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HOMOGENIZATION OF WEAKLY COERCIVE INTEGRAL FUNCTIONALS IN THREE-DIMENSIONAL LINEAR ELASTICITY

by Marc Briane & Antonio Jesús Pallares Martín

Abstract. — This paper deals with the homogenization through Γ-convergence of weakly coercive integral energies, with the oscillating density $\mathbb{L}(x/\varepsilon)\nabla v{:}\nabla v$, in three-dimensional elasticity. The energies are weakly coercive in the sense where the classical functional coercivity satisfied by the periodic tensor $\mathbb{L}$:

$$\int_{\mathbb{R}^3} \mathbb{L}(y)\nabla v{:}\nabla v\,dy \geqslant \Lambda(\mathbb{L}) \int_{\mathbb{R}^3} |\nabla v|^2\,dy,$$

for any smooth function v with compact support in $\mathbb{R}^3$, with $\Lambda(\mathbb{L}) > 0$, is replaced by the relaxed condition $\Lambda(\mathbb{L}) \geqslant 0$. We prove that the homogenized tensor $\mathbb{L}^0$ remains strongly elliptic, or equivalently $\Lambda(\mathbb{L}^0) > 0$, for any tensor $\mathbb{L} = \mathbb{L}(y_1)$ satisfying the pointwise inequality:

$$\mathbb{L}(y)M{:}M + D{:}\mathrm{Cof}(M) \geqslant 0, \quad \text{a.e. } y \in \mathbb{R}^3,\ \forall\, M \in \mathbb{R}^{3\times 3},$$

adding a quadratic null-Lagrangian for some matrix $D \in \mathbb{R}^{3\times 3}$, and assuming the periodic functional coercivity $\Lambda_{\rm per}(\mathbb{L}) > 0$ (using smooth test functions v with periodic gradients). However, we derive rigorously the loss of strong ellipticity for the homogenized tensor, which is based on a Γ-convergence result under the sole assumption $\Lambda(\mathbb{L}) \geqslant 0$, and on a rank-two lamination.

Résumé (Homogénéisation de fonctionnelles intégrales faiblement coercives en élasticité linéaire tridimensionnelle)

Dans cet article on étudie la Γ-convergence d'énergies intégrales faiblement coercives, de densité oscillante $\mathbb{L}(x/\varepsilon)\nabla v{:}\nabla v$, en élasticité tridimensionnelle. Les énergies sont faiblement coercives du fait que la coercivité fonctionnelle classique satisfaite par le tenseur périodique $\mathbb{L}$:

$$\int_{\mathbb{R}^3} \mathbb{L}(y)\nabla v{:}\nabla v\,dy \geqslant \Lambda(\mathbb{L}) \int_{\mathbb{R}^3} |\nabla v|^2\,dy,$$

pour toute fonction régulière v à support compact dans $\mathbb{R}^3$, avec $\Lambda(\mathbb{L}) > 0$, est remplacée par la condition relaxée $\Lambda(\mathbb{L}) \geqslant 0$. On montre que le tenseur homogénéisé $\mathbb{L}^0$ reste fortement elliptique ou, de manière équivalente, $\Lambda(\mathbb{L}^0) > 0$, pour tout tenseur $\mathbb{L} = \mathbb{L}(y_1)$ vérifiant l'inégalité ponctuelle :

$$\mathbb{L}(y)M{:}M + D{:}\mathrm{Cof}(M) \geqslant 0, \quad \text{p.p. } y \in \mathbb{R}^3,\ \forall\, M \in \mathbb{R}^{3\times 3},$$

par l'addition d'un lagrangien nul pour une matrice $D \in \mathbb{R}^{3\times 3}$ donnée, et en supposant la coercivité fonctionnelle périodique $\Lambda_{\rm per}(\mathbb{L}) > 0$ (obtenue avec des fonctions test v de gradient périodique). Cependant, on obtient une perte d'ellipticité du tenseur homogénéisé, fondée sur un résultat de Γ-convergence sous la seule hypothèse $\Lambda(\mathbb{L}) \geqslant 0$, et sur une lamination de rang 2.

Mathematical subject classification (2010). — 35B27, 74B05, 74Q15.

Keywords. — Linear elasticity, ellipticity, homogenization, Γ-convergence, lamination.

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Contents

1. Introduction

In this paper, for a bounded domain Ω of $\mathbb{R}^3$ and for a periodic symmetric tensor-valued function $\mathbb{L} = \mathbb{L}(y)$, we study the homogenization of the elasticity energy

$$v \in H^1_0(\Omega;\mathbb{R}^3) \longmapsto \int_\Omega \mathbb{L}(x/\varepsilon)\nabla v \cdot \nabla v \, dx \quad \text{as } \varepsilon \longrightarrow 0, \tag{1.1}$$

especially when the tensor $\mathbb{L}$ is weakly coercive (see below). It is shown in [12, 5] that for any periodic symmetric tensor-valued function $\mathbb{L} = \mathbb{L}(y)$ satisfying the functional coercivity, i.e.,

$$\Lambda(\mathbb{L}) := \inf\left\{\int_{\mathbb{R}^3} \mathbb{L}\nabla v{:}\nabla v \, dy,\ v \in C^\infty_c(\mathbb{R}^3;\mathbb{R}^3),\ \int_{\mathbb{R}^3} |\nabla v|^2 \, dy = 1\right\} > 0, \tag{1.2}$$

and for any $f \in H^{-1}(\Omega;\mathbb{R}^3)$, the elasticity system

$$\begin{cases} -\operatorname{div}\big(\mathbb{L}(x/\varepsilon)\nabla u^\varepsilon\big) = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

H-converges as $\varepsilon \to 0$ in the sense of Murat-Tartar [10] to the elasticity system with the so-called homogenized tensor $\mathbb{L}^0$ defined by

$$\mathbb{L}^0 M{:}M := \inf\left\{\int_{Y_3} \mathbb{L}(M + \nabla v){:}(M + \nabla v) \, dy,\ v \in H^1_{\rm per}(Y_3;\mathbb{R}^3)\right\} \quad \text{for } M \in \mathbb{R}^{3\times 3}. \tag{1.4}$$

Equivalently, under the functional coercivity (1.2) the energy (1.1) Γ-converges for the weak topology of $H^1_0(\Omega;\mathbb{R}^3)$ (see Definition 1.2) to the functional

$$v \in H^1_0(\Omega;\mathbb{R}^3) \longmapsto \int_\Omega \mathbb{L}^0\nabla v{:}\nabla v \, dx. \tag{1.5}$$

The functional coercivity (1.2), which is a nonlocal condition satisfied by the symmetric tensor $\mathbb{L}$, is implied by the very strong ellipticity, i.e., the local condition

$$\alpha_{\rm vse}(\mathbb{L}) := \operatorname*{ess\text{-}inf}_{y\in\mathbb{R}^3}\big(\min\{\mathbb{L}(y)M{:}M,\ M \in \mathbb{R}^{3\times 3}_s,\ |M| = 1\}\big) > 0, \tag{1.6}$$

and the converse is not true in general. Moreover, condition (1.2) implies strong ellipticity, i.e.,

$$\alpha_{\rm se}(\mathbb{L}) := \operatorname*{ess\text{-}inf}_{y\in\mathbb{R}^3}\big(\min\{\mathbb{L}(y)(a\otimes b){:}(a\otimes b),\ a,b \in \mathbb{R}^3,\ |a| = |b| = 1\}\big) > 0, \tag{1.7}$$

but contrary to the scalar case, the converse is not true in general.

Here, we focus on the case where the tensor $\mathbb{L}$ is weakly coercive, i.e., relaxing the condition $\Lambda(\mathbb{L}) > 0$ by $\Lambda(\mathbb{L}) \geqslant 0$. In this case the homogenization of the elasticity system (1.3) associated with the energy (1.1) is badly posed in general, since one has no a priori L^2-bound on the stress tensor ∇u^ε (assuming the existence of a solution u^ε to the elasticity system (1.3)) due to the loss of coercivity. However, it was shown by Geymonat et al. [7] that the previous Γ-convergence result still holds when $\Lambda(\mathbb{L}) \geqslant 0$, under the extra condition of periodic functional coercivity, i.e.,

$$\Lambda_{\rm per}(\mathbb{L}) := \inf \left\{ \int_{Y_3} \mathbb{L}\nabla v{:}\nabla v \, dy, \ v \in H^1_{\rm per}(Y_3;\mathbb{R}^3), \ \int_{Y_3} |\nabla v|^2 \, dy = 1 \right\} > 0. \tag{1.8}$$

Furthermore, using the Murat-Tartar 1*-convergence for tensors which depend only on one direction[(1)] (see [9] in the conductivity case, see [8, §3] and [2, Lem. 3.1] in the elasticity case), Gutiérrez [8, Prop. 1] derived in two and three dimensions a 1-periodic rank-one laminate with two isotropic phases whose tensor is

$$\mathbb{L}_1(y_1) = \chi(y_1)\,\mathbb{L}_a + \big(1 - \chi(y_1)\big)\,\mathbb{L}_b \quad \text{for } y_1 \in \mathbb{R}, \tag{1.9}$$

which is strongly elliptic, i.e., $\alpha_{\rm se}(\mathbb{L}_1) > 0$, and only weakly coercive, i.e., $\Lambda(\mathbb{L}_1) \geqslant 0$, but such that the homogenized tensor $\mathbb{L}_1^*$ induced by 1*-convergence, which is shown to agree with the Γ-limit $\mathbb{L}_1^0$ of formula (1.4) (see the proof of Theorem 3.9, step 4), is not strongly elliptic, i.e., $\alpha_{\rm se}(\mathbb{L}_1^*) = 0$. However, the 1*-convergence process used by Gutiérrez in [8] needs to have a priori L^2-bounds for the sequence of deformations, a property which is not compatible with the weak coercivity assumption. Therefore, Gutiérrez' approach is not a H-convergence process applied to the elasticity system (1.3). Francfort and the first author [2] obtained in dimension two a similar loss of ellipticity through a homogenization process using the Γ-convergence approach of [7] from a more generic (with respect to (1.9)) 1-periodic isotropic tensor $\mathbb{L} = \mathbb{L}(y_1)$ satisfying

$$\Lambda(\mathbb{L}) = 0, \quad \Lambda_{\rm per}(\mathbb{L}) > 0 \quad \text{and} \quad \alpha_{\rm se}(\mathbb{L}^0) = 0. \tag{1.10}$$

They also showed that Gutiérrez' lamination is the only one among rank-one laminates which implies such a loss of strong ellipticity.

The aim of the paper is to extend the result of [2] to dimension three, namely justifying the loss of ellipticity of [8] by a homogenization process. The natural idea is to find as in [2] a 1-periodic isotropic tensor $\mathbb{L} = \mathbb{L}(y_1)$ satisfying (1.10). Firstly, in order to check the relaxed functional coercivity $\Lambda(\mathbb{L}) \geqslant 0$, we apply the translation method used in [2], which consists in adding to the elastic energy density a suitable null Lagrangian such that the following pointwise inequality holds for some matrix $D \in \mathbb{R}^{3\times3}$:

$$\mathbb{L}M{:}M + D{:}\,\mathrm{Cof}(M) \geqslant 0, \quad \forall\, M \in \mathbb{R}^{3\times3}. \tag{1.11}$$

(1) Recall that the 1*-convergence enables us to obtain the homogenized tensor $\mathbb{L}^0$ from a coercive local tensor $\mathbb{L} = \mathbb{L}(y_1)$ thanks to a cascade of weak convergences based on the fact that there are no oscillations with respect to the variables $y_2, \ldots, y_d$.

Note that in dimension two the translation method reduces to adding the term $d\det(M)$ with one coefficient d, rather than a (3×3)-matrix D in dimension three. But surprisingly, and contrary to the two-dimensional case of [2], we prove (see Theorem 3.3) that for any 1-periodic tensor $\mathbb{L}=\mathbb{L}(y_1)$, condition (1.11) combined with $\Lambda_{\rm per}(\mathbb{L})>0$ actually implies that $\alpha_{\rm se}(\mathbb{L}^0)>0$, making impossible the loss of ellipticity through homogenization. This specificity was already observed by Gutiérrez [8] in the particular case of isotropic two-phase rank-one laminates (1.9), where certain regimes satisfied by the Lamé coefficients of the isotropic phases $\mathbb{L}_a,\mathbb{L}_b$ are not compatible with the desired equality $\alpha_{\rm se}(\mathbb{L}^0)=0$.

To overcome this difficulty Gutiérrez [8] considered a rank-two laminate obtained by mixing in the direction y_2 the homogenized tensor $\mathbb{L}_1^*$, in the sense of 1^*-convergence (see above), of the local tensor $\mathbb{L}_1(y_1)$ defined by (1.9), with a very strongly elliptic isotropic tensor $\mathbb{L}_c$. In the present context we derive a similar loss of ellipticity by rank-two lamination, but justifying it through homogenization still using a Γ-convergence procedure (see Theorem 3.9). However, the proof is rather delicate, since we have to choose the isotropic materials a,b,c so that the 1-periodic rank-one laminate tensor $\mathbb{L}_2$ in the direction y_2 obtained after the first rank-one lamination of $\mathbb{L}_a,\mathbb{L}_b$ in the direction y_1, namely

$$\mathbb{L}_2(y_2)=\chi_2(y_2)\,\mathbb{L}_1^*+\big(1-\chi_2(y_2)\big)\,\mathbb{L}_c\quad\text{for } y_2\in\mathbb{R},\tag{1.12}$$

satisfies

$$\Lambda(\mathbb{L}_2)\geqslant 0\quad\text{and}\quad\alpha_{\rm se}(\mathbb{L}_2^0)=0,\tag{1.13}$$

where $\mathbb{L}_2^0$ is the homogenized tensor defined by formula (1.4) with $\mathbb{L}=\mathbb{L}_2$. Moreover, the condition $\Lambda(\mathbb{L}_2)\geqslant 0$ without $\Lambda_{\rm per}(\mathbb{L}_2)>0$ (which seems very intricate to check) forces us to extend the Γ-convergence result of [7, Th. 3.1(i)]. However, Braides and the first author have proved (see Theorem 2.4) that the Γ-convergence result for the energy (1.1) holds true under the sole condition $\Lambda(\mathbb{L})\geqslant 0$.

The paper is divided in two sections. In the first section we prove the Γ-convergence result for (1.1) under the assumption $\Lambda(\mathbb{L})\geqslant 0$, and without the condition $\Lambda_{\rm per}(\mathbb{L})>0$. The second section is devoted to the main results of the paper: In Section 3.1 we prove the strong ellipticity of the homogenized tensor $\mathbb{L}^0$ for any isotropic tensor $\mathbb{L}=\mathbb{L}(y_1)$ satisfying both the two conditions (1.11) (which implies $\Lambda(\mathbb{L})\geqslant 0$) and $\Lambda_{\rm per}(\mathbb{L})>0$. In Section 3.2 we show the loss ellipticity by homogenization using a suitable rank-two laminate tensor $\mathbb{L}_2$ of type (1.12), and the Γ-convergence result under the sole condition $\Lambda(\mathbb{L}_2)\geqslant 0$. Finally, the Appendix is devoted to the proof of Theorem 2.2 which provides sufficient conditions on a periodic tensor $\mathbb{L}$ for satisfying the periodic coercivity $\Lambda_{\rm per}(\mathbb{L})>0$.

Notation

- The space dimension is denoted by $N\geqslant 2$, but most of the time it will be $N=3$.
- $\mathbb{R}_s^{N\times N}$ denotes the set of the symmetric matrices in $\mathbb{R}^{N\times N}$.
- I_N denotes the identity matrix of $\mathbb{R}^{N\times N}$.

• For any $M \in \mathbb{R}^{N\times N}$, M^T denotes the transposed of M, and M^s denotes the symmetrized matrix of M.

• : denotes the Frobenius inner product in $\mathbb{R}^{N\times N}$, i.e., $M{:}M' := \mathrm{tr}(M^T M')$ for $M, M' \in \mathbb{R}^{N\times N}$.

• $\mathscr{L}_s(\mathbb{R}^{N\times N})$ denotes the space of the symmetric tensors $\mathbb{L}$ on $\mathbb{R}^{N\times N}$ satisfying

$$\mathbb{L}M = \mathbb{L}M^s \in \mathbb{R}^{N\times N}_s \quad \text{and} \quad \mathbb{L}M{:}M' = \mathbb{L}M'{:}M, \quad \forall\, M, M' \in \mathbb{R}^{N\times N}_s.$$

In terms of the entries $\mathbb{L}_{ijkl}$ of $\mathbb{L}$, this is equivalent to $\mathbb{L}_{ijkl} = \mathbb{L}_{jikl} = \mathbb{L}_{klij}$ for any $i, j, k, l \in \{1, \dots, N\}$.

• $\mathbb{I}_s$ denotes the unit tensor of $\mathscr{L}_s(\mathbb{R}^{N\times N})$ defined by $\mathbb{I}_s M := M^s$ for $M \in \mathbb{R}^{N\times N}$.

• M_{ij} denotes the (i, j) entry of the matrix $M \in \mathbb{R}^{N\times N}$.

• $\widetilde{M^{ij}}$ denotes the $(N-1)\times(N-1)$-matrix resulting from deleting the i-th row and the j-th column of the matrix $M \in \mathbb{R}^{N\times N}$ for $i, j \in \{1, \dots, N\}$.

• $\mathrm{Cof}(M)$ denotes the cofactors matrix of $M \in \mathbb{R}^{N\times N}$, i.e., the matrix with entries $(\mathrm{Cof}M)_{ij} = (-1)^{i+j} \det(\widetilde{M^{ij}})$ for $i, j \in \{1, \dots, N\}$.

• $\mathrm{adj}(M)$ denotes the adjugate matrix of $M \in \mathbb{R}^{N\times N}$, i.e., $\mathrm{adj}(M) = (\mathrm{Cof}\, M)^T$.

• $Y_N := [0, 1)^N$ denotes the unit cube of $\mathbb{R}^N$.

Let $\mathbb{L} \in L^\infty_{\mathrm{per}}\big(Y_N; \mathscr{L}_s(\mathbb{R}^{N\times N})\big)$ be a Y_N-periodic symmetric tensor-valued function. In the whole paper we will use the following ellipticity constants related to the tensor $\mathbb{L}$ (see [7, §3] for further details):

• $\alpha_{\mathrm{se}}(\mathbb{L})$ denotes the best ellipticity constant for $\mathbb{L}$, i.e.,

$$\alpha_{\mathrm{se}}(\mathbb{L}) := \operatorname*{ess\,inf}_{y\in Y_N}\big(\min\{\mathbb{L}(y)(a\otimes b){:}(a\otimes b),\ a, b \in \mathbb{R}^N,\ |a| = |b| = 1\}\big).$$

• $\alpha_{\mathrm{vse}}(\mathbb{L})$ denotes the best constant of very strong ellipticity of $\mathbb{L}$, i.e.,

$$\alpha_{\mathrm{vse}}(\mathbb{L}) := \operatorname*{ess\,inf}_{y\in Y_N}\big(\min\{\mathbb{L}(y)M{:}M,\ M \in \mathbb{R}^{N\times N}_s, |M| = 1\}\big).$$

• $\Lambda(\mathbb{L})$ denotes the global functional coercivity constant for $\mathbb{L}$, i.e.,

$$\Lambda(\mathbb{L}) := \inf\left\{\int_{\mathbb{R}^N} \mathbb{L}\nabla v{:}\nabla v\, dy,\ v \in C^\infty_c(\mathbb{R}^N; \mathbb{R}^N),\ \int_{\mathbb{R}^N} |\nabla v|^2\, dy = 1\right\}.$$

• $\Lambda_{\mathrm{per}}(\mathbb{L})$ denotes the functional coercivity constant of $\mathbb{L}$ with respect to Y_N-periodic deformations, i.e.,

$$\Lambda_{\mathrm{per}}(\mathbb{L}) := \inf\left\{\int_{Y_N} \mathbb{L}\nabla v{:}\nabla v\, dy,\ v \in H^1_{\mathrm{per}}(Y_N; \mathbb{R}^N),\ \int_{Y_N} |\nabla v|^2\, dy = 1\right\}.$$

Remark 1.1

• The very strong ellipticity implies the strong ellipticity, i.e., for any tensor $\mathbb{L}$,

$$\alpha_{\mathrm{vse}}(\mathbb{L}) > 0 \implies \alpha_{\mathrm{se}}(\mathbb{L}) > 0.$$

• According to [7, Th. 3.3(i)], if $\alpha_{\mathrm{se}}(\mathbb{L}) > 0$, then the following inequalities hold:

$$\Lambda(\mathbb{L}) \leqslant \Lambda_{\mathrm{per}}(\mathbb{L}) \leqslant \alpha_{\mathrm{se}}(\mathbb{L}). \tag{1.14}$$

• Using a Fourier transform we get that for any constant tensor $\mathbb{L}_0$,

$$\alpha_{\mathrm{se}}(\mathbb{L}_0) > 0 \iff \Lambda(\mathbb{L}_0) > 0.$$

In the sequel will always assume the strong ellipticity of the tensor $\mathbb{L}$, i.e., $\alpha_{\rm se}(\mathbb{L}) > 0$.

We conclude this section with the following sequential definition of Γ-convergence (see, e.g. [4, 1]):

Definition 1.2. — Let X be a reflexive and separable Banach space endowed with the weak topology $\sigma(X, X')$, and let $\mathscr{F}^\varepsilon{:}X \to \mathbb{R}$ be a ε-indexed sequence of functionals. The sequence $\mathscr{F}^\varepsilon$ is said to Γ-converge to the functional $\mathscr{F}^0{:}X \to \mathbb{R}$ for the weak topology of X, and we denote $\mathscr{F}^\varepsilon \xrightarrow{\Gamma\text{-}X} \mathscr{F}^0$, if for any $u \in X$,

- $\forall\, u_\varepsilon \longrightarrow u$, $\mathscr{F}^0(u) \leqslant \liminf_{\varepsilon\to 0} \mathscr{F}^\varepsilon(u_\varepsilon)$,
- $\exists\, \overline{u}_\varepsilon \longrightarrow u$, $\mathscr{F}^0(u) = \lim_{\varepsilon\to 0} \mathscr{F}^\varepsilon(\overline{u}_\varepsilon)$.

Such a sequence $\overline{u}_\varepsilon$ is called a recovery sequence.

Note that the weak topology of X is metrizable on closed balls of X. This will be used in the proof of Theorem 2.4 below.

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2. The Γ-convergence results

It is stated in [12, Ch. 6, §11] that the first homogenization result in linear elasticity can be found in Duvaut's work (unavailable reference). It claims that if the periodic tensor $\mathbb{L}$ is very strongly elliptic, i.e., $\alpha_{\rm vse}(\mathbb{L}) > 0$, then the solution $u^\varepsilon \in H^1_0(\Omega;\mathbb{R}^3)$ to the elasticity system (1.3) satisfies

$$\left\{\begin{array}{l} u^\varepsilon \longrightarrow u \quad \text{weakly in } H^1_0(\Omega;\mathbb{R}^3), \\ \mathbb{L}(x/\varepsilon)\nabla u^\varepsilon \longrightarrow \mathbb{L}^0\nabla u \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{3\times 3}), \\ -\operatorname{div}(\mathbb{L}^0\nabla u) = f, \end{array}\right. \tag{2.1}$$

for some $f \in H^{-1}(\Omega;\mathbb{R}^3)$, where $\mathbb{L}^0$ is given by

$$\mathbb{L}^0 M{:}M := \inf\left\{\int_{Y_3} \mathbb{L}(M+\nabla v){:}(M+\nabla v)\,dy,\ v \in H^1_{\rm per}(Y_3;\mathbb{R}^3)\right\} \quad \text{for } M \in \mathbb{R}^{3\times 3}, \tag{2.2}$$

which is attained when $\Lambda_{\rm per}(\mathbb{L}) > 0$. The previous homogenization result actually holds under the weaker assumption of functional coercivity, i.e., $\Lambda(\mathbb{L}) > 0$, as shown in [5].

Otherwise, from the point of view of the elastic energy consider the functionals

$$\mathscr{F}^\varepsilon(v) := \int_\Omega \mathbb{L}(x/\varepsilon)\nabla v{:}\nabla v\,dx, \tag{2.3}$$

$$\mathscr{F}^0(v) := \int_\Omega \mathbb{L}^0\nabla v{:}\nabla v\,dx \quad \text{for } v \in H^1(\Omega,\mathbb{R}^3). \tag{2.4}$$

Then, the following homogenization result [7, Th. 3.4(i)] through the Γ-convergence of energy (2.3), allows us to relax the very strong ellipticity of $\mathbb{L}$.

Theorem 2.1 (Geymonat et al. [7]). — *Under the conditions*

$$\Lambda(\mathbb{L}) \geqslant 0 \quad \textit{and} \quad \Lambda_{\rm per}(\mathbb{L}) > 0,$$

one has

$$\mathscr{F}^\varepsilon \xrightarrow{\Gamma\text{-}H^1_0(\Omega;\mathbb{R}^3)} \mathscr{F}^0,$$

for the weak topology of $H^1_0(\Omega;\mathbb{R}^3)$*, where* $\mathbb{L}^0$ *is given by* (2.2).

2.1. Generic examples of tensors satisfying $\Lambda(\mathbb{L}) \geqslant 0$ and $\Lambda_{\rm per}(\mathbb{L}) > 0$

Reference [2] provides a class of isotropic strongly elliptic tensors for which Theorem 2.1 applies. However, this work is restricted to dimension two. We are going to extend the result [2, Th. 2.2] to dimension three.

Let us assume that there exist $p \in \mathbb{N}$ phases Z_i, $i = 1,\dots,p$ satisfying

$$\begin{cases} Z_i \text{ is open, connected and Lipschitz for any } i \in \{1,\dots,p\}, \\ Z_i \cap Z_j = \varnothing \quad \forall\, i \neq j \in \{1,\dots,p\}, \\ \overline{Y}_3 = \bigcup_{i=1}^p \overline{Z}_i, \end{cases} \tag{2.5}$$

such that the tensor $\mathbb{L}$ satisfies

$$\begin{cases} \mathbb{L}(y)M = \lambda(y)\,\mathrm{tr}(M)I_3 + 2\mu(y)M, \quad \forall\, y \in Y_3,\ \forall\, M \in \mathbb{R}^{3\times3}_s, \\ \lambda(y) = \lambda_i,\ \mu(y) = \mu_i \text{ in } Z_i, \quad \forall\, i \in \{1,\dots,p\}, \\ \mu_i > 0,\ 2\mu_i + \lambda_i > 0, \quad \forall\, i \in \{1,\dots,p\}. \end{cases} \tag{2.6}$$

We further assume the existence of $d > 0$ such that

$$-\min_{i=1,\dots,p}\{2\mu_i + 3\lambda_i\} \leqslant d \leqslant 4\min_{i=1,\dots,p}\{\mu_i\}. \tag{2.7}$$

Now, we define the following subsets of indexes

$$\begin{cases} I := \{i \in \{1,\dots,p\}\colon d = 4\mu_i\}, \\ J := \{j \in \{1,\dots,p\}\colon 2\mu_j + 3\lambda_j = -d\}, \\ K := \{1,\dots,p\} \setminus (I \cup J). \end{cases} \tag{2.8}$$

Note that the three previous sets are disjoint. This is true, since we have $4\mu_i > -(2\mu_i + 3\lambda_i)$ due to $2\mu_i + \lambda_i > 0$.

In this framework, we are able to prove the following theorem which is an easy extension of the two-dimensional result of [2, Th. 2.2]. For the reader convenience the proof is given in the Appendix.

Theorem 2.2. — *Let $\mathbb{L}$ be the tensor defined by* (2.6) *and* (2.7)*. Then we have $\Lambda(\mathbb{L}) \geqslant 0$. We also have $\Lambda_{\rm per}(\mathbb{L}) > 0$ provided that one of the two following conditions is fulfilled by the sets defined in* (2.8)*:*

Case 1. — For each $j \in J$, there exist intervals $(a_j^-, a_j^+), (b_j^-, b_j^+) \subset [0,1]$ such that

$$(a_j^-, a_j^+) \times (b_j^-, b_j^+) \times \{0,1\} \subset \partial Z_j, \quad \text{or}$$
$$(a_j^-, a_j^+) \times \{0,1\} \times (b_j^-, b_j^+) \subset \partial Z_j, \quad \text{or}$$
$$\{0,1\} \times (a_j^-, a_j^+) \times (b_j^-, b_j^+) \subset \partial Z_j.$$

Case 2. — For each $j \in J$, there exists $k \in K$ with $\mathscr{H}^2(\partial Z_j \cap \partial Z_k) > 0$, where $\mathscr{H}^2$ denotes the 2-dimensional Hausdorff measure.

Remark 2.3. — Recently, Francfort and Gloria have proved in [6, Th. 2.1] that in dimension two a two-phase periodic isotropic tensor $\mathbb{L}(y)$ with Lamé's coefficients $\lambda(y)$, $\mu(y)$ composed of inclusions of a very strong elliptic material 1, i.e., with $\mu_1 > 0$, $\lambda_1 + \mu_1 > 0$, imbedded in a strongly (but not very strongly) elliptic connected material 2, i.e., with $\mu_2 > 0$, $\lambda_2 + 2\mu_2 > 0$, such that $\mu_1 = -(\lambda_2 + \mu_2)$, satisfies $\Lambda(\mathbb{L}) \geqslant 0$ and $\Lambda_{\rm per}(\mathbb{L}) > 0$.

Using similar arguments as the ones of the proofs of [2, Th. 2.2] and Theorem 2.2, we can check that the two-dimensional result [6, Th. 2.1] can be extended in dimension three to a two-phase periodic isotropic tensor composed a very strong elliptic material 1 (see Remark 3.2 below) imbedded in a strongly (but not very strongly) elliptic connected material 2, satisfying

$$\mu_1 > 0,\ 3\lambda_1 + 2\mu_1 > 0, \quad \mu_2 > 0,\ \lambda_2 + 2\mu_2 > 0 \quad \text{and} \quad 4\mu_1 = -(3\lambda_2 + 2\mu_2).$$

2.2. Relaxation of condition $\Lambda_{\rm per}(\mathbb{L}) > 0$. — According to Theorem 2.1 the Γ-convergence of the functional (2.3) holds true if both $\Lambda(\mathbb{L}) \geqslant 0$ and $\Lambda_{\rm per}(\mathbb{L}) > 0$. However, the following theorem due to Braides and the first author shows that in N-dimensional elasticity for $N \geqslant 2$, the Γ-convergence result still holds under the sole assumption $\Lambda(\mathbb{L}) \geqslant 0$.

Theorem 2.4 (Braides & Briane). — *Let Ω be a bounded open subset of $\mathbb{R}^N$, and let $\mathbb{L}$ be a bounded Y_N-periodic symmetric tensor-valued function in $L^\infty_{\rm per}\big(Y_N; \mathscr{L}_s(\mathbb{R}^{N\times N})\big)$ such that*

$$\Lambda(\mathbb{L}) \geqslant 0. \tag{2.9}$$

Then, we have

$$\mathscr{F}^\varepsilon \xrightarrow{\Gamma\text{-}H_0^1(\Omega;\mathbb{R}^N)} \mathscr{F}^0, \tag{2.10}$$

for the weak topology of $H_0^1(\Omega;\mathbb{R}^N)$ in the sense of Definition 1.2*, where $\mathscr{F}^0$ is given by* (2.4) *with the tensor $\mathbb{L}^0$ defined by* (2.2)*.*

Proof. — For $\delta > 0$, set $\mathbb{L}_\delta := \mathbb{L} + \delta\,\mathbb{I}_s$ where $\mathbb{I}_s$ is the unit symmetric tensor, and let $\mathscr{F}^\varepsilon_\delta$ be the functional defined by (2.3) with $\mathbb{L}_\delta$ in place of $\mathbb{L}$. We claim that

$$\Lambda(\mathbb{L}_\delta) > 0. \tag{2.11}$$

To prove it consider $v \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$ and take $R > 0$ such that $\operatorname{supp} v \subset B(0,R)$. Then, by (2.9) we have

$$\int_{\mathbb{R}^N} \mathbb{L}_\delta \nabla v{:}\nabla v \, dy = \int_{B(0,R)} \mathbb{L}\nabla v{:}\nabla v \, dy + \delta \int_{B(0,R)} \mathbb{I}_s \nabla v{:}\nabla v \, dy \geqslant \delta \int_{B(0,R)} |e(v)|^2 \, dy,$$

and since the constant of the Korn inequality is invariant by homothecy, there exists a constant $\alpha > 0$ independent of R such that $\Lambda(\mathbb{L}_\delta) \geqslant \delta\,\alpha > 0$.

Thanks to (2.11) we can apply Theorem 2.1 with the functional $\mathscr{F}_\delta^\varepsilon$. Hence, $\mathscr{F}_\delta^\varepsilon \xrightarrow{\ \Gamma\ } \mathscr{F}_\delta^0$ for the weak topology of $H_0^1(\Omega;\mathbb{R}^N)$, where

$$\mathscr{F}_\delta^0(u) := \int_\Omega \mathbb{L}_\delta^0 \nabla u{:}\nabla u \, dx \quad \text{for } u \in H_0^1(\Omega,\mathbb{R}^N),$$

and $\mathbb{L}_\delta^0$ is given by (2.2) with $\mathbb{L} = \mathbb{L}_\delta$.

On the one hand, since the weak topology of $H_0^1(\Omega;\mathbb{R}^N)$ is metrizable on the closed balls of $H_0^1(\Omega;\mathbb{R}^N)$, we can apply the compactness of Γ-convergence on these balls. Fix $k \in \mathbb{N}$, and denote d_k any metric inducing the $H_0^1(\Omega;\mathbb{R}^N)$-weak topology on the ball $B_k \subset H_0^1(\Omega;\mathbb{R}^N)$ centered on 0 and of radius k. Then, there exists a subsequence ε_j such that $\mathscr{F}^{\varepsilon_j}$ $\Gamma(d_k)$-converges to some functional $\mathscr{F}^{0,k}$ in B_k. Let $u \in B_k$, and let u_{ε_j} be a recovery sequence for $\mathscr{F}^{\varepsilon_j}$ which converges to u in (B_k, d_k). Since $\mathscr{F}^{\varepsilon_j} \leqslant \mathscr{F}_\delta^{\varepsilon_j}$ which Γ-converges to $\mathscr{F}_\delta^0$ in the sense of Definition 1.2, we have

$$\begin{aligned}
\mathscr{F}^{0,k}(u) = \big(\Gamma(d_k)\text{-}\lim \mathscr{F}^{\varepsilon_j}\big)(u) &\leqslant \mathscr{F}_\delta^0(u) \\
&\leqslant \liminf_{\varepsilon_j \to 0} \int_\Omega \mathbb{L}_\delta(x/\varepsilon_j)\nabla u_{\varepsilon_j}{:}\nabla u_{\varepsilon_j} \, dx \\
&\leqslant \liminf_{\varepsilon_j \to 0} \int_\Omega \mathbb{L}(x/\varepsilon_j)\nabla u_{\varepsilon_j}{:}\nabla u_{\varepsilon_j} \, dx + O(\delta) \\
&= \big(\Gamma(d_k)\text{-}\lim \mathscr{F}^{\varepsilon_j}\big)(u) + O(\delta) = \mathscr{F}^{0,k}(u) + O(\delta),
\end{aligned}$$

which implies that $\mathscr{F}_\delta^0(u)$ converges to $\mathscr{F}^{0,k}(u)$ as $\delta \to 0$. Hence, the $\Gamma(d_k)$-limit $\mathscr{F}^{0,k}$ of $\mathscr{F}^{\varepsilon_j}$ is equal to $\mathscr{F}^0 := \lim_{\delta\to 0} \mathscr{F}_\delta^0$ in B_k, and is thus independent both on k and on the subsequence ε_j. Then, repeating the previous argument, any subsequence of $\mathscr{F}^\varepsilon$ has a further subsequence which $\Gamma(d_k)$-converges to $\mathscr{F}^0$ in B_k. Therefore, thanks to the Urysohn property (see, e.g. [1, Proposition 1.44]) we can conclude that the whole sequence $\mathscr{F}^\varepsilon$ $\Gamma(d_k)$-converges to $\mathscr{F}^0 = \lim_{\delta\to 0} \mathscr{F}_\delta^0$ in B_k. Finally, since any sequence converging weakly in $H_0^1(\Omega;\mathbb{R}^N)$ belongs to some ball B_k as well as its limit, it follows that the Γ-lim inf and Γ-lim sup properties of Definition 1.2 hold true for the sequence $\mathscr{F}^\varepsilon$.

On the other hand, let $\mathbb{L}^0$ be given by (2.2). For $\eta > 0$ and for $M \in \mathbb{R}^{N\times N}$, consider a function φ_η in $H^1_{\mathrm{per}}(Y_N;\mathbb{R}^N)$ such that

$$\int_{Y_N} \mathbb{L}(y)(M + \nabla\varphi_\eta){:}(M + \nabla\varphi_\eta) \, dy \leqslant \mathbb{L}^0 M{:}M + \eta.$$

We then have

$$\begin{aligned}\mathbb{L}^0M{:}M &\leqslant \mathbb{L}^0_\delta M{:}M\\ &\leqslant \int_{Y_N} \mathbb{L}_\delta(y)(M+\nabla\varphi_\eta){:}(M+\nabla\varphi_\eta)\,dy\\ &\leqslant \int_{Y_N} \mathbb{L}(y)(M+\nabla\varphi_\eta){:}(M+\nabla\varphi_\eta)\,dy + O_\eta(\delta).\end{aligned}$$

Hence, making δ tend to 0 for a fixed η, we obtain

$$\begin{aligned}\mathbb{L}^0M{:}M &\leqslant \liminf_{\delta\to0}\ (\mathbb{L}^0_\delta M{:}M)\\ &\leqslant \limsup_{\delta\to0}\ (\mathbb{L}^0_\delta M{:}M)\\ &\leqslant \int_{Y_N} \mathbb{L}(y)(M+\nabla\varphi_\eta){:}(M+\nabla\varphi_\eta)\,dy\\ &\leqslant \mathbb{L}^0M{:}M+\eta.\end{aligned}$$

Due to the arbitrariness of η, we get that $\mathbb{L}^0_\delta$ converges to $\mathbb{L}^0$ as $\delta\to0$.

Therefore, by the Lebesgue dominated convergence theorem we conclude that for any $u\in H^1_0(\Omega;\mathbb{R}^N)$,

$$\mathscr{F}^0(u)=\lim_{\delta\to0}\mathscr{F}^0_\delta(u)=\lim_{\delta\to0}\int_\Omega \mathbb{L}^0_\delta\nabla u{:}\nabla u\,dx=\int_\Omega \mathbb{L}^0\nabla u{:}\nabla u\,dx. \qquad \square$$

3. Loss of ellipticity in three-dimensional linear elasticity through the homogenization of a laminate

In this section we will construct an example of a three-dimensional strong elliptic material $\mathbb{L}$ which is weakly coercive, i.e., $\Lambda(\mathbb{L})\geqslant 0$, but for which the strong ellipticity is lost through homogenization. Firstly, let us recall the following result due to Gutiérrez [8].

Proposition 3.1 (Gutiérrez [8]). — *For any strongly, but not semi-very strongly elliptic isotropic material, referred to as material a, i.e., whose tensor $\mathbb{L}_a$ satisfies $\alpha_{\rm vse}(\mathbb{L}_a)<0$, there are very strongly elliptic isotropic materials such that if we laminate them with material a, in appropriately chosen proportions and directions, we generate an effective elasticity tensor that is not strongly elliptic.*

Remark 3.2 (Isotropic tensors). — The elasticity tensor $\mathbb{L}\in L^\infty\big(Y_3;\mathscr{L}_s(\mathbb{R}^{3\times3})\big)$ of an isotropic material is given by

$$\mathbb{L}(y)M=\lambda(y)\,\mathrm{tr}(M)I_3+2\mu(y)M,\quad \text{for } y\in Y_3 \text{ and } M\in\mathbb{R}^{3\times3}_s,$$

where λ and μ are the Lamé coefficients of $\mathbb{L}$. As a consequence, we have

$$\alpha_{\rm se}(\mathbb{L})=\operatorname*{ess\text{-}inf}_{y\in Y_3}\big(\min\{\mu(y),2\mu(y)+\lambda(y)\}\big),$$

$$\alpha_{\rm vse}(\mathbb{L})=\operatorname*{ess\text{-}inf}_{y\in Y_3}\big(\min\{\mu(y),2\mu(y)+3\lambda(y)\}\big).$$

Here is a summary of the proof of Proposition 3.1. Consider two isotropic, homogeneous tensors $\mathbb{L}_a$ and $\mathbb{L}_b$ such that $\mathbb{L}_a$ is strongly elliptic, i.e.,

$$\lambda_a + 2\mu_a > 0, \quad \mu_a > 0,$$

but not semi-very strongly elliptic, i.e.,

$$3\lambda_a + 2\mu_a < 0.$$

and such that $\mathbb{L}_b$ is very strongly elliptic, i.e.,

$$3\lambda_b + 2\mu_b > 0, \quad \mu_b > 0.$$

Considering the rank-one laminate in the direction y_1 mixing $\mathbb{L}_a$ with volume fraction $\theta_1 \in (0,1)$ and $\mathbb{L}_b$ with volume fraction $(1-\theta_1)$, Gutiérrez [8] proved that the effective tensor $\mathbb{L}_1^*$ in the sense of Murat-Tartar 1*-convergence (see, e.g. [8, §3]) satisfies the following properties:

- If $0 \leqslant \mu_a + \lambda_a$, then
$$\alpha_{\rm se}(\mathbb{L}_1^*) > 0.$$
- If $-\mu_b \leqslant \mu_a + \lambda_a < 0$, then
$$\alpha_{\rm se}(\mathbb{L}_1^*) \begin{cases} = 0 & \text{if } \mu_b = -\mu_a - \lambda_a, \\ \geqslant 0 & \text{if } -\mu_a - \lambda_a < \mu_b \leqslant -\frac{1}{4}(2\mu_a + 3\lambda_a), \\ > 0 & \text{if } -\frac{1}{4}(2\mu_a + 3\lambda_a) < \mu_b. \end{cases}$$
- The case $\mu_a + \lambda_a < -\mu_b$ is disposed of, since $\mathbb{L}_1^*$ does not even satisfy the Legendre-Hadamard condition.

In the case where $\alpha_{\rm se}(\mathbb{L}_1^*) > 0$, Gutiérrez (see [8, §5.2]) performed a second lamination in the direction y_2 mixing the anisotropic material generated by the first lamination with volume fraction $\theta_2 \in (0,1)$, and a suitable very strongly elliptic isotropic material $(\mathbb{L}_c, \mu_c, \lambda_c)$ with volume fraction $(1-\theta_2)$. In this way he derived a rank-two laminate of effective tensor $\mathbb{L}_2^*$ which is not strongly elliptic.

In this section we will try to find a general class of periodic laminates for which the strong ellipticity is lost through homogenization. To this end we will extend to dimension three the rank-one lamination approach of [2] performed in dimension two. However, the outcome is surprisingly different from that of the two-dimensional case of [2]. Indeed, we will prove in the first subsection that it is not possible to lose strong ellipticity by a rank-one lamination through homogenization following the two-dimensional approach of [2]. This is the reason why we will perform a second lamination in the second part of the section.

3.1. Rank-one lamination. — In this subsection we are going to focus on the rank-one lamination. As noted before, in the two-dimensional case of [2] it was proved a necessary and sufficient condition for a general rank-one laminate to lose strong ellipticity. Mimicking the same approach in dimension three we obtain the following quite different result.

Theorem 3.3. — *Let $\mathbb{L} \in L^\infty_{\rm per}\big(Y_1; \mathscr{L}_s(\mathbb{R}^{3\times 3})\big)$ be a Y_1-periodic isotropic tensor-valued function which is strongly elliptic, i.e., $\alpha_{\rm se}(\mathbb{L}) > 0$. Assume that $\Lambda_{\rm per}(\mathbb{L}) > 0$ and that there exists a constant matrix $D \in \mathbb{R}^{3\times 3}$ such that*

$$\mathbb{L}(y_1)M{:}M + D{:}\,\mathrm{Cof}(M) \geqslant 0, \quad \textit{a.e. } y_1 \in Y_1, \quad \forall\, M \in \mathbb{R}^{3\times 3}. \tag{3.1}$$

Then, the homogenized tensor $\mathbb{L}^0$ defined by (2.2) is strongly elliptic, i.e., $\alpha_{\rm se}(\mathbb{L}^0) > 0$.

Remark 3.4. — In dimension two for any periodic function $\varphi \in H^1_{\rm per}(Y_2;\mathbb{R}^2)$, the only null Lagrangian (up to a multiplicative constant) is the determinant of $\nabla\varphi$. Although the two-dimensional case seems a priori more restrictive than the three-dimensional case from an algebraic point of view, the two-dimensional Theorem 3.1 of [2] shows that for a suitable isotropic tensor $\mathbb{L} = \mathbb{L}(y_1)$, satisfying for some constant $d \in \mathbb{R}$, the condition

$$\mathbb{L}(y_1)M{:}M + d\det(M) \geqslant 0, \quad \text{a.e. in } Y_1,\ \forall\, M \in \mathbb{R}^{2\times 2}, \tag{3.2}$$

it is possible to lose strong ellipticity through homogenization. On the contrary, the three-dimensional Theorem 3.3 shows that it is not possible to lose strong ellipticity under condition (3.1) which is the natural three-dimensional extension of (3.2).

Remark 3.5. — Observe that condition (3.1) implies that $\mathbb{L}$ is weakly coercive, i.e., $\Lambda(\mathbb{L}) \geqslant 0$, but the converse is not true in general. Therefore, it might be possible to find a weakly coercive, strongly elliptic isotropic tensor $\mathbb{L} = \mathbb{L}(y_1)$ for which the strong ellipticity is lost. However, we have not succeeded in deriving such a tensor.

Remark 3.6. — In the proof of Proposition 3.1 Gutiérrez implicitly proved the result of Theorem 3.3 when the matrix D has the form $D = dI_3$ and $\mathbb{L}$ is of the type

$$\mathbb{L}(y_1) = \chi(y_1)\,\mathbb{L}_a + \big(1 - \chi(y_1)\big)\,\mathbb{L}_b.$$

Moreover, it is worth mentioning that the cases for which Guitiérrez obtained the loss of ellipticity with a rank-one lamination do not contradict Theorem 3.3, since in those cases condition (3.1) does not hold.

The rest of this subsection is devoted to the proof of Theorem 3.3. For any Y_1-periodic tensor-valued function $\mathbb{L} \in L^\infty_{\rm per}\big(Y_1; \mathscr{L}_s(\mathbb{R}^{3\times 3})\big)$ which is strongly elliptic, i.e., $\alpha_{\rm se}(\mathbb{L}) > 0$, define for a.e. $y_1 \in Y_1$, the y_1-dependent inner product

$$(\xi,\eta) \in \mathbb{R}^3 \times \mathbb{R}^3 \longmapsto \mathbb{L}(y_1)(\xi \otimes e_1){:}(\eta \otimes e_1).$$

It is indeed an inner product because $\alpha_{\rm se}(\mathbb{L}) > 0$. The matrix-valued function

$$\begin{aligned} L(y_1) &= \begin{pmatrix} l_1(y_1) & l_{12}(y_1) & l_{13}(y_1) \\ l_{12}(y_1) & l_2(y_1) & l_{23}(y_1) \\ l_{13}(y_1) & l_{23}(y_1) & l_3(y_1) \end{pmatrix} \\ &:= \begin{pmatrix} \mathbb{L}(y_1)(e_1\otimes e_1){:}(e_1\otimes e_1) & \mathbb{L}(y_1)(e_1\otimes e_1){:}(e_2\otimes e_1) & \mathbb{L}(y_1)(e_1\otimes e_1){:}(e_3\otimes e_1) \\ \mathbb{L}(y_1)(e_1\otimes e_1){:}(e_2\otimes e_1) & \mathbb{L}(y_1)(e_2\otimes e_1){:}(e_2\otimes e_1) & \mathbb{L}(y_1)(e_2\otimes e_1){:}(e_3\otimes e_1) \\ \mathbb{L}(y_1)(e_1\otimes e_1){:}(e_3\otimes e_1) & \mathbb{L}(y_1)(e_2\otimes e_1){:}(e_3\otimes e_1) & \mathbb{L}(y_1)(e_3\otimes e_1){:}(e_3\otimes e_1) \end{pmatrix} \end{aligned} \tag{3.3}$$

is therefore symmetric positive definite.

Similarly to [2, Lem. 3.3] the next result provides an estimate which is a direct consequence of condition (3.1) with a matrix of the type $D = dI_3$. Observe that for the moment we are not assuming that the tensor $\mathbb{L}$ is isotropic.

Lemma 3.7. — *Let $\mathbb{L} \in L^\infty_{\rm per}(Y_1; \mathscr{L}_s(\mathbb{R}^{3\times 3}))$ be a Y_1-periodic bounded tensor-valued function with $\Lambda_{\rm per}(\mathbb{L}) > 0$. Assume the existence of a constant $d \in \mathbb{R}$ such that $\mathbb{L}$ satisfies condition (3.1) with $D = dI_3$. Then, we have*

$$\mathbb{L}(y_1)M{:}M \geqslant Q(M), \quad \text{a.e. in } Y_1,\ \forall\, M \in \mathbb{R}^{3\times 3},\ M \text{ rank-one}, \tag{3.4}$$

where

$$\begin{aligned} Q(M) := {} & \frac{\det(\widetilde{L}^{11})}{\det(L)}\Big(\mathbb{L}M{:}(e_1\otimes e_1) + \frac{d}{2}M_{33} + \frac{d}{2}M_{22}\Big)^2 \\ & + \frac{\det(\widetilde{L}^{22})}{\det(L)}\Big(\mathbb{L}M{:}(e_2\otimes e_1) - \frac{d}{2}M_{12}\Big)^2 + \frac{\det(\widetilde{L}^{33})}{\det(L)}\Big(\mathbb{L}M{:}(e_3\otimes e_1) - \frac{d}{2}M_{13}\Big)^2 \\ & - \frac{2\det(\widetilde{L}^{12})}{\det(L)}\Big(\mathbb{L}M{:}(e_1\otimes e_1) + \frac{d}{2}M_{33} + \frac{d}{2}M_{22}\Big)\Big(\mathbb{L}M{:}(e_2\otimes e_1) - \frac{d}{2}M_{12}\Big) \\ & + \frac{2\det(\widetilde{L}^{13})}{\det(L)}\Big(\mathbb{L}M{:}(e_1\otimes e_1) + \frac{d}{2}M_{33} + \frac{d}{2}M_{22}\Big)\Big(\mathbb{L}M{:}(e_3\otimes e_1) - \frac{d}{2}M_{13}\Big) \\ & - \frac{2\det(\widetilde{L}^{23})}{\det(L)}\Big(\mathbb{L}M{:}(e_2\otimes e_1) - \frac{d}{2}M_{12}\Big)\mathbb{L}M{:}(e_3\otimes e_1) - \frac{d}{2}M_{13}\Big). \end{aligned}$$

Furthermore, if $\mathbb{L}^0$ is the homogenized tensor of $\mathbb{L}$, then $\alpha_{\rm se}(\mathbb{L}^0) = 0$ if and only if there exists a rank-one matrix M such that

$$\mathbb{L}(y_1)M{:}M = Q(M), \quad \text{a.e. in } Y_1, \tag{3.5}$$

together with

$$\begin{aligned} \int_{Y_1} \frac{\det(\widetilde{L}^{13})}{\det(L)}(t)\Big(\mathbb{L}(t)M{:}(e_1\otimes e_1) + \frac{d}{2}M_{22} + \frac{d}{2}M_{33}\Big)dt \\ = \int_{Y_1}\Big[\frac{\det(\widetilde{L}^{23})}{\det(L)}(t)\Big(\mathbb{L}(t)M{:}(e_2\otimes e_1) - \frac{d}{2}M_{12}\Big) \\ - \frac{\det(\widetilde{L}^{33})}{\det(L)}(t)\Big(\mathbb{L}(t)M{:}(e_3\otimes e_1) - \frac{d}{2}M_{13}\Big)\Big]dt, \end{aligned} \tag{3.6.a}$$

$$\begin{aligned} \int_{Y_1} \frac{\det(\widetilde{L}^{12})}{\det(L)}(t)\Big(\mathbb{L}(t)M{:}(e_1\otimes e_1) + \frac{d}{2}M_{22} + \frac{d}{2}M_{33}\Big)dt \\ = \int_{Y_1}\Big[\frac{\det(\widetilde{L}^{22})}{\det(L)}(t)\Big(\mathbb{L}(t)M{:}(e_2\otimes e_1) - \frac{d}{2}M_{12}\Big) \\ - \frac{\det(\widetilde{L}^{23})}{\det(L)}(t)\Big(\mathbb{L}(t)M{:}(e_3\otimes e_1) - \frac{d}{2}M_{13}\Big)\Big]dt, \end{aligned} \tag{3.6.b}$$

$$\begin{aligned} \int_{Y_1} \frac{\det(\widetilde{L}^{11})}{\det(L)}(t)\Big(\mathbb{L}(t)M{:}(e_1\otimes e_1) + \frac{d}{2}M_{22} + \frac{d}{2}M_{33}\Big)dt \\ = \int_{Y_1}\Big[\frac{\det(\widetilde{L}^{12})}{\det(L)}(t)\Big(\mathbb{L}(t)M{:}(e_2\otimes e_1) - \frac{d}{2}M_{12}\Big) \\ - \frac{\det(\widetilde{L}^{13})}{\det(L)}(t)\Big(\mathbb{L}(t)M{:}(e_3\otimes e_1) - \frac{d}{2}M_{13}\Big)\Big]dt. \end{aligned} \tag{3.6.c}$$

Finally, we state a corollary of the previous result in the particular case of isotropic tensors.

Lemma 3.8. — *Let $\mathbb{L} \in L^\infty_{\rm per}(Y_1; \mathscr{L}_s(\mathbb{R}^{3\times 3}))$ be a Y_1-periodic bounded isotropic tensor-valued function with $\Lambda_{\rm per}(\mathbb{L}) > 0$. Assume that there exists a constant $d \in \mathbb{R}$ such that the Lamé coefficients of $\mathbb{L}(y_1)$ satisfy*

$$\max\{0, -2\mu(y_1) - 3\lambda(y_1)\} \leqslant d \leqslant 4\mu(y_1) \quad \text{for a.e. } y_1 \text{ in } Y_1. \tag{3.7}$$

Then, the homogenized tensor $\mathbb{L}^0$ defined by (2.2) is strongly elliptic.

Thanks to the previous lemmas, we are now able to demonstrate the main result of this section.

Proof of Theorem 3.3. — Firstly, assume that (3.1) is satisfied with the matrix D being of the type $D = dI_3$ for some $d \in \mathbb{R}$. This is equivalent to condition (3.7), as it was proved by Gutiérrez in [8, §4.2]. By virtue of Lemma 3.8, $\mathbb{L}^0$ is strongly elliptic, which concludes the proof in this case.

In the sequel we will show that if there exists a constant matrix $D \in \mathbb{R}^{3\times 3}$ such that condition (3.1) is fulfilled, then there exists a constant $d \in \mathbb{R}$ such that (3.1) holds with $D = dI_3$. This combined with Lemma 3.8 implies that $\mathbb{L}^0$ is strongly elliptic.

Assume that (3.1) holds for some matrix $D \in \mathbb{R}^{3\times 3}$, namely for any $M \in \mathbb{R}^{3\times 3}$, we have a.e. in Y_1,

$$\begin{aligned}
0 \leqslant\ & \lambda(M_{11}+M_{22}+M_{33})^2 \\
& + 2\mu\Big(M_{11}^2+M_{22}^2+M_{33}^2 \\
& \qquad + 2\Big[\Big(\frac{M_{12}+M_{21}}{2}\Big)^2 + \Big(\frac{M_{13}+M_{31}}{2}\Big)^2 + \Big(\frac{M_{23}+M_{32}}{2}\Big)^2\Big]\Big) \\
& + D_{11}(M_{22}M_{33} - M_{23}M_{32}) \\
& - D_{12}(M_{21}M_{33} - M_{23}M_{31}) + D_{13}(M_{21}M_{32} - M_{22}M_{31}) \\
& - D_{21}(M_{12}M_{33} - M_{13}M_{32}) \\
& + D_{22}(M_{11}M_{33} - M_{13}M_{31}) - D_{23}(M_{11}M_{32} - M_{12}M_{31}) \\
& + D_{31}(M_{12}M_{23} - M_{13}M_{22}) \\
& - D_{32}(M_{11}M_{23} - M_{13}M_{21}) + D_{33}(M_{11}M_{22} - M_{12}M_{21}).
\end{aligned}$$

The previous condition is equivalent to the matrix (3.10) below being positive semi-definite a.e. in Y_1. In particular, this implies that the following matrices are positive semi-definite a.e. in Y_1:

$$\begin{pmatrix} \mu & \mu - D_{ii}/2 \\ \mu - D_{ii}/2 & \mu \end{pmatrix} \quad \text{for } i = 1,2,3, \tag{3.8}$$

$$B := \begin{pmatrix} \lambda+2\mu & \lambda+D_{33}/2 & \lambda+D_{22}/2 \\ \lambda+D_{33}/2 & \lambda+2\mu & \lambda+D_{11}/2 \\ \lambda+D_{22}/2 & \lambda+D_{11}/2 & \lambda+2\mu \end{pmatrix}. \tag{3.9}$$

$$\begin{pmatrix}
\lambda+2\mu & \lambda+\frac{D_{33}}{2} & \lambda+\frac{D_{22}}{2} & 0 & 0 & 0 & 0 & -\frac{D_{32}}{2} & -\frac{D_{23}}{2} \\
\lambda+\frac{D_{33}}{2} & \lambda+2\mu & \lambda+\frac{D_{11}}{2} & 0 & 0 & -\frac{D_{31}}{2} & \frac{D_{13}}{2} & 0 & 0 \\
\lambda+\frac{D_{22}}{2} & \lambda+\frac{D_{11}}{2} & \lambda+2\mu & -\frac{D_{21}}{2} & -\frac{D_{12}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{D_{21}}{2} & \mu & \mu-\frac{D_{33}}{2} & 0 & \frac{D_{23}}{2} & \frac{D_{31}}{2} & 0 \\
0 & 0 & -\frac{D_{12}}{2} & \mu-\frac{D_{33}}{2} & \mu & \frac{D_{32}}{2} & 0 & 0 & \frac{D_{13}}{2} \\
0 & -\frac{D_{31}}{2} & 0 & 0 & \frac{D_{32}}{2} & \mu & \mu-\frac{D_{22}}{2} & 0 & \frac{D_{21}}{2} \\
0 & -\frac{D_{13}}{2} & 0 & \frac{D_{23}}{2} & 0 & \mu-\frac{D_{22}}{2} & \mu & \frac{D_{12}}{2} & 0 \\
-\frac{D_{32}}{2} & 0 & 0 & \frac{D_{31}}{2} & 0 & 0 & \frac{D_{12}}{2} & \mu & \mu-\frac{D_{11}}{2} \\
-\frac{D_{23}}{2} & 0 & 0 & 0 & \frac{D_{13}}{2} & \frac{D_{21}}{2} & 0 & \mu-\frac{D_{11}}{2} & \mu
\end{pmatrix}$$

Matrix (3.10).

Now, we will prove that there exists $i \in \{1,2,3\}$ such that

$$-\operatorname*{ess\text{-}inf}_{y_1\in Y_1}\{2\mu(y_1)+3\lambda(y_1)\} \leqslant D_{ii} \leqslant 4\operatorname*{ess\text{-}inf}_{y_1\in Y_1}\{\mu(y_1)\}. \tag{3.11}$$

Note that we can assume

$$\operatorname*{ess\text{-}inf}_{y_1\in Y_1}\{2\mu(y_1)+3\lambda(y_1)\} < 0. \tag{3.12}$$

Otherwise, since the matrix (3.8) is positive semi-definite, or equivalently

$$0 \leqslant D_{ii} \leqslant 4\operatorname*{ess\text{-}inf}_{y_1\in Y_1}\{\mu(y_1)\} \quad \text{for } i=1,2,3, \tag{3.13}$$

condition (3.11) holds immediately.

We assume by contradiction that (3.11) is violated for any $i=1,2,3$. Since the matrix B defined by (3.9) is positive semi-definite, we get for any $i=1,2,3$,

$$\begin{vmatrix} \lambda+2\mu & \lambda+D_{ii}/2 \\ \lambda+D_{ii}/2 & \lambda+2\mu \end{vmatrix} \geqslant 0 \quad \text{a.e. in } Y_1,$$

which is equivalent to

$$-4\operatorname*{ess\text{-}inf}_{y_1\in Y_1}\{\mu(y_1)+\lambda(y_1)\} \leqslant D_{ii} \leqslant 4\operatorname*{ess\text{-}inf}_{y_1\in Y_1}\{\mu(y_1)\} \quad \text{for } i=1,2,3.$$

Since by assumption (3.11) is not satisfied for any $i=1,2,3$ and (3.13) holds, then the previous condition yields

$$-4\operatorname*{ess\text{-}inf}_{y_1\in Y_1}\{\mu(y_1)+\lambda(y_1)\} \leqslant D_{ii} < -\operatorname*{ess\text{-}inf}_{y_1\in Y_1}\{2\mu(y_1)+3\lambda(y_1)\} \quad \text{for } i=1,2,3. \tag{3.14}$$

Set $d := \max_{i=1,2,3}\{D_{ii}\}$. By (3.14) there exists $\varepsilon > 0$ such that

$$d + \varepsilon < -\operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{2\mu(y_1) + 3\lambda(y_1)\}. \tag{3.15}$$

Define the set $P_\varepsilon \subset Y_1$ by

$$P_\varepsilon := \Big\{x_1 \in Y_1 : 2\mu(x_1) + 3\lambda(x_1) < \operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{2\mu(y_1) + 3\lambda(y_1)\} + \varepsilon\Big\}.$$

It is clear that $|P_\varepsilon| > 0$, and from (3.15) and the definition of P_ε we obtain

$$d + \varepsilon < -\operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{2\mu(y_1) + 3\lambda(y_1)\} < -\big(2\mu(x_1) + 3\lambda(x_1)\big) + \varepsilon \quad \text{a.e. } x_1 \in P_\varepsilon,$$

which leads to

$$\lambda(x_1) + \frac{d}{2} < -\frac{1}{2}\big(\lambda(x_1) + 2\mu(x_1)\big) < 0 \quad \text{a.e. } x_1 \in P_\varepsilon. \tag{3.16}$$

Since the matrix B from (3.9) is positive semi-definite, its determinant is non-negative a.e. in Y_1. In particular we have

$$\begin{aligned} 0 \leqslant \det\big(B(x_1)\big) = \big(\lambda(x_1) + 2\mu(x_1)\big)^3 & \\ + 2\Big(\lambda(x_1) + \frac{D_{11}}{2}\Big)\Big(\lambda(x_1) + \frac{D_{22}}{2}\Big)\Big(\lambda(x_1) + \frac{D_{33}}{2}\Big) & \\ - \big(\lambda(x_1) + 2\mu(x_1)\big)\Big[\Big(\lambda(x_1) + \frac{D_{11}}{2}\Big)^2 & \\ + \Big(\lambda(x_1) + \frac{D_{22}}{2}\Big)^2 + \Big(\lambda(x_1) + \frac{D_{33}}{2}\Big)^2\Big], & \end{aligned} \tag{3.17}$$

a.e. $x_1 \in P_\varepsilon$. Then, it follows that

$$\begin{aligned} \det\big(B(x_1)\big) \leqslant \big(\lambda(x_1) + 2\mu(x_1)\big)^3 + 2\Big(\lambda(x_1) + \frac{d}{2}\Big)^3 & \\ - 3\big(\lambda(x_1) + 2\mu(x_1)\big)\Big(\lambda(x_1) + \frac{d}{2}\Big)^2 \quad \text{a.e. } x_1 \in P_\varepsilon. & \end{aligned} \tag{3.18}$$

To derive a contradiction let us show that the right-hand side of inequality (3.18) is negative. By (3.16) we get

$$4\Big(\lambda(x_1) + \frac{d}{2}\Big)^2 > \big(\lambda(x_1) + 2\mu(x_1)\big)^2 \quad \text{a.e. } x_1 \in P_\varepsilon,$$

which, multiplying by $\lambda(x_1) + 2\mu(x_1) > 0$, leads to

$$\big(\lambda(x_1) + 2\mu(x_1)\big)^3 - 4\big(\lambda(x_1) + 2\mu(x_1)\big)\Big(\lambda(x_1) + \frac{d}{2}\Big)^2 < 0 \quad \text{a.e. } x_1 \in P_\varepsilon.$$

Again using (3.16) we deduce that

$$2\Big(\lambda(x_1) + \frac{d}{2}\Big)^3 < -\big(\lambda(x_1) + 2\mu(x_1)\big)\Big(\lambda(x_1) + \frac{d}{2}\Big)^2 \quad \text{a.e. } x_1 \in P_\varepsilon.$$

Adding the two last inequalities we obtain

$$\big(\lambda(x_1)+2\mu(x_1)\big)^3+2\Big(\lambda(x_1)+\frac{d}{2}\Big)^3-3\big(\lambda(x_1)+2\mu(x_1)\big)\Big(\lambda(x_1)+\frac{d}{2}\Big)^2 < 0 \quad \text{a.e. } x_1 \in P_\varepsilon,$$

which by (3.18) implies that $\det(B) < 0$ in P_ε, a contradiction with (3.17).

Therefore, condition (3.11) is satisfied by $D_{ii} \geqslant 0$ (due to (3.13)) for some $i = 1, 2, 3$. Hence, condition (3.7) holds with $d = D_{ii}$, or equivalently (3.1) is satisfied by the matrix $D_{ii} I_3$, which concludes the proof. □

Now, let us prove the auxiliary results of the section.

Proof of Lemma 3.7. — Let $M \in \mathbb{R}^{3\times 3}$ be a rank-one matrix. Then, we have $\det(M) = 0$, and $\operatorname{adj}_{ii}(M) = 0$ for $i = 1, 2, 3$. Therefore, we get

$$\begin{aligned} \mathbb{L}^0 M{:}M &= \min\left\{ \int_{Y_3} \mathbb{L}(M + \nabla\varphi){:}(M + \nabla\varphi)\, dy{:}\varphi \in H^1_{\mathrm{per}}(Y_3;\mathbb{R}^3) \right\} \\ &= \min\left\{ \int_{Y_3} \big(\mathbb{L}(M + \nabla\varphi)(M + \nabla\varphi) \right. \\ &\qquad \left. + dI_3 \colon \operatorname{Cof}(M + \nabla\varphi){:}\varphi \in H^1_{\mathrm{per}}(Y_3;\mathbb{R}^3) \big)\, dy \right\} \geqslant 0. \end{aligned} \tag{3.19}$$

Take $\varphi = \varphi(y_1) = (\varphi_1, \varphi_2, \varphi_3) \in C^1_{\mathrm{per}}(Y_1;\mathbb{R}^3)$. Then, the matrix

$$\nabla\varphi = \varphi' \otimes e_1 = \varphi_1'(e_1 \otimes e_1) + \varphi_2'(e_2 \otimes e_1) + \varphi_3'(e_3 \otimes e_1),$$

is either a rank-one or the null matrix. Also, note that

$$\operatorname{adj}_{ij}(M) = (-1)^{i+j} \det(\widetilde{M}^{ji}).$$

Considering the previous expressions, from (3.1) it follows that

$$\begin{aligned} 0 &\leqslant \mathbb{L}(M + \nabla\varphi){:}(M + \nabla\varphi) + d\textstyle\sum_{i=1}^3 \operatorname{adj}_{ii}(M + \nabla\varphi) \\ &= \mathbb{L}M{:}M + 2\mathbb{L}M{:}(e_1 \otimes e_1)\varphi_1' + 2\mathbb{L}M{:}(e_2 \otimes e_1)\varphi_2' \\ &\qquad + 2\mathbb{L}M{:}(e_3 \otimes e_1)\varphi_3' + l_1(\varphi_1')^2 + 2l_{12}\varphi_1'\varphi_2' \\ &\qquad + 2l_{13}\varphi_1'\varphi_3' + l_2(\varphi_2')^2 + 2l_{23}\varphi_2'\varphi_3' + l_2(\varphi_3')^2 \\ &\qquad + d(M_{33}\varphi_1' - M_{13}\varphi_3' + M_{22}\varphi_1' - M_{12}\varphi_2') \\ &= \mathbb{L}M{:}M + l_1(\varphi_1')^2 + l_2(\varphi_2')^2 + l_3(\varphi_3') \\ &\qquad + 2l_{12}\varphi_1'\varphi_2' + 2l_{13}\varphi_1'\varphi_3' + 2l_{23}\varphi_2'\varphi_3' \\ &\qquad + \big[2\mathbb{L}M{:}(e_1 \otimes e_1) + d(M_{33} + dM_{22})\big]\varphi_1' \\ &\qquad + \big[2\mathbb{L}M{:}(e_2 \otimes e_1) - dM_{12}\big]\varphi_2' \\ &\qquad + \big[2\mathbb{L}M{:}(e_3 \otimes e_1) - dM_{13}\big]\varphi_3'. \end{aligned}$$

For the previous equalities we have used that

$$\operatorname{adj}_{ii}(A + B) = \operatorname{adj}_{ii}(A) + \operatorname{adj}_{ii}(B) + \operatorname{Cof}(\widetilde{A}^{ii}){:}\widetilde{B}^{ii}.$$

The purpose is to rewrite the last expression as the sum of squares. With that in mind, one obtains

$$
\begin{aligned}
0 \leqslant{}& \mathbb{L}(M+\nabla\varphi){:}(M+\nabla\varphi) + dI_3{:}\,\mathrm{Cof}(M+\nabla\varphi)\\
={}& \mathbb{L}M{:}M - Q(M)\\
&+ l_1\Big[\varphi_1' + \frac{l_{12}}{l_1}\varphi_2' + \frac{l_{13}}{l_1}\varphi_3' + \frac{1}{l_1}\Big(\mathbb{L}M{:}(e_1\otimes e_1) + \frac{d}{2}M_{22} + \frac{d}{2}M_{33}\Big)\Big]^2\\
&+ \frac{\det(\widetilde{L}^{33})}{l_1}\Big[\varphi_2' + \frac{\det(\widetilde{L}^{23})}{\det(\widetilde{L}^{33})}\varphi_3'\\
&\qquad - \frac{l_{12}}{\det(\widetilde{L}^{33})}\Big(\mathbb{L}M{:}(e_1\otimes e_1) + \frac{d}{2}M_{22} + \frac{d}{2}M_{33}\Big)\\
&\qquad\qquad + \frac{l_1}{\det(\widetilde{L}^{33})}\Big(\mathbb{L}M{:}(e_2\otimes e_1) - \frac{d}{2}M_{12}\Big)\Big]^2\\
&+ \frac{\det(L)}{\det(\widetilde{L}^{33})}\Big[\varphi_3' + \frac{\det(\widetilde{L}^{13})}{\det(L)}\Big(\mathbb{L}M{:}(e_1\otimes e_1) + \frac{d}{2}M_{22} + \frac{d}{2}M_{33}\Big)\\
&\qquad - \frac{\det(\widetilde{L}^{23})}{\det(L)}\Big(\mathbb{L}M{:}(e_2\otimes e_1) - \frac{d}{2}M_{12}\Big)\\
&\qquad\qquad + \frac{\det(\widetilde{L}^{33})}{\det(L)}\Big(\mathbb{L}M{:}(e_3\otimes e_1) - \frac{d}{2}M_{13}\Big)\Big]^2.
\end{aligned}
\tag{3.20}
$$

Since φ_1', φ_2' and φ_3' can be chosen arbitrarily, the three square brackets in the previous equality can be equated to 0 at any Lebesgue point $y_1 \in Y_1$ of $\mathbb{L}$, and thus (3.4) holds. Using a density argument the previous equality also holds a.e. in Y_1, for any $\varphi \in H^1_{\mathrm{per}}(Y_1;\mathbb{R}^3)$.

Now, we are going to prove the second part of Lemma 3.7. Assume $\mathbb{L}^0$ is not strongly elliptic. Then, there exists a rank-one matrix M such that $\mathbb{L}^0M{:}M = 0$. Taking into account expressions (3.19), the minimizer v_M associated with $\mathbb{L}^0M{:}M$ (see [2, Lem. 3.2]) satisfies $v_M = v_M(y_1)$ and

$$
\begin{aligned}
0 = \mathbb{L}^0M{:}M &= \int_{Y_1} \mathbb{L}(t)(M + v_M'(t)\otimes e_1){:}(M + v_M'(t)\otimes e_1)dt\\
&= \int_{Y_1} \big[\mathbb{L}(t)(M+\nabla v_M(t)){:}(M+\nabla v_M(t)) + dI_3{:}\,\mathrm{Cof}(M+\nabla v_M)\big]dt.
\end{aligned}
$$

The first inequality in (3.20) implies that the integrand of the previous expression must be pointwisely 0, and thus the inequality in (3.20) for $\varphi = v_M$ is actually an equality. From this we deduce

$$
\mathbb{L}M{:}M = Q(M),
$$

and

$$
\begin{cases}
0 = (v'_M)_1 + \dfrac{l_{12}}{l_1}(v'_M)_2 + \dfrac{l_{13}}{l_1}(v'_M)_3 \\
\qquad\qquad + \dfrac{1}{l_1}\Big(\mathbb{L}M{:}(e_1\otimes e_1) + \dfrac{d}{2}M_{22} + \dfrac{d}{2}M_{33}\Big), \\
0 = (v'_M)_2 + \dfrac{\det(\widetilde{L}^{23})}{\det(\widetilde{L}^{33})}(v'_M)_3 \\
\qquad\qquad - \dfrac{l_{12}}{\det(\widetilde{L}^{33})}\Big(\mathbb{L}M{:}(e_1\otimes e_1) + \dfrac{d}{2}M_{22} + \dfrac{d}{2}M_{33}\Big) \\
\qquad\qquad + \dfrac{l_1}{\det(\widetilde{L}^{33})}\Big(\mathbb{L}M{:}(e_2\otimes e_1) - \dfrac{d}{2}M_{12}\Big), \\
0 = (v'_M)_3 + \dfrac{\det(\widetilde{L}^{13})}{\det(L)}\Big(\mathbb{L}M{:}(e_1\otimes e_1) + \dfrac{d}{2}M_{22} + \dfrac{d}{2}M_{33}\Big) \\
\qquad\qquad - \dfrac{\det(\widetilde{L}^{23})}{\det(L)}\Big(\mathbb{L}M{:}(e_2\otimes e_1) - \dfrac{d}{2}M_{12}\Big) \\
\qquad\qquad + \dfrac{\det(\widetilde{L}^{33})}{\det(L)}\Big(\mathbb{L}M{:}(e_3\otimes e_1) - \dfrac{d}{2}M_{13}\Big).
\end{cases}
\tag{3.21}
$$

Since v_M is Y_1-periodic, we have

$$
\int_{Y_1} (v'_M)_i \, dy_1 = 0 \quad i = 1, 2, 3.
$$

Integrating the third equality in (3.21) we obtain (3.6.a). Replacing $(v'_M)_3$ in the second equality of (3.21), we end up getting (3.6.b). Finally, replacing $(v'_M)_2$ and $(v'_M)_3$ in the first equality of (3.21) it yields (3.6.c).

Conversely, let us assume that equalities (3.5) and (3.6.a)–(3.6.c) hold. Considering (3.6.a), taking into account that the all the integrands belong to $L^\infty(Y_1)$, there exists a function $\varphi_3 \in W^{1,\infty}_{\mathrm{per}}(Y_1)$ such that, a.e. in Y_1, it holds

$$
\begin{aligned}
0 = \varphi'_3 &+ \frac{\det(\widetilde{L}^{13})}{\det(L)}\Big(\mathbb{L}M{:}(e_1\otimes e_1) + \frac{d}{2}M_{22} + \frac{d}{2}M_{33}\Big) \\
&- \frac{\det(\widetilde{L}^{23})}{\det(L)}\Big(\mathbb{L}M{:}(e_2\otimes e_1) - \frac{d}{2}M_{12}\Big) + \frac{\det(\widetilde{L}^{33})}{\det(L)}\Big(\mathbb{L}M{:}(e_3\otimes e_1) - \frac{d}{2}M_{13}\Big).
\end{aligned}
$$

Repeating the argument with (3.6.b) and (3.6.c), we get the existence of functions φ_2 and φ_1 in $W^{1,\infty}_{\mathrm{per}}(Y_1)$ respectively, such that

$$
\begin{aligned}
\varphi'_2 + \frac{\det(\widetilde{L}^{23})}{\det(\widetilde{L}^{33})}\varphi'_3 - \frac{l_{12}}{\det(\widetilde{L}^{33})}&\Big(\mathbb{L}M{:}(e_1\otimes e_1) + \frac{d}{2}M_{22} + \frac{d}{2}M_{33}\Big) \\
&+ \frac{l_1}{\det(\widetilde{L}^{33})}\Big(\mathbb{L}M{:}(e_2\otimes e_1) - \frac{d}{2}M_{12}\Big) = 0, \\
\varphi'_1 + \frac{l_{12}}{l_1}\varphi'_2 + \frac{l_{13}}{l_1}\varphi'_3 + \frac{1}{l_1}&\Big(\mathbb{L}M{:}(e_1\otimes e_1) + \frac{d}{2}M_{22} + \frac{d}{2}M_{33}\Big) = 0.
\end{aligned}
$$

These three equalities together with (3.5) imply the equality in (3.20), and thus by (3.19) it follows that

$$0 = \int_{Y_1} \big(\mathbb{L}(M+\nabla\varphi){:}(M+\nabla\varphi) + dI_3{:}\,\mathrm{Cof}(M+\nabla\varphi)\big)\,dy_1 \geqslant \mathbb{L}^0 M{:}M \geqslant 0,$$

which shows that $\mathbb{L}^0$ is not strongly elliptic.

Finally, due to the equality $\mathbb{L}^0 M{:}M = \mathbb{L}^0 M^T{:}M^T$, conditions (3.5) and (3.6) are equivalent to the similar equalities replacing M by M^T. □

Proof of Lemma 3.8. — Since $\mathbb{L}$ is isotropic, condition (3.7) is equivalent to the condition (3.1) with $D = dI_3$. As a consequence, (3.7) implies $\Lambda(\mathbb{L}) \geqslant 0$. By [7, Cor. 3.5], we have $\alpha_{\mathrm{se}}(\mathbb{L}^0) \geqslant \Lambda(\mathbb{L})$. Therefore, we get that $\alpha_{\mathrm{se}}(\mathbb{L}^0) \geqslant 0$.

Assume that $\mathbb{L}^0$ is not strongly elliptic, i.e., $\alpha_{\mathrm{se}}(\mathbb{L}^0) = 0$. Then, there exists a rank-one matrix $M := \xi \otimes \eta$ in $\mathbb{R}^{3\times 3}$, with $\xi, \eta \in \mathbb{R}^3 \setminus \{0\}$, such that $\mathbb{L}^0 M{:}M = 0$.

Since $\mathbb{L}$ is isotropic, the matrix L defined in (3.3) is

$$L = \begin{pmatrix} \lambda+2\mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

Moreover, the following equalities hold

$$\begin{aligned} M_{ij} &= \xi_i\eta_j \quad i,j \in \{1,2,3\}, \\ \mathbb{L}M{:}(e_1\otimes e_1) &= (\lambda+2\mu)\xi_i\eta_1 + \lambda(\xi_2\eta_2+\xi_3\eta_3), \\ \mathbb{L}M{:}(e_2\otimes e_1) &= \mu(\xi_1\eta_2+\xi_2\eta_1), \\ \mathbb{L}M{:}(e_3\otimes e_1) &= \mu(\xi_1\eta_{\cdot}+\xi_3\eta_1), \\ \mathbb{L}M{:}M &= (\lambda+\mu)(\xi{:}\eta)^2 + \mu|\xi|^2|\eta|^2. \end{aligned}$$

Because $\mathbb{L}^0 M{:}M = 0$, from equalities (3.5) and (3.6) in Lemma 3.7 we obtain a.e. in Y_1

$$\begin{aligned} &(\lambda+\mu)(\xi{:}\eta)^2 + \mu|\xi|^2|\eta|^2 \\ &\qquad = \frac{1}{\lambda+2\mu}\Big[(\lambda+2\mu)\xi_1\mu_1 + \lambda(\xi_2\eta_2+\xi_3\eta_3) + \frac{d}{2}(\xi_2\eta_2+\xi_3\eta_3)\Big]^2 \\ &\qquad\quad + \frac{1}{\mu}\Big[\mu(\xi_1\eta_2+\xi_2\eta_1) - \frac{d}{2}\xi_1\eta_2\Big]^2 + \frac{1}{\mu}\Big[\mu(\xi_1\eta_3+\xi_3\eta_1) - \frac{d}{2}\xi_1\eta_3\Big]^2, \end{aligned} \tag{3.22}$$

together with

$$0 = \xi_1\eta_3 + \xi_3\eta_1 - \frac{\xi_1\eta_3}{2}\int_{Y_1}\frac{d}{\mu}(t)\,dt, \tag{3.23}$$

$$0 = \xi_1\eta_2 + \xi_2\eta_1 - \frac{\xi_1\eta_2}{2}\int_{Y_1}\frac{d}{\mu}(t)\,dt, \tag{3.24}$$

$$0 = \xi_1\eta_1 + (\xi_2\eta_2+\xi_3\eta_3)\int_{Y_1}\frac{\lambda+d/2}{\lambda+2\mu}(t)\,dt. \tag{3.25}$$

After some calculations, from (3.22) we get

$$\frac{(\lambda+2\mu)^2-(\lambda+d/2)^2}{\lambda+2\mu}(\xi_2\eta_2+\xi_3\eta_3)^2+\mu(\xi_2\eta_3-\xi_3\eta_2)^2 + \frac{d(\mu-d/4)}{\mu}\xi_1^2(\eta_2^2+\eta_3^2)=0 \quad \text{a.e. in } Y_1. \tag{3.26}$$

Observe that, since $\mathbb{L}$ is isotropic and (strictly) strongly elliptic in Y_1, we have

$$\mu>0,\ 2\mu+\lambda>0 \quad \text{a.e. in } Y_1,$$

which implies that

$$(\lambda+2\mu)^2-\Big(\lambda+\frac{d}{2}\Big)^2\geqslant 0 \quad \text{a.e. in } Y_1.$$

Hence, taking into account assumption (3.7), equality (3.26) implies the following three conditions:

$$\Big[(\lambda+2\mu)^2-\Big(\lambda+\frac{d}{2}\Big)^2\Big](\xi_2\eta_2+\xi_3\eta_3)^2=0 \quad \text{a.e. in } Y_1, \tag{3.27}$$

$$\xi_2\eta_3=\xi_3\eta_2, \tag{3.28}$$

$$d\Big(\mu-\frac{d}{4}\Big)\xi_1^2(\eta_2^2+\eta_3^2)=0 \quad \text{a.e. in } Y_1. \tag{3.29}$$

We will now prove by contradiction that we cannot have $d=4\mu$ a.e. in Y_1. Otherwise, equalities (3.23), (3.24) and (3.25) can be written as

$$\begin{cases} 0=\xi_1\eta_3-\xi_3\eta_1,\\ 0=\xi_1\eta_2-\xi_2\eta_1,\\ 0=\xi_1\eta_1+\xi_2\eta_2+\xi_3\eta_3. \end{cases} \tag{3.30}$$

Under these conditions, if $\eta_1\neq 0$, then the first and second equalities of (3.30) lead to

$$\xi_3=\eta_3\frac{\xi_1}{\eta_1},\quad \xi_2=\eta_2\frac{\xi_1}{\eta_1}.$$

Replacing ξ_2 and ξ_3 in the third equality in (3.30), we obtain

$$\xi_1(\eta_1^2+\eta_2^2+\eta_3^2)=0.$$

Since $\eta\neq 0$, we get $\xi_1=0$. This implies that $\xi_2=\xi_3=0$, a contradiction with $\xi\neq 0$. Therefore, we necessarily have $\eta_1=0$. Moreover, using the two first equalities of (3.30) and the fact that $\eta\neq 0$, we obtain $\xi_1=0$. As a consequence, (3.30) reduces to

$$\xi_2\eta_2+\xi_3\eta_3=0. \tag{3.31}$$

If $\eta_2\neq 0$, then using (3.28) we get

$$\xi_3=\xi_2\frac{\eta_3}{\eta_2},$$

and replacing ξ_3 in the previous equality, it yields

$$\xi_2(\eta_2^2+\eta_3^2)=0.$$

Again, since $\eta\neq 0$, we have $\xi_2=0$. Using (3.28) and the assumption $\eta_2\neq 0$, it follows that $\xi_3=0$, again a contradiction with $\xi,\eta\neq 0$. Thus, we necessarily have

$\eta_2 = 0$. Taking into account that $\eta_1 = \eta_2 = 0$ we have $\eta_3 \neq 0$, hence from (3.31) we deduce that $\xi_3 = 0$. Now (3.28) is written as $\xi_2\eta_3 = 0$. However, recall that $\xi_1 = \xi_3 = \eta_1 = \eta_2 = 0$. This implies that either $\xi = 0$ or $\eta = 0$, a contradiction.

We have just shown that the set $\{d < 4\mu\}$ has a positive Lebesgue measure. Similarly, we can check that $d > 0$. Using (3.27) and (3.29) together with $0 < d \leqslant 4\mu$, we deduce that

$$\xi_2\eta_2 + \xi_3\eta_3 = \xi_1^2(\eta_2^2 + \eta_3^2) = 0,$$

which combined with (3.25) also gives $\xi_1\eta_1 = 0$. As above, using the three previous equalities, (3.23), (3.24) and (3.28), we get a contradiction with the fact that $\xi, \eta \neq 0$. Therefore, we have proved that $\mathbb{L}^0$ is strongly elliptic if (3.7) holds for some d. □

3.2. Rank-two lamination. — In the proof of Proposition 3.1 for dimension three [8, §5.2], Gutiérrez performed a rank-one laminate mixing a strongly elliptic but not semi-very strongly isotropic material $\mathbb{L}_a$, and a very strongly elliptic isotropic material $\mathbb{L}_b$. However, as it was noted at the beginning of the section, there are some cases for which the strong ellipticity of the homogenized tensor is not lost after this first lamination. In fact, our Theorem 3.3 shows that for a general rank-one laminate, it is not possible to lose the strong ellipticity through homogenization if there exists a matrix $D \in \mathbb{R}^{3\times3}$ satisfying condition (3.1). As done in [8], we need to perform a second lamination with a third material $\mathbb{L}_c$ which can be very strongly elliptic, in order to lose the strong ellipticity in those cases.

Our purpose is to justify Gutiérrez' approach using formally 1^*-convergence (see [8, §3]), by a homogenization procedure using the Γ-convergence result of Theorem 2.4.

Theorem 3.9. — *For any strongly elliptic but not semi-very strongly elliptic isotropic tensor $\mathbb{L}_a$ whose Lamé coefficients satisfy*

$$4\mu_a + 3\lambda_a > 0, \tag{3.32}$$

there exist two very strongly elliptic isotropic tensors $\mathbb{L}_b, \mathbb{L}_c$ and volume fractions $\theta_1, \theta_2 \in (0,1)$ such that the tensor $\mathbb{L}_2$ obtained by laminating in the direction y_2 the effective tensor $\mathbb{L}_1^*$ – firstly obtained by laminating in the direction y_1 the tensors $\mathbb{L}_a$, $\mathbb{L}_b$ with proportions θ_1, $1-\theta_1$ – and the tensor $\mathbb{L}_c$ with proportions θ_2 and $1-\theta_2$ respectively, namely

$$\mathbb{L}_2(y_2) := \chi_2(y_2)\,\mathbb{L}_1^* + \big(1 - \chi_2(y_2)\big)\,\mathbb{L}_c \quad \text{for } y_2 \in Y_1, \tag{3.33}$$

where χ_2 is a 1-periodic characteristic function with $\int_0^1 \chi_2(t)\,dt = \theta_2$, satisfies

$$\Lambda(\mathbb{L}_2) = 0, \tag{3.34}$$

and

$$\int_\Omega \mathbb{L}_2(x_2/\varepsilon)\nabla v{:}\nabla v\,dx \xrightarrow{\Gamma\text{-}H_0^1(\Omega)^3} \int_\Omega \mathbb{L}_2^0\nabla v{:}\nabla v\,dx, \tag{3.35}$$

where the homogenized tensor $\mathbb{L}_2^0$ is not strongly elliptic, i.e.,

$$\alpha_{\rm se}(\mathbb{L}_2^0) = 0. \tag{3.36}$$

Remark 3.10. — Theorem 3.9 shows that for certain strongly elliptic but not very strongly elliptic isotropic tensors, namely those whose Lamé parameters satisfy (3.32), it is possible to find two very strongly elliptic isotropic tensors for which the homogenization process through Γ-convergence using a rank-two lamination leads to the loss of ellipticity of the effective tensor.

Proof of Theorem 3.9. — We divide the proof into four steps.

Step 1. Choice of $\mathbb{L}_a$, $\mathbb{L}_b$, θ_1, θ_2. — Let $\mathbb{L}_a$ be a strongly elliptic but not semi-very strongly elliptic isotropic tensor satisfying (3.32). Our aim is to find two very strongly isotropic tensors $\mathbb{L}_b, \mathbb{L}_c$ and two volume fractions θ_1, θ_2 such that the strong ellipticity is lost through homogenization using a rank-two lamination.

Let $\chi_1, \chi_2 : \mathbb{R} \to \{0,1\}$ be two 1-periodic characteristic functions such that

$$\int_{Y_1} \chi_1(y_1)\,dy_1 = \theta_1 \quad \text{and} \quad \int_{Y_1} \chi_2(y_2)\,dy_2 = \theta_2,$$

where $\theta_1, \theta_2 \in (0,1)$ will be chosen later.

The 1^*-convergence procedure of [8, §5.2] applied to the tensor

$$\mathbb{L}_1(y_1) := \chi_1(y_1)\,\mathbb{L}_a + \big(1 - \chi_1(y_1)\big)\,\mathbb{L}_b \quad \text{for } y_1 \in Y_1, \tag{3.37}$$

yields a non-isotropic effective tensor $\mathbb{L}_1^*$. The computations of [8, §5.2] lead to an explicit expression of the tensor $\mathbb{L}_1^*$ whose non-zero entries are

$$\begin{aligned}
&(\mathbb{L}_1^*)_{1111} = \frac{1}{A},\\
&(\mathbb{L}_1^*)_{1122} = (\mathbb{L}_1^*)_{2211} = (\mathbb{L}_1^*)_{1133} = (\mathbb{L}_1^*)_{3311} = \frac{B}{A},\\
&(\mathbb{L}_1^*)_{1212} = (\mathbb{L}_1^*)_{1221} = (\mathbb{L}_1^*)_{2112} = (\mathbb{L}_1^*)_{2121} = \frac{1}{E},\\
&(\mathbb{L}_1^*)_{1313} = (\mathbb{L}_1^*)_{1331} = (\mathbb{L}_1^*)_{3113} = (\mathbb{L}_1^*)_{3131} = \frac{1}{E},\\
&(\mathbb{L}_1^*)_{2222} = \frac{B^2}{A} + 2(C+D),\\
&(\mathbb{L}_1^*)_{2233} = (\mathbb{L}_1^*)_{3322} = \frac{B^2}{A} + 2D,\\
&(\mathbb{L}_1^*)_{2323} = (\mathbb{L}_1^*)_{2332} = (\mathbb{L}_1^*)_{3223} = (\mathbb{L}_1^*)_{3232} = C,\\
&(\mathbb{L}_1^*)_{3333} = \frac{B^2}{A} + 2(C+D),
\end{aligned} \tag{3.38}$$

where

$$\begin{aligned}
&A = \frac{\theta_1}{2\mu_a + \lambda_a} + \frac{1-\theta_1}{2\mu_b + \lambda_b}, \quad B = \frac{\theta_1\lambda_a}{2\mu_a + \lambda_a} + \frac{(1-\theta_1)\lambda_b}{2\mu_b + \lambda_b},\\
&C = \theta_1\mu_a + (1-\theta_1)\mu_b, \qquad D = \frac{\theta_1\mu_a\lambda_a}{2\mu_a + \lambda_a} + \frac{(1-\theta_1)\mu_b\lambda_b}{2\mu_b + \lambda_b},\\
&E = \frac{\theta_1}{\mu_a} + \frac{1-\theta_1}{\mu_b}.
\end{aligned} \tag{3.39}$$

Now, let us specify the choice of the two very strongly elliptic isotropic tensors $\mathbb{L}_b$, $\mathbb{L}_c$, and the volume fractions θ_1, θ_2. For the Lamé parameters of material c we denote $\lambda_c = \alpha_c \mu_c$ as done in [8]. We assume that

$$-\frac{1}{4}(2\mu_a + 3\lambda_a) \leqslant \mu_b < \frac{\mu_a(2\mu_a + 3\lambda_a)}{3\lambda_a}, \tag{3.40}$$

$$\lambda_b > \frac{2\mu_b^2 \lambda_a}{\mu_a(2\mu_a + 3\lambda_a) - 3\mu_b\lambda_a}, \tag{3.41}$$

$$\theta_1 = \frac{-\lambda_b(2\mu_a + \lambda_a)}{2(\mu_b\lambda_a - \mu_a\lambda_b)}, \tag{3.42}$$

$$\alpha_c \geqslant \frac{-D}{C+D}, \tag{3.43}$$

$$\mu_c = C\frac{\alpha_c(C+2D)}{D(1+\alpha_c)}, \tag{3.44}$$

$$\theta_2 = \frac{\alpha_c(C+D)}{\alpha_c(C+D) - D(2+\alpha_c)}. \tag{3.45}$$

Observe that, thanks to the first inequality in (3.40), the tensor $\mathbb{L}_1$ given by (3.37) satisfies $\Lambda(\mathbb{L}_1) \geqslant 0$ (see [8, §4.2]). Hence, by Theorem 3.3 the homogenized tensor $\mathbb{L}_1^0 = \mathbb{L}_1^*$ (see Step 4 below) is strongly elliptic. This justifies the first lamination from the point of view of homogenization through Γ-convergence.

To conclude the first step, let us check that the previous conditions satisfy the assumptions of Theorem 3.9. The tensor $\mathbb{L}_a$ is strongly elliptic but not semi-very strongly elliptic, i.e.,

$$\mu_a > 0, \quad 2\mu_a + 3\lambda_a < 0,$$

which implies that $\mu_b > 0$. The fact that necessarily $\lambda_a < 0$ together with (3.40) implies that $\lambda_b > 0$ thanks to (3.41), and thus $\mathbb{L}_b$ is very strongly elliptic. The volume fraction θ_1 clearly belongs to $(0,1)$, since (3.42) reads as

$$\theta_1 = \frac{\lambda_b(2\mu_a + \lambda_a)}{\lambda_b(2\mu_a + \lambda_a) - \lambda_a(2\mu_b + \lambda_b)}.$$

The choice of θ_1 implies that in (3.39)

$$B = 0. \tag{3.46}$$

In addition, $C + D > 0$ as it was proved in [8, App. C] and $C + 2D < 0$ by (3.40), (3.41) and (3.42). This also implies that $D < 0$. Thanks to the previous inequalities we have $\theta_2 \in (0,1)$, $\alpha_c > 0$ and $\mu_c > 0$, which implies that $\mathbb{L}_c$ is very strongly elliptic.

Step 2. $\Lambda(\mathbb{L}_2) \geqslant 0$. — To get $\Lambda(\mathbb{L}_2) \geqslant 0$ we will prove that for

$$D := \begin{pmatrix} 4\mu_c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we have

$$\mathbb{L}_2(y_2)M{:}M + D{:}\operatorname{Cof}(M) \geqslant 0 \quad \text{a.e. } y_2 \in Y_1, \text{ for all } M \in \mathbb{R}^{N\times N}. \tag{3.47}$$

We need to prove that the previous inequality holds in each homogeneous phase of $\mathbb{L}_2$. Firstly, for the phase $\mathbb{L}_c$ which is isotropic and very strongly elliptic, we get for any $M \in \mathbb{R}^{3\times 3}$,

$$\begin{aligned}
\mathbb{L}_c M{:}M + D{:}\operatorname{Cof}(M)
&= 2\mu_c\Big[M_{11}^2 + M_{22}^2 + M_{33}^2 \\
&\qquad + 2\Big(\frac{M_{12}+M_{21}}{2}\Big)^2 + 2\Big(\frac{M_{13}+M_{31}}{2}\Big)^2 + 2\Big(\frac{M_{23}+M_{32}}{2}\Big)^2\Big] \\
&\qquad + \lambda_c(M_{11}+M_{22}+M_{33})^2 + 4\mu_c(M_{22}M_{33} - M_{23}M_{32}) \\
&= (\lambda_c + 2\mu_c)(M_{11}^2 + M_{22}^2 + M_{33}^2) \\
&\qquad + 2\lambda_c(M_{11}M_{22} + M_{11}M_{33}) + 2(\lambda_c + 2\mu_c)M_{22}M_{33} \\
&\qquad + \mu_c(M_{12}+M_{21})^2 + \mu_c(M_{31}+M_{13})^2 + \mu_c(M_{23}-M_{32})^2.
\end{aligned}$$

This quantity is non-negative for any $M \in \mathbb{R}^{3\times 3}$, since the following matrix is positive semi-definite:

$$\begin{pmatrix} \lambda_c + 2\mu_c & \lambda_c & \lambda_c \\ \lambda_c & \lambda_c + 2\mu_c & \lambda_c + 2\mu_c \\ \lambda_c & \lambda_c + 2\mu_c & \lambda_c + 2\mu_c \end{pmatrix},$$

due to the strong ellipticity of $\mathbb{L}_c$. Therefore, the desired inequality holds for the homogeneous phase $\mathbb{L}_c$.

Secondly, we need to check the same inequality for the phase with $\mathbb{L}_1^*$. By (3.38) we have for $M \in \mathbb{R}^{3\times 3}$,

$$\begin{aligned}
\mathbb{L}_1^* M{:}M + D{:}\operatorname{Cof}(M) &= \frac{1}{A}M_{11}^2 + \Big[\frac{B^2}{A} + 2(C+D)\Big](M_{22}^2 + M_{33}^2) \\
&\qquad + 2\frac{B}{A}(M_{11}M_{22} + M_{11}M_{33}) \\
&\qquad + 2\Big[\frac{B^2}{A} + 2D + 2\mu_c\Big](M_{22}M_{33}) \\
&\qquad + \frac{1}{E}(M_{12}+M_{21})^2 + \frac{1}{E}(M_{13}+M_{31})^2 \\
&\qquad + C(M_{23}^2 + M_{32}^2) + 2(C - 2\mu_c)M_{23}M_{32}.
\end{aligned}$$

Since $E \geqslant 0$, this quantity is non-negative for any $M \in \mathbb{R}^{3\times 3}$ if the following two matrices are positive semi-definite:

$$\begin{pmatrix} 1/A & B/A & B/A \\ B/A & B^2/A + 2(C+D) & B^2/A + 2D + 2\mu_c \\ B/A & B^2/A + 2D + 2\mu_c & B^2/A + 2(C+D) \end{pmatrix}, \tag{3.48}$$

$$\begin{pmatrix} C & C - 2\mu_c \\ C - 2\mu_c & C \end{pmatrix}. \tag{3.49}$$

Since $C \geqslant 0$, the matrix (3.49) is positive semi-definite if and only if $\mu_c \leqslant C$. Taking into account that $\mu_c \leqslant C$, we can check that the matrix (3.48) is positive semi-definite

if $-(C+2D)\leqslant\mu_c$. Therefore, the matrices (3.48) and (3.49) are positive semi-definite if

$$-(C+2D)\leqslant\mu_c\leqslant C. \tag{3.50}$$

By the definition (3.44) of μ_c, we deduce that the first inequality of (3.50) holds if and only if

$$\frac{\alpha_c C}{-D(1+\alpha_c)}\geqslant 1,$$

which is satisfied due to inequality (3.43). For the second inequality of (3.50), we need to check that (see (3.44))

$$\frac{\alpha_c(C+2D)}{D(1+\alpha_c)}\leqslant 1,$$

or equivalently,

$$\alpha_c\geqslant\frac{D}{C+D}.$$

This is true since $\alpha_c>0$ by (3.43) and $\frac{D}{C+D}<0$. Therefore, condition (3.47) holds true, and consequently

$$\Lambda(\mathbb{L}_2)\geqslant 0. \tag{3.51}$$

Step 3. $\mathbb{L}_2$ *loses the strong ellipticity through homogenization*. — On the one hand, due to $\Lambda(\mathbb{L}_2)\geqslant 0$, by virtue of Theorem 2.4 the Γ-convergence (3.35) holds with the homogenized tensor $\mathbb{L}_2^0$ which is given by the minimization formula (2.2) replacing $\mathbb{L}$ by $\mathbb{L}_2$.

On the other hand, following Gutiérrez' 1^*-convergence procedure we obtain a homogenized tensor $\mathbb{L}_2^*$ such that (see [8, §5.2] for the expression of $\mathbb{L}_2^*$)

$$\mathbb{L}_2^*(e_3\otimes e_3){:}(e_3\otimes e_3)=I_1+\frac{G_1^2}{F_1},$$

where by (3.46),

$$\begin{aligned}
I_1&=4(1-\theta_2)\,\frac{1+\alpha_c}{2+\alpha_c}+2\theta_2 C\,\frac{C+2D}{C+D},\\
G_1&=(1-\theta_2)\,\frac{\alpha_c}{2+\alpha_c}+\theta_2\,\frac{D}{C+D},\\
F_1&\neq 0.
\end{aligned}$$

It is not difficult to check that the choice of $\mathbb{L}_b$, $\mathbb{L}_c$, θ_1, θ_2 leads to $I_1=G_1=0$, which yields

$$\mathbb{L}_2^*(e_3\otimes e_3){:}(e_3\otimes e_3)=0. \tag{3.52}$$

To conclude the proof it is enough to show that

$$\mathbb{L}_2^*=\mathbb{L}_2^0. \tag{3.53}$$

Indeed, thanks to $\mathbb{L}_2^*=\mathbb{L}_2^0$ equality (3.52) implies the loss of ellipticity (3.36), and (3.36) implies $\Lambda(\mathbb{L}_2)\leqslant 0$. This combined with (3.51) finally shows the desired lost of functional coercivity (3.34).

Step 4. $\mathbb{L}_2^* = \mathbb{L}_2^0$. — By formally using 1^*-convergence in terms of [2, Lem. 3.1], Gutiérrez's computations for the tensor $\mathbb{L}_2^*$ in [8, §5.2] can be written as

$$\begin{cases} A^{-1}[\mathbb{L}_2^*] = \displaystyle\int_0^1 A^{-1}[\mathbb{L}_2](t)\,dt, \\ A_{im}^{-1}[\mathbb{L}_2^*](\mathbb{L}_2^*)_{2mkl} = \displaystyle\int_0^1 \big(A_{im}^{-1}[\mathbb{L}_2](t)(\mathbb{L}_2)_{2mkl}(t)\big)\,dt, \\ (\mathbb{L}_2^*)_{ijkl} - (\mathbb{L}_2^*)_{ij2m}A_{mn}^{-1}[\mathbb{L}_2^*](\mathbb{L}_2^*)_{2nkl} \\ \qquad\qquad = \displaystyle\int_0^1 \big((\mathbb{L}_2)_{ijkl}(t) - (\mathbb{L}_2)_{ij2m}(t)A_{mn}^{-1}[\mathbb{L}_2](t)(\mathbb{L}_2)_{2nkl}(t)\big)\,dt, \end{cases} \tag{3.54}$$

where in the present context, for any $\mathbb{L} \in L^\infty_{\rm per}(Y_1; \mathscr{L}_s(\mathbb{R}^{3\times 3}))$, $A[\mathbb{L}] \in L^\infty_{\rm per}(Y_1; \mathbb{R}^{3\times 3}_s)$ is defined by

$$A[\mathbb{L}](y_2)\xi := [\mathbb{L}(y_2)(\xi \otimes e_2)]e_2 \quad \text{for } y_2 \in Y_1 \text{ and } \xi \in \mathbb{R}^3.$$

By focusing on the first equality of (3.54) we have

$$A^{-1}[\mathbb{L}_2^*] = \int_0^1 A^{-1}[\mathbb{L}_2](t)\,dt = \theta_2 A^{-1}[\mathbb{L}_1^*] + (1-\theta_2)A^{-1}[\mathbb{L}_c], \tag{3.55}$$

where all the quantities are finite. Now, similarly to the proof of Theorem 2.4 we consider the perturbation of $\mathbb{L}_2$ defined by

$$\mathbb{L}_\delta := \mathbb{L}_2 + \delta\,\mathbb{I}_s \quad \text{for } \delta > 0. \tag{3.56}$$

On the one hand, due to $\Lambda(\mathbb{L}_\delta) > 0$ (which by (1.14) implies $0 < \Lambda_{\rm per}(\mathbb{L}_\delta) \leqslant \alpha_{\rm se}(\mathbb{L}_\delta)$), thanks to [2, Lem. 3.2] the 1^*-limit $\mathbb{L}_\delta^*$ of $\mathbb{L}_\delta$ and the homogenized tensor $\mathbb{L}_\delta^0$ of $\mathbb{L}_\delta$ defined by (2.2) agree. Then, applying [2, Lem. 3.1] with $\mathbb{L}_\delta$ we get that

$$A^{-1}[\mathbb{L}_\delta^*] = \int_0^1 A^{-1}[\mathbb{L}_\delta](t)dt = \theta_2 A^{-1}[\mathbb{L}_1^* + \delta\,\mathbb{I}_s] + (1-\theta_2)A^{-1}[\mathbb{L}_c + \delta\,\mathbb{I}_s]. \tag{3.57}$$

Observe that we have

$$A[\mathbb{L}_1^* + \delta\,\mathbb{I}_s] \geqslant A[\mathbb{L}_1^*], \qquad A[\mathbb{L}_1^* + \delta\,\mathbb{I}_s] \longrightarrow A[\mathbb{L}_1^*] \quad \text{as } \delta \longrightarrow 0,$$

where the previous inequality must be understood in the sense of the quadratic forms. This combined with the fact that both $\mathbb{L}_1^* + \delta\,\mathbb{I}_s$ and $\mathbb{L}_1^*$ are strongly elliptic tensors (which implies that the previous matrices are positive definite), yields

$$A^{-1}[\mathbb{L}_1^* + \delta\,\mathbb{I}_s] \leqslant A^{-1}[\mathbb{L}_1^*],$$

and thus,

$$A^{-1}[\mathbb{L}_1^* + \delta\,\mathbb{I}_s] \longrightarrow A^{-1}[\mathbb{L}_1^*] \quad \text{as } \delta \longrightarrow 0.$$

Similarly, we have

$$A^{-1}[\mathbb{L}_c + \delta\,\mathbb{I}_s] \longrightarrow A^{-1}[\mathbb{L}_c] \quad \text{as } \delta \longrightarrow 0.$$

Hence, from the two previous convergences and taking into account (3.55), (3.57), we deduce that

$$A^{-1}[\mathbb{L}_\delta^*] \longrightarrow A^{-1}[\mathbb{L}_2^*] \quad \text{as } \delta \longrightarrow 0.$$

On the other hand, following the proof of Theorem 2.4 we have

$$\mathbb{L}^*_\delta = \mathbb{L}^0_\delta \longrightarrow \mathbb{L}^0_2 \quad \text{as } \delta \longrightarrow 0.$$

Therefore, we obtain the equality

$$A^{-1}[\mathbb{L}^0_2] = A^{-1}[\mathbb{L}^*_2]. \tag{3.58}$$

Using similar arguments, we can prove that $\mathbb{L}^0_2$ and $\mathbb{L}^*_2$ satisfy for any $i,j,k,l \in \{1,2,3\}$,

$$A^{-1}_{im}[\mathbb{L}^*_2](\mathbb{L}^*_2)_{2mkl} = A^{-1}_{im}[\mathbb{L}^0_2](\mathbb{L}^0_2)_{2mkl}, \tag{3.59}$$

$$(\mathbb{L}^*_2)_{ijkl} - (\mathbb{L}^*_2)_{ij2m}A^{-1}_{mn}[\mathbb{L}^*_2](\mathbb{L}^*_2)_{2nkl} = (\mathbb{L}^0_2)_{ijkl} - (\mathbb{L}^0_2)_{ij2m}A^{-1}_{mn}[\mathbb{L}^0_2](\mathbb{L}^0_2)_{2nkl}. \tag{3.60}$$

Since the set of equalities (3.54) completely determine the tensor $\mathbb{L}^*_2$, equalities (3.58), (3.59), (3.60) thus imply the desired equality (3.53), which concludes the proof. □

Appendix

Proof of Theorem 2.2. — We simply adapt the proof of [2, Th. 2.2] to dimension 3.

Firstly, let us prove the first part of the theorem, i.e., $\Lambda(\mathbb{L}) \geqslant 0$. The quasi-affinity of the cofactors (see [3]) reads as

$$\int_{Y_3} \mathrm{adj}_{ii}(\nabla v)\, dy = 0, \quad \forall\, v \in C^\infty_c(\mathbb{R}^3;\mathbb{R}^3),\ \forall\, i \in \{1,2,3\}. \tag{3.61}$$

As a consequence, for any $d \in \mathbb{R}$, the definition of $\Lambda(\mathbb{L})$ can be rewritten as

$$\Lambda(\mathbb{L}) = \inf\Big\{ \int_{\mathbb{R}^3} \Big[\mathbb{L}e(v){:}e(v) + d\textstyle\sum_{i=1}^3 \mathrm{adj}_{ii}(\nabla v)\Big] dy,\ v \in C^\infty_c(\mathbb{R}^3;\mathbb{R}^3)\Big\}.$$

If we compute the integrand in the previous infimum, we obtain

$$\begin{aligned}\Lambda(\mathbb{L}) = \inf\Big\{ \int_{\mathbb{R}^3} \Big[& P(y;\partial_1 v_1, \partial_2 v_2, \partial_3 v_3) + Q(y;\partial_3 v_2, \partial_2 v_3) \\ & + Q(y;\partial_3 v_1, \partial_1 v_3) + Q(\partial_2 v_1, \partial_1 v_2)\Big] dy,\ v \in C^\infty_c(\mathbb{R}^3;\mathbb{R}^3)\Big\},\end{aligned} \tag{3.62}$$

where

$$\begin{cases} P(y;a,b,c) := \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} \lambda+2\mu & \lambda+d/2 & \lambda+d/2 \\ \lambda+d/2 & \lambda+2\mu & \lambda+d/2 \\ \lambda+d/2 & \lambda+d/2 & \lambda+2\mu \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \\ Q(y;a,b) := \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} \mu & \mu-d/2 \\ \mu-d/2 & \mu \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \end{cases}$$

We can check that condition (2.7) with $d \geqslant 0$ implies that the quadratic forms P and Q are non negative. Hence, the integrand in (3.62) is pointwisely non-negative, and thus $\Lambda(\mathbb{L}) \geqslant 0$.

Now, let us prove that $\Lambda_{\rm per}(\mathbb{L}) > 0$. By the definition of $\Lambda_{\rm per}(\mathbb{L})$ and using the same argument as before, we have

$$\Lambda_{\rm per}(\mathbb{L}) = \inf\Big\{\int_{Y_3}\Big[\mathbb{L}e(v){:}e(v) + d\sum_i \mathrm{adj}_{ii}(\nabla v)\Big]dy, \\ v \in H^1_{\rm per}(Y_3;\mathbb{R}^3), \int_{Y_3}|\nabla v|^2\,dy = 1\Big\}.$$

Similar computations lead to

$$\Lambda_{\rm per}(\mathbb{L}) = \inf\Big\{\int_{Y_3}\big[P(y;\partial_1 v_1,\partial_2 v_2,\partial_3 v_3) + Q(y;\partial_3 v_2,\partial_2 v_3) \\ + Q(y;\partial_3 v_1,\partial_1 v_3) + Q(\partial_2 v_1,\partial_1 v_2)\big]dy\Big\}. \tag{3.63}$$

Take $y \in Z_i, i \in I$. Then, using that $4\mu_i = d$, we have

$$P(y;a,b,c) = (\lambda_i + 2\mu_i)(a+b+c)^2 \geqslant 0,$$

and

$$Q(y;a,b) = \mu_i(a-b)^2 \geqslant 0.$$

For $y \in Z_j, j \in J$, using that $2\mu_j + 3\lambda_j = -d$, we get

$$P(y;a,b,c) = \Big(\mu_j + \frac{\lambda_j}{2}\Big)\big[(a-b)^2 + (a-c)^2 + (b-c)^2\big] \geqslant 0,$$

and

$$Q(y;a,b) = d\Big(\mu_j + \frac{d}{4}\Big) \geqslant 0.$$

Finally, for $y \in Z_k$, $k \in K$, since $-(2\mu_k + 3\lambda_k) < d < 4\mu_k$, it is easy to see that the quadratic forms P and Q are positive semi-definite. Therefore, we have just proved that there exists $\alpha > 0$ such that

$$\begin{aligned} &P(y;a,b,c) \geqslant \alpha(a+b+c)^2, \\ &Q(y;a,b) \geqslant \alpha(a-b)^2, \end{aligned} \qquad y \in Z_i,\ i \in I, \tag{3.64}$$

$$\begin{aligned} &P(y;a,b,c) \geqslant \alpha[(a-b)^2 + (a-c)^2 + (b-c)^2], \\ &Q(y;a,b) \geqslant \alpha(a^2+b^2), \end{aligned} \qquad y \in Z_j,\ j \in J, \tag{3.65}$$

$$\begin{aligned} &P(y;a,b,c) \geqslant \alpha(a^2+b^2+c^2), \\ &Q(y;a,b) \geqslant \alpha(a^2+b^2), \end{aligned} \qquad y \in Z_k,\ k \in K, \tag{3.66}$$

which implies that $\Lambda_{\rm per}(\mathbb{L}) \geqslant 0$.

Assume by contradiction that $\Lambda_{\rm per}(\mathbb{L}) = 0$. In this case there exists a sequence $v^n \in H^1_{\rm per}(Y_3;\mathbb{R}^3)$ with

$$\int_{Y_3} v^n\,dy = 0,$$

such that

$$\int_{Y_3}|\nabla v^n|^2\,dy = 1, \quad \forall\, n \in \mathbb{N}, \tag{3.67}$$

together with

$$\int_{Y_3}\mathbb{L}(y)e(v^n){:}e(v^n)\,dy \longrightarrow 0.$$

By the Poincaré-Wirtinger inequality, v^n is bounded in $L^2(Y_3;\mathbb{R}^3)$. Moreover, by (3.63) we have

$$\int_{Y_3}\Big[P(y;\partial_1 v_1^n,\partial_2 v_2^n,\partial_3 v_3^n)+\sum_{i<j}Q(y;\partial_j v_i^n,\partial_i v_j^n)\Big]dy\longrightarrow 0. \tag{3.68}$$

Take $k\in K$. Using (3.66) we get

$$\int_{Z_k}\Big[P(y;\partial_1 v_1^n,\partial_2 v_2^n,\partial_3 v_3^n)+\sum_{i<j}Q(y;\partial_j v_i^n,\partial_i v_j^n)\Big]dy\geqslant\alpha\int_{Z_k}|\nabla v^n|^2\,dy.$$

Then, using (3.68) and the fact that both P and Q are non negative, it follows that

$$\int_{Z_k}|\nabla v^n|^2\,dy\longrightarrow 0\quad\forall\,k\in K,$$

and therefore

$$\lim_{n\to\infty}\sum_{k\in K}\int_{Z_k}\sum_{q,r=1,2,3}(\partial_r v_q^n)^2\,dy=0. \tag{3.69}$$

Next, take $j\in J$. By (3.65) we obtain

$$\int_{Z_j}\Big[P(y;\partial_1 v_1^n,\partial_2 v_2^n,\partial_3 v_3^n)+\sum_{i<k}Q(y;\partial_k v_i^n,\partial_i v_k^n)\Big]dy \\ \geqslant\alpha\int_{Z_j}\sum_{i<k}\Big[(\partial_i v_i^n-\partial_k v_k^n)^2+(\partial_k v_i^n)^2+(\partial_i v_k^n)^2\Big]dy.$$

Again using (3.68) and the non-negativity of P and Q we get

$$\lim_{n\to\infty}\int_{Z_j}\Big[(\partial_i v_i^n-\partial_k v_k^n)^2+(\partial_k v_i^n)^2+(\partial_i v_k^n)^2\Big]=0\quad\text{for } i,k\in\{1,2,3\},\ i<k. \tag{3.70}$$

From (3.70) and the continuity of the operator $\partial_1{:}L^2(Z_j)\to H^{-1}(Z_j)$ we deduce that

$$\begin{cases}\partial_2(\partial_1 v_1^n)=\partial_1(\partial_2 v_1^n)\longrightarrow 0\quad\text{strongly in } H^{-1}(Z_j),\\ \partial_1(\partial_1 v_1^n)=\partial_1(\partial_1 v_1^n-\partial_2 v_2^n)+\partial_2(\partial_1 v_2^n)\longrightarrow 0\quad\text{strongly in } H^{-1}(Z_j).\end{cases} \tag{3.71}$$

By (3.67) we also have

$$\partial_1 v_1^n\text{ is bounded in } L^2(Z_j). \tag{3.72}$$

However, thanks to Korn's Lemma (see, e.g. [11]) the following norms are equivalent in $L^2(Z_j)$:

$$\begin{cases}\|\nabla\cdot\|_{H^{-1}(Z_j;\mathbb{R}^3)}+\|\cdot\|_{H^{-1}(Z_j)},\\ \|\cdot\|_{L^2(Z_j)}.\end{cases}$$

Hence, from estimates (3.71), (3.72) and the compact embedding of L^2 into H^{-1}, it follows that

$$\partial_1 v_1^n\text{ is strongly convergent in } L^2(Z_j).$$

Furthermore, by (3.71) and the fact that Z_j is connected for all j, there exists $c_j\in\mathbb{R}$ such that

$$\partial_1 v_1^n\longrightarrow c_j\text{ strongly in } L^2(Z_j),$$

which combined with (3.70) yields

$$\nabla v^n \longrightarrow c_j I_3 \text{ strongly in } L^2(Z_j)^3.$$

Since v^n is bounded in $L^2(Y_3;\mathbb{R}^3)$, we can conclude that there exists $V_j \in \mathbb{R}^3$ such that

$$v^n \longrightarrow v := c_j y + V_j \quad \text{strongly in } H^{-1}(Z_j;\mathbb{R}^3). \tag{3.73}$$

- In Case 1, by the periodicity of the limit $c_j y + V_j$ it is necessary to have $c_j = 0$.
- In Case 2, since Z_k is connected, by (3.69) there exists a constant c_k such that v_n converges to $\chi_{Z_j} v + \chi_{Z_k} c_k$ strongly in $H^1(Z_j \cup Z_k)$. Hence, since the sets Z_j and Z_k are regular, the trace of v must be equal to c_k a.e. on $\partial Z_j \cap \partial Z_k$. Therefore, the only way for $c_j y + V_j$ to remain constant on a set of non-null $\mathscr{H}^2$-measure is to have $c_j = 0$.

In both cases this implies that ∇v^n converges strongly to 0 in $L^2(Z_j;\mathbb{R}^{3\times 3})$, and thus

$$\lim_{n\to\infty} \sum_{j\in J} \int_{Z_j} \sum_{r,q=1,2,3} (\partial_q v_r^n)^2 dy = 0. \tag{3.74}$$

Finally, take $i \in I$. By (3.64) we have

$$\begin{aligned}\int_{Z_i} \Big[P(y;\partial_1 v_1^n, \partial_2 v_2^n, \partial_3 v_3^n) + \sum_{r<q} Q(y;\partial_q v_r^n, \partial_r v_q^n)\Big] dy \\ \geqslant \alpha \int_{Z_i} \Big[(\partial_1 v_1^n + \partial_2 v_2^n + \partial_3 v_3^n)^2 + (\partial_2 v_1^n - \partial_1 v_2^n)^2 \\ + (\partial_3 v_1^n - \partial_1 v_3^n)^2 + (\partial_3 v_2^n - \partial_2 v_3^n)^2\Big] dy.\end{aligned}$$

By virtue of (3.68) we also have

$$\begin{aligned}\int_{Z_i} \Big[(\partial_1 v_1^n + \partial_2 v_2^n + \partial_3 v_3^n)^2 + (\partial_2 v_1^n - \partial_1 v_2^n)^2 \\ + (\partial_3 v_1^n - \partial_1 v_3^n)^2 + (\partial_3 v_2^n - \partial_2 v_3^n)^2\Big] dy \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.75}$$

Limits (3.74), (3.69) combined with (3.61) yield

$$\lim_{n\to\infty} \sum_{i\in I} \int_{Z_i} \sum_{r=1}^{3} \mathrm{adj}_{rr}(\nabla v^n) dy = 0.$$

Therefore, upon subtracting twice this quantity to the sum over $i \in I$ of (3.75) we conclude that

$$\lim_{n\to\infty} \sum_{i\in I} \int_{Z_i} \sum_{r,q=1}^{3} (\partial_q v_r^n)^2 dy = 0. \tag{3.76}$$

Finally, limits (3.74), (3.69) and (3.76) contradict condition (3.67). The proof is thus complete. □

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