\Gamma < G$ be a lattice embedded with dense projections.

Let M be a Γ -von Neumann algebra endowed with a faithful normal ucp Γ -map $\Phi : M \rightarrow L^\infty(B)$. Denote by $\sigma : \Gamma \curvearrowright M$ the action by automorphisms. Following [BBHP22, Def. 5.3], an element $x \in M$ is said to be G_1 -continuous if for every sequence $(\gamma_n)_n$ in Γ such that $p_1(\gamma_n) \rightarrow e$ in G_1 , we have $\sigma_{\gamma_n}(x) \rightarrow x$ $*$ -strongly in M . The subset $M_1 \subset M$ of all G_1 -continuous elements in M forms a Γ -invariant von Neumann subalgebra for which the action $\Gamma \curvearrowright M_1$ extends to a continuous action $G \curvearrowright M$ such that G_2 acts trivially (see [BBHP22, Th. 5.5]).

Our main result relies on the explicit computation of the subalgebra of continuous elements in the specific context of noncommutative boundaries. More precisely, we regard $L(\Gamma \curvearrowright B)$ as a Γ -von Neumann algebra via the conjugation action $\Gamma \curvearrowright L(\Gamma \curvearrowright B)$. The canonical conditional expectation $E : L(\Gamma \curvearrowright B) \rightarrow L^\infty(B)$ is a faithful normal ucp Γ -map. We denote by $u_\gamma \in L(\Gamma) \subset L(\Gamma \curvearrowright B)$ for $\gamma \in \Gamma$ the unitaries implementing the action $\Gamma \curvearrowright (B, \nu_B)$.

THEOREM 2.1. — *For every $i \in \{1, 2\}$, the von Neumann subalgebra of G_i -continuous elements in $L(\Gamma \curvearrowright B)$ is equal to $L(\Gamma_i \curvearrowright B_i)$, where $\Gamma_i = p_i^{-1}(\mathcal{ZZ}(G_i)) \cap \Gamma < \Gamma$.*

Note that this theorem extends [BBHP22, Lem. 5.4] to the whole group measure space von Neumann algebra $L(\Gamma \curvearrowright B)$. The general approach is similar to [BBHP22, Lem. 5.4] and relies on the following refinement of Peterson’s result [Pet14, Lem. 5.1]. We essentially follow his proof, with a little more care to ensure our extra conditions.

LEMMA 2.2. — *As in the statement of Theorem 2.1, set $\Gamma_1 = p_1^{-1}(\mathcal{ZZ}(G_1)) \cap \Gamma$. Let $\gamma \in \Gamma \setminus \Gamma_1$ and $E \subset B_2$ be a nonnull measurable subset. Then there exists a sequence $(\gamma_n)_n$ in Γ such that $\nu_{B_2}(p_2(\gamma_n)E) \rightarrow 1$, $p_1(\gamma_n) \rightarrow e$ and $(\gamma_n \gamma \gamma_n^{-1})_n$ are pairwise distinct in Γ .*

Proof. — Set $g = p_1(\gamma) \in G_1 \setminus \mathcal{ZZ}(G_1)$. Then the closed subgroup $\mathcal{Z}_{G_1}(g)$ has empty interior in G_1 . Since G_1 is a lcsc group, we may choose a compatible proper right invariant metric $d : G_1 \times G_1 \rightarrow \mathbb{R}_+$ (see e.g. [Str74]).

Set $\gamma_0 = e$. We construct by induction a sequence $(\gamma_n)_{n \in \mathbb{N}^*}$ in Γ such that for every $n \in \mathbb{N}^*$, the element $\gamma_n \in \Gamma$ satisfies the following three conditions:

- (i) $1 - \nu_{B_2}(p_2(\gamma_n)E) \leq 1/n$,
- (ii) $d(p_1(\gamma_n), e) \leq 1/n$ and
- (iii) $p_1(\gamma_n) \notin \bigcup_{j=0}^{n-1} p_1(\gamma_j)\mathcal{Z}_{G_1}(g)$.

Let $n \geq 0$ and assume that we have constructed $\gamma_0, \dots, \gamma_n \in \Gamma$ satisfying the above three conditions. Let us construct $\gamma_{n+1} \in \Gamma$ that satisfies the above three conditions. Since the action $G_2 \curvearrowright (B_2, \nu_{B_2})$ is nonsingular, we may choose a compact neighborhood $\mathcal{X}_2 \subset G_2$ of $e \in G_2$ such that

$$\forall k \in \mathcal{X}_2, \quad \|k_*\nu_{B_2} - \nu_{B_2}\|_1 \leq \frac{1}{2(n+1)}.$$

Set $\mathcal{F} = \bigcup_{j=0}^n p_1(\gamma_j)\mathcal{Z}_{G_1}(g)$ and note that \mathcal{F} is a closed set with empty interior in G_1 . Consider the open ball $B_{G_1}(e, 1/2(n+1))$ of center e and radius $1/2(n+1)$ in G_1 . Since $B_{G_1}(e, 1/2(n+1)) \cap \mathcal{F}^c$ is a nonempty open set, we may choose a closed

ball $\mathcal{C} = \overline{B}_{G_1}(o, r) \subset B_{G_1}(e, 1/2(n+1)) \cap \mathcal{F}^c$ of center $o \in G_1$ and radius $0 < r \leq 1/2(n+1)$ in G_1 . Since \mathcal{C} is compact and since \mathcal{F} is closed, we have $\alpha = d(\mathcal{C}, \mathcal{F}) > 0$. Note that $\alpha \leq d(\mathcal{C}, e) \leq 1/2(n+1)$. Next, define $\mathcal{K}_1 = \overline{B}_{G_1}(e, \alpha/2)$ to be the closed ball of center $e \in G_1$ and radius $\alpha/2$ in G_1 . For every $y \in \mathcal{F}$, every $c \in \mathcal{C}$ and every $k \in \mathcal{K}_1$, we have

$$d(kc, y) \geq d(c, y) - d(kc, c) = d(c, y) - d(k, e) \geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} > 0.$$

Thus, we have $\mathcal{K}_1\mathcal{C} \subset \mathcal{F}^c$. Moreover, for every $k \in \mathcal{K}_1$ and every $c \in \mathcal{C}$, we have

$$d(kc, e) \leq d(kc, c) + d(c, e) = d(k, e) + d(c, e) \leq \frac{\alpha}{2} + \frac{1}{2(n+1)} \leq \frac{1}{n+1}.$$

Denote by $m \in \text{Prob}(G/\Gamma)$ the unique G -invariant Borel probability measure. Following the proof of [Pet14, Lem. 5.1], since the pmp action $G_2 \curvearrowright (G/\Gamma, m)$ is ergodic, Kakutani's random ergodic theorem implies that for $\mu_2^{\otimes \mathbb{N}}$ -almost every $(\omega_j)_j \in G_2^{\mathbb{N}}$ and m -almost every $z\Gamma \in G/\Gamma$, we have

$$\lim_N \frac{1}{N+1} \sum_{j=0}^N \mathbf{1}_{(\mathcal{K}_1 \times \mathcal{K}_2)\Gamma}(\omega_j^{-1} \cdots \omega_0^{-1} z) = m((\mathcal{K}_1 \times \mathcal{K}_2)\Gamma) > 0.$$

Since $m((\mathcal{C} \times \mathcal{K}_2)\Gamma) > 0$, we may choose $z \in \mathcal{C} \times \mathcal{K}_2$ such that for $\mu_2^{\otimes \mathbb{N}}$ -almost every $(\omega_j)_j \in G_2^{\mathbb{N}}$, the intersection $\{\omega_j^{-1} \cdots \omega_0^{-1} \mid j \in \mathbb{N}\} \cap (\mathcal{K}_1 \times \mathcal{K}_2)\Gamma z^{-1}$ is infinite. Since (B_2, ν_{B_2}) is the (G_2, μ_2) -Furstenberg-Poisson boundary, for $\mu_2^{\otimes \mathbb{N}}$ -almost every $\omega = (\omega_j)_j \in G_2^{\mathbb{N}}$, the limit measure $(\nu_{B_2})_\omega$ is a Dirac mass (see [BS06, Th. 2.14]). Moreover, we have $\nu_{B_2} = \int_{G_2^{\mathbb{N}}} (\nu_{B_2})_\omega d\mu_2^{\otimes \mathbb{N}}(\omega)$ (see [BS06, Th. 2.10]). Therefore, we have

$$\mu_2^{\otimes \mathbb{N}}(\{(\omega_j)_j \in G_2^{\mathbb{N}} \mid \lim_j \nu_{B_2}(\omega_j^{-1} \cdots \omega_0^{-1} p_2(z)E) = 1\}) = \nu_{B_2}(p_2(z)E) > 0.$$

We may choose $(\omega_j)_j \in G_2^{\mathbb{N}}$ such that the intersection $\{\omega_j^{-1} \cdots \omega_0^{-1} \mid j \in \mathbb{N}\} \cap (\mathcal{K}_1 \times \mathcal{K}_2)\Gamma z^{-1}$ is infinite and $\lim_j \nu_{B_2}(\omega_j^{-1} \cdots \omega_0^{-1} p_2(z)E) = 1$. We may then choose $j \in \mathbb{N}$, $h \in \mathcal{K}_1 \times \mathcal{K}_2$ and $\gamma_{n+1} \in \Gamma$ such that $\omega_j^{-1} \cdots \omega_0^{-1} = h\gamma_{n+1}z^{-1}$ and

$$1 - \nu_{B_2}(\omega_j^{-1} \cdots \omega_0^{-1} p_2(z)E) \leq \frac{1}{2(n+1)}.$$

Then $p_1(\gamma_{n+1}) = p_1(h^{-1})p_1(z) \in \mathcal{K}_1\mathcal{C}$ and so $p_1(\gamma_{n+1}) \notin \bigcup_{j=0}^n p_1(\gamma_j)\mathcal{F}_{G_1}(g)$ and $d(p_1(\gamma_{n+1}), e) \leq 1/n+1$. Moreover, we have

$$\begin{aligned} 1 - \nu_{B_2}(p_2(\gamma_{n+1})E) &= 1 - \nu_{B_2}(p_2(h^{-1})\omega_j^{-1} \cdots \omega_0^{-1} p_2(z)E) \\ &= 1 - \nu_{B_2}(\omega_j^{-1} \cdots \omega_0^{-1} p_2(z)E) + \|p_2(h)_* \nu_{B_2} - \nu_{B_2}\|_1 \\ &\leq \frac{1}{2(n+1)} + \frac{1}{2(n+1)} = \frac{1}{n+1}. \end{aligned}$$

Thus, the element $\gamma_{n+1} \in \Gamma$ satisfies the above three conditions.

The sequence $(\gamma_n)_n$ that we have constructed by induction satisfies the conclusion of the lemma. \square

Let (X, ν) be a standard probability space. For every measurable function $f : X \rightarrow \mathbb{C}$, we set

$$\|f\|_\nu = \left(\int_X |f(x)|^2 d\nu(x) \right)^{1/2}.$$

Proof of Theorem 2.1. — We may assume that $i = 1$. Set $\mathcal{B} = L(\Gamma \curvearrowright B)$ and denote by $\mathcal{B}_1 \subset \mathcal{B}$ the von Neumann subalgebra of all G_1 -continuous elements in \mathcal{B} . Set $\varphi = \nu_B \circ E \in \mathcal{B}_*$.

Observe that $L(\Gamma_1 \curvearrowright B_1) \subset \mathcal{B}_1$. Indeed, it is obvious that $L^\infty(B_1) \subset \mathcal{B}_1$. Let now $\gamma \in \Gamma_1$ so that $p_1(\gamma) \in \mathcal{L}(G_1)$. Let $(\gamma_n)_n$ be a sequence in Γ such that $p_1(\gamma_n) \rightarrow e$. Since $\mathcal{L}_{G_1}(p_1(\gamma)) < G_1$ is open, there exists $n_0 \in \mathbb{N}$ such that $p_1(\gamma_n) \in \mathcal{L}_{G_1}(p_1(\gamma))$ for every $n \geq n_0$. Since $p_1|_\Gamma : \Gamma \rightarrow G_1$ is injective, we have $\gamma_n \in \mathcal{L}_\Gamma(\gamma)$ for every $n \geq n_0$. This implies that $u_\gamma \in \mathcal{B}_1$. Altogether, we obtain $L(\Gamma_1 \curvearrowright B_1) \subset \mathcal{B}_1$.

Next, we prove that $\mathcal{B}_1 \subset L(\Gamma_1 \curvearrowright B_1)$. Firstly, we show that $\mathcal{B}_1 \subset L(\Gamma_1 \curvearrowright B)$. By contraposition, let $x \in \mathcal{B} \setminus L(\Gamma_1 \curvearrowright B)$. Write $x = \sum_{\gamma \in \Gamma} x_\gamma u_\gamma$ for the Fourier expansion of $x \in \mathcal{B}$ where $x_\gamma = E(xu_\gamma^*)$ for every $\gamma \in \Gamma$. Since $x \notin L(\Gamma_1 \curvearrowright B)$, there exists $\gamma \in \Gamma \setminus \Gamma_1$ such that $x_\gamma \neq 0$. We may regard $x_\gamma \in L^\infty(B) = L^\infty(B_2, L^\infty(B_1))$ as a measurable function $f : B_2 \rightarrow L^\infty(B_1)$. Since $x_\gamma \neq 0$, the measurable function $f : B_2 \rightarrow L^\infty(B_1)$ possesses a nonzero essential value $y \in L^\infty(B_1)$. Set $\varepsilon = \|y\|_{\nu_{B_1}}/2 > 0$. Then the measurable subset $E = \{b \in B_2 \mid \|f(b) - y\|_{\nu_{B_1}} < \varepsilon\}$ is nonnull. Choose a sequence $(\gamma_n)_n$ in Γ that satisfies the conclusion of Lemma 2.2 for $\gamma \in \Gamma \setminus \Gamma_1$ and $E \subset B_2$. Then

$$\begin{aligned} \forall n \in \mathbb{N}, \quad \|u_{\gamma_n} x u_{\gamma_n}^* - x\|_\varphi^2 &= \sum_{h \in \Gamma} \|\sigma_{\gamma_n}(x_h) - x_{\gamma_n h \gamma_n^{-1}}\|_{\nu_B}^2 \\ &\geq \|\sigma_{\gamma_n}(x_\gamma) - x_{\gamma_n \gamma \gamma_n^{-1}}\|_{\nu_B}^2. \end{aligned}$$

On the one hand, since the elements $(\gamma_n \gamma \gamma_n^{-1})_n$ are pairwise distinct in Γ and since $\|x\|_\varphi^2 = \sum_{h \in \Gamma} \|x_h\|_{\nu_B}^2 \geq \sum_{n \in \mathbb{N}} \|x_{\gamma_n \gamma \gamma_n^{-1}}\|_{\nu_B}^2$, we have

$$\lim_n \|x_{\gamma_n \gamma \gamma_n^{-1}}\|_{\nu_B}^2 = 0.$$

On the other hand, since $p_1(\gamma_n) \rightarrow e$, since $\nu_{B_2}(p_2(\gamma_n)E) \rightarrow 1$ and since $\|f(b)\|_{\nu_{B_1}} \geq \|y\|_{\nu_{B_1}} - \|f(b) - y\|_{\nu_{B_1}} \geq \varepsilon$ for every $b \in E$, we have

$$\begin{aligned} \liminf_n \|\sigma_{\gamma_n}(x_\gamma)\|_{\nu_B}^2 &= \liminf_n \int_{B_2} \|\sigma_{p_1(\gamma_n)}(f(p_2(\gamma_n)^{-1}b))\|_{\nu_{B_1}}^2 d\nu_{B_2}(b) \\ &= \liminf_n \int_{B_2} \|f(p_2(\gamma_n)^{-1}b)\|_{\nu_{B_1} \circ p_1(\gamma_n)}^2 d\nu_{B_2}(b) \\ &\geq \liminf_n \int_{p_2(\gamma_n)E} \|f(p_2(\gamma_n)^{-1}b)\|_{\nu_{B_1} \circ p_1(\gamma_n)}^2 d\nu_{B_2}(b) \geq \varepsilon^2. \end{aligned}$$

Altogether, this implies that $\liminf_n \|u_{\gamma_n} x u_{\gamma_n}^* - x\|_\varphi^2 \geq \varepsilon^2$ and so $x \notin \mathcal{B}_1$. This shows that $\mathcal{B}_1 \subset L(\Gamma_1 \curvearrowright B)$.

Secondly, we show that $\mathcal{B}_1 \subset L(\Gamma_1 \curvearrowright B_1)$. Indeed, let $x \in \mathcal{B}_1$. The previous paragraph shows that $x \in L(\Gamma_1 \curvearrowright B)$ and so we may write $x = \sum_{\gamma \in \Gamma_1} x_\gamma u_\gamma$, where

$x_\gamma = E(xu_\gamma^*)$ for every $\gamma \in \Gamma_1$. Since the faithful normal conditional expectation $E : \mathcal{B} \rightarrow L^\infty(B)$ is Γ -equivariant, we have that $E(\mathcal{B}_1)$ is contained in the von Neumann subalgebra of G_1 -continuous elements in $L^\infty(B)$, which is equal to $L^\infty(B_1)$ by [BBHP22, Lem. 5.4]. Since $L(\Gamma_1) \subset \mathcal{B}_1$, we have $x_\gamma = E(xu_\gamma^*) \in L^\infty(B_1)$ for every $\gamma \in \Gamma_1$. Then [Suz20, Cor. 3.4] further implies that $\mathcal{B}_1 \subset L(\Gamma_1 \curvearrowright B_1)$. Thus, we have $\mathcal{B}_1 = L(\Gamma_1 \curvearrowright B_1)$. \square

Next, we further assume that $\mathcal{Q}\mathcal{Z}(G_i) = \mathcal{Z}(G_i)$ for every $i \in \{1, 2\}$ and that $\mathcal{Z}(\Gamma)$ is finite. Set $\Lambda = \Gamma/\mathcal{Z}(\Gamma)$ and denote by $\pi : \Gamma \rightarrow \Lambda$ the quotient homomorphism. Consider the well-defined ergodic action $\Lambda \curvearrowright (B, \nu_B)$. We may regard $L(\Lambda \curvearrowright B)$ as a Γ -von Neumann algebra via the quotient homomorphism $\pi : \Gamma \rightarrow \Lambda$ and the conjugation action $\Lambda \curvearrowright L(\Lambda \curvearrowright B)$. Moreover, the canonical conditional expectation $E : L(\Lambda \curvearrowright B) \rightarrow L^\infty(B)$ is a faithful normal ucp Γ -map. We derive the following result that will be used in the proof of Theorem A.

COROLLARY 2.3. — *For every $i \in \{1, 2\}$, the von Neumann subalgebra of G_i -continuous elements in $L(\Lambda \curvearrowright B)$ is equal to $L^\infty(B_i)$. Moreover, the action $\Gamma \curvearrowright L(\Lambda \curvearrowright B)$ is ergodic.*

Proof. — The second assertion follows from the first one since a Γ -invariant element in $L(\Lambda \curvearrowright B)$ is both G_1 -continuous and G_2 -continuous, hence must be contained in $L^\infty(B_1) \cap L^\infty(B_2) = \mathbb{C}1$.

To prove the first assertion, we observe that with the above notation, $L(\Lambda \curvearrowright B) \cong zL(\Gamma \curvearrowright B)$, where $z = \frac{1}{|\mathcal{Z}(\Gamma)|} \sum_{h \in \mathcal{Z}(\Gamma)} u_h \in \mathcal{Z}(L(\Gamma \curvearrowright B))$. Let us be more explicit about this identification. Since $\mathcal{Z}(\Gamma)$ acts trivially on B , the projection z is indeed central in $L(\Gamma \curvearrowright B)$. Given $g \in \Gamma$, the element zu_g only depends on $\pi(g) \in \Lambda$. Then the map

$$L(\Lambda \curvearrowright B) \longrightarrow zL(\Gamma \curvearrowright B) : \quad au_{\pi(g)} \longmapsto zau_g, \quad a \in L^\infty(B), g \in \Gamma,$$

is easily seen to extend to the desired von Neumann algebra isomorphism

$$\Theta : L(\Lambda \curvearrowright B) \longrightarrow zL(\Gamma \curvearrowright B).$$

Note that Θ is Γ -equivariant.

Let $i \in \{1, 2\}$. Since $\mathcal{Q}\mathcal{Z}(G_i) = \mathcal{Z}(G_i)$ and since $p_i|_\Gamma : \Gamma \rightarrow G_i$ is injective, we have $\Gamma_i = p_i^{-1}(\mathcal{Z}(G_i)) = \mathcal{Z}(\Gamma)$. It is obvious that all the elements of $L^\infty(B_i)$ are G_i -continuous in $L(\Lambda \curvearrowright B)$. Conversely, let $x \in L(\Lambda \curvearrowright B)$ be a G_i -continuous element. Then $\Theta(x) \in zL(\Gamma \curvearrowright B)$ is G_i -continuous in $L(\Gamma \curvearrowright B)$. Applying Theorem 2.1, we obtain that $\Theta(x) \in L(\mathcal{Z}(\Gamma) \curvearrowright B_i) \cap zL(\Gamma \curvearrowright B) = zL^\infty(B_i)$. Thus, we have $x \in L^\infty(B_i)$. This finishes the proof of the corollary. \square

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem A. — Regard $\mathcal{B} = L(\Lambda \curvearrowright B)$ as a Γ -von Neumann algebra and denote by $E : \mathcal{B} \rightarrow L^\infty(B)$ the canonical Γ -equivariant faithful normal conditional expectation. By Corollary 2.3, the action $\Gamma \curvearrowright \mathcal{B}$ is ergodic. Let $L(\Lambda) \subset M \subset \mathcal{B}$ be an

intermediate von Neumann subalgebra. Then $M \subset \mathcal{B}$ is a Γ -invariant von Neumann subalgebra and the action $\Gamma \curvearrowright M$ is ergodic. Consider the faithful normal ucp Γ -map $\Phi = \mathbb{E}|_M : M \rightarrow L^\infty(B)$. If $\Phi : M \rightarrow L^\infty(B)$ is invariant, then for every $x \in M$ and every $\lambda \in \Lambda$, we have $\mathbb{E}(xu_\lambda^*) \in \mathbb{C}1$. Then we infer that $M = L(\Lambda)$ (see e.g. [Suz20, Cor. 3.4]).

Next, assume that $\Phi : M \rightarrow L^\infty(B)$ is not invariant. Then [BBHP22, Th. 5.8] implies that there exists $i \in \{1, \dots, d\}$ such that the Γ -invariant von Neumann subalgebra $M_i \subset M$ of all G_i -continuous elements in M is nontrivial. Moreover, the action $\Gamma \curvearrowright M_i$ extends to a continuous action $G_i \curvearrowright M_i$ and the ucp map $\Phi|_{M_i} : M_i \rightarrow L^\infty(B_i)$ is G_i -equivariant and not invariant. Since $M \subset \mathcal{B}$ is Γ -invariant, we have $M_i \subset \mathcal{B}_i$ where $\mathcal{B}_i \subset \mathcal{B}$ is the Γ -invariant von Neumann subalgebra of all G_i -continuous elements in \mathcal{B} . Corollary 2.3 implies that $M_i \subset L^\infty(B_i)$. Then $M_i \subset L^\infty(B_i)$ is a nontrivial G_i -invariant von Neumann subalgebra and so there exists a nontrivial (G_i, μ_i) -boundary (D_i, ν_{D_i}) such that $M_i = L^\infty(D_i)$. Then we have $L(\Lambda \curvearrowright D_i) \subset M \subset \mathcal{B}$. Since the pair (G_i, μ_i) satisfies the boundary freeness condition and since the restriction $p_i|_\Gamma : \Gamma \rightarrow G_i$ is injective, it follows that the nonsingular action $\Lambda \curvearrowright (D_i, \nu_{D_i})$ is essentially free. Then a combination of Bader-Shalom's factor theorem [BS06, Th. 1.7] and Suzuki's result [Suz20, Th. 3.6] implies that for every $j \in \{1, \dots, d\}$, there exists a unique (G_j, μ_j) -boundary (C_j, ν_j) such that with $(C, \nu_C) = \prod_{j=1}^d (C_j, \nu_{C_j})$, we have $M = L(\Lambda \curvearrowright C)$. \square

REMARK 3.1. — We point out that the analogous statement of Theorem A for $L(\Gamma \curvearrowright B)$ is false in the case when $\mathcal{Z}(\Gamma)$ is nontrivial. Indeed, the presence of the central projection $z \in \mathcal{Z}(L(\Gamma \curvearrowright B))$ as defined in the proof of Corollary 2.3 produces pathological intermediate subalgebras such as $zL(\Gamma) \oplus (1-z)L(\Gamma \curvearrowright B)$. Nevertheless, our strategy still allows to classify intermediate subalgebras in this case.

Next, we prove a general noncommutative factor theorem for higher rank lattices that will imply Theorem B.

We introduce the following notation. Let $d \geq 1$. For every $i \in \{1, \dots, d\}$, let k_i be a local field and \mathbf{G}_i a k_i -isotropic almost k_i -simple linear algebraic k_i -group and set $G_i = \mathbf{G}_i(k_i)$. Denote by $G_i^+ < G_i$ the subgroup generated by the subgroups $U(k_i)$ where U runs through the set of unipotent k_i -split subgroups of \mathbf{G}_i (see [Mar91, Prop. I.5.4(i)]). Choose a minimal parabolic k_i -subgroup $\mathbf{P}_i < \mathbf{G}_i$ and set $P_i = \mathbf{P}_i(k_i)$. Set $G = \prod_{i=1}^d G_i$, $G^+ = \prod_{i=1}^d G_i^+$, $P = \prod_{i=1}^d P_i$ and endow the homogeneous space G/P with its unique G -invariant measure class. Observe that we have $G = G^+ \cdot P$ and so the action $G^+ \curvearrowright G/P$ is transitive (see [Mar91, Prop. I.5.4(vi)]). For every $i \in \{1, \dots, d\}$, denote by $p_i : G \rightarrow G_i$ and by $\widehat{p}_i : G \rightarrow \prod_{j \neq i} G_j$ the canonical homomorphisms.

Let $\Gamma < G$ be a lattice such that for every $i \in \{1, \dots, d\}$, we have $\prod_{j \neq i} G_j^+ < \widehat{p}_i(\Gamma)$ and $p_i|_\Gamma : \Gamma \rightarrow G_i$ is injective. We point out that $\Gamma < G$ need not be embedded with dense projections. Note that $\mathcal{Z}(\Gamma) < \mathcal{Z}(G) < P$ (see [Mar91, Lem. II.6.3(I)]). Set

$\Lambda = \Gamma / \mathcal{L}(\Gamma)$ and denote by $\pi : \Gamma \rightarrow \Lambda$ the quotient homomorphism. Consider the well-defined ergodic action $\Lambda \curvearrowright G/P$ and its associated group measure space von Neumann algebra $L(\Lambda \curvearrowright G/P)$. We may regard $L(\Lambda \curvearrowright G/P)$ as a Γ -von Neumann algebra via the quotient homomorphism $\pi : \Gamma \rightarrow \Lambda$ and the conjugation action $\Lambda \curvearrowright L(\Lambda \curvearrowright G/P)$.

LEMMA 3.2. — *The action $\Gamma \curvearrowright L(\Lambda \curvearrowright G/P)$ is ergodic.*

Proof. — Choose a Borel probability measure $\mu \in \text{Prob}(G)$ of the form $\mu = \psi \cdot m_G$, where m_G is a Haar measure on G and $\psi : G \rightarrow \mathbb{R}_+$ is a compactly supported continuous function such that the set $\{g \in G \mid \psi(g) > 0 \text{ and } \psi(g^{-1}) > 0\}$ generates G . Since G/P is compact, there exists a Borel probability measure $\nu \in \text{Prob}(G/P)$ such that $(G/P, \nu)$ is a (G, μ) -space. Moreover, $\nu \in \text{Prob}(G/P)$ belongs to the unique G -invariant measure class (see e.g. [NZ99, Lem. 1.1]).

By [Mar91, Prop. VI.4.1], there exists a fully supported probability measure $\mu_\Gamma \in \text{Prob}(\Gamma)$ such that $(G/P, \nu)$ is a (Γ, μ_Γ) -space. Denote by $\mu_\Lambda \in \text{Prob}(\Lambda)$ the pushforward measure of μ_Γ under the quotient homomorphism $\pi : \Gamma \rightarrow \Lambda$. Then $(G/P, \nu)$ is a (Λ, μ_Λ) -space. Since Λ has infinite conjugacy classes, [KP22, Lem. 2.6] implies that the conjugation action $\Lambda \curvearrowright L(\Lambda \curvearrowright G/P)$ is ergodic. Thus, the action $\Gamma \curvearrowright L(\Lambda \curvearrowright G/P)$ is ergodic. \square

The following theorem is a noncommutative analogue of Margulis' factor theorem for all higher rank lattices (see [Mar91, Th. IV.2.11]). It is a more general version of Theorem B.

THEOREM 3.3. — *Keep the same notation as above. Let $L(\Lambda) \subset M \subset L(\Lambda \curvearrowright G/P)$ be an intermediate von Neumann subalgebra.*

Then for every $i \in \{1, \dots, d\}$, there exists a unique parabolic k_i -subgroup $\mathbf{P}_i < \mathbf{Q}_i < \mathbf{G}_i$ such that with $Q = \prod_{i=1}^d \mathbf{Q}_i(k_i)$, we have $M = L(\Lambda \curvearrowright G/Q)$. In particular, the rank $\sum_{i=1}^d \text{rk}_{k_i}(\mathbf{G}_i)$ is an invariant of the inclusion $L(\Lambda) \subset L(\Lambda \curvearrowright G/P)$.

Proof. — Regard $\mathcal{B} = L(\Lambda \curvearrowright G/P)$ as a Γ -von Neumann algebra and denote by $E : \mathcal{B} \rightarrow L^\infty(G/P)$ the canonical Γ -equivariant faithful normal conditional expectation. By Lemma 3.2, the action $\Gamma \curvearrowright \mathcal{B}$ is ergodic. Let $L(\Lambda) \subset M \subset \mathcal{B}$ be an intermediate von Neumann subalgebra and set $\Phi = E|_M : M \rightarrow L^\infty(G/P)$. Using the same notation and following the same argument as in the first paragraph of the proof of Theorem A, we conclude that if $\Phi : M \rightarrow L^\infty(G/P)$ is invariant, then $M = L(\Lambda)$. Next, assume that $\Phi : M \rightarrow L^\infty(G/P)$ is not invariant. There are two cases to consider.

Firstly, assume that $d = 1$. Let $k = k_1$ and $\mathbf{G} = \mathbf{G}_1$ such that $\text{rk}_k(\mathbf{G}) \geq 2$. We proceed as in the proof of [BBH21, Th. D]. By [BBH21, Th. 5.4] (see also [BH21, Th. B]), there exist a proper parabolic k -subgroup $\mathbf{P} < \mathbf{Q} < \mathbf{G}$ and a Γ -equivariant unital normal embedding $\iota : L^\infty(G/Q) \rightarrow M$ such that $\Phi \circ \iota : L^\infty(G/Q) \rightarrow L^\infty(G/P)$ is the canonical embedding with $Q = \mathbf{Q}(k)$. Then we have $L(\Lambda \curvearrowright G/Q) \subset M$. Since $\Lambda \curvearrowright G/Q$ is essentially free (see [BBHP22, Lem. 6.2]), a combination of [Mar91,

Th. IV.2.11] and [Suz20, Th. 3.6] shows that there exists a unique parabolic k -subgroup $\mathbf{Q} < \mathbf{R} < \mathbf{G}$ such that $M = \mathbf{L}(\Lambda \curvearrowright G/R)$ with $R = \mathbf{R}(k)$.

Secondly, assume that $d \geq 2$. Since $\Gamma < G$ need not be embedded with dense projections, we cannot apply Theorem A. However, we may proceed as in the proof of [BBHP22, Prop. 6.1]. Upon permuting the indices, letting $H_1 = \overline{p_1(\Gamma)} < G_1$ and $H_2 = \widehat{p_1(\Gamma)} < \prod_{j=2}^d G_j$, we have that $\Gamma < H_1 \times H_2$ is a lattice embedded with dense projections and the von Neumann subalgebra $M_1 \subset M$ of all H_1 -continuous elements in M is nontrivial. Set $B_1 = G_1/P_1$ and $B_2 = \prod_{j=2}^d G_j/P_j$. Since $G_1^+ < H_1$ and $\prod_{j=2}^d G_j^+ < H_2$, as explained in the proof of [BBHP22, Prop. 6.1] and using [BS06, Cor. 5.2], for every $i \in \{1, 2\}$, we may choose Borel probability measures $\mu_i \in \text{Prob}(H_i)$ and $\nu_i \in \text{Prob}(B_i)$ such that (B_i, ν_i) is the (H_i, μ_i) -Furstenberg-Poisson boundary. Since $G_1^+ < H_1$ and $\prod_{j=2}^d G_j^+ < H_2$, using [Mar91, Th. I.1.5.6(i)] and [CM11, Prop. 4.3], we infer that for every $i \in \{1, 2\}$, $\mathcal{QZ}(H_i) = \mathcal{Z}(H_i)$. Moreover, [BBHP22, Lem. 6.2] implies that for every $i \in \{1, 2\}$, the pair (H_i, μ_i) satisfies the boundary freeness condition. We may now apply Theorem A to obtain the conclusion.

For every $i \in \{1, \dots, d\}$, there are $2^{\text{rk}_{k_i}(\mathbf{G}_i)}$ intermediate parabolic k_i -subgroups $\mathbf{P}_i < \mathbf{Q}_i < \mathbf{G}_i$. The NFT implies that there are $2^{\sum_{i=1}^d \text{rk}_{k_i}(\mathbf{G}_i)}$ intermediate von Neumann subalgebras $\mathbf{L}(\Lambda) \subset M \subset \mathbf{L}(\Lambda \curvearrowright G/P)$. Thus, the rank $\sum_{i=1}^d \text{rk}_{k_i}(\mathbf{G}_i)$ is an invariant of the inclusion $\mathbf{L}(\Lambda) \subset \mathbf{L}(\Lambda \curvearrowright G/P)$. \square

Proof of Theorem C. — For every $i \in \{1, \dots, d\}$, we have $\mathcal{QZ}(G_i) = \{e\}$ (see [BM00a, Lem. 3.1.1 & Prop. 3.1.2]). Moreover, the nonsingular action $\Gamma \curvearrowright \partial\mathbf{T}_i$ is essentially free (see [BBHP22, Prop. 6.4]). This condition is sufficient to apply the proof of Theorem A to obtain the conclusion. \square

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