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DECAY ESTIMATES FOR LARGE VELOCITIES IN THE BOLTZMANN EQUATION WITHOUT CUTOFF

by Cyril Imbert, Clément Mouhot & Luis Silvestre

Abstract. — We consider solutions $f = f(t, x, v)$ to the full (spatially inhomogeneous) Boltzmann equation with periodic spatial conditions $x \in \mathbb{T}^d$, for hard and moderately soft potentials *without the angular cutoff assumption*, and under the a priori assumption that the main hydrodynamic fields, namely the local mass $\int f \, \mathrm{d}v$ and local energy $\int f|v|^2 \, \mathrm{d}v$ and local entropy $\int f \ln f \, \mathrm{d}v$, are controlled along time. We establish quantitative estimates of *propagation* in time of "pointwise polynomial moments", i.e., $\sup_{x,v} f(t, x, v)(1 + |v|)^q$, $q \geqslant 0$. In the case of hard potentials, we also prove *appearance* of these moments for all $q \geqslant 0$. In the case of moderately soft potentials, we prove the *appearance* of low-order pointwise moments. All these conditional bounds are uniform as t goes to $+\infty$, conditionally to the bounds on the hydrodynamic fields being uniform.

Résumé (Décroissance aux grandes vitesses pour les solutions de l'équation de Boltzmann sans troncature angulaire)

Cet article considère des solutions a priori $f = f(t, x, v)$ de l'équation de Boltzmann sans hypothèse d'homogénéité spatiale et avec conditions périodiques $x \in \mathbb{T}^d$, pour des interactions de type potentiels durs ou modérément mous *sans troncature angulaire*. Sous l'hypothèse a priori que les champs hydrodynamiques associés à la solution : masse locale $\int f \, \mathrm{d}v$, énergie locale $\int f|v|^2 \, \mathrm{d}v$, entropie locale $\int f \ln f \, \mathrm{d}v$, restent bornés au cours du temps, nous montrons des bornes sur les « moments polynomiaux ponctuels » $\sup_{x,v} f(t, x, v)(1 + |v|^q)$, $q \geqslant 0$. Ces moments sont propagés dans le cas des potentiels modérément mous, et apparaissent dans le cas des potentiels durs. Dans le cas des potentiels modérément mous, nous montrons également l'apparition de moments ponctuels d'ordre bas. Toutes ces bornes conditionnelles sont uniformes en temps grand, dès lors que les bornes sur les champs hydrodynamiques sont elles-mêmes uniformes en temps grand.

Contents

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1. Introduction

1.1. The Boltzmann equation. — The Boltzmann equation models the evolution of rarefied gases, described through the probability density of the particles in the phase space. It sits at a mesoscopic scale between the hydrodynamic equations (e.g. the compressible Euler or Navier Stokes equations) describing the evolution of observable quantities on a large scale, and the complicated dynamical system describing the movement of the very large number of molecules in the gas. Fluctuations around steady state, on a large scale, follow incompressible Navier-Stokes equations under the appropriate limit.

This probability density of particles is a non-negative function $f = f(t,x,v)$ defined on a given time interval $I \in \mathbb{R}$ and $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$ and it solves the integro-differential *Boltzmann equation*

$$\partial_t f + v \cdot \nabla_x f = Q(f,f). \tag{1.1}$$

The bilinear *Boltzmann collision operator* $Q(f_1,f_2)$ is defined as

$$Q(f_1,f_2) := \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \Big[f_1(v'_*)f_2(v') - f_1(v_*)f_2(v)\Big] B(|v-v_*|,\cos\theta)\,\mathrm{d}v_*\,\mathrm{d}\sigma,$$

where B is the *collision kernel* and the *pre-collisional velocities* v'_* and v' are given by (see Figure 1.1)

$$v' := \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \quad \text{and} \quad v'_* := \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

The so-called *deviation angle* θ is the angle between the pre- and post-collisional relative velocities (observe that $|v-v_*| = |v'-v'_*|$):

$$\cos\theta := \frac{v-v_*}{|v-v_*|}\cdot\frac{v'-v'_*}{|v'-v'_*|} \quad \text{and} \quad \sigma := \frac{v'-v'_*}{|v'-v'_*|}$$
$$\sin(\theta/2) := \frac{v'-v}{|v'-v|}\cdot\sigma \quad \text{and} \quad \cos(\theta/2) := \frac{v'-v_*}{|v'-v_*|}\cdot\sigma.$$

The precise form of the collision kernel B depends on the molecular interaction [19]. For all *long-range interactions*, that is all interactions apart from the *hard spheres model*, it is singular at $\theta \sim 0$, i.e., small deviation angles that correspond to *grazing collisions*. Keeping this singularity dictated by physics in the mathematical analysis has come to be known quite oddly as a *non-cutoff* assumption. In dimension $d=3$, when this long-range interaction derives from a power-law repulsive force $F(r) = Cr^{-\alpha}$ with $\alpha \in (2,+\infty)$, then B is given by (see [19] and [52, Chap. 1])

$$B(r,\cos\theta) = r^\gamma b(\cos\theta) \quad \text{with} \quad b(\cos\theta) \sim_{\theta\sim 0} \mathrm{cst}\,\theta^{-(d-1)-2s}$$

with $\gamma = (\alpha-5)/(\alpha-1) \in (-d,1)$, $C>0$ and $s = 1/(\alpha-1) \in (0,1)$. The singularity of b at *grazing collisions* $\theta \sim 0$ is the legacy of long-range interactions.

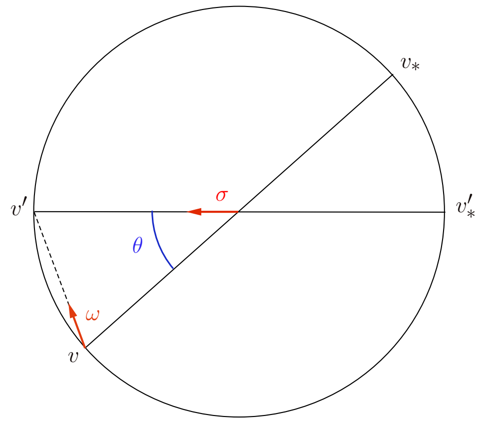

Figure 1. The geometry of the binary collision.

The assumption above is vaguely formulated as far as the singularity is concerned, the more precise formulation that we will use in this paper is

$$\begin{cases} B(r,\cos\theta) = r^{\gamma} b(\cos\theta) \\ b(\cos\theta) = (\sin\theta/2)^{-(d-2)+\gamma}(\tan\theta/2)^{-(\gamma+2s+1)}\widetilde{b}(\cos\theta) \end{cases} \tag{1.2}$$

with some smooth $\widetilde{b}$ satisfying $0 < \widetilde{b}_0 \leqslant \widetilde{b} \leqslant \widetilde{b}_1$ for constants $\widetilde{b}_0$, $\widetilde{b}_1 > 0$. The precise mixture of sinus and tangent functions to model the singularity is made for technical convenience and is no loss of generality: it is easy to check using the symmetry of the collision process $v' \leftrightarrow v'_*$ that physical collision kernels satisfy it. It corresponds to the technical condition (3.1) in [49].

We consider collision kernels satisfying (1.2) in general dimension $d \geqslant 2$ and with general exponents $\gamma \in (-d, 2]$ and $s \in (0,1)$ that are not necessarily derived from the inverse power-law formula above. The hard spheres interactions play the role of the limit case $\alpha \to \infty$ ($\gamma = 1$ and integrable b).

It is standard terminology in dimension $d = 3$ to denote respectively:

– the case $\alpha = 5$ ($\gamma = 0$ and $2s = 1/2$) as *Maxwell molecules* [42],
– the case $\alpha \in (5, +\infty)$ ($\gamma \in (0,1)$ and $2s \in (0, 1/2)$) as *hard potentials*,
– the case $\alpha \in [3, 5)$ ($\gamma \in [-1, 0)$ and $2s \in (1/2, 1]$) as *moderately soft potentials*,
– the case $\alpha \in (2, 3)$ ($\gamma \in (-3, -1)$ and $2s \in (1, 2)$) as *very soft potentials*.

The limit $s \to 1$ is called the *grazing collision limit*, and in this limit the Boltzmann collision operator converges to the *Landau-Coulomb collision operator*. It turns out that the threshold between moderately and very soft potentials corresponds to $\gamma + 2s = 0$. We therefore denote by *moderately soft potentials*, in any dimension $d \geqslant 2$, the case $\gamma + 2s \in [0, 2]$.

1.2. The question at hand. — The global well-posedness for solutions to the inhomogeneous Boltzmann equation is an outstanding open problem. Since it is a more detailed model than the Euler and Navier-Stokes equations, and it includes these equations as limits in certain scalings, one can expect that it will share some of the (currently intractable) difficulties of these hydrodynamic models. Even in the spatially homogeneous case, the Cauchy problem is shown to be well-posed without perturbative assumptions only in the case of moderately soft potentials [23]. Given that global well-posedness seems out of reach at present time, our more realistic goal is to show that for suitable initial data $f(0,x,v) = f_0(x,v)$, the equation (1.1) has a unique smooth solution for as long as its associated hydrodynamic quantities stay under control. Morally, this neglects the hydrodynamic difficulties of the model and concentrates on the difficulties that are intrinsic to the kinetic representation of the fluid.

Let us state the longer-term conjecture. Consider the following hydrodynamic quantities

$$
\begin{aligned}
&\text{(mass density)} && M(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v,\\
&\text{(energy density)} && E(t,x) := \int_{\mathbb{R}^d} f(t,x,v)|v|^2\,\mathrm{d}v,\\
&\text{(entropy density)} && H(t,x) := \int_{\mathbb{R}^d} f\ln f(t,x,v)\,\mathrm{d}v.
\end{aligned}
$$

Conjecture (conditional regularisation). — *Consider any solution*

$$f \in L^\infty([0,T], L^1(\mathbb{R}^d \times \mathbb{R}^d))$$

to (1.1) *on a time interval* $[0,T]$ *for some* $T \in (0,+\infty]$, *such that the hydrodynamic fields of* f *remain controlled on this time interval: more precisely assume that for all points* (t,x), *the mass density is bounded below and above* $0 < m_0 \leqslant M(t,x) \leqslant M_0$, *the energy density is bounded above* $E(t,x) \leqslant E_0$ *and the entropy density is bounded above* $H(t,x) \leqslant H_0$ *(for constants* m_0, M_0, E_0, $H_0 > 0$*). Then this solution is bounded and smooth on* $(0,T]$.

Remarks 1.1

(1) Observe that the contraposition of this statement means that any finite-time blow-up in solutions to the Boltzmann equation with long-range interactions must include a blow-up in the hydrodynamic quantity (local mass, energy or entropy diverging at some position), or the creation of vacuum (local mass vanishing at some position). In other words, one of the hydrodynamic bounds above has to degenerate as $t \uparrow T^-$.

(2) There are two natural ways in which this conjecture can be strengthened or weakened:

(a) Strengthening the statement: the blow-up scenario through the creation of vacuum is likely to be ruled out by further work, which means that the lower bound assumption on the mass could be removed. Mixing in velocity through collisions combined with transport effects generate lower bounds in many settings, see [47, 27, 14, 13], and the assumption was indeed removed for the related Landau equation with moderately soft potentials in [35]. We might also expect that the pointwise bounds could be replaced with an $L^p_t(L^q_x)$ bound for E, M and H, similar to the Prodi-Serrin condition for Navier-Stokes equations.

(b) Weakening the statement: more regularity or decay could be assumed on the initial data, as long as it is propagated conditionally to the hydrodynamic bounds assumed on the solution. This would slightly weaken the conjecture but the contraposed conclusion would remain unchanged: any blow-up must occur at the level of the hydrodynamic quantities.

1.3. Known results of conditional regularisation in kinetic theory

1.3.1. *The Boltzmann equation with long-range interactions.* — In [5], the authors prove that if the solution f has five derivatives in L^2, with respect to all variables t, x and v, weighted by $\langle v\rangle^q := (1+|v|^2)^{q/2}$ for arbitrarily large powers q, and in addition the mass density is bounded below, then the solution f is C^∞. Note also that stability (uniqueness) holds under such $H^5_{x,v}(\langle v\rangle^q)$ regularity. Note also the previous partial result [24] and the subsequent follow-up papers [2, 36, 4, 3, 45] in the spatially homogeneous case, with less assumption on the initial data. Our goal however is to reduce the regularity assumed on the solution as close to the minimal hydrodynamic bounds as possible.

The natural strategy we follow goes through the following steps:

(1) *A pointwise estimate in* $L^\infty((0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$: observe that hydrodynamic quantities only control v-integrals on the solution.

(2) *A decay estimate for large velocities*: the non-compact velocity space is a source of mathematical difficulties in the Boltzmann theory, and badly thermalised solutions (e.g. spikes of high-velocity particles) break regularity estimates. Such decay can be searched in L^1 (moment estimates) or L^∞ spaces as in this paper.

(3) *A regularisation estimate in Hölder spaces*: this is where the hypoelliptic nature of the equation enters the strategy, and such a regularity estimate is in the spirit of De Giorgi-Nash-Moser theory.

(4) *Schauder estimates to obtain higher regularity by bootstrap*: this is a standard principle for quasilinear equations that regularity can be bootstrapped in C^k Hölder spaces, but the non-local integral nature of the collision operator creates new interesting difficulties.

The first step was completed in [49]. The main result in the present paper is the completion of the second step, i.e., decay estimate for large velocities. The third step, i.e., the regularisation in C^α was completed in [38]. The bootstrap mechanism to obtain higher regularity is the piece of the puzzle that currently remains unsolved. In future work, we intend to address the forth step using the Schauder estimates from [39].

1.3.2. *The Landau equation*. — This program of "conditional regularisation" following the four steps above has already been carried out for the inhomogeneous Landau equation with moderately soft potentials, which corresponds to the limit of the Boltzmann equation as $s \to 1$, when furthermore $\gamma \in [-2, 0]$. The L^∞ estimate, as well as Gaussian upper bounds, were obtained in [16] (first and second steps). The regularisation estimate in Hölder spaces was obtained in [30] (third step). The fourth step was completed in [34] in the form of Schauder estimates for kinetic parabolic equations. The regularity of solutions of the Landau equation is iteratively improved using Schauder estimates up to C^∞ regularity. In the physical case of the Landau-Coulomb equation (playing the role of the limit case $\alpha = 2$, $\gamma = -3$, $s = 1$ in dimension 3), the conjecture is still open: the L^∞ bound is missing (see however partial results in this direction in [50]), and the Schauder estimates [34] do not cover this case even though this last point is probably only a milder technical issue.

An important inspiration we draw from the case of the Landau equation is that the iterative gain of regularity in the spirit of [34] require a solution that decays, as $|v| \to \infty$, faster than any algebraic power rate $|v|^{-q}$. We expect the same general principle to apply to the Boltzmann equation.

1.4. Main result. — We consider in this paper strong (classical) solutions to the Boltzmann equation in the torus $x \in \mathbb{T}^d$ (periodic spatial boundary conditions) with decay $O((1 + |v|)^{-\infty})$, i.e., polynomial of any order.

Definition 1.2 (Classical solutions to the Boltzmann equation with rapid decay)
Given $T \in (0, +\infty]$, we say that a function $f : [0, T] \times \mathbb{T}^d \times \mathbb{R}^d \to [0, +\infty)$ is a *classical solution to the Boltzmann equation* (1.1) *with rapid decay* if

– the function f is differentiable in t and x and twice differentiable in v everywhere;
– the equation (1.1) holds classically at every point in $[0, T] \times \mathbb{T}^d \times \mathbb{R}^d$;
– for any $q > 0$, $(1 + |v|)^q f(t, x, v)$ is uniformly bounded on $[0, T] \times \mathbb{T}^d \times \mathbb{R}^d$.

We chose the setting of classical solutions. This is natural because we work under a priori assumptions (the hydrodynamic bounds), and moreover the only theory of existence of weak solutions available in the case of long-range interactions is the theory of "*renormalized solution with defect measure*" [6], that extends the notion of renormalized solutions of DiPerna and P.-L. Lions [25], and these very weak solutions are too weak to be handled by the methods of this paper. The rapid polynomial decay we impose at large velocities is a qualitative assumption that we make for technical reasons: just like the periodicity in x, it is used to guarantee the existence of a first contact point in the argument of maximum principle. It is specially needed in the case $\gamma > 0$. However the estimates in the conclusion of our theorem do not depend on the decay rate as $|v| \to \infty$ that is initially assumed for the solution (otherwise, the theorem would obviously be empty). We discuss in Section 5 how to relax this qualitative assumption.

Theorem 1.3 (Pointwise moment bounds for the Boltzmann with hard or moderately soft potentials)

Let $\gamma \in (-2,2)$ *and* $s \in (0,1)$ *satisfy* $\gamma + 2s \in [0,2]$ *and* f *be a solution of the Boltzmann equation* (1.1) *as in Definition* 1.2 *such that* $f(0,x,v) = f_0(x,v)$ *in* $\mathbb{T}^d \times \mathbb{R}^d$ *and*

$$\forall\,(t,x) \in [0,T] \times \mathbb{T}^d, \quad 0 < m_0 \leqslant M(t,x) \leqslant M_0,\ E(t,x) \leqslant E_0 \text{ and } H(t,x) \leqslant H_0 \tag{1.3}$$

holds true for some positive constants m_0, M_0, E_0, H_0. *It was then proved in* [49] *that* f *satisfies an* L^∞ *a priori estimate depending on these constants (see Theorem* 4.1 *recalled later); we establish here the following more precise decay estimates at large velocities.*

(1) Propagation of "pointwise moments" for moderately soft and hard potentials. *There exists* q_0 *depending on* d, s, γ, m_0, M_0, E_0, H_0 *such that if* $q \geqslant q_0$ *and* $f_0 \leqslant C(1+|v|)^{-q}$ *for some* $C > 0$ *then there exists a constant* N *depending on* C, m_0, M_0, E_0, H_0, q, d, γ *and* s, *such that*

$$\forall\, t \in [0,T],\ x \in \mathbb{T}^d,\ v \in \mathbb{R}^d, \quad f(t,x,v) \leqslant N\,(1+|v|)^{-q}\,.$$

(2) Appearance of "pointwise moments" for hard potentials. *If additionally* $\gamma \in (0,2)$ *then, for any* $q > 0$ *there exists a constant* N *depending on* $m_0, M_0, E_0, H_0, q, d, \gamma$ *and* s *and a power* $\beta > 0$ *depending on* d, q, γ *and* s *such that*

$$\forall\, t \in (0,T],\ x \in \mathbb{T}^d,\ v \in \mathbb{R}^d, \quad f(t,x,v) \leqslant N\left(1+t^{-\beta}\right)(1+|v|)^{-q}\,.$$

(3) Appearance of lower order "pointwise moments" for moderately soft potentials. *For all* $\gamma \in (-2,0]$, *there exists a constant* N *depending on* $m_0, M_0, E_0, H_0, d, \gamma$ *and* s *such that*

$$\forall\, t \in (0,T],\ x \in \mathbb{T}^d,\ v \in \mathbb{R}^d, \quad f(t,x,v) \leqslant N(1+t^{-d/2s})\,(1+|v|)^{-d-1-d\gamma/2s}\,.$$

Remarks 1.4

(1) In the third point (3), the "order" of the pointwise decay is lower than what would ensure integrability in the energy bound $\int f|v|^2\,\mathrm{d}v < +\infty$ since $d+1+\gamma d/2s < d+1$. Since $\gamma+2s \geqslant 0$ (moderately soft potentials), $d+1+\gamma d/2s \geqslant 1$. More precisely, in dimension 3 and for an inverse power-law interaction force $Cr^{-\alpha}$ this is $(3\alpha-7)/2$ with $\alpha \in [3,5)$. However this bound is locally (in v) stronger than the energy bound as it is pointwise, and it does not depend on norms on derivatives through an interpolation argument.

(2) In the proof of point (2), our reasoning provides $\beta = d/(2s) + q/\gamma$ if q is large, without claim of optimality.

(3) As discussed in Section 5, see Theorem 5.2, the qualitative assumption of rapid decay can be relaxed entirely for (3) and for (2) for $q = d+1$ and also for (1) when $\gamma \leqslant 0$ and q large enough. Finally for (2) it can be weakened to $(1+|v|^{q_0})f$ uniformly bounded on $t \in [0,T]$, $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$ for some q_0 large enough.

(4) It is conceivable that some versions points (1) and (2) of Theorem 1.3 should hold in the cutoff case, probably with stronger conclusions. We are interested here in

the non-cutoff model, so we have not investigated this problem. Note however that point (3) is likely to be false in the cutoff case, i.e., to be of a strictly non-cutoff nature.

(5) Note that our conditional bounds are uniform as $t \to +\infty$, provided that the bounds on the hydrodynamic fields are also uniform as $t \to +\infty$. The latter is known for spatially homogeneous solutions, and in this case our result implies uniform in time bounds on the pointwise moments in the case of moderately soft potentials which is new to the best of our knowledge (and improves on the previous results [21, 22]).

1.5. Decay at large velocity in the Boltzmann theory. — The study of the decay at large velocity is central in the study of solutions to the Boltzmann equation, and has a long history. Such decay is necessary for instance to prove that appropriate weak solutions satisfy the conservation of the kinetic energy (second moment), and more generally appears in any regularity estimate.

1.5.1. *Moment estimates (weighted L^1 estimates).* — Measuring the decay at large velocity in terms of moments, i.e., weighted integral $\int f|v|^q \, \mathrm{d}v$, is a natural step in view of the fact that the velocity space is unbounded and the collision operator integrates over all velocities. The study of moments was initiated for Maxwellian potentials ($\gamma = 0$) in the spatially homogeneous case in [37, 51]: closed systems of exact differential equations are derived for polynomial moments and their propagation in time is shown, without any possibility of appearance. In the case of hard potentials ($\gamma > 0$) with angular cutoff (playing the role of "$s = 0$") and spatial homogeneity (no x dependency), the study of moments relies on the so-called *Povzner identities* [48]:

– Elmroth [26] used them to prove that if any moment $q > 2$ exists initially, then they remain bounded for all times.

– Desvillettes [21] then showed that all moments are generated as soon as one moment of order $q > 2$ exists initially.

– Finally [44, 40] proved that even the condition on one moment of order $s > 2$ can be dispensed with and only the conservation of the energy is required; it was later extended to the spatially homogeneous hard potentials without cutoff in [53].

– Then Bobylev [10], through some clever refinement of the Povzner inequality and infinite summation, proved, for spatially homogeneous hard potentials with cutoff, the propagation of (integral) exponential tail estimates $\int f e^{C|v|^b} \, \mathrm{d}v$ with $b \in (0, 2]$ and C small enough if $b = 2$.

– This result was extended in [12] to more general collision kernels, that remains variants of hard potentials with cutoff.

– Finally the Bobylev's argument was improved to obtain *generation* of (integral) exponential tail estimates $\int f e^{C|v|^b} \, \mathrm{d}v$ with $b \in (0, \gamma]$ in [43, 7].

– The case of measure-valued solutions in the spatially homogeneous hard potentials with cutoff is considered in [41], and the case of the Boltzmann-Nordheim equation for bosons was addressed in [15].

Let us also mention two important extensions of these methods:

– In the case of spatially homogeneous moderately soft potentials with cutoff, Desvillettes [21] proved for $\gamma \in (-1, 0)$ that initially bounded polynomial moments grow at most linearly with time and it is explained in [52] that the method applies to $\gamma \in [-2, 0)$. This was later improved [22] into bounds uniform in time thanks to the convergence to equilibrium.

– In [31, §5], the appearance and propagation of polynomial moments $L^1_v L^\infty_x(1 + |v|^q)$ is proved for the the spatially inhomogeneous Boltzmann equation in $x \in \mathbb{T}^d$ for hard spheres, as well as the appearance and propagation of exponential moments $L^1_v W^{3,1}_x(e^{c|v|})$. All these results assume bounds on the hydrodynamic quantities similar to what is assumed in this paper.

1.5.2. *Pointwise decay (weighted L^∞ estimates)*. — In the spatially homogeneous setting (with cutoff), the study of pointwise decay goes back to Carleman [17, 18], where it was first studied for radially symmetric solutions $f = f(t, |v|)$, and was further developed in [9] (see also the L^p bounds in [32, 33]). The first exponential pointwise bound was obtained in [28] and the latter paper pioneered the use of the comparison principle for the Boltzmann equation: the authors obtain pointwise Gaussian upper and lower bounds. The method was extended in [11], and in [29] (using estimates from [8]) where the authors prove exponential (but not exactly Gaussian) upper bound for the space homogeneous non-cutoff Boltzmann equation.

Regarding the spatially inhomogeneous setting, it is mentioned in [52, Chap. 2, §2] that: "*In the case of the full, spatially-inhomogeneous Boltzmann equation there is absolutely no clue of how to get such [moment] estimates. This would be a major breakthrough in the theory*". This refers to unconditional moment bounds, and, as the result in [31, §5] mentioned above shows, it is expected that some of these estimates can be extended to the space inhomogeneous case under the assumptions that the hydrodynamic quantities stay under control. However, moment estimates obtained using Povzner inequalities would, in the most optimistic scenario, involve an upper bound on a weighted integral quantity with respect to x and v. There seems to be no natural procedure to imply pointwise upper bounds from them. Indeed, in order to apply methods similar to [28, 11, 29] to the space inhomogeneous case, we would first need strong $L^\infty_{t,x} L^1_v(\omega)$ moments for some weight ω (probably exponential). In this paper, we bypass any analysis of moment estimates by obtaining pointwise upper bounds directly.

1.6. Strategy of proof. — The proof of the main theorem consists in proving that the solution satisfies $f(t, x, v) < g(t, v)$ with $g(t, v) := N(t)(1 + |v|)^{-q}$, for different choices of the function $N : (0, +\infty) \to (0, +\infty)$ and $q \in (0, +\infty)$. The appearance of pointwise bounds requires $N(t) \to +\infty$ as $t \to 0$, whereas the function $N(t)$ is bounded near $t = 0$ for proving propagation of pointwise bounds. Without loss of generality, it is convenient in order to simplify calculations to use instead the barrier $g(t, v) = N(t) \min(1, |v|^{-q})$.

We then ensure that the comparison is satisfied initially and look for the first time $t_0 > 0$ when the inequality is invalidated. We prove the existence of a first contact point (t_0, x_0, v_0) such that $f(t_0, x_0, v_0) = g(t_0, v_0)$, and search for a contradiction at this point. The existence of this first contact point follows from the rapid decay assumption in Definition 1.2 and the compactness of the spatial domain.

At the first contact point we have $f(t_0, x_0, v_0) = g(t_0, v_0)$. Since the right hand side does not depend on x, we must have $\nabla_x f(t_0, x_0, v_0) = 0$. We also deduce that $\partial_t f(t_0, x_0, v_0) \geqslant \partial_t g(t_0, v_0)$. Therefore, since f solves the equation (1.1), we have

$$\partial_t g(t_0, v_0) \leqslant \partial_t f(t_0, x_0, v_0) = Q(f, f)(t_0, x_0, v_0). \tag{1.4}$$

We then decompose the collision operator (using the so-called *Carleman representation*, see (2.1) and (3.2) below) into $Q = \mathcal{G} + \mathcal{B} + Q_{\mathrm{ns}}$ where $\mathcal{G}$ is the "good" term, that is to say negative at large velocities, $\mathcal{B}$ is the "bad" term, treated as a positive error term at large velocities, and finally Q_{ns} is a remaining "non-singular / lower order term" where the angular singularity has been removed by the so-called "cancellation lemma". The core of the proof then consists in proving that the "good" negative term dominates over all the other terms at large velocities, hence yielding a contradiction.

Remark that the only purpose of the rapid decay assumption in Definition 1.2 is to obtain this first contact point (t_0, x_0, v_0). In Section 5 we explore a setting in which we can relax this qualitative assumption: we add a small correction term to the function $g(t, v)$ in order to ensure the inequality $f < g$ for large values of v; we recover a large part of Theorem 1.3, but run into technical problems when $\gamma > 0$ (see Theorem 5.2).

1.7. Open questions. — Here are some natural questions that remain unanswered and are natural problems to investigate in the future:

– Our result says, for some range of parameters, that the rate of decay of the solution $f(t, x, v)$ is faster than any power function $|v|^{-q}$ as $|v| \to \infty$. The most desirable result would be to obtain the appearance of exponential upper bounds on $\sup_{x,v} f e^{C|v|^\kappa}$ or the propagation of Gaussian upper bound $\sup_{x,v} f e^{C|v|^2}$ as in [28] or [16]. This seems to require new techniques.

– Another open problem regards the range of parameters γ, s for which the bounds hold. This work is restricted to moderately soft potentials $\gamma + 2s \in [0, 2]$. The case $\gamma + 2s \in (-1, 0)$ (very soft potentials) is of great interest but seems out of reach with the current methods and requires new ideas. The non-physical range $\gamma + 2s > 2$ presents a difficulty in that the energy estimate is insufficient to control the kernel K_f defined in (2.3).

– It would also be interesting to relax the qualitative assumption of rapid decay in Definition 1.2 of the solutions we use. We explore this question in Section 5. In the case $\gamma \leqslant 0$, we recover essentially the same result as in Theorem 1.3 without assuming the rapid decay at infinity of solutions provided that $\gamma + 2s < 1$. In the case $\gamma > 0$, we can always generate decay of the form $f \leqslant N|v|^{-d-1}$. However, in order to obtain an upper bound that decays with a higher power, we need to make the qualitative assumption that $\lim_{|v|\to\infty} |v|^{q_0} f(t, x, v) = 0$ for some power q_0 that depends on all the other parameters. We would naturally expect the estimates in Theorem 1.3 to hold for solutions with only the energy decay.

1.8. Organisation of the article. — Section 2 reviews quickly results from previous works that are used in the proof of the main result. The collision operator is divided into different pieces which are estimated successively in Section 3. Section 4 contains the proof of the main result. Finally Section 5 discusses how to relax the assumption of rapid decay.

1.9. Notation. — For two real numbers $a, b \in \mathbb{R}$, we write $a \wedge b$ for their minimum. Moreover, $a \lesssim b$ means that $a \leqslant Cb$ with C only depending on dimension, γ, s and hydrodynamic quantities m_0, M_0, E_0, H_0. The notation $a \lesssim_q b$ means that C may additionally depend on the parameter q. Constants C_q, R_q also depend on q, and can be large. The constant c_q is "explicit" in Proposition 3.1. We sometimes use the shorthand $f' = f(v')$, $f'_* = f(v'_*)$, $f = f(v)$, $f_* = f(v_*)$. We will also denote classically: $w := v' - v$, $\omega = w/|w|$, $u := v'_* - v$, $\widehat{u} = u/|u|$.

2. Preliminaries results

2.1. Cancellation lemma and Carleman representation. — We split the Boltzmann collision operator in two, along an idea introduced in [1], and use the so-called *cancellation lemma* from the latter paper to estimate the non-singular part. Then, in the remaining singular part, we change variables to the so-called *Carleman representation* introduced in [17, 18] (see also [52, §4.6] for a review, and developed in the non-cutoff case in [49, §4]). Given a velocity v, the possible binary collisions can be parametrised (i) by $v_* \in \mathbb{R}^d$, $\sigma \in \mathbb{S}^{d-1}$ which is sometimes called the "σ-representation" given in the previous section, (ii) by $v_* \in \mathbb{R}^d$ and $\omega \in \mathbb{S}^{d-1}$ with $\omega := (v' - v)/|v' - v|$ (see Figure 1.1) which is sometimes called the "ω-representation", and (iii) by $v' \in \mathbb{R}^d$ and $v'_* \in v + (v' - v)^\perp$ (the $(d-1)$-dimensional hyperplane) which is called the "Carleman representation" to acknowledge its introduction in [17] in the radial case. This alternative Carleman representation is used, as in previous works, in order to write the Boltzmann collision operator as an integral singular Markov generator applied to its second argument, with an explicit kernel depending on its first argument.

The splitting is (without caring for now about convergences of integrals):

$$
\begin{aligned}
&Q(f_1, f_2)(v) \\
&\quad = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \Big[f_1(v'_*) f_2(v') - f_1(v_*) f_2(v) \Big] B \, \mathrm{d}v_* \, \mathrm{d}\sigma \\
&\quad = \int f_1(v'_*) \big[f_2(v') - f_2(v) \big] B \, \mathrm{d}v_* \, \mathrm{d}\sigma + f_2(v) \int \big[f_1(v'_*) - f_1(v_*) \big] B \, \mathrm{d}v_* \, \mathrm{d}\sigma \\
&\quad =: Q_{\mathrm{s}}(f_1, f_2) + Q_{\mathrm{ns}}(f_1, f_2),
\end{aligned}
\tag{2.1}
$$

where "s" stands for "singular" and "ns" stands for "non-singular".

Let us first consider the non-singular part Q_{ns}. Given $v \in \mathbb{R}^d$, the change of variables $(v_*, \sigma) \mapsto (v'_*, \sigma)$ has Jacobian $\mathrm{d}v'_* \mathrm{d}\sigma = 2^{d-1} (\cos\theta/2)^2 \, \mathrm{d}v_* \mathrm{d}\sigma$, which yields (same calculation as [1, Lem. 1])

$$Q_{\mathrm{ns}}(f_1,f_2)(v) = f_2(v)\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \big[f_1(v'_*) - f_1(v_*)\big] B \,\mathrm{d}v_* \,\mathrm{d}\sigma =: f_2(v)(f_1 * S)(v)$$

with

$$\begin{aligned} S(u) &:= \big|\mathbb{S}^{d-2}\big| \int_0^{\pi/2} (\sin\theta)^{d-2}\Big[(\cos\theta/2)^{-d} B\Big(\frac{|u|}{\cos\theta/2},\cos\theta\Big) - B(|u|,\cos\theta)\Big]\,\mathrm{d}\theta \\ &= \big|\mathbb{S}^{d-2}\big|\,|u|^\gamma \int_0^{\pi/2} (\sin\theta)^{d-2}\big[(\cos\theta/2)^{-d-\gamma} - 1\big] b(\cos\theta)\,\mathrm{d}\theta \\ &=: C_S|u|^\gamma, \end{aligned}$$

where we have used the precise form (1.2) of the collision kernel in the second line. The constant $C_S > 0$ is finite and only depends on b, d, and γ. In short, the cancellation lemma is a kind of discrete integration by parts where the singularity of the fractional derivative is pushed onto the kernel itself.

Let us consider the singular part Q_{s}. We change variables (Carleman representation) according to $(v_*,\sigma) \mapsto (v', v'_*)$ as described above. The Jacobian is $\mathrm{d}v_*\mathrm{d}\sigma = 2^{d-1}|v-v'|^{-1}|v-v_*|^{-(d-2)}\,\mathrm{d}v'\mathrm{d}v'_*$ (see for instance [49, Lem. A.1]):

$$Q_{\mathrm{s}}(f_1,f_2)(v) = \mathrm{p.\,v.}\int_{\mathbb{R}^d} K_{f_1}(v,v')\big[f_2(v') - f_2(v)\big]\,\mathrm{d}v', \tag{2.2}$$

where

$$\begin{aligned} K_{f_1}(v,v') &:= \frac{1}{|v'-v|}\int_{v'_*\in v+(v'-v)^\perp} f_1(v'_*)|v-v_*|^{\gamma-(d-2)} b(\cos\theta)\,\mathrm{d}v'_* \\ &:= \frac{1}{|v'-v|^{d+2s}}\int_{v'_*\in v+(v'-v)^\perp} f_1(v'_*)|v-v'_*|^{\gamma+2s+1}\widetilde{b}(\cos\theta)\,\mathrm{d}v'_*, \end{aligned} \tag{2.3}$$

where we have used the assumption (1.2) and in particular the fact that

$$b(\cos\theta) = |v-v'|^{-(d-1)-2s}|v-v_*|^{(d-2)-\gamma}|v-v'_*|^{\gamma+2s+1}\widetilde{b}(\cos\theta).$$

The notation p. v. denotes the Cauchy principal value around the point v. Note that it is needed only when $s \geqslant [1/2,1)$.

One can reverse the order of the integration variables to get the alternative formula

$$Q_{\mathrm{s}}(f_1,f_2) = \mathrm{p.\,v.}\int_{v'_*\in\mathbb{R}^d} f_1(v'_*)|v-v'_*|^{\gamma+2s}\int_{v'\in v+(v-v'_*)^\perp} \frac{\big[f_2(v') - f_2(v)\big]\widetilde{b}}{|v'-v|^{d-1+2s}}\,\mathrm{d}v'. \tag{2.4}$$

Note that we have used the following standard manipulation:

$$\begin{aligned} \int_{u\in\mathbb{R}^d}\int_{w\perp u} F(u,w)\,\mathrm{d}w\,\mathrm{d}u &= \int_{u,w\in\mathbb{R}^d} F(u,w)\delta(w\cdot\widehat{u})\,\mathrm{d}w\,\mathrm{d}u \\ &= \int_{u,w\in\mathbb{R}^d} F(u,w)\delta(|w|(\widehat{w}\cdot\widehat{u}))\,\mathrm{d}w\,\mathrm{d}u = \int_{u,w\in\mathbb{R}^d} F(u,w)|w|^{-1}\delta(\widehat{w}\cdot\widehat{u})\,\mathrm{d}w\,\mathrm{d}u \\ &= \int_{u,w\in\mathbb{R}^d} F(u,w)\,\frac{|u|}{|w|}\,\delta(u\cdot\widehat{w})\,\mathrm{d}u\,\mathrm{d}w = \int_{w\in\mathbb{R}^d}\int_{u\perp w}\frac{|u|}{|w|}\,F(u,w)\,\mathrm{d}u\,\mathrm{d}w. \end{aligned}$$

2.2. Lower bound induced by the hydrodynamic bounds. — It is classical that, for each x, the controls on the local mass, energy and entropy, and the non-vacuum condition, together imply that the mass is bounded below and cannot concentrate in a zero-measure set: hence it implies pointwise lower bound on non zero-measure sets.

Lemma 2.1 (Lower bound on a set with positive measure). — *Under the assumption* (1.3)*, there exists $R_0 > 0$ such that for all $t \in [0,T]$ and $x \in \mathbb{T}^d$ there exists a set $\mathcal{D} = \mathcal{D}(t,x) \subset B_{R_0}$ such that*

$$\forall\, v \in \mathcal{D}(t,x),\ f(t,x,v) \geqslant c_0 \quad \textit{and} \quad |\mathcal{D}(t,x)| \geqslant \mu > 0$$

for R_0 and μ only depending on M_0, m_0, E_0, H_0 and dimension.

Proof. — The proof is elementary and can be found for instance in [49, Lem. 4.6]. It follows from the classical fact that the entropy bound implies the non-concentration estimate

$$\int_A f(t,x,v)\,\mathrm{d}v \lesssim_{M_0,H_0} \varphi(|A|) \quad \text{with } \varphi(r) = \ln(1+r) + \left[\ln\left(r^{-1}\right)\right]^{-1}$$

and A a Borel set and $|A|$ its Lebesgue measure. The energy bound provides tightness and prevents the mass from being arbitrarily far from the origin. $\square$

2.3. The cone of non-degeneracy. — We recall from [49, 38] the following more subtle result.

Lemma 2.2 (Cone of non-degeneracy). — *Consider a non-negative function f satisfying* (1.3)*. Then there are constants $c_0, C_0, \mu, \mu' > 0$ (depending on d, γ, s, m_0, M_0, E_0 and H_0) such that for any $t \in [0,T]$, $x \in \mathbb{T}^d$ and $v \in \mathbb{R}^d$, there exists a cone of directions $\Xi = \Xi(t,x,v)$ that is symmetric (i.e., $\Xi(t,x,v) = -\Xi(t,x,v)$) and so that*

$$\forall\, v \in \mathbb{R}^d,\ \forall\, v' \in v + \Xi(t,x,v), \quad K_f(v,v') \geqslant c_0 (1+|v|)^{1+2s+\gamma} |v-v'|^{-d-2s}$$

and

$$\Xi(t,x,v) \subset \left\{ w \in \mathbb{R}^d \mid |v \cdot w| \leqslant C_0 |w| \right\}$$

and for any $r > 0$

$$\frac{\mu r^d}{(1+|v|)} \leqslant |\Xi(t,x,v) \cap B_r| \leqslant \frac{\mu' r^d}{(1+|v|)}. \tag{2.5}$$

Proof. — The proof is gathered from [49, Lem. 4.8] and [49, Lem. 7.1]. Observe that by changing the order of integration, for any integrable function F on $\mathbb{R}^d$:

$$\int_{\omega \in \mathbb{S}^{d-1}} \int_{\{u \in \mathbb{R}^d,\ u \perp \omega\}} F(u)\,\mathrm{d}u\,\mathrm{d}\omega = \left|\mathbb{S}^{d-2}\right| \int_{\mathbb{R}^d} \frac{F(u)}{|u|}\,\mathrm{d}u.$$

We distinguish the two cases $|v| \leqslant 2R_0$ and $|v| > 2R_0$, where R_0 is the upper bound on elements of $\mathcal{D}(t,x)$ in Lemma 2.1.

Case when $|v| \leqslant 2R_0$. — We estimate

$$\int_{\omega\in\mathbb{S}^{d-1}}\int_{\{u\in\mathbb{R}^d,\ u\perp\omega\}}\mathbb{1}_{\mathcal{D}(t,x)}(v+u)\,\mathrm{d}u\,\mathrm{d}\omega = \left|\mathbb{S}^{d-2}\right|\int_{\mathbb{R}^d}\mathbb{1}_{\mathcal{D}(t,x)}(v+u)|u|^{-1}\,\mathrm{d}u$$
$$= \left|\mathbb{S}^{d-2}\right|\int_{v'_*\in\mathcal{D}(t,x)}|v'_*-v|^{-1}\,\mathrm{d}v'_*,$$

which is bounded below by some positive constant $\delta_0 > 0$ independent of $t \in \mathbb{R}$, $x \in \mathbb{T}^{d-1}$, $v \in B_{2R_0}$. Because of the upper bound

$$\int_{\{u\in\mathbb{R}^d,\ u\perp\omega\}}\mathbb{1}_{\mathcal{D}(t,x)}(v+u)\,\mathrm{d}u \leqslant \int_{\{u\in\mathbb{R}^d,\ |u|\leqslant R_0\}}\mathrm{d}u \lesssim_{R_0} 1 \tag{2.6}$$

following from the boundedness of $\mathcal{D}(t,x)$, we deduce that there exists $\mu_0, \lambda_0 > 0$ such that for all $t \in \mathbb{R}$, $x \in \mathbb{T}^{d-1}$ and $v \in B_{2R_0}$, there exists a set $\Xi(t,x,v)\cap\mathbb{S}^{d-1}$ of unit vectors ω such that $|\Xi(t,x,v)\cap\mathbb{S}^{d-1}| \geqslant \mu_0$ and

$$\forall\,\omega\in\Xi(t,x,v)\cap\mathbb{S}^{d-1},\qquad \int_{\{u\in\mathbb{R}^d,\ u\perp\omega\}}\mathbb{1}_{\mathcal{D}(t,x)}(v+u)\,\mathrm{d}u\,\mathrm{d}\omega \geqslant \lambda_0$$

for some $\lambda_0 > 0$. Since the integrand above is even as a function of ω, the cone $\Xi(t,x,v)$ can be chosen symmetric.

Case when $|v| > 2R_0$. — We estimate

$$\int_{\omega\in\mathbb{S}^{d-1}}\int_{\{u\in\mathbb{R}^d,\ u\perp\omega\}}\mathbb{1}_{\mathcal{D}(t,x)}(v+u)\,\mathrm{d}u\,\mathrm{d}\omega = \left|\mathbb{S}^{d-2}\right|\int_{v'_*\in\mathcal{D}(t,x)}|v'_*-v|^{-1}\,\mathrm{d}v'_*$$
$$\geqslant \frac{\delta_1}{1+|v|}$$

for some positive constant $\delta_1 > 0$ independent of $t \in \mathbb{R}$, $x \in \mathbb{T}^{d-1}$, $v \in B^c_{2R_0}$. Given the upper bound (2.6) on the one hand and the fact that the support of the function

$$\omega \longmapsto \int_{\{u\in\mathbb{R}^d,\ u\perp\omega\}}\mathbb{1}_{\mathcal{D}(t,x)}(v+u)\,\mathrm{d}u$$

is included in the set $\{\omega\in\mathbb{S}^{d-1} : |\omega\cdot v| \leqslant C\}$ for some constant $C > 0$, we deduce that there exists $\mu_1, \lambda_1 > 0$ such that for all $t \in \mathbb{R}$, $x \in \mathbb{T}^{d-1}$ and $v \in B_{2R_0}$, there exists a set $\Xi(t,x,v)\cap\mathbb{S}^{d-1}$ of unit vectors ω such that

$$\left|\Xi(t,x,v)\cap\mathbb{S}^{d-1}\right| \geqslant \frac{\mu_1}{1+|v|}$$

and

$$\forall\,\omega\in\Xi(t,x,v)\cap\mathbb{S}^{d-1},\qquad \int_{\{u\in\mathbb{R}^d,\ u\perp\omega\}}\mathbb{1}_{\mathcal{D}(t,x)}(v+u)\,\mathrm{d}u\,\mathrm{d}\omega \geqslant \lambda_1.$$

Since the integrand above is even as a function of ω, the cone $\Xi(t,x,v)$ can be chosen symmetric. It lies by construction in the equatorial region required.

The cone $\Xi(t,x,v)$ built above satisfies the statement. $\square$

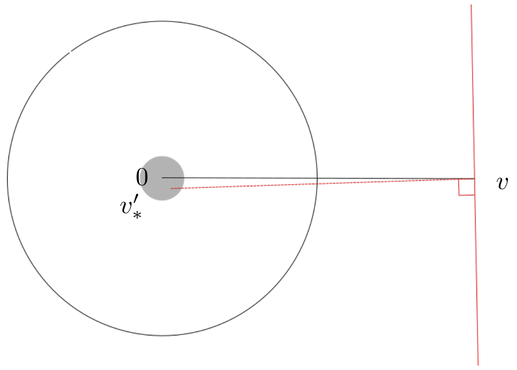

Figure 2. The good term: $\mathcal{G}$ corresponds to the integration over the red (plain) line ($\{v' \in \mathbb{R}^d : v' - v \perp w\}$). The grey ball is of radius $c_q|v|$ while the larger ball is of radius $|v|/2$.

3. Technical estimates on the collision operator

We consider a contact point where

$$f(t,x,v') \leqslant g(t,v') \text{ for all } v' \in \mathbb{R}^d \quad \text{and} \quad f(t,x,v) = g(t,v). \tag{3.1}$$

Since the collision operator does not act on the t and x variables, we omit them in most of this section to keep calculations uncluttered.

3.1. Estimates of the collision operator at the first contact point.

— In order to estimate the singular part $Q_{\mathrm{s}}(f,f)$ of the collision operator, we split it into a "good" term, negative at large velocities, and a "bad" term, treated as a smaller error at large velocities: define $c_1(q) = q^{-1}/20$ and

$$Q_{\mathrm{s}}(f,f) = \mathcal{G}(f,f) + \mathcal{B}(f,f) \tag{3.2}$$

with

$$\mathcal{G}(f_1,f_2) = \text{p. v.} \int_{|v'_*| \leqslant c_1(q)|v|} f_1(v'_*)|v - v'_*|^{\gamma+2s} \int_{v' \in v + (v-v'_*)^\perp} \frac{\big[f_2(v') - f_2(v)\big]\widetilde{b}}{|v-v'|^{d-1+2s}} \,\mathrm{d}v' \,\mathrm{d}v'_*$$

$$\mathcal{B}(f_1,f_2) = \text{p. v.} \int_{|v'_*| > c_1(q)|v|} f_1(v'_*)|v - v'_*|^{\gamma+2s} \int_{v' \in v + (v-v'_*)^\perp} \frac{\big[f_2(v') - f_2(v)\big]\widetilde{b}}{|v-v'|^{d-1+2s}} \,\mathrm{d}v' \,\mathrm{d}v'_*.$$

Note first that this decomposition is based on the representation (2.4) but the order of integration will sometimes be reversed back to the representation (2.2), depending on technical convenience. Note second that the idea behind this decomposition is to isolate the "good" configurations when v'_* is close enough to zero, where the bulk of the mass is located. Note finally that under assumption (3.1) on f one has $Q(f,f) \leqslant Q(f,g)$ and similarly $Q_{\mathrm{s}}(f,f) \leqslant Q_{\mathrm{s}}(f,g)$ and $\mathcal{G}(f,f) \leqslant \mathcal{G}(f,g)$ and $\mathcal{B}(f,f) \leqslant \mathcal{B}(f,g)$. We bound from below $\mathcal{G}(f,g)$ for large q and $\mathcal{G}(f,f)$ for not-so-large q, and we bound from above $\mathcal{B}(f,f)$ and $Q_{\mathrm{ns}}(f,f)$ successively in the next subsections.

3.2. Lower bound on the good term $\mathcal{G}$ for large q

Proposition 3.1 (Estimate of $\mathcal{G}(f,g)$ useful for large q). — *Let f be a non-negative function satisfying* (1.3) *and $g = N\min(1,|v|^{-q})$, $q \geqslant 0$. Then there exists a radius $R_q \geqslant 1$ so that*

$$\forall\, v \in \mathbb{R}^d \mid |v| \geqslant R_q, \quad \mathcal{G}(f,g)(v) \lesssim -(1+q)^s |v|^\gamma g(v).$$

We first estimate from above the inner integral in the following lemma.

Lemma 3.2. — *Let $q \geqslant 0$ and $g(v) = N\min(1,|v|^{-q})$. Then for all $v \in \mathbb{R}^d$ such that $|v| \geqslant 2$ and all $v'_* \in \mathbb{R}^d$ such that $|v'_*| < c_1(q)|v|$, we have (with a constant uniform in $q \geqslant 0$)*

$$\int_{v' \in\, v + (v - v'_*)^\perp} \big[g(v') - g(v)\big] \frac{\widetilde{b}(\cos\theta)}{|v'-v|^{d-1+2s}} \,\mathrm{d}v' \lesssim -(1+q)^s N |v|^{-2s-q}.$$

Proof. — We first prove that this integral is non-positive when $(v'-v)$ is small enough with respect to v. We then prove that when $(v'-v)$ is large enough in proportion to v then $|v'|$ is larger than $|v|$ (in this step we use the assumption $|v'_*| < c_1(q)|v|$) which gives an explicit negative upper bound due to the decay of g. The geometric interpretation is simple: when v'_* is small with respect to v and v is large, then the cone of possible directions for $(v'-v)$ is close to orthogonal to v, and when $(v'-v)$ is not too small v' leaves $B(0,|v|)$.

Define $c_2(q) := (1+q)^{-1/2}/20$ (note the different asymptotic behaviour as compared to $c_1(q)$) and assume first that $|v-v'| < 4c_2(q)|v|$. Then $|v'| \geqslant (1-4c_2(q))|v| \geqslant 1$ since $|v| \geqslant 2$, and $g(v') = N|v'|^{-q}$. By assumption $|v| \geqslant 1$ and thus $g(v) = N|v|^{-q}$. The integration kernel is invariant under rotation around the axis $(vv*)$ and therefore

$$\begin{aligned}\int_{v' \in\, v + (v - v'_*)^\perp} &\big[g(v') - g(v)\big] \frac{\widetilde{b}(\cos\theta)}{|v'-v|^{d-1+2s}} \,\mathrm{d}v' \\ &= \int_{v' \in\, v + (v - v'_*)^\perp} \Big[g(v') - g(v) - \nabla g(v)\cdot(v'-v)\Big] \frac{\widetilde{b}(\cos\theta)}{|v'-v|^{d-1+2s}} \,\mathrm{d}v'.\end{aligned}$$

Taylor expand the integrand: there is some $\theta \in (0,1)$ and $v'_\theta := v + \theta(v'-v)$ so that

$$\begin{aligned}\Big[g(v') - g(v) - \nabla g(v)\cdot(v'-v)\Big] &= \frac{1}{2} D^2 g(v'_\theta)(v'-v)\cdot(v'-v) \\ &= \frac{N}{2}|v'_\theta|^{-q-2} q \Big(\frac{(q+2)}{|v'_\theta|^2}|v'_\theta \cdot (v'-v)|^2 - |v'-v|^2\Big). \qquad (3.3)\end{aligned}$$

Since $(v-v') \perp (v-v'_*)$ and $|v-v'| \leqslant 4c_2(q)|v|$ and $|v'_*| \leqslant c_1(q)|v|$, we have

$$\begin{aligned}|v'_\theta \cdot (v'-v)| &\leqslant |v\cdot(v'-v)| + |v'-v|^2 = |v'_* \cdot (v'-v)| + |v'-v|^2 \\ &\leqslant c_1(q)|v||v'-v| + |v'-v|^2 \leqslant (c_1(q) + 4c_2(q))\,|v||v'-v|\end{aligned}$$

and $|v'_\theta| \geqslant (1-4c_2(q))|v|$. We deduce

$$\text{(3.4)}\quad \Big[g(v') - g(v) - \nabla g(v)\cdot(v'-v)\Big] \\ \leqslant \frac{qN}{2|v'_\theta|^{q+2}}\Big((q+2)\Big(\frac{c_1(q)+4c_2(q)}{1-4c_2(q)}\Big)^2 - 1\Big)|v'-v|^2 \leqslant 0$$

since

$$\Big(\frac{c_1(q)+4c_2(q)}{1-4c_2(q)}\Big)^2 \leqslant \frac{1}{9(q+1)} \implies (q+2)\Big(\frac{5c_q}{1-4c_q}\Big)^2 \leqslant \frac{1}{3}$$

uniformly for $q \geqslant 0$, due to the smallness assumptions on $c_1(q) \leqslant c_2(q)$.

When $|v'-v| \geqslant 4c_2(q)|v|$, then (using the smallness and orthogonality properties as before):

$$\begin{aligned}|v'|^2 &= |v|^2 + |v'-v|^2 + 2v'\cdot(v'-v) = |v|^2 + |v'-v|^2 + 2v'_*\cdot(v'-v)\\ &\geqslant |v|^2 + |v'-v|^2 - 2c_q|v||v'-v| \geqslant |v|^2 + |v'-v|\left(|v'-v| - 2c_1(q)|v|\right)\\ &\geqslant |v|^2 + |v'-v|2c_2(q)|v| \geqslant (1+8c_2(q)^2)|v|^2,\end{aligned}$$

where we used $c_1(q) \leqslant c_2(q)$, and in particular,

$$\text{(3.5)}\qquad g(v') - g(v) \leqslant -N\left[1 - \left(1+8c_2(q)^2\right)^{-q/2}\right]|v|^{-q} \lesssim -N|v|^{-q}.$$

The last inequality uses $1-(1+8c_2(q)^2)^{-q/2} \to 1 - e^{-4/19^2} > 0$ as $q\to\infty$. (This is where we use that $c_2(q) = O(q^{-1/2})$ rather than $Q(q^{-1})$ like $c_1(q)$.)

We deduce from (3.4) and (3.5) that, if $|v'_*| \leqslant c_1(q)|v|$ and $|v| \geqslant 2$ then

$$\begin{aligned}&\int_{v'\in v+(v-v'_*)^\perp} \big[g(v')-g(v)\big]\widetilde{b}(\cos\theta)\frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}}\\ &\qquad\leqslant \int_{v'\in v+(v-v'_*)^\perp} \big[g(v')-g(v)\big]\widetilde{b}(\cos\theta)\chi_{\{|v-v'|\geqslant 4c_2(q)|v|\}}\frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}}\\ &\qquad\lesssim -N|v|^{-q}\int_{v'\in v+(v-v'_*)^\perp}\widetilde{b}(\cos\theta)\chi_{\{|v-v'|\geqslant 4c_2(q)|v|\}}\frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}}\\ &\qquad\lesssim -Nc_2(q)^{-2s}|v|^{-q-2s} \approx -q^sN|v|^{-q-2s}.\end{aligned}$$

This achieves the proof of the lemma. □

We can now prove Proposition 3.1.

Proof of Proposition 3.1. — We estimate $\mathcal{G}(f,g)$ using Lemma 3.2.

$$\begin{aligned}\mathcal{G}(f,g) &= \text{p.v.}\int_{|v'_*|\leqslant c_1(q)|v|} f(v'_*)|v-v'_*|^{\gamma+2s}\int_{v'\in v+(v-v'_*)^\perp}\frac{\big[g(v')-g(v)\big]\widetilde{b}}{|v-v'|^{d-1+2s}}\,\mathrm{d}v'\,\mathrm{d}v'_*\\ &\lesssim -q^sN|v|^{-q-2s}\int_{|v'_*|\leqslant c_1(q)|v|} f(v'_*)|v-v'_*|^{\gamma+2s}\,\mathrm{d}v'_*,\\ &\lesssim -q^sN|v|^{-q+\gamma}.\end{aligned}$$

The last inequality follows from the lower bound on $\widetilde{b}$ and choosing $R_q \geqslant R_0 c_1(q)^{-1}$, where R_0 is the radius of Lemma 2.1: then the lower bound of Lemma 2.1 implies a lower bound on

$$\int_{|v'_*|\leqslant c_1(q)|v|} f(v'_*)|v-v'_*|^{\gamma+2s}\,\mathrm{d}v'_* \gtrsim |v|^{\gamma+2s}\int_{|v'_*|\leqslant R_0} f(v'_*)\,\mathrm{d}v'_* \gtrsim |v|^{\gamma+2s}$$

since $|v-v'_*| \gtrsim |v|$ on the domain of integration (and given $|v|\geqslant R_q$). □

3.3. Lower bound on the good term $\mathcal{G}$ for not-so-large q. — The coercivity constant $(1+q)^s$ in the previous estimate is not large enough to dominate other bad and non-singular terms for not-so-large q: we therefore prove a second estimate inspired by the study of the L^∞ norm in [49].

Proposition 3.3 (Estimate of $\mathcal{G}(f,f)$ for not-so-large q). — *Assume f satisfies* (3.1) *for $g = N\min(1,|v|^{-q})$ with $q\geqslant 0$. Then there exists $R_q\geqslant 1$ so that*

$$\forall\, v\in\mathbb{R}^d \mid |v|\geqslant R_q,\quad \mathcal{G}(f,f)(v) \lesssim_q -g(v)^{1+2s/d}|v|^{\gamma+2s+2s/d}.$$

Remark 3.4. — Note that the constant here depends on q, but we do not track this dependency since this proposition will be used for not-so-large values $q\in[0,d+1]$.

Proof. — We first claim that the estimate this proposition is implied by the previous Proposition 3.1 whenever $|v|\geqslant 1$ and $N|v|^{-q}\lesssim_q |v|^{-(d+1)}$. Indeed for such choice of q and v one has

$$q^s|v|^\gamma g(v) \gtrsim_q g(v)^{1+2s/d}|v|^{\gamma+2s+2s/d}.$$

Consider now the case where $g(v)\geqslant C_q|v|^{-(d+1)}$ for a constant $C_q>0$ large enough (depending on q) to be chosen later. We proceed as in the proof of Proposition 3.1, but refine it in that we estimate the difference $f(v')-g(v')$:

$$\begin{aligned}
&\mathcal{G}(f,f)(v)\\
&\quad\leqslant \mathrm{p.v.}\int_{|v'_*|\leqslant c_1(q)|v|} f(v'_*)|v-v'_*|^{\gamma+2s}\left\{\int_{v'\in v+(v-v'_*)^\perp}\frac{[f(v')-g(v)]\,\widetilde{b}}{|v-v'|^{d-1+2s}}\,\mathrm{d}v'\right\}\mathrm{d}v'_*,\\
&\quad= \mathrm{p.v.}\int_{|v'_*|\leqslant c_1(q)|v|} f(v'_*)|v-v'_*|^{\gamma+2s}\left\{\int_{v'\in v+(v-v'_*)^\perp}\frac{[g(v')-g(v)]\,\widetilde{b}}{|v-v'|^{d-1+2s}}\,\mathrm{d}v'\right.\\
&\qquad\qquad\left.+\int_{v'\in v+(v-v'_*)^\perp}\frac{[f(v')-g(v')]\,\widetilde{b}}{|v-v'|^{d-1+2s}}\,\mathrm{d}v'\right\}\mathrm{d}v'_*
\end{aligned}$$

(note that second principal value is well-defined since $f(v)=g(v)$).

The first of the two inner integral terms is negative because of Lemma 3.2. Thus

$$\begin{aligned}
&\mathcal{G}(f,f)(v)\\
&\quad\leqslant \mathrm{p.v.}\int_{|v'_*|\leqslant c_1(q)|v|} f(v'_*)|v-v'_*|^{\gamma+2s}\left\{\int_{v'\in v+(v-v'_*)^\perp}\frac{[f(v')-g(v')]\,\widetilde{b}}{|v'-v|^{d-1+2s}}\,\mathrm{d}v'\right\}\mathrm{d}v'_*,\\
&\quad\leqslant \mathrm{p.v.}\int_{\mathbb{R}^d}[f(v')-g(v')]\,K_{\overline{f}}(v,v')\,\mathrm{d}v'\leqslant 0.
\end{aligned}$$

where we have exchanged the order of integration and where $K_{\overline{f}}$ denotes the kernel (2.3) with the truncated $\overline{f}(v'_*) := f(v'_*)\mathbb{1}_{|v'_*|\leqslant c_1(q)|v|}$. If $|v|$ is sufficiently large, the estimates in Lemma 2.2 hold for $K_{\overline{f}}$ as well since f and $\overline{f}$ share comparable bounds on their hydrodynamic quantities.

Let us estimate the measure of points $w \in \Xi(t,x,v)$, the cone from Lemma 2.2, such that $f(v+w) \geqslant g(v)/2$. Note that for sufficiently large $|v|$, whenever $w \in \Xi(t,x,v)$, the almost-orthogonality condition in Lemma 2.2 implies

$$|v+w|^2 \geqslant |v|^2 + |w|^2 - 2C_0|w| \geqslant \frac{|v|^2}{2}$$

and therefore

$$\Big|\Big\{w \in \Xi(t,x,v) : f(v+w) \geqslant \frac{g(v)}{2}\Big\}\Big| \leqslant \frac{2}{g(v)} \int_{w\in\Xi(t,x,v)} f(v+w)\,\mathrm{d}w \leqslant \frac{4E_0}{|v|^2 g(v)}.$$

The estimate (2.5) from Lemma 2.2 implies that we can pick $r>0$ such that

$$|\Xi(t,x,v) \cap B_r| = \frac{4^2 E_0}{|v|^2 g(v)}.$$

The corresponding r is given by $r \approx \big(|v|^{-1} g(v)^{-1}\big)^{1/d}$ and for this choice of r we have

$$|\Xi(t,x,v) \cap B_r| \geqslant 4\Big|\Big\{w \in \Xi : f(v+w) \geqslant \frac{g(v)}{2}\Big\}\Big|.$$

This implies that three fourth of the $w \in \Xi(t,x,v) \cap B_r$ satisfy $f(v+w) \leqslant g(v)/2$.

Going back to our estimate on $\mathcal{G}$, we restrict the domain of integration (since the integral is non-positive)

$$\mathcal{G}(f,f)(v) \leqslant \int_{\Xi(t,x,v)\cap B_r \cap \{f(v+w)\leqslant g(v)/2\}} \Big[\frac{g(v)}{2} - g(v+w)\Big] K_{\overline{f}}(v,v')\,\mathrm{d}v'.$$

This is a useful estimate when $g(v+w) > g(v)/2$ with $w \in \Xi(t,x,v)\cap B_r$. Recall that we assume that $g(v) \gtrsim C_q |v|^{-d-1}$ for an arbitrarily large constant C_q. Let us pick C_q large so that if $g(v) \gtrsim C_q |v|^{-d-1}$ then

$$r \lesssim \big(|v|^{-1} g(v)^{-1}\big)^{1/d} \leqslant \big[1-(3/4)^{1/q}\big]|v|,$$

so that $g(v+w) \geqslant \frac{3}{4} g(v)$ for $w \in B_r$. Note that the latter inequality is always satisfied when $q=0$ without extra-condition on v, and for large q the constant $C_q = O(q)$.

Therefore, we get

$$\begin{aligned}\mathcal{G}(f,f) &\leqslant -\frac{g(v)}{4} \int_{\Xi(t,x,v)\cap B_r\cap\{f(v+w)\leqslant g(v)/2\}} K_{\overline{f}}(v,v')\,\mathrm{d}v', \\ &\lesssim -g(v)|v|^{\gamma+2s+1} r^{-d-2s} \Big|\Xi(t,x,v)\cap B_r \cap \Big\{f(v+w) \leqslant \frac{g(v)}{2}\Big\}\Big|\end{aligned}$$

where we have used the estimate on the kernel of Lemma 2.2. We use now

$$\Big|\Xi(t,x,v)\cap B_r \cap \Big\{f(v+w) \leqslant \frac{g(v)}{2}\Big\}\Big| \geqslant \frac{3}{4}\,|\Xi(t,x,v)\cap B_r| \approx r^d |v|^{-1}$$

that follows from our choice of r to deduce

$$\mathcal{G}(f,f)(v) \lesssim -g(v)|v|^{\gamma+2s} r^{-2s} = -g(v)^{1+2s/d}|v|^{\gamma+2s+2s/d}$$

which concludes the proof. $\square$

Remark 3.5. — Here, we interpret in terms of the collision process on v, v', v_*, v'_* the two last estimates for the good term.

The first estimate given by Proposition 3.1 is generated by the angles θ such that

$$|\sin(\theta/2)| = \frac{|v' - v|}{|v'_* - v|} \geqslant \frac{c_2(q)}{1 + c_1(q)} = O(q^{-1/2}).$$

Hence, in some sense, the singularity is not used fundamentally. It is only used to get a constant larger and larger for $q \to +\infty$, because of the q^s factor coming for r_q^{-2s} in the proof of Lemma 3.2.

The second estimate given by Proposition 3.3 is genuinely non-cutoff in nature. Indeed, it is adapted from [49] where the nonlinear maximum principle for singular integral operators in the spirit of [20] is used. In particular, the higher exponent on $|v|$ in Proposition 3.3 is crucial in order to dominate the bad and non-singular terms for not-so-large q.

3.4. Lower bound on the good term $\mathcal{G}$ for $q = 0$ and small v

Proposition 3.6 (Estimate of $\mathcal{G}(f, f)$ for $q = 0$ and small v)

Assume f satisfies (3.1) *for $g = m$ (constant function $m = m(t_0)$ at the time of contact). Then we have*

$$\mathcal{G}(f, f)(v) \lesssim -m^{1+2s/d}.$$

Proof. — The proof is a variant of the previous one. We start from

$$\mathcal{G}(f, f)(v) \leqslant \mathrm{p.\,v.} \int_{\mathbb{R}^d} [f(v') - m]\, K_{\overline{f}}(v, v')\, \mathrm{d}v' \leqslant 0.$$

where $m = g(t_0)$ is the upper bound barrier at the contact point.

We then use that

$$\Big|\Big\{w \in \Xi(t, x, v) \,:\, f(v + w) \geqslant \frac{m}{2}\Big\}\Big| \leqslant \frac{2}{m} \int_{w \in \Xi(t,x,v)} f(v + w)\, \mathrm{d}w \leqslant \frac{2M_0}{m}$$

and using again equation (2.5) of Lemma 2.2, we can pick $r > 0$ such that

$$|\Xi(t, x, v) \cap B_r| = \frac{8M_0}{m} \quad \text{with} \quad r \approx \big(|v|^{-1} m^{-1}\big)^{1/d}$$

and for this choice of r we have

$$|\Xi(t, x, v) \cap B_r| \geqslant 4\Big|\Big\{w \in \Xi(t, x, v) \,:\, f(v + w) \geqslant \frac{m}{2}\Big\}\Big|.$$

This implies that three fourth of the $w \in \Xi(v) \cap B_r$ satisfy $f(v + w) \leqslant m/2$, and

$$\begin{aligned}
\mathcal{G}(f, f) &\leqslant -\frac{m}{2} \int_{\Xi(t,x,v) \cap B_r \cap \{f(v+w) \leqslant m/2\}} K_{\overline{f}}(v, v')\, \mathrm{d}v' \\
&\leqslant -\frac{m}{2} \int_{\Xi(t,x,v) \cap B_r \cap \{f(v+w) \leqslant m/2\}} K_{\overline{f}}(v, v')\, \mathrm{d}v' \\
&\lesssim -m|v|^{\gamma+2s+1} r^{-d-2s} \Big|\Xi(t, x, v) \cap B_r \cap \Big\{f(v + w) \leqslant \frac{m}{2}\Big\}\Big|,
\end{aligned}$$

where we have used the estimate on the kernel $K_{\overline{f}}$ of Lemma 2.2. We use now

$$\left|\Xi(t,x,v)\cap B_r\cap\left\{f(v+w)\leqslant\frac{g(v)}{2}\right\}\right|\geqslant\frac{3}{4}\,|\Xi(t,x,v)\cap B_r|\approx r^d|v|^{-1}$$

that follows from our choice of r to deduce

$$\mathcal{G}(f,f)(v)\lesssim -m|v|^{\gamma+2s}r^{-2s}\lesssim -m^{1+2s/d},$$

which concludes the proof. □

3.5. Upper bound on the bad term $\mathcal{B}$ for large q. — We decompose further the bad term (see Figure 3)

$$\mathcal{B}(f,f)=\mathcal{B}_1(f,f)+\mathcal{B}_2(f,f)+\mathcal{B}_3(f,f) \tag{3.6}$$

with

$$\begin{aligned}
&\mathcal{B}_1(f_1,f_2)(v)\\
&:=\mathrm{p.\,v.}\int_{v'_*\in\mathbb{R}^d}\widetilde{\chi}(v'_*)f_1(v'_*)|v-v'_*|^{\gamma+2s}\int_{v'\in v+(v-v'_*)^{\perp}}\chi_1(v')\frac{\big[f_2(v')-f_2(v)\big]\widetilde{b}}{|v-v'|^{d-1+2s}}\,\mathrm{d}v'\,\mathrm{d}v'_*.\\
&\mathcal{B}_2(f_1,f_2)(v)\\
&:=\int_{v'\in\mathbb{R}^d}\chi_2(v')\frac{\big[f_2(v')-f_2(v)\big]}{|v'-v|^{d+2s}}\int_{v'_*\in v+(v-v')^{\perp}}\widetilde{\chi}(v'_*)f_1(v'_*)|v-v'_*|^{\gamma+2s+1}\widetilde{b}\,\mathrm{d}v'_*\,\mathrm{d}v'.\\
&\mathcal{B}_3(f_1,f_2)(v)\\
&:=\int_{v'\in\mathbb{R}^d}\chi_3(v')\frac{\big[f_2(v')-f_2(v)\big]}{|v'-v|^{d+2s}}\int_{v'_*\in v+(v-v')^{\perp}}\widetilde{\chi}(v'_*)f_1(v'_*)|v-v'_*|^{\gamma+2s+1}\widetilde{b}\,\mathrm{d}v'_*\,\mathrm{d}v'
\end{aligned}$$

with $\widetilde{\chi}(v'_*):=\mathbb{1}_{\{|v'_*|\geqslant c_1(q)|v|\}}$ (inherited from good/bad decomposition) and the v'-integration domain is decomposed as follows:

$$\chi_1(v'):=\mathbb{1}_{\{|v'|>|v|/2\}},\quad \chi_2(v'):=\mathbb{1}_{\{|v'|<c_3(q)|v|\}},\quad \chi_3(v')=\mathbb{1}_{\{c_3(q)|v|\leqslant|v'|<|v|/2\}}$$

with $c_3(q):=(1/2)(1+q)^{-1}$.

It is intentional that the first term in the decomposition is written with the $\int_{v'_*}\int_{v'}$ representation, while the second and third is written with the $\int_{v'}\int_{v'_*}$ representation. This corresponds to the respective representations used to estimate each term below.

Observe that when $f\leqslant g$ with contact at v, one has $\mathcal{B}_1(f,f)\leqslant\mathcal{B}_1(f,g)$.

Proposition 3.7 (Estimate of $\mathcal{B}_1(f,g)$ for all $q\geqslant 0$). — *Let f be non-negative and satisfy* (1.3). *Let $g=N\min(1,|v|^{-q})$ with $q\geqslant 0$. For $|v|\geqslant 2$,*

$$\mathcal{B}_1(f,g)(v)\lesssim(1+q)^2 2^q|v|^{\gamma-2}g(v)$$

with constant uniform in q.

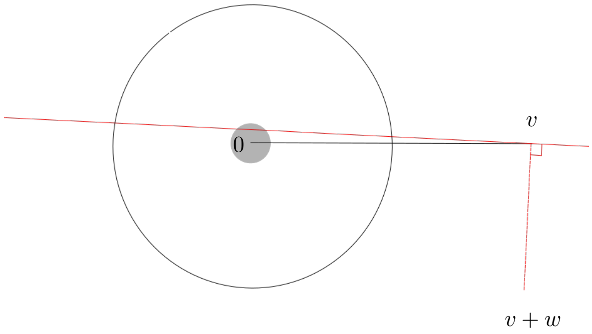

Figure 3. The bad terms: $\mathcal{B}_1$ corresponds to the integration over the intersection of the line with the exterior of the balls, $\mathcal{B}_2$ corresponds to the integration over the intersection of the line with the grey ball, and $\mathcal{B}_3$ over the intersection of the line with the ring.

Proof. — Since $|v| \geqslant 2$ and (restriction of the domain χ_1) $|v'| > |v|/2 > 1$, we have $g(v) = N|v|^{-q}$ and $g(v') = N|v'|^{-q}$ and we further decompose the inner integral as

$$\begin{aligned} I_1(v,v'_*) &:= \mathrm{p.\,v.} \int_{v' \in v + (v-v'_*)^\perp} \chi_1(v') \frac{\big[g(v') - g(v)\big]\widetilde{b}}{|v'-v|^{d-1+2s}} \,\mathrm{d}v' \\ &= \mathrm{p.\,v.} \int \chi_1(v') \chi_{\{|v-v'|<|v|/2\}} \cdots + \mathrm{p.\,v.} \int \chi_1(v') \chi_{\{|v-v'| \geqslant |v|/2\}} \cdots \\ &=: I_{1,<} + I_{1,>}. \end{aligned}$$

The term $I_{1,<}$ is estimated following the same argument as in (3.3). We subtract by symmetry the term $\nabla g(v) \cdot (v'-v)$ that vanishes after integration

$$I_{1,<}(v,v'_*) = \int_{v' \in v + (v-v'_*)^\perp} \chi_1(v') \chi_{\{|v-v'|<|v|/2\}} \frac{\big[g(v') - g(v) - \nabla g(v) \cdot (v'-v)\big]\widetilde{b}}{|v'-v|^{d-1+2s}} \,\mathrm{d}v'$$

and use (3.3) with $|v'_\theta| = |v + \theta(v'-v)| \geqslant |v|/2$:

$$\begin{aligned} I_{1,<}(v,v'_*) &\lesssim 2^q N \int_{v' \in v + (v-v'_*)^\perp} \chi_1(v') \chi_{\{|v-v'|<|v|/2\}} |v|^{-q-2} |v'-v|^2 \frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}} \\ &\lesssim 2^q N |v|^{-q-2s} \lesssim 2^q |v|^{-2s} g(v). \end{aligned}$$

The term $I_{1,>}$ is even simpler: the singularity is removed by the restriction $|v'-v| \geqslant |v|/2 > 1$ and the integrability at infinity is provided by the kernel:

$$\begin{aligned} I_{1,>}(v,v'_*) &\leqslant N \int_{v' \in v + (v-v'_*)^\perp} \chi_1(v') \chi_{\{|v-v'| \geqslant |v|/2\}} |v'|^{-q} \frac{\widetilde{b}(\cos\theta)}{|v'-v|^{d-1+2s}} \,\mathrm{d}v' \\ &\lesssim 2^q N |v|^{-q-2s} \leqslant 2^q |v|^{-2s} g(v). \end{aligned}$$

We deduce that $I_1(v,v'_*) \lesssim 2^q |v|^{-2s} g(v)$ and we compute

$$\begin{aligned}
\mathcal{B}_1(f,g)(v) &= \text{p. v.} \int_{v'_* \in \mathbb{R}^d} \widetilde{\chi}(v'_*) f(v'_*) |v-v'_*|^{\gamma+2s} I_1(v,v'_*)\, \mathrm{d}v'_* \\
&\lesssim 2^q g(v) |v|^{-2s} \int_{v'_* \in \mathbb{R}^d} \widetilde{\chi}(v'_*) f(v'_*) |v-v'_*|^{\gamma+2s}\, \mathrm{d}v'_* \\
&\lesssim c_1(q)^{-(\gamma+2s)} 2^q g(v) |v|^{-2s} \int_{v'_* \in \mathbb{R}^d} \widetilde{\chi}(v'_*) f(v'_*) |v'_*|^{\gamma+2s}\, \mathrm{d}v'_* \\
&\lesssim c_1(q)^{-2} 2^q g(v) |v|^{\gamma-2} \int_{v'_* \in \mathbb{R}^d} f(v'_*)(1+|v'_*|^2)\, \mathrm{d}v'_*,
\end{aligned}$$

where we have used in the last lines the fact that, under the restriction $|v'_*| \geqslant c_1(q)|v|$ imposed by $\widetilde{\chi}$, we have

$$|v-v'_*|^{\gamma+2s} \lesssim c_1(q)^{-(\gamma+2s)}(1+|v'_*|)^{\gamma+2s} \lesssim c_1(q)^{-2}|v|^{\gamma+2s-2}(1+|v'_*|)^2.$$

We deduce, since $c_1(q) = O(q^{-1})$ that

$$\mathcal{B}_1(f,g)(v) \lesssim (1+q)^2 2^q g(v) |v|^{\gamma-2}(M_0+E_0) \lesssim (1+q)^2 2^q g(v)|v|^{\gamma-2},$$

which concludes the proof. $\square$

We recall that the next two estimates are based on the $\int_{v'} \int_{v'_*}$ representation.

Proposition 3.8 (Estimate of $\mathcal{B}_2(f,f)$ for large q). — *Let f be a non-negative function satisfying* (1.3). *Let $g = N\min(1,|v|^{-q})$ with $q > d+\gamma+2s$. Assume $f \leqslant g$ for all $v \in \mathbb{R}^d$. Then for $|v| \geqslant 2$,*

$$\mathcal{B}_2(f,f)(v) \lesssim \frac{|v|^\gamma}{q-(d+\gamma+2s)} g(v).$$

Proof. — We first estimate from above the inner integral

$$\begin{aligned}
I_2(v,v') &:= \int_{v'_* \in v+(v-v')^\perp} \widetilde{\chi}(v'_*) f(v'_*) |v-v'_*|^{\gamma+2s+1} \widetilde{b}(\cos\theta)\, \mathrm{d}v'_*, \\
&\leqslant \int_{v'_* \in v+(v-v')^\perp} \widetilde{\chi}(v'_*) g(v'_*) |v-v'_*|^{\gamma+2s+1} \widetilde{b}(\cos\theta)\, \mathrm{d}v'_*.
\end{aligned}$$

We have for $|v'| < c_3(q)|v|$:

$$\begin{aligned}
|v'_*| &\geqslant |v'_*-v'| - c_3(q)|v| \\
&\geqslant \left[|v-v'|^2 + |v-v'_*|^2\right]^{1/2} - c_3(q)|v| \\
&\geqslant \left[(1-c_3(q))^2|v|^2 + |v-v'_*|^2\right]^{1/2} - c_3(q)|v| \\
&\geqslant \left[\frac{1}{2}|v|^2 + |v-v'_*|^2\right]^{1/2} - c_3(q)|v| \\
&\geqslant \left(1-\sqrt{2}c_3(q)\right)\left[\frac{1}{2}|v|^2 + |v-v'_*|^2\right]^{1/2}.
\end{aligned}$$

Then we write ($|v'_*| > 1$ for v large enough from above, and thus $g(v'_*) = N|v'_*|^{-q}$):

$$
\begin{aligned}
I_2(v,v') &\lesssim \int_{v'_*\in v+(v-v')^\perp} \widetilde{\chi}(v'_*) g(v'_*) |v-v'_*|^{\gamma+2s+1} \widetilde{b}(\cos\theta)\,\mathrm{d}v'_*\\
&\lesssim N \int_{v'_*\in v+(v-v')^\perp} \widetilde{\chi}(v'_*) |v'_*|^{-q} |v-v'_*|^{\gamma+2s+1}\,\mathrm{d}v'_*\\
&\lesssim N\big(1-\sqrt{2}c_3(q)\big)^{-q} \int_{v'_*\in v+(v-v')^\perp} \widetilde{\chi}(v'_*) \Big[\frac{1}{2}|v|^2 + |v-v'_*|^2\Big]^{-q/2} |v-v'_*|^{\gamma+2s+1}\,\mathrm{d}v'_*\\
&\lesssim N\big(1-\sqrt{2}c_3(q)\big)^{-q} \int_{u\in\mathbb{R}^{d-1}} \Big[\frac{1}{2}|v|^2 + |u|^2\Big]^{-q/2} |u|^{\gamma+2s+1}\,\mathrm{d}u\\
&\lesssim N\Big(1-\sqrt{2}c_3(q)\Big)^{-q} |v|^{-q+\gamma+2s+d} \bigg(\int_{\widehat{u}\in\mathbb{R}^{d-1}} \Big[\frac{1}{2} + |\widehat{u}|^2\Big]^{-q/2} |\widehat{u}|^{\gamma+2s+1}\,\mathrm{d}\widehat{u}\bigg),
\end{aligned}
$$

where the last integral is finite when $q-\gamma-2s-1 > d-1$, as assumed, with

$$
\bigg(\int_{\mathbb{R}^{d-1}} \Big[\frac{1}{2} + |\widehat{u}|^2\Big]^{-q/2} |\widehat{u}|^{\gamma+2s+1}\,\mathrm{d}\widehat{u}\bigg) \lesssim \frac{1}{q-(\gamma+2s+d)}.
$$

Hence

$$
I_2(v,v') \lesssim N \frac{\big(1-\sqrt{2}c_3(q)\big)^{-q}}{q-(d+\gamma+2s)} |v|^{-q+\gamma+2s+d}.
$$

We plug our estimate on I_2 into the formula for $\mathcal{B}_2$ (using the control of $|v-v'|^{-1} \leqslant (1-c_3(q))^{-1}|v|^{-1}$ over the restriction χ_2):

$$
\begin{aligned}
\mathcal{B}_2(f,f)(v) &= \int_{\mathbb{R}^d} \chi_2(v') \frac{\big[f(v')-f(v)\big]}{|v-v'|^{d+2s}} I_2(v,v')\,\mathrm{d}v',\\
&\leqslant N\left((1-c_3(q)\right)^{-d-2s} |v|^{-d-2s} \left(\int_{v'\in\mathbb{R}^d} f(v')\,\mathrm{d}v'\right) \sup_{v'} I_2(v,v')\\
&\lesssim \frac{N\left(1-c_3(q)\right)^{-d-2s}\left(1-\sqrt{2}c_3(q)\right)^{-q}}{q-(d+\gamma+2s)} |v|^{-q+\gamma}.
\end{aligned}
$$

The choice of $c_3(q) = (1/2)(1+q)^{-1}$ shows that the factor

$$
\left(1-c_3(q)\right)^{-d-2s}\left(1-\sqrt{2}c_3(q)\right)^{-q}
$$

is uniformly bounded for $q \geqslant 0$, which concludes the proof. □

Proposition 3.9 (Estimate of $\mathcal{B}_3(f,f)$ for large q). — *Let f be a non-negative function satisfying* (1.3). *Assume $f \leqslant g$ for all $v\in\mathbb{R}^d$, where $g = N\min(1,|v|^{-q})$ for $q > d+\gamma+2s$. Then for all $|v| \geqslant 2$,*

$$
\mathcal{B}_3(f,f)(v) \lesssim (1+q)^2\Big((1+q)^{q-(d-1)} + \frac{1}{q-(d+\gamma+2s)}\Big)|v|^{\gamma-2} g(v).
$$

Proof. — We first estimate

$$
\begin{aligned}
I_3(v,v') &:= \int_{v'_*\in v+(v-v')^\perp} \widetilde{\chi}(v'_*) f(v'_*) |v-v'_*|^{\gamma+2s+1} \widetilde{b}(\cos\theta)\,\mathrm{d}v'_*,\\
&\leqslant \int_{v'_*\in v+(v-v')^\perp} \widetilde{\chi}(v'_*) g(v'_*) |v-v'_*|^{\gamma+2s+1} \widetilde{b}(\cos\theta)\,\mathrm{d}v'_*
\end{aligned}
$$

under the conditions $c_3(q)|v| < |v'| < |v|/2$ and $|v'_*| \geqslant c_1(q)|v|$ imposed by χ_3 and $\widetilde{\chi}$. We change variable $v'_* = v + |v|\widetilde{u}$ and bound from above (denoting $\widehat{v} := v/|v|$)

$$I_3(v,v') \lesssim N|v|^{\gamma+2s+d-q} \int_{\widetilde{u}\in(v-v')^\perp} \widetilde{\chi}(v'_*)\, |\widehat{v}+\widetilde{u}|^{-q}\, |\widetilde{u}|^{\gamma+2s+1}\, \mathrm{d}\widetilde{u}.$$

The restriction $\widetilde{\chi}(v'_*)$ imposes $|\widehat{v}+\widetilde{u}| \geqslant c_1(q) > 0$. Close to the singularity $|\widehat{v}+\widetilde{u}| \sim c_1(q)$, then $|\widetilde{u}| \sim 1$ and the integral in $\widetilde{u}$ is controlled by $O(c_1(q)^{d-1-q})$. At large $\widetilde{u}$, the integral is finite provided that $q > d+\gamma+2s$:

$$\int_{\widetilde{u}\in(v-v')^\perp} \widetilde{\chi}(v'_*)\, |\widehat{v}+\widetilde{u}|^{-q}\, |\widetilde{u}|^{\gamma+2s+1}\, \mathrm{d}\widetilde{u} \lesssim c_1(q)^{d-1-q} + \frac{1}{q-(d+\gamma+2s)}.$$

We finally plug this estimate into the formula for $\mathcal{B}_3$:

$$\begin{aligned}
&\mathcal{B}_3(f,f)(v)\\
&\quad\leqslant \int_{v'\in\mathbb{R}^d} \chi_3(v') \frac{\big[f(v')-f(v)\big]}{|v'-v|^{d+2s}} I_3(v,v')\,\mathrm{d}v'\\
&\quad\lesssim Nc_3(q)^{-2}\Big(c_q^{d-1-q} + \frac{1}{q-(d+\gamma+2s)}\Big)|v|^{\gamma-2-q}\Big(\int_{v'\in\mathbb{R}^d} f(v')(1+|v'|^2)\,\mathrm{d}v'\Big),
\end{aligned}$$

where we have used the restriction $|v'| < |v|/2$ on $\chi_3(v')$ to deduce $|v-v'| \sim |v|$, and the restriction $|v'| > c_3(q)|v|$ to deduce $1 \leqslant c_3(q)^{-2}|v|^{-2}|v'|^2$. From the assumption on the mass and energy of f we get finally

$$\mathcal{B}_3(f,f)(v) \lesssim c_3(q)^{-2}\Big(c_1(q)^{d-1-q} + \frac{1}{q-(d+\gamma+2s)}\Big)|v|^{\gamma-2}g(v)$$

which concludes the proof. □

3.6. Upper bound on the bad term $\mathcal{B}$ for not-so-large q

Proposition 3.10 (Estimate of $\mathcal{B}_2(f,g)+\mathcal{B}_3(f,g)$ for not-so-large q)

Let f be a non-negative function satisfying (1.3). *Let $g = N\min(1,|v|^{-q})$ and $q\in[0,d+1]$. Then for $|v|\geqslant 2$,*

$$(\mathcal{B}_2+\mathcal{B}_3)(f,g)(v) \lesssim \begin{cases} |v|^{\gamma-(d+1-q)}g(v) & \text{if } q > d-1,\\ |v|^{\gamma-2}\ln(1+|v|)g(v) & \text{if } q = d-1,\\ |v|^{\gamma-2}g(v) & \text{if } q < d-1.\end{cases}$$

Proof. — Denote $\mathcal{B}_{2+3} := \mathcal{B}_2+\mathcal{B}_3$ and $\chi_{2+3}(v') := \chi_2(v')+\chi_3(v') = \mathbb{1}_{\{|v'|<|v|/2\}}$. Then

$$\begin{aligned}
&\mathcal{B}_{2+3}(f,g)(v)\\
&\quad= \int_{|v'_*|>c_1(q)|v|} f(v'_*)|v-v'_*|^{\gamma+2s}\Big\{\int_{v'\in v+(v-v'_*)^\perp} \chi_{2+3}(v')\frac{[g(v')-g(v)]\,\widetilde{b}}{|v'-v|^{d-1+2s}}\,\mathrm{d}v'\Big\}\,\mathrm{d}v'_*.
\end{aligned}$$

We use that $g(v') - g(v) \leqslant 2^q N(1+|v'|)^{-q}$ in order to write

$$
\begin{aligned}
&\mathcal{B}_{2+3}(f,g)(v) \\
&\lesssim 2^q \int_{|v'_*|>c_1(q)|v|} f(v'_*)|v-v'_*|^{\gamma+2s}\Big\{\int_{v'\in v+(v-v'_*)^\perp} \chi_{2+3}(v')\frac{N(1+|v'|)^{-q}}{|v'-v|^{d-1+2s}}\Big\}\,\mathrm{d}v'\,\mathrm{d}v'_*, \\
&\lesssim 2^q N|v|^{-d+1-2s}\times \\
&\qquad \times \int_{|v'_*|>c_1(q)|v|} f(v'_*)|v-v'_*|^{\gamma+2s}\Big\{\int_{v'\in v+(v-v'_*)^\perp} \chi_{2+3}(v')(1+|v'|)^{-q}\,\mathrm{d}v'\Big\}\,\mathrm{d}v'_*.
\end{aligned}
$$

We get

$$
\int_{v'\in v+(v-v'_*)^\perp} \chi_{2+3}(v')(1+|v'|)^{-q}\,\mathrm{d}v' \lesssim \Theta(v) \quad \text{with} \quad \Theta(v) := \begin{cases} 1 & \text{if } q>d-1,\\ \ln(1+|v|) & \text{if } q=d-1,\\ |v|^{d-1-q} & \text{if } q<d-1.\end{cases}
$$

We deduce that

$$
\begin{aligned}
\mathcal{B}_{2+3}(f,g)(v) &\lesssim N|v|^{-d+1-2s}\Theta(v)\int_{|v'_*|>c_1(q)|v|} f(v'_*)|v-v'_*|^{\gamma+2s}\,\mathrm{d}v'_*, \\
&\leqslant N|v|^{-d+1-2s}\Theta(v)\Big(\max_{|v'_*|>c_1(q)|v|}\frac{|v-v'_*|^\gamma}{|v'_*|^2}\Big)\int_{|v'_*|>c_1(q)|v|} f(v'_*)\left(1+|v'_*|^2\right)\,\mathrm{d}v'_*, \\
&\lesssim N|v|^{-d-1+\gamma}\Theta(v),
\end{aligned}
$$

which concludes the proof. □

3.7. Upper bound on the non-singular (lower order) term Q_{ns}

Proposition 3.11 (Estimate of $Q_{\mathrm{ns}}(f,f)$)
Assume f satisfies (3.1) *with $g = N\min(1,|v|^{-q})$. Then for $\gamma \geqslant 0$*

$$Q_{\mathrm{ns}}(f,f)(v) \lesssim (1+|v|)^\gamma g(v),$$

while for $\gamma<0$,

$$Q_{\mathrm{ns}}(f,f) \lesssim 2^{-q\gamma/d} g(v)^{1-\gamma/d} + (1+|v|)^\gamma g(v).$$

Moreover, when $q=0$ and $\gamma\in(-d,0)$, by using the uniform bound on the local entropy it is possible to weaken slightly the dependency on $g(v)$ as follows: there is a function $\psi=\psi(r)$ on $\mathbb{R}^*_+$ that goes to zero as $r\to+\infty$ such that

$$Q_{\mathrm{ns}}(f,f) \lesssim g(v)^{1-\gamma/d}\psi(g(v)) + (1+|v|)^\gamma g(v). \tag{3.7}$$

The function ψ is explicit from the proof and depends on M_0 and H_0.

Proof. — We first deal with the easier case $\gamma \geqslant 0$:

$$\begin{aligned}
Q_{\mathrm{ns}}(f,f)(v) &= C_S f(v) \int_{\mathbb{R}^d} f(v-v_*)|v_*|^\gamma \,\mathrm{d}v_* \\
&\lesssim g(v) \int_{\mathbb{R}^d} f(v-v_*)\left(|v-v_*|^\gamma + |v_*|^\gamma\right) \mathrm{d}v_* \\
&\lesssim g(v) \int_{\mathbb{R}^d} f(v-v_*)\left(|v-v_*|^2 + 1 + |v|^\gamma\right) \mathrm{d}v_*, \\
&\lesssim g(v)\left[E_0 + (1+|v|^\gamma) M_0\right] \\
&\lesssim |v|^\gamma g(v),
\end{aligned}$$

where we have used $|v| \geqslant 1$.

We now turn to the case $\gamma < 0$ and pick $r < |v|/2$ and write

$$\begin{aligned}
&Q_{\mathrm{ns}}(f,f)(v) \\
&\quad = C_S f(v) \int_{|v-v_*|<r} f(v_*)|v-v_*|^\gamma \,\mathrm{d}v_* + C_S f(v) \int_{|v-v_*|>r} f(v_*)|v-v_*|^\gamma \,\mathrm{d}v_* \\
&\quad \lesssim 2^q g(v)^2 \int_{|v-v_*|<r} |v-v_*|^\gamma \,\mathrm{d}v_* + g(v) r^\gamma \int_{\mathbb{R}^n} f(v_*)\,\mathrm{d}v_* \\
&\quad \lesssim 2^q g(v)^2 r^{d+\gamma} + M_0 r^\gamma,
\end{aligned}$$

where we have used in the first integral the fact that $|v_*| \geqslant |v|/2$, and $|v| \geqslant 1$. The optimisation in r gives, for

$$r := \min\Big[\Big(\frac{M_0}{2^q g(v)}\Big)^{1/d}, \frac{|v|}{2}\Big],$$

the estimate

$$Q_{\mathrm{ns}}(f,f)(v) \lesssim 2^{-q\gamma/d} g(v)^{1-\gamma/d}$$

when $g(v) \geqslant M_0 2^{-q}(|v|/2)^{-d}$, and otherwise it gives

$$Q_{\mathrm{ns}}(f,f)(v) \lesssim g(v)(1+|v|)^\gamma,$$

which concludes the proof of the second inequality.

Let us finally consider the proof of the third refined inequality in the case $\gamma < 0$. We use again the classical fact that the entropy bound implies the non-concentration estimate

$$\int_A f(t,x,v)\,\mathrm{d}v \lesssim_{M_0,H_0} \varphi(|A|) \qquad \text{with } \varphi(r) = \ln(1+r) + \left[\ln\left(r^{-1}\right)\right]^{-1}$$

and A a Borel set. Split the integral as

$$Q_{\mathrm{ns}}(f,f)(v) = C_S f(v) \int_{|v-v_*|<r_1} \cdots + C_S f(v) \int_{r_1 \leqslant |v-v_*|<r_2} \cdots + C_S f(v) \int_{|v-v_*|>r_2} \cdots$$

with (for $g(v)$ large enough, otherwise the previous estimate is sufficient):

$$r_1 := \Big(\frac{M_0}{g(v)}\Big)^{1/d} \Big[\varphi\Big(\frac{M_0}{g(v)}\Big)\Big]^{-1/2\gamma} < r_2 := \Big(\frac{M_0}{g(v)}\Big)^{1/d} \Big[\varphi\Big(\frac{M_0}{g(v)}\Big)\Big]^{1/2\gamma}$$

and apply the L^∞ bound in the first term, the non-concentration estimate in the second term and the L^1 bound on the third term to get

$$Q_{\mathrm{ns}}(f,f) \lesssim g(v)^{1-\gamma/d}\Big[\varphi\Big(\frac{M_0}{g(v)}\Big)\Big]^{\min(1/2,(d+\gamma)/|\gamma|)} + (1+|v|)^\gamma g(v).$$

This concludes the proof of this third inequality. □

4. Maximum principle and proof of the upper bounds

4.1. The strategy. — We recall that the strategy is to prove that the solution f remains below a certain barrier function g ensuring the upper bound $N(t)(1+|v|)^{-|q|}$ for $q \geqslant 0$ and $N(t)$ a function of time that is either constant (for propagation of pointwise moments) or singular (for the appearance of pointwise moments) at $t \to 0$.

We consider a first contact point (t_0, x_0, v_0) such that (3.1) holds true. Recall that the existence of this first contact point is guaranteed by the rapid decay assumption in Definition 1.2 and the compactness of the spatial domain. At this point, the inequality (1.4) would hold. We use the fine structure of the collision operator $Q(f,f)$ to obtain that it is "negative enough" at large velocities. Concretely, we prove that the negative "good part" $\mathcal{G}$ dominates the other "bad" and non-singular parts of the collision operator at large velocities. Note that, for higher pointwise moments, the not-so-large velocities are controlled thanks to the L^∞ bound in Theorem 4.1.

We start by revisiting the L^∞ bound of [49] in order to include the minor technical extensions needed for this paper.

4.2. The L^∞ bound from [49]. — The first proof of the L^∞ bound for solutions satisfying (1.3) was obtained by the third author in [49, Th. 1.2]. We state here a slightly refined version.

Theorem 4.1 (L^∞ bound). — *Let $\gamma \in \mathbb{R}$ and $s \in (0,1)$ satisfy $\gamma + 2s \in [0,2]$ and f be a non-negative solution of the Boltzmann equation* (1.1) *such that* (1.3) *holds true for some positive constants m_0, M_0, E_0, H_0. Then*

$$\forall\, t \in (0,T], \qquad \|f(t,\cdot)\|_{L^\infty} \leqslant N_\infty\big(1 + t^{-d/2s}\big)$$

for a positive constant N_∞ only depending on m_0, M_0, E_0, H_0, dimension, γ and s.

Moreover, if $\|f_0\|_{L^\infty} < N$ for $N \geqslant N_\infty$, then $\|f(t,\cdot)\|_{L^\infty} < N$ for $t \in [0,T]$.

Remark 4.2. — With respect to [49, Th. 1.2]: the marginal improvements are the inclusion of the borderline case $\gamma + 2s = 0$ and the fact that if the initial data is bounded, the L^∞ bound is uniform as $t \downarrow 0^+$. We provide a detailed proof below for self-containedness and because of these small variations.

Proof. — Without loss of generality, it is enough to show the inequality holds for $t \in (0,1]$. We consider the barrier $g(t,v) := N_\infty t^{-d/2s}$ and consider the equation (1.4) at a first contact point $t_0 \in (0,1]$ and $v_0 \in \mathbb{R}^d$. It is enough to prove that for N_∞ large enough

$$Q(f,f)(t_0,x_0,v_0) < -\frac{d}{2s} N_\infty t_0^{-(d/2s)-1}. \tag{4.1}$$

Observe that when g is constant in v, at the contact point the bad term satisfies

$$\mathcal{B}(f,f)(t_0,x_0,v_0) \leqslant 0$$

and can be discarded. We then apply Proposition 3.3 for $|v_0| \geqslant 1$ and Proposition 3.6 for $|v_0| \leqslant 1$ to get

$$\mathcal{G}(f,f)(t_0,x_0,v_0) \lesssim -N_\infty^{1+2s/d} t_0^{-(d/2s)-1}(1+|v_0|)^{\gamma+2s+2s/d}$$

and Proposition 3.11 (valid for all v) to get

$$Q_{\mathrm{ns}}(f,f)(t_0,x_0,v_0) \lesssim N_\infty t_0^{-d/2s}(1+|v_0|)^{\gamma} + \mathbb{1}_{\gamma<0} N_\infty^{1-\gamma/d} t_0^{-(d/2s)+\gamma/2s}. \tag{4.2}$$

In case $\gamma+2s>0$, the exponents of N_∞ and t_0^{-1} in the first negative equation are strictly greater than those in second positive equation. Moreover, the exponent $\gamma+2s+2s/d$ of $|v_0|$ is strictly greater than all the other exponents $\gamma-2$, γ and 0. Therefore by choosing N_∞ large enough, we deduce that

$$Q(f,f)(t_0,x_0,v_0) \leqslant -\frac{1}{2} N_\infty^{1+2s/d} t_0^{-(d/2s)-1}(1+|v_0|)^{\gamma+2s+2s/d}$$

and taking N_∞ even greater if necessary, this contradicts (4.1).

The case $\gamma+2s=0$ (and thus $\gamma<0$) is treated similarly but since the inequality (4.2) is now too weak to show that $Q_{\mathrm{ns}}(f,f)$ is dominated by $\mathcal{G}(f,f)$ for large N_∞, we use instead the refined inequality (3.7) from Proposition 3.11 to get

$$Q_{\mathrm{ns}}(f,f)(t_0,x_0,v_0) \lesssim \big(N_\infty t^{-d/2s}\big)^{1+\gamma/d} \psi\big(N_\infty t^{-d/2s}\big).$$

With this inequality, we recover that $Q_{\mathrm{ns}}(f,f)$ is dominated by $\mathcal{G}(f,f)$ for N_∞ sufficiently large and the contradiction follows as before.

We finally prove the propagation of the L^∞ bound when it is finite initially. If $\|f_0\|_{L^\infty} < N$ for some $N \geqslant N_\infty$, we pick $t_0 \in (0,1)$ such that $N_\infty t_0^{-\frac{d}{2s}} = N$. By the same reasoning as before, we obtain

$$f(t,x,v) < N_\infty (t+t_0)^{-d/2s}.$$

In particular, $f(t,x,v) < N$ for $t \in (0,1-t_0)$. This allows us to extend the upper bound for a fixed period of time. Iterating this, we extend it for all time. □

Remark 4.3. — Here we present some further interpretation of the cone of non-degeneracy and the L^∞ bound. The cone of Lemma 2.1 is a cone of direction for $(v'-v)$, i.e., the so-called "ω" vector of the "ω-representation" (see [52, §4.6]):

$$A(v) := \big\{\omega \in \mathbb{S}^{d-1} \text{ s.t. } |\{v'_* \,:\, (v'_*-v_*)\cdot\omega = 0 \ \&\ f(v'_*) > c_0 \ \&\ |v'_*| < R_0\}| > \delta\big\}.$$

(The variable v' remains to be integrated independently of this cone.) This is a set of directions where the kernel is bounded below in the Carleman representation. The fact that the set where f is bounded below can be some complicated Borel set in a ball near zero does not change fundamentally the argument, which would be very similar if $f \geqslant \ell\chi_{B_r}$. The set $A(v)$ is $\{\omega \in \mathbb{S}^{d-1} : |\omega \cdot v/|v|| \lesssim |v|^{-1}\}$ or a non-zero measure-proportion of this set, hence $|\omega \cdot v| \lesssim R_0$ or a non-zero proportion of this set of directions.

The goal of this cone of direction is to find configurations so that v'_* is brought back near 0 in a zone where f is bounded below, in order to bound from below the "coefficients" of the operator, i.e., the kernel.

Then this set of directions $A(v)$ creates a cone $v' \in \Xi(v)$ centred at v and of angles of order $1/|v|$ close to orthogonal to $v/|v|$. Then in [49], see Theorem 4.1 above, the part of this cone where $f < (1/2)\max f$ is bounded below using the Chebychev inequality and the mass and energy bounds. That is: the assumptions imply that f is, for a significant amount of the large velocities, far from its maximum, i.e., less than $(\max f)/2$. On this part of the cone, the coercivity of $Q_1(f,f)$ is recovered, and together with the bounds from above on $Q_{\mathrm{ns}}(f,f)$, gives the contradiction and the L^∞ barrier.

4.3. Appearance of pointwise moments when $\gamma > 0$. — Let us now prove the appearance of pointwise moments (second part of the theorem), when assuming furthermore that $\gamma > 0$ and restricting without loss of generality to $t \in [0,1]$. Consider $g(v) = N(t)\min(1,|v|^{-q})$ where $N(t) = N_0 t^{-\beta}$ and $\beta = (q/\gamma) + d/2s$, and with $q > d+1$ to be chosen large enough later. We recall that the existence of the first contact point is granted by our assumptions on the solution (periodic condition in x and rapid qualitative decay in v).

At the first contact point $g(t_0,v_0) = f(t_0,x_0,v_0)$ and Theorem 4.1 implies that

$$N(t_0)\min\big(1,|v_0|^{-q}\big) = g(t_0,v_0) = f(t_0,x_0,v_0) \leqslant N_\infty(1+t_0)^{-d/2s} \leqslant N_\infty t_0^{-d/2s},$$

where N_∞ is the constant in Theorem 4.1 and we have used $t_0 \in [0,1]$. It shows that $|v_0| \gtrsim N_0^{1/q} t_0^{-\gamma}$ can be made large by choosing N_0 large enough. In particular we can apply again Propositions 3.1, 3.7, 3.8, 3.9 and 3.11 to get

$$\begin{aligned}
\mathcal{G}(f,f)(t_0,x_0,v_0) &\lesssim -q^s|v_0|^\gamma g(v_0) && \text{from Proposition 3.1,}\\
\mathcal{B}_1(f,f)(t_0,x_0,v_0) &\lesssim q^2 2^q |v_0|^{\gamma-2} g(v_0) && \text{from Proposition 3.7,}\\
\mathcal{B}_2(f,f)(t_0,x_0,v_0) &\lesssim \frac{1}{q}\,|v_0|^\gamma g(v_0) && \text{from Proposition 3.8,}\\
\mathcal{B}_3(f,f)(t_0,x_0,v_0) &\lesssim q|v_0|^{\gamma-2} g(v_0) && \text{from Proposition 3.9,}\\
Q_{\mathrm{ns}}(f,f)(t_0,x_0,v_0) &\lesssim |v_0|^\gamma g(v_0) && \text{from Proposition 3.11,}
\end{aligned}$$

and therefore by choosing q large enough (independently of N_0) we deduce

$$Q(f,f)(t_0,x_0,v_0) \lesssim \left[-q^s|v_0|^\gamma + |v_0|^\gamma + |v_0|^{\gamma-2}\right] g(v_0) \lesssim -q^s|v_0|^\gamma g(v_0),$$

which yields the inequality

$$-\Big(\frac{q}{\gamma} + \frac{d}{2s}\Big)\frac{g(v_0)}{t_0} = \partial_t g(t_0,v_0) \leqslant Q(f,f)(t_0,x_0,v_0) \leqslant -Cq^s|v_0|^\gamma g(v_0)$$

for some constant $C > 0$ at the contact point. Since $|v_0| \gtrsim N_0^{1/q} t_0^{-\gamma}$ we deduce that

$$\Big(\frac{q}{\gamma} + \frac{d}{2s}\Big) \geqslant C' q^s N_0^{\gamma/q},$$

which is a contradiction for N_0 large enough. This shows that the contact point does not exist and concludes the proof of the appearance of pointwise moments.

Remark 4.4. — Note that this proof only uses the first estimate on the good term (Proposition 3.1), and therefore does not fully exploit the non-cutoff nature of the collision operator.

4.4. Propagation of pointwise moments. — We consider the setting and assumptions of Theorem 1.3 and prove first the propagation of pointwise moments (first part of the theorem). Consider $g(v) = N_0 \min(1, |v|^{-q})$ with $q > d + 1$ to be chosen large enough later. At the first contact point $g(t_0, v_0) = f(t_0, x_0, v_0)$ and Theorem 4.1 implies that $N_0 \min(1, |v_0|^{-q}) \leqslant N_\infty$ which shows that $|v_0| \gtrsim N_0^{1/q}$ can be made large by choosing N_0 large enough. Apply Propositions 3.1, 3.7, 3.8, 3.9 and 3.11 apply at this contact point:

$$\begin{aligned}
\mathcal{G}(f,f)(t_0,x_0,v_0) &\lesssim -q^s |v_0|^\gamma g(v_0) && \text{from Proposition 3.1,}\\
\mathcal{B}_1(f,f)(t_0,x_0,v_0) &\lesssim q^2 2^q |v_0|^{\gamma-2} g(v_0) && \text{from Proposition 3.7,}\\
\mathcal{B}_2(f,f)(t_0,x_0,v_0) &\lesssim \frac{1}{q}\, |v_0|^\gamma g(v_0) && \text{from Proposition 3.8,}\\
\mathcal{B}_3(f,f)(t_0,x_0,v_0) &\lesssim q |v_0|^{\gamma-2} g(v_0) && \text{from Proposition 3.9,}\\
Q_{\mathrm{ns}}(f,f)(t_0,x_0,v_0) &\lesssim |v_0|^\gamma g(v_0) + \mathbb{1}_{\gamma<0} 2^{-q\gamma/d} g(v_0)^{1-\gamma/d} && \text{from Proposition 3.11.}
\end{aligned}$$

We choose q large enough (independently of N_0) so that

$$\mathcal{G}(f,f) + \mathcal{B}_2(f,f) + |v_0|^\gamma g(v_0) \lesssim -q^s |v_0|^\gamma g(v_0).$$

For large $|v_0|$ (ensured by our choice of N_0, that depends on q), we get

$$\begin{aligned}
Q(f,f)(t_0,x_0,v_0) &= \mathcal{G}(f,f)(t_0,x_0,v_0) + \mathcal{B}_1(f,f)(t_0,x_0,v_0)\\
&\qquad + \mathcal{B}_2(f,f)(t_0,x_0,v_0) + \mathcal{B}_3(f,f)(t_0,x_0,v_0) + Q_{\mathrm{ns}}(f,f)(t_0,x_0,v_0)\\
&\lesssim -q^s |v_0|^\gamma g(v_0) < 0
\end{aligned}$$

which contradicts the inequality $0 = \partial_t g(t_0, v_0) \leqslant \partial_t f(t_0, x_0, v_0) = Q(f,f)(t_0, x_0, v_0)$ at this contact point. This shows that the contact point does not exist and concludes the proof of the propagation of pointwise moments.

When $\gamma < 0$ and $q \geqslant d+1$ large enough the only additional difficulty is the second term on the right hand side of the control on Q_{ns}. But

$$g(v_0)^{1-\gamma/d} \lesssim N_0^{1-\gamma/d} |v_0|^{-q+q/d\gamma}$$

and the exponent of $|v_0|$ is strictly lower than that of $\mathcal{G}(f,f)$, uniformly in $q \geqslant d+1$, so is dominated by $\mathcal{G}(f,f)$ by taking $|v_0|$ large enough (through N_0 large enough). Finally taking q large enough yields the same contradiction as before.

4.5. Appearance of low pointwise moments for $\gamma \leqslant 0$. — Consider as before $g(v) = N(t)\min(1,|v|^{-q})$ with $q \geqslant 0$ to be restricted later, and $N(t) = N_0 t^{-d/2s}$ and $N_0 = N_0(m_0, M_0, E_0, H_0, \gamma, s, d)$ is a large constant to be determined below. As before it is sufficient to prove that the conclusion holds for $t \in (0,1]$.

At the first contact point $g(t_0, v_0) = f(t_0, x_0, v_0)$ and Theorem 4.1 implies that $N_0 \min(1,|v_0|^{-q}) \leqslant N_\infty$ which shows that $|v_0| \gtrsim N_0^{1/q}$ can be made large by choosing N_0 large enough. Apply Propositions 3.3, 3.7, 3.8, 3.10 and 3.11 at this contact point (note that we do not track the dependency in q since it is bounded here):

$$\begin{aligned} \mathcal{G}(f,f)(t_0,x_0,v_0) &\lesssim -|v_0|^{(\gamma+2s)+2s/d} g(v_0)^{1+2s/d} && \text{from Proposition 3.3,}\\ \mathcal{B}_1(f,f)(t_0,x_0,v_0) &\lesssim |v_0|^{\gamma-2} g(v_0) && \text{from Proposition 3.7,}\\ (\mathcal{B}_2+\mathcal{B}_3)(f,f)(t_0,x_0,v_0) &\lesssim |v_0|^{\gamma} g(v_0) && \text{from Proposition 3.10,}\\ Q_{\mathrm{ns}}(f,f)(t_0,x_0,v_0) &\lesssim |v_0|^{\gamma} g(v_0) + g(v_0)^{1-\gamma/d} && \text{from Proposition 3.11.} \end{aligned}$$

To check that the first negative term dominates the other terms (i.e., is larger than, say, twice all the other terms for N_0 large enough), there are three *independent* conditions to check: (1) that the (negative) exponent of $|v_0|$ is strictly greater in this negative term than the corresponding exponents in all the other terms, and (2) that the (positive) power of $N(t)$ is strictly greater in this negative term than the corresponding exponents in all the other terms, and finally (3) that the exponent of $|v_0|$ in the negative term is greater or equal than that of the barrier, i.e., q. Note in particular that the two first conditions must be checked independently since $|v_0|$ can be possibly be much larger than $N_0^{1/q}$. As far as (1) is concerned, check that

$$\begin{aligned} &\gamma + 2s + \frac{2s}{d} - q - q\frac{2s}{d} > \gamma - 2 - q && \text{for all } q \in [0, 3(d+1)) \supset [0, d+1),\\ &\gamma + 2s + \frac{2s}{d} - q - q\frac{2s}{d} > \gamma - q && \text{for all } q \in [0, d+1),\\ &\gamma + 2s + \frac{2s}{d} - q - q\frac{2s}{d} > -q + q\frac{\gamma}{d} && \text{for all } q \geqslant \Big[0, d + \frac{2s}{\gamma+2s}\Big) \supset [0, d+1). \end{aligned}$$

As far as (2) is concerned, check that

$$\begin{aligned} &1 + \frac{2s}{d} > 1 && \text{for all } q \in \mathbb{R}_+ \supset [0, d+1),\\ &1 + \frac{2s}{d} \geqslant 1 - \frac{\gamma}{d} && \text{for all } q \in \mathbb{R}_+ \supset [0, d+1), \end{aligned}$$

where we have used $\gamma + 2s \geqslant 0$ in the last inequality. As far as (3) is concerned, check that

$$\gamma + 2s + \frac{2s}{d} - q - q\frac{2s}{d} \geqslant -q \qquad \text{for all } q \in \Big[0, d+1+\frac{d\gamma}{2s}\Big].$$

We thus impose the most restrictive condition $q = d + 1 + d\gamma/2s$ if $\gamma < 0$. In the limit case $\gamma = 0$, observe however that if the condition (1) above is saturated (same exponents of $|v_0|$) and the condition (3) is satisfied, but the condition (2) is strict (strictly greater exponent of $N(t)$ is the negative term), we can still prove that the negative term dominates by taking N_0 large enough. This proves in all cases that, by choosing N_0 and thus $|v_0|$ large enough:

$$\begin{aligned}Q(f,f)(t_0,x_0,v_0) &\lesssim -|v_0|^{(\gamma+2s)+2s/d} g(v)^{1+2s/d} \\ &\lesssim -N(t)^{1+2s/d}|v_0|^{-d-1-d\gamma/2s} \approx -N(t)^{2s/d}N'(t)|v|^{-d-1-d\gamma/2s},\end{aligned}$$

which contradicts $\partial_t g(t_0,v_0) \leqslant Q(f,f)(t_0,x_0,v_0)$ at the contact point by picking N_0 large enough.

5. Relaxing partially the qualitative rapid decay assumption

This section discusses various ways of weakening the qualitative assumptions made on the initial data in 1.2. Observe first that if a clean local existence and stability theory was available in $H^k(\langle v\rangle)$ for hard potentials, and some k,q, then it would be possible to use the generation of L^1 moments conditionally to hydrodynamic bounds in the style of [31, §5.3.1], together with interpolation, to deduce the qualitative pointwise moments. In the case of soft potentials such a local existence and stability theory is available in [46] but the propagation of L^1 moments conditionally to hydrodynamic bounds is not available: if it was, an approximation argument on the initial data (truncating its support) could be performed. We postpone this discussion to another work.

Meanwhile we discuss here how to weaken the qualitative decay assumed in Definition 1.2 by approximation argument on the barrier $g(t,v)$ used in the maximum principle arguments.

5.1. Solutions without rapid decay and statement

Definition 5.1 (Classical solutions to the Boltzmann equation with mild decay)
Given $T \in (0,+\infty]$, we say that a function $f : [0,T] \times \mathbb{T}^d \times \mathbb{R}^d \to [0,+\infty)$ is a *classical solution to the Boltzmann equation* (1.1) *with mild decay* if

- the function f is differentiable in t and x and twice differentiable in v everywhere;
- the equation (1.1) holds classically at every point;
- The limit $\lim_{|v|\to\infty} f(t,x,v) = 0$ holds uniformly in $t \in [0,T]$ and $x \in \mathbb{T}^d$.

Theorem 5.2 (Pointwise moment bounds revisited). — *Let $\gamma \in \mathbb{R}$ and $s \in (0,1)$ satisfy $\gamma+2s \in [0,2]$ and f be a solution of the Boltzmann equation* (1.1) *as in Definition* 5.1 *such that $f(0,x,v) = f_0(x,v)$ in $\mathbb{T}^d \times \mathbb{R}^d$ and* (1.3) *holds. Then*

(1) *If $\gamma \in (-2,0]$ and $q = d+1+\gamma d/2s$ if $\gamma < 0$ or $q \in [0,q+1)$, then there exists $N > 0$ depending on m_0, M_0, E_0, H_0, d and s such that*

$$\forall\, t \in (0,T],\ x \in \mathbb{T}^d,\ v \in \mathbb{R}^d, \quad f(t,x,v) \leqslant N\big(1+t^{-d/2s}\big)\min(1,|v|^{-q}).$$

(2) *If $\gamma > 0$ there exists a constant $N > 0$ depending on m_0, M_0, E_0, H_0, d and s, and a power $\beta > 0$ such that*

$$\forall\, t \in (0,T],\ x \in \mathbb{T}^d,\ v \in \mathbb{R}^d, \quad f(t,x,v) \leqslant N\left(1+t^{-\beta}\right)\min(1,|v|^{-d-1}).$$

(3) *If $\gamma \leqslant 0$ and $\gamma + 2s < 1$, there exists q_0 depending on $d, s, \gamma, m_0, M_0, E_0, H_0$ such that for all $q \geqslant q_0$ and $f_0 \leqslant C \min(1, |v|^{-q})$ then there exists N depending on $C, m_0, M_0, E_0, H_0, q, d$ and s such that*

$$\forall\, t \in [0,T],\ x \in \mathbb{T}^d,\ v \in \mathbb{R}^d, \quad f(t,x,v) \leqslant N \min(1, |v|^{-q}).$$

(4) *If $\gamma > 0$, there exists $q_0 > 0$ such that if*

$$\lim_{|v| \to \infty} |v|^{-q_0} f(t,x,v) = 0$$

holds uniformly in $t \in [0,T]$ and $x \in \mathbb{T}^d$, then for all $q > 0$, there exists constants N and $\beta > 0$ depending on m_0, M_0, E_0, H_0, q, d and s such that

$$\forall\, t \in (0,T],\ x \in \mathbb{T}^d,\ v \in \mathbb{R}^d, \quad f(t,x,v) \leqslant N \left(1 + t^{-\beta}\right) \min(1, |v|^{-q}).$$

Remark 5.3

(1) Note that for $\gamma > 0$, we know from part (2) that for all $t > 0$,

$$\lim_{|v| \to \infty} |v|^{-q} f(t,x,v) = 0,$$

for any $q < d+1$. The assumption in part (4) would be automatically true if $q_0 < d+1$. Unfortunately, it is hard to compute q_0 explicitly from our proof.

(2) The only purpose of the technical assumption $\gamma + 2s < 1$ in (3) is to handle the error term – see $\varepsilon |v|^{-d-1+\varepsilon}$ in (5.2) below. It is most likely not necessary. It is certainly not necessary for the a priori estimate if we knew that our solution decays faster than $|v|^{-d-2}$ at infinity.

The proof follows the same pattern as before. The only new difficulty is to prove the existence of the first contact point, and avoid the situation where it would appear asymptotically as $|v| \to \infty$. To this purpose we modify the barrier functions used in Section 4 by adding arbitrarily small correctors: $\widetilde{g} = g + e$. The correctors are related to the decay known on g, in order to ensure the existence of the first contact point.

– For parts (1), (2) and (3) with $q \leqslant d+1$, we use a constant corrector

$$\widetilde{g}(t,v) = N(t) \left(1 \wedge |v|^{-q}\right) + e \quad \text{with } e = \varepsilon > 0. \tag{5.1}$$

– For part (3) with $q > d+1$ and $\gamma \leqslant 0$, we use

$$\widetilde{g}(t,v) = N(t) \left(1 \wedge |v|^{-q}\right) + e(t,v) \quad \text{with } e(t,v) := \varepsilon(t) \left(1 \wedge |v|^{-d-1+\eta}\right) \tag{5.2}$$

for certain choices of $N(t)$ and $\varepsilon(t)$ and $\eta > 0$.

– For part (4) with $\gamma > 0$, it is enough to consider $q > q_0$ and we use

$$g(t,v) = N(t) \left(1 \wedge |v|^{-q}\right) + e(t,v) \quad \text{with} \quad e(t,v) := \varepsilon(t) \left(1 \wedge |v|^{-q_0}\right) \tag{5.3}$$

for certain choices of $N(t)$ and $\varepsilon(t)$.

5.2. Technical estimates on the collision operator. — The following results are variations of the corresponding results in Section 3 when taking into account the correctors to the barrier function. We define the decomposition $Q_{\mathrm{s}} = \mathcal{G} + \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3$ as before in (3.2)-(3.6).

Proposition 5.4 (Estimate of $\mathcal{G}(f,g)$ useful for large q). — *Let f be a non-negative function satisfying* (1.3) *and g given by* (5.1), (5.2) *or* (5.3) *with $q \geqslant 0$. Then there exists a radius $R_q = C_R(1+q)$ so that*

$$\forall\,|v| \geqslant R_q, \quad \mathcal{G}(f,g)(v) \lesssim \begin{cases} -q^s N|v|^{\gamma-q} & \text{if } g \text{ is as in (5.1)},\\ -q^s N|v|^{\gamma-q} - \varepsilon(t)|v|^{\gamma-(d+1)+\eta} & \text{if } g \text{ is as in (5.2)},\\ -q^s N|v|^{\gamma-q} - q_0^s\varepsilon(t)|v|^{\gamma-q_0} & \text{if } g \text{ is as in (5.3)}. \end{cases}$$

where the constants $C_R > 0$ and in the latter inequality are independent of q.

Proof. — It is a straight forward modification of Proposition 3.1 adding an extra correction term. In the case g is as in (5.1), note that the extra terms $+\varepsilon$ will cancel out in the upper bound for $(f(v') - f(v))$. ☐

Proposition 5.5 (Estimate of $\mathcal{G}(f,f)$ useful for not-so-large q)

Assume f satisfies (3.1) *for of the form* (5.1) *or* (5.2) *and $q \geqslant 0$. Then there exists $R_q = C_R(1+q)$ so that*

$$\forall\, v \in \mathbb{R}^d \mid |v| \geqslant R_q, \quad \mathcal{G}(f,g)(v) \lesssim_q -g(v)^{1+2s/d}|v|^{\gamma+2s+2s/d}.$$

Proof. — We follow the same ideas as in the proof of Proposition 3.3. We first analyse the range of values of v where the inequality follows from Proposition 5.4.

If g is given by (5.1), then the estimate of Proposition 5.5 derives from Proposition 5.4 for $|v| \geqslant 1$ and $\varepsilon \in (0,1)$ such that

$$\frac{\varepsilon}{3} \leqslant N|v|^{-q} \lesssim_q |v|^{-(d+1)}.$$

Indeed it implies $\varepsilon^{1+2s/d} \lesssim N|v|^{-q-2s-2s/d}$ and $(N|v|^{-q})^{1+2s/d} \lesssim N|v|^{-q-2s-2s/d}$ which in turn yields

$$g(v)^{1+2s/d} \lesssim N|v|^{-q-2s-2s/d}$$

and then the conclusion follows from Proposition 5.4.

If now g is given by (5.2) or (5.3), then the estimate of Proposition 5.5 derives from Proposition 5.4 as soon as $|v| \geqslant 1$, $\varepsilon \in (0,1)$ and $g(v) \lesssim_q |v|^{-(d+1)}$. Indeed, we then have $g(v)^{2s/d}|v|^{2s+2s/d} \lesssim_q 1$ and the conclusion follows.

We are left with two cases: (1) when $g(v) \gtrsim_q |v|^{-(d+1)}$ with g given by (5.1), (5.2) or (5.3), or (2) when $g(v) < \varepsilon/3$ and g is of the form (5.1).

In both cases, we argue as in the proof of Proposition 3.1. We pick $r > 0$ such that

$$|\Xi(t,x,v) \cap B_r| = \frac{4^2 E_0}{|v|^2 g(v)}.$$

and deduce

$$\mathcal{G}(f,g)(v) \leqslant \int_{\Xi(t,x,v)\cap B_r\cap\{f(v+w)\leqslant g(v)/2\}} \Big[\frac{g(v)}{2} - g(v+w)\Big] K_{\overline{f}}(v,v+w)\,\mathrm{d}w.$$

As in Proposition 3.1, this is a useful estimate if $g(v+w) > g(v)/2$ for $w \in \Xi \cap B_r$. If $g(v) \gtrsim_q |v|^{-d-1}$, we end the proof as in Proposition 3.1 (by choosing an appropriately large constant C_q). If g is given by (5.1) and $N|v|^{-q} \leqslant \varepsilon/3$, then we have for all $w \in \mathbb{R}^d$ that $g(v+w) \geqslant 3g(v)/4$. This also allows to continue the proof and conclude. □

Proposition 5.6 (Estimate of $\mathcal{B}_1(f,g)$ for all $q \geqslant 0$). — *Let f be a non-negative function satisfying* (1.3). *Let g be of the form* (5.1), (5.2) *or* (5.3) *with $q \geqslant 0$. Then for $|v| \geqslant 2$,*

$$\mathcal{B}_1(f,g)(v) \lesssim (1+q)^2 2^q |v|^{\gamma-2} g(v)$$

with constant uniform in q.

Proof. — It is a straight forward adaptation of the proof of Proposition 3.7. □

Proposition 5.7 (Estimate of $\mathcal{B}_2(f,f)$ for large q). — *Let f be a non-negative function satisfying* (1.3). *Assume $f \leqslant g$ for all $v \in \mathbb{R}^d$ and either g is of the form* (5.2) *with $\gamma + 2s < 1 - \eta$, or g is of the form* (5.3) *with $q_0 > d + \gamma + 2s$. Assume further that $q > \gamma + 2s + d$. Then for $|v| \geqslant 2$*

$$\mathcal{B}_2(f,f) \lesssim \begin{cases} \dfrac{1}{q-(d+\gamma+2s)} |v|^{-q+\gamma} + \varepsilon |v|^{-d-1+\eta+\gamma} & \text{if } g \text{ is as in (5.2)}, \\[2ex] \dfrac{1}{q-(d+\gamma+2s)} |v|^{-q+\gamma} + \varepsilon \dfrac{1}{q_0-(d+\gamma+2s)} |v|^{-q_0+\gamma} & \text{if } g \text{ is as in (5.3)}. \end{cases}$$

Proof. — It is the result of the same computation as in the proof of Proposition 3.8 but with the extra correction terms. The purpose of the assumptions $\gamma + 2s < 1 - \eta$ or $q_0 > d + \gamma + 2s$ is to make sure the tail of the integral

$$\int_{v'_* \in v + (v'-v)^\perp} g(v'_*) |v - v_*|^{\gamma+2s+1} \,\mathrm{d}v'_*$$

is convergent (which was also the purpose of the assumption $q > \gamma + 2s - d$). □

Proposition 5.8 (Estimate of $\mathcal{B}_3(f,f)$ for large q). — *Let f be a non-negative function satisfying* (1.3). *Assume $f \leqslant g$ for all $v \in \mathbb{R}^d$ and g of the form* (5.2) *or* (5.3) *and $q > d + \gamma + 2s$. Then for all $|v| \geqslant 2$*

$$\mathcal{B}_3(f,f)(v) \lesssim_q \frac{1}{q-(d+\gamma+2s)} |v|^{\gamma-2} g(v).$$

Remark 5.9. — The dependency in q of the constant is explicit and can be tracked from the proof below.

Proof. — The proof is similar to that of Proposition 3.9 but takes the extra corrector term into account. Define for $|v'| < |v|/2$:

$$I_3(v,v') := \int_{v'_* \in v + (v'-v)^\perp} \widetilde{\chi}(v'_*) g(v'_*) |v - v'_*|^{\gamma+2s+1} \widetilde{b}(\cos\theta)\, \mathrm{d}v'_*$$

and decompose $v'_* = |v|(\widehat{v} + \widetilde{u})$ and calculate as before (the restriction $\widetilde{\chi}$ imposes $|\widehat{v} + \widetilde{u}| > c_1(q)$)

$$\begin{aligned} I_3(v,v') &\lesssim N|v|^{-q+\gamma+2s+d} \int_{\widetilde{u}\in(v'-v)^\perp} \widetilde{\chi}(v'_*)\, |\widehat{v}+\widetilde{u}|^{-q}\, |\widetilde{u}|^{\gamma+2s+1}\, \mathrm{d}\widetilde{u} \\ &+ \varepsilon \begin{cases} |v|^{-(d+1)+\eta+\gamma+2s+d} \displaystyle\int_{\substack{\widetilde{u}\in(v'-v)^\perp \\ |\widehat{v}+\widetilde{u}|>c_q}} |\widehat{v}+\widetilde{u}|^{-(d+1)+\eta}\, |\widetilde{u}|^{\gamma+2s+1}\, \mathrm{d}\widetilde{u} & \text{if } g \text{ is as in (5.2)}, \\ |v|^{-q_0+\gamma+2s+d} \displaystyle\int_{\substack{\widetilde{u}\in(v'-v)^\perp \\ |\widehat{v}+\widetilde{u}|>c_q}} |\widehat{v}+\widetilde{u}|^{-q_0}\, |\widetilde{u}|^{\gamma+2s+1}\, \mathrm{d}\widetilde{u} & \text{if } g \text{ is as in (5.3)}. \end{cases} \end{aligned}$$

This implies the following estimates:

$$\mathcal{B}_3(f,f)(v) \lesssim \begin{cases} C_q N |v|^{-q+\gamma-2} + \varepsilon |v|^{-(d+1)+\eta+\gamma-2} & \text{when } g \text{ is as in (5.2)}, \\ C_q N |v|^{-q+\gamma-2} + C_{q_0} \varepsilon |v|^{-q_0+\gamma-2} & \text{when } g \text{ is as in (5.3)}, \end{cases}$$

which concludes the proof. □

Proposition 5.10 (Estimate of $\mathcal{B}_2(f,g) + \mathcal{B}_3(f,g)$ for not-so-large q)
Let f be a non-negative function satisfying (1.3) *and g be of the form* (5.1) *and $q \in [0, d+1]$. Then for $|v| \geqslant 2$,*

$$(\mathcal{B}_2 + \mathcal{B}_3)(f,g)(v) \lesssim \begin{cases} N|v|^{-d-1+\gamma} & \textit{if } q > d-1, \\ N|v|^{-d-1+\gamma} \ln(1+|v|) & \textit{if } q = d-1, \\ N|v|^{-q-2+\gamma} & \textit{if } q < d-1. \end{cases}$$

Proof. — The proof is identical to that of Proposition 3.10. Note that the extra constant corrector term ε cancels out in the estimate $f(v') - f(v) \leqslant g(v') - g(v)$. □

Proposition 5.11 (Estimate of $Q_{\mathrm{ns}}(f,f)$). — *Assume f satisfies* (3.1) *with g of the form* (5.1) *or* (5.2). *Then for $\gamma \geqslant 0$*

$$Q_{\mathrm{ns}}(f,f)(v) \lesssim (1+|v|)^\gamma g(v),$$

while for $\gamma < 0$,

$$Q_{\mathrm{ns}}(f,f) \lesssim C_q g(v)^{1-\gamma/d} + (1+|v|)^\gamma g(v)$$

for some constant C_q depending on q.

Proof. — In the case $\gamma \geqslant 0$, the estimate $Q_{\mathrm{ns}}(f,f) \lesssim |v|^\gamma f(v)$ implies the result follows for any form of the function g. In the case $\gamma < 0$, the proof of Proposition 3.11 applies as soon as $g(v') \leqslant C_q g(v)$ whenever $|v' - v| < |v|/2$. This property is satisfied for all the variants of the function g given by (5.1), (5.2) or (5.3). □

5.3. Proof of Theorem 5.2

5.3.1. *Proof of part* (1). — It is identical to the proof of part (3) in Theorem 1.3 but using

$$\widetilde{g}(t,v)=N(t)\min(1,|v|^{-q})+\varepsilon$$

for $\varepsilon>0$ arbitrarily small. We apply Propositions 5.4, 5.6, 5.10 and 5.11 instead of Propositions 3.1, 3.7, 3.10 and 3.11 and we arrive to the same set of inequalities that imply the contradiction.

5.3.2. *Proof of part* (2). — We use the same estimates as for part (1), which are not the same as the ones used for part (2) in Theorem 1.3. Set $\widetilde{g}(t,v)=N(t)\min(1,|v|^{-q})+\varepsilon$ and $N(t)=N_0t^{-d/2s}$, where N_0 is a large constant depending on m_0,M_0,E_0,H_0,γ,s and d, to be determined below, and ε is arbitrarily small. Apply Propositions 5.4, 5.6, 5.10 and 5.11 at the point of contact (t_0,x_0,v_0), for $|v_0|$ large enough:

$$\begin{aligned}\mathcal{G}(f,f)(t_0,x_0,v_0)&\lesssim-|v_0|^{(\gamma+2s)+2s/d}g(v_0)^{1+2s/d}&&\text{from Proposition 5.5,}\\ \mathcal{B}_1(f,f)(t_0,x_0,v_0)&\lesssim|v_0|^{\gamma-2}g(v_0)&&\text{from Proposition 5.6,}\\ (\mathcal{B}_2+\mathcal{B}_3)(f,f)(t_0,x_0,v_0)&\lesssim|v_0|^{\gamma}g(v_0)&&\text{from Proposition 5.10,}\\ Q_{\mathrm{ns}}(f,f)(t_0,x_0,v_0)&\lesssim|v_0|^{\gamma}g(v_0)&&\text{from Proposition 5.11.}\end{aligned}$$

As before $|v_0|$ large can be imposed by taking N large, and the first negative term dominates all other at large $|v_0|$ which contradicts $\partial_t\widetilde{g}(t_0,v_0)\leqslant Q(f,f)(t_0,x_0,v_0)$ and concludes the proof.

5.3.3. *Proof of part* (3) *in the case* $\gamma\leqslant 0$ *and* $q=d+1$. — Consider a function $\widetilde{g}$ of the form (5.1) with $q=d+1$ and $\varepsilon>0$ arbitrarily small and $N=N_0$ large enough so that $\widetilde{g}(0,v)\geqslant f(0,x,v)$ everywhere (using the L^∞ bound on f). The first contact (t_0,x_0,v_0), such that (3.1) holds true, exists because f goes to zero as $|v|\to+\infty$. Using the L^∞ bound and picking N large enough, we can force $|v_0|$ to be arbitrarily large, and we can apply Propositions 5.5, 5.6, 5.10 and 5.11:

$$\begin{aligned}\mathcal{G}(f,f)(t_0,x_0,v_0)&\lesssim-|v_0|^{\gamma+2s+2s/d}g(v_0)^{1+2s/d}&&\text{from Proposition 5.5,}\\ \mathcal{B}_1(f,f)(t_0,x_0,v_0)&\lesssim|v_0|^{\gamma-2}g(v_0)&&\text{from Proposition 5.6,}\\ (\mathcal{B}_2+\mathcal{B}_3)(f,f)(t_0,x_0,v_0)&\lesssim|v_0|^{\gamma}g(v_0)&&\text{from Proposition 5.10,}\\ Q_{\mathrm{ns}}(f,f)(t_0,x_0,v_0)&\lesssim|v_0|^{\gamma}g(v_0)+C_qg(v_0)^{1-\gamma/d}&&\text{from Proposition 5.11.}\end{aligned}$$

Since $|v_0|^{(\gamma+2s)+2s/d}\gtrsim(|v_0|^{-q}+\varepsilon)^{-(\gamma+2s)/(d+1)-2s/d(d+1)}$ uniformly as $\varepsilon\to0$ ($|v_0|$ is not close to zero), the negative term dominates the term $g(v_0)^{1-\gamma/d}$ by taking N_0 large enough, and we deduce for some constants $K,C>0$

$$\begin{aligned}Q(f,f)(t_0,x_0,v_0)&\leqslant-K|v_0|^{\gamma+2s+2s/d}g(v_0)^{1+2s/d}+C|v_0|^{\gamma}g(t,v),\\ &\leqslant|v|^{\gamma}g(t,v)(-KN_0^{2s/d}+C).\end{aligned}$$

We choose N_0 large enough to achieve the contradiction $Q(f,f)(t_0,x_0,v_0)<0$.

5.3.4. *Proof of part* (3) *in the case* $\gamma \leqslant 0$ *and* q *large*. — Having proved already that (3) holds when $q = d+1$, we now use the corrected barrier $\widetilde{g}$ as in (5.2). The previous subsubsection implies then $f(t,x,v) < g(t,v)$ when v is sufficiently large and therefore there is a first contact point (t_0, x_0, v_0). Take $\varepsilon(t) = \varepsilon_0 e^{C_\varepsilon t}$ in (5.2), for $\varepsilon_0 > 0$ arbitrarily small. As before we impose $|v_0|$ large thanks to the L^∞ by choosing N_0 large enough, and we now apply Propositions 5.4, 5.5, 5.6, 5.7, 5.8 and 5.11. Following the same computations as in the proof of part (1) of Theorem 1.3, the principal terms cancel out and we are left with the terms derived from the correction term $\varepsilon(t)|v_0|^{-d-1+\eta}$. We get

$$Q(f,f)(t_0,x_0,v_0) \leqslant C\varepsilon(t)|v_0|^{\gamma+d+1-\eta}.$$

Since $\gamma \leqslant 0$ and $|v_0|$ is large, we have $Q(f,f)(t_0,x_0,v_0) < C_\varepsilon \varepsilon(t)$ for some $C_\varepsilon > 0$. We plug C_ε in the corrector $\varepsilon(t) = \varepsilon_0 e^{C_\varepsilon t}$ and achieve the contradiction.

5.3.5. *Proof of part* (4). — We now use $\widetilde{g}$ as in (5.3) with $N(t) = N_0 t^{-\beta}$ with $\beta := (q/\gamma) - d/2s$ and $\varepsilon(t) = \varepsilon_0 t^{-\beta_0}$ with $\beta_0 := q_0/\gamma - d/2s$ and N_0 large enough and ε_0 arbitrarily small and the exponents q and q_0 large enough, to be chosen later. The first contact point (t_0, x_0, v_0) exists because of the convergence $|v|^{q_0} f(t,x,v) \to 0$ as $|v| \to +\infty$ and the corrector term. We impose $|v_0|$ large enough by taking N_0 large enough, and we apply Propositions 5.4, 5.6, 5.7, 5.8 and 5.11:

$$\begin{aligned}
\mathcal{G}(f,f)(t_0,x_0,v_0) &\lesssim -q^s N(t)|v_0|^{-q+\gamma} - q_0^s \varepsilon(t)|v_0|^{-q_0+\gamma} && \text{from Proposition 5.4,}\\
\mathcal{B}_1(f,f)(t_0,x_0,v_0) &\lesssim_{q,q_0} N(t)|v_0|^{-q+\gamma-2} + \varepsilon(t)|v_0|^{-q_0+\gamma-2} && \text{from Proposition 5.6,}\\
\mathcal{B}_2(f,f)(t_0,x_0,v_0) &\lesssim \frac{1}{q} N(t)|v_0|^{-q+\gamma} + \frac{1}{q_0}\varepsilon(t)|v_0|^{-q_0+\gamma} && \text{from Proposition 5.7,}\\
\mathcal{B}_3(f,f)(t_0,x_0,v_0) &\lesssim_{q,q_0} N(t)|v_0|^{-q+\gamma-2} + \varepsilon(t)|v_0|^{-q_0+\gamma-2} && \text{from Proposition 5.8,}\\
Q_{\mathrm{ns}}(f,f)(t_0,x_0,v_0) &\lesssim N(t)|v_0|^{-q+\gamma} + \varepsilon(t)|v_0|^{-q_0+\gamma} && \text{from Proposition 5.11.}
\end{aligned}$$

The first negative term dominates all other term when q and q_0 and $|v_0|$ are sufficiently large and we deduce

$$Q(f,f)(t_0,x_0,v_0) \lesssim -q^s N(t)|v_0|^{-q+\gamma} - q_0^s \varepsilon(t)|v_0|^{-q_0+\gamma}.$$

We use that $|v_0| \gtrsim_q N_0^{1/q} t^{-1/\gamma}$ to get

$$Q(f,f)(t_0,x_0,v_0) \lesssim -q^s t^{-\beta-1}|v_0|^{-q} - q_0^s \varepsilon_0 t^{\beta-1}|v_0|^{-q_0},$$

which yields a contradiction for $q \geqslant q_0$ large enough, and finishes the proof.

References

[1] R. Alexandre, L. Desvillettes, C. Villani & B. Wennberg – "Entropy dissipation and long-range interactions", *Arch. Rational Mech. Anal.* **152** (2000), no. 4, p. 327–355.

[2] R. Alexandre & M. El Safadi – "Littlewood-Paley theory and regularity issues in Boltzmann homogeneous equations. I. Non-cutoff case and Maxwellian molecules", *Math. Models Methods Appl. Sci.* **15** (2005), no. 6, p. 907–920.

[3] ———, "Littlewood-Paley theory and regularity issues in Boltzmann homogeneous equations. II. Non cutoff case and non Maxwellian molecules", *Discrete Contin. Dynam. Systems* **24** (2009), no. 1, p. 1–11.

[4] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu & T. Yang – "Uncertainty principle and kinetic equations", *J. Funct. Anal.* **255** (2008), no. 8, p. 2013–2066.

[5] ______, "Regularizing effect and local existence for the non-cutoff Boltzmann equation", *Arch. Rational Mech. Anal.* **198** (2010), no. 1, p. 39–123.

[6] R. Alexandre & C. Villani – "On the Boltzmann equation for long-range interactions", *Comm. Pure Appl. Math.* **55** (2002), no. 1, p. 30–70.

[7] R. Alonso, J. A. Cañizo, I. Gamba & C. Mouhot – "A new approach to the creation and propagation of exponential moments in the Boltzmann equation", *Comm. Partial Differential Equations* **38** (2013), no. 1, p. 155–169.

[8] R. Alonso, I. M. Gamba & M. Tasković – "Exponentially-tailed regularity and time asymptotic for the homogeneous Boltzmann equation", 2017, `arXiv:1711.06596`.

[9] L. Arkeryd – "L^∞ estimates for the space-homogeneous Boltzmann equation", *J. Statist. Phys.* **31** (1983), no. 2, p. 347–361.

[10] A. V. Bobylev – "Moment inequalities for the Boltzmann equation and applications to spatially homogeneous problems", *J. Statist. Phys.* **88** (1997), no. 5-6, p. 1183–1214.

[11] A. V. Bobylev & I. M. Gamba – "Upper Maxwellian bounds for the Boltzmann equation with pseudo-Maxwell molecules", *Kinet. and Relat. Mod.* **10** (2017), no. 3, p. 573–585.

[12] A. V. Bobylev, I. M. Gamba & V. A. Panferov – "Moment inequalities and high-energy tails for Boltzmann equations with inelastic interactions", *J. Statist. Phys.* **116** (2004), no. 5-6, p. 1651–1682.

[13] M. Briant – "Instantaneous exponential lower bound for solutions to the Boltzmann equation with Maxwellian diffusion boundary conditions", *Kinet. and Relat. Mod.* **8** (2015), no. 2, p. 281–308.

[14] ______, "Instantaneous filling of the vacuum for the full Boltzmann equation in convex domains", *Arch. Rational Mech. Anal.* **218** (2015), no. 2, p. 985–1041.

[15] M. Briant & A. Einav – "On the Cauchy problem for the homogeneous Boltzmann-Nordheim equation for bosons: local existence, uniqueness and creation of moments", *J. Statist. Phys.* **163** (2016), no. 5, p. 1108–1156.

[16] S. Cameron, L. Silvestre & S. Snelson – "Global a priori estimates for the inhomogeneous Landau equation with moderately soft potentials", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **35** (2018), no. 3, p. 625–642.

[17] T. Carleman – "Sur la théorie de l'équation intégrodifférentielle de Boltzmann", *Acta Math.* **60** (1933), no. 1, p. 91–146.

[18] ______, *Problèmes mathématiques dans la théorie cinétique des gaz*, Publ. Sci. Inst. Mittag-Leffler, vol. 2, Almqvist & Wiksells Boktryckeri Ab, Uppsala, 1957.

[19] C. Cercignani – *The Boltzmann equation and its applications*, Applied Math. Sciences, vol. 67, Springer-Verlag, New York, 1988.

[20] P. Constantin & V. Vicol – "Nonlinear maximum principles for dissipative linear nonlocal operators and applications", *Geom. Funct. Anal.* **22** (2012), no. 5, p. 1289–1321.

[21] L. Desvillettes – "Some applications of the method of moments for the homogeneous Boltzmann and Kac equations", *Arch. Rational Mech. Anal.* **123** (1993), no. 4, p. 387–404.

[22] L. Desvillettes & C. Mouhot – "Large time behavior of the a priori bounds for the solutions to the spatially homogeneous Boltzmann equations with soft potentials", *Asymptot. Anal.* **54** (2007), no. 3-4, p. 235–245.

[23] ______, "Stability and uniqueness for the spatially homogeneous Boltzmann equation with long-range interactions", *Arch. Rational Mech. Anal.* **193** (2009), no. 2, p. 227–253.

[24] L. Desvillettes & B. Wennberg – "Smoothness of the solution of the spatially homogeneous Boltzmann equation without cutoff", *Comm. Partial Differential Equations* **29** (2004), no. 1-2, p. 133–155.

[25] R. J. DiPerna & P.-L. Lions – "On the Cauchy problem for Boltzmann equations: global existence and weak stability", *Ann. of Math. (2)* **130** (1989), no. 2, p. 321–366.

[26] T. Elmroth – "Global boundedness of moments of solutions of the Boltzmann equation for forces of infinite range", *Arch. Rational Mech. Anal.* **82** (1983), no. 1, p. 1–12.

[27] F. Filbet & C. Mouhot – "Analysis of spectral methods for the homogeneous Boltzmann equation", *Trans. Amer. Math. Soc.* **363** (2011), no. 4, p. 1947–1980.

[28] I. M. Gamba, V. Panferov & C. Villani – "Upper Maxwellian bounds for the spatially homogeneous Boltzmann equation", *Arch. Rational Mech. Anal.* **194** (2009), no. 1, p. 253–282.

[29] I. M. Gamba, N. Pavlović & M. Tasković – "On pointwise exponentially weighted estimates for the Boltzmann equation", *SIAM J. Math. Anal.* **51** (2019), no. 5, p. 3921–3955.

[30] F. Golse, C. Imbert, C. Mouhot & A. F. Vasseur – "Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **19** (2019), no. 1, p. 253–295.

[31] M. P. Gualdani, S. Mischler & C. Mouhot – *Factorization of non-symmetric operators and exponential H-theorem*, Mém. Soc. Math. France (N.S.), vol. 153, Société Mathématique de France, Paris, 2017.

[32] T. Gustafsson – "L^p-estimates for the nonlinear spatially homogeneous Boltzmann equation", *Arch. Rational Mech. Anal.* **92** (1986), no. 1, p. 23–57.

[33] ———, "Global L^p-properties for the spatially homogeneous Boltzmann equation", *Arch. Rational Mech. Anal.* **103** (1988), no. 1, p. 1–38.

[34] C. Henderson & S. Snelson – "C^∞ smoothing for weak solutions of the inhomogeneous Landau equation", 2017, `arXiv:1707.05710`.

[35] C. Henderson, S. Snelson & A. Tarfulea – "Local existence, lower mass bounds, and smoothing for the Landau equation", *J. Differential Equations* **266** (2019), no. 2-3, p. 1536–1577.

[36] Z. Huo, Y. Morimoto, S. Ukai & T. Yang – "Regularity of solutions for spatially homogeneous Boltzmann equation without angular cutoff", *Kinet. and Relat. Mod.* **1** (2008), no. 3, p. 453–489.

[37] E. Ikenberry & C. Truesdell – "On the pressures and the flux of energy in a gas according to Maxwell's kinetic theory. I", *J. Rational Mech. Anal.* **5** (1956), p. 1–54.

[38] C. Imbert & L. Silvestre – "Weak Harnack inequality for the Boltzmann equation without cut-off", 2017, to appear in *J. Eur. Math. Soc. (JEMS)*, `HAL:hal-01357047`.

[39] ———, "The Schauder estimate for kinetic integral equations", 2018, `arXiv:1812.11870`.

[40] X. Lu – "Conservation of energy, entropy identity, and local stability for the spatially homogeneous Boltzmann equation", *J. Statist. Phys.* **96** (1999), no. 3-4, p. 765–796.

[41] X. Lu & C. Mouhot – "On measure solutions of the Boltzmann equation, part I: moment production and stability estimates", *J. Differential Equations* **252** (2012), no. 4, p. 3305–3363.

[42] J. C. Maxwell – "On the dynamical theory of gases", *J. Philos. Trans. Roy. Soc. London* **157** (1867), p. 49–88.

[43] S. Mischler & C. Mouhot – "Cooling process for inelastic Boltzmann equations for hard spheres. II. Self-similar solutions and tail behavior", *J. Statist. Phys.* **124** (2006), no. 2-4, p. 703–746.

[44] S. Mischler & B. Wennberg – "On the spatially homogeneous Boltzmann equation", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **16** (1999), no. 4, p. 467–501.

[45] Y. Morimoto, S. Ukai, C.-J. Xu & T. Yang – "Regularity of solutions to the spatially homogeneous Boltzmann equation without angular cutoff", *Discrete Contin. Dynam. Systems* **24** (2009), no. 1, p. 187–212.

[46] Y. Morimoto & T. Yang – "Local existence of polynomial decay solutions to the Boltzmann equation for soft potentials", *Anal. Appl. (Singap.)* **13** (2015), no. 6, p. 663–683.

[47] C. Mouhot – "Quantitative lower bounds for the full Boltzmann equation. I. Periodic boundary conditions", *Comm. Partial Differential Equations* **30** (2005), no. 4-6, p. 881–917.

[48] A. J. Povzner – "On the Boltzmann equation in the kinetic theory of gases", *Mat. Sb. (N.S.)* **58 (100)** (1962), p. 65–86.

[49] L. Silvestre – "A new regularization mechanism for the Boltzmann equation without cut-off", *Comm. Math. Phys.* **348** (2016), no. 1, p. 69–100.

[50] ———, "Upper bounds for parabolic equations and the Landau equation", *J. Differential Equations* **262** (2017), no. 3, p. 3034–3055.

[51] C. Truesdell – "On the pressures and the flux of energy in a gas according to Maxwell's kinetic theory. II", *J. Rational Mech. Anal.* **5** (1956), p. 55–128.

[52] C. Villani – "A review of mathematical topics in collisional kinetic theory", in *Handbook of mathematical fluid dynamics, Vol. I*, North-Holland, Amsterdam, 2002, p. 71–305.

[53] B. Wennberg – "The Povzner inequality and moments in the Boltzmann equation", *Rend. Circ. Mat. Palermo (2) Suppl.* **45** (1996), p. 673–681, Proceedings of the VIII Int. Conf. on Waves and Stability in Continuous Media, Part II (Palermo, 1995).

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Cyril Imbert, CNRS & Department of Mathematics and Applications, École Normale Supérieure (Paris)
45 rue d'Ulm, 75005 Paris, France
E-mail : Cyril.Imbert@ens.fr
Url : https://cyril-imbert.blog/

Clément Mouhot, University of Cambridge, DPMMS, Centre for Mathematical Sciences,
Wilberforce road, Cambridge CB3 0WA, UK
E-mail : C.Mouhot@dpmms.cam.ac.uk
Url : https://www.dpmms.cam.ac.uk/~cm612/

Luis Silvestre, Mathematics Department, University of Chicago,
Chicago, Illinois 60637, USA
E-mail : luis@math.uchicago.edu
Url : http://math.uchicago.edu/~luis/