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ON A COUNTER-EXAMPLE TO QUANTITATIVE JACOBIAN BOUNDS

by Yves Capdeboscq

Abstract. — This note provides a counter-example to the local positivity of the Jacobian determinant for solutions of the conductivity equation in dimension 3. It shows that the sign of the determinant cannot be imposed by an a priori choice of boundary data in $H^{1/2}(\partial\Omega)$ depending only on the upper and lower bound of the conductivity, even locally. The argument uses a scalar two-phase conductivity constructed by Briane, Milton & Nesi [11, 10].

Résumé (Sur un contre-exemple aux bornes quantitatives du jacobien)

Cette note fournit un contre-exemple à la positivité locale du déterminant jacobien des solutions de l'équation de conduction en dimension 3. On montre que le signe du déterminant ne peut pas être imposé par un choix a priori de données au bord dans $H^{1/2}(\partial\Omega)$ dépendant seulement des bornes inférieure et supérieure de la conductivité, même localement. L'argument utilise une conductivité scalaire à deux phases construite par Briane, Milton & Nesi [11, 10].

Contents

1. Introduction

Let B_1 be the unit open ball centred at the origin in $\mathbb{R}^d$. Given $\gamma \in L^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$ such that $1 \leqslant \gamma(x) \leqslant \beta$ for a.e. $x \in B_1$, consider $U = \left(u^1, \dots, u^d\right)$ whose components are solutions of the Dirichlet boundary value problems

$$\begin{aligned} -\operatorname{div}\left(\gamma D u^i\right) &= 0 \quad \text{in } B_1,\\ u^i &= \phi^i \quad \text{on } \partial B_1, \end{aligned}$$

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where $\phi = (\phi^1, \ldots, \phi^d)$ is an homeomorphism from ∂B_1 onto a Jordan curve Γ which is the boundary of a bounded convex domain. The vector valued map U is called the γ-harmonic extension of ϕ.

When $d = 2$ it is known that without further assumption on γ this implies that $\det DU > 0$ almost everywhere in B_1 – for more precise bounds see [1, 2]. When ϕ is also a $C^{1,\alpha}$ diffeomorphism from B_ρ with $\rho > 1$ and $\gamma \in C^{0,\alpha}$ then $\det DU$ is bounded below by a positive constant on B_1, see [7]. For harmonic maps, this fact follows from the Radó-Kneser-Choquet Theorem [13, Chapter 3.1].

The Radó-Kneser-Choquet Theorem (and its extensions) does not hold when $d = 3$, see [24, 18, 11, 13]. The recent paper [3] contains a review of the numerous pathologies and open problems that appear in this case. All currently available counter-examples are based on specific boundary data : this is natural, as for example the boundary data $x = (x^1, x^2, x^3)$ has an harmonic extension (which is itself) of determinant 1 in B_1. In any dimension $d \geqslant 3$, for a fixed sufficiently regular conductivity γ, it is known that there exists a set of Dirichlet data $(\phi^1, \ldots, \phi^N)$ with $N \in \{d, d+1, 2d+1\}$ such that the rank of $[Du^1, \ldots, Du^N]$ is d over all the whole domain, and a positive determinant constraint is satisfied locally by d of the associated γ-harmonic maps $(u^1, \ldots, u^N)$, see section 3.

In the context of coupled, or hybrid, inverse problems, it is desirable to be able to choose the Dirichlet data independently of the conductivity γ, which is an unknown of the problem.

When $d = 3$, extrapolating from the existence results mentioned above for fixed conductivities, one could think that given a priori bounds on the conductivity, $1 \leqslant \gamma(x) \leqslant \beta$ in B_1, and possibly assuming that γ is sufficiently regular, if a large enough variety of boundary conditions is used, the positivity of the determinant can be guaranteed locally for any such γ. Indeed, this is true if $\beta - 1$ is small enough by a perturbation argument.

This note shows that if β is larger than some universal constant δ_0 defined in Lemma 2 then in any C^1 bounded open domain $\Omega \subset \mathbb{R}^3$, no Dirichlet data in $H^{1/2}(\partial\Omega)^3$ can enforce a local determinant constraint for every real two-phase constant conductivities (or all real C^∞ conductivities) satisfying $1 \leqslant \gamma(x) \leqslant \beta$.

The proof is constructive. It uses a function γ introduced in [11] which is defined over $[0,1]^3$ and repeated periodically, together with a regularity result in the theory of homogenization proved in [19]. The argument is that any Dirichlet data whose harmonic extension would satisfy locally a given positivity determinant constraint has a $\gamma(n\cdot)$-harmonic extension whose determinant changes sign locally, for n large enough. The frequency n depends on the positive lower bound for the determinant and the size of the open set where the constraint is imposed, but not on the boundary condition.

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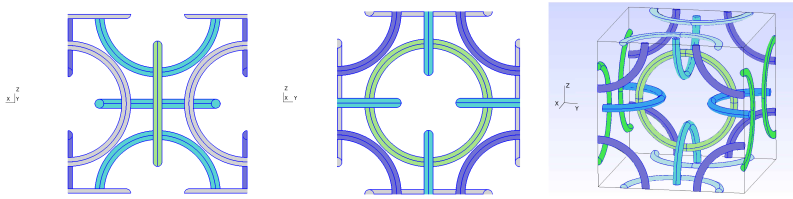

Figure 2.1. Sketch of the periodic set. The various colors were added for readability, to highlight which elements correspond to the same ring, once the cell is repeated periodically. Q is the union of all rings.

2. Main result

Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with C^1 boundary. Let Ω' a sub-domain of Ω, such that $d\,(\Omega', \partial\Omega) > 0$.

Given $\phi = \big(\phi^1, \phi^2, \phi^3\big)$, with $\phi^i \in H^{1/2}\,(\partial\Omega; \mathbb{R})$ for $i = 1, 2, 3$, let $U_0 = (u_0^1, u_0^2, u_0^3) \in H^1(\Omega; \mathbb{R}^3)$ be the harmonic extension of ϕ, that is, for $i = 1, 2, 3$, $u_0^i \in H^1(\Omega)$ is the unique solution of

$$\begin{aligned} \Delta u_0^i &= 0 \quad \text{in } \Omega, \\ u_0^i &= \phi^i \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Let γ be a piecewise constant 1-periodic function γ such that in $Y = [0,1]^3$ we have

$$\gamma\,(y) = 1 + (\delta - 1)\,\chi_Q(y) \quad \text{for all } y \in Y,$$

where χ_Q is the Y-periodic characteristic function Q, made of rotations and translations of a scaled copy of the tori (that is, circular annuli whose cross-section is a disk), with cubic symmetry. An illustration of one such Q is given in Figure 2.1.

This type of construction was originally introduced in [11]; the variant presented here was introduced in [10] (see also [16]). The value of δ is fixed, and decided by Lemma 2 below.

Write $U_n = (u_n^1, u_n^2, u_n^3)$ where for $i = 1, 2, 3$, $u_n^i \in H^1(\Omega; \mathbb{R}^3)$ is the solution of

$$\begin{aligned} \operatorname{div}\big(\gamma\,(nx)\,Du_n^i\big) &= 0 \quad \text{in } \Omega, \\ u_n^i &= \phi^i \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

Note that due to cubic symmetry the corresponding effective (or homogenized) conductivity is a positive real number and therefore as $n \to \infty$,

$$U_n \rightharpoonup U_0 \quad \text{weakly in } H^1(\Omega).$$

Given $\rho > 0$, $x_0 \in \Omega'$ such that $B_\rho(x_0) \subset \Omega'$, and $\lambda > 0$ let

$$A(\phi, x_0, \rho, \lambda) := \Big\{\phi \in H^{1/2}(\partial\Omega; \mathbb{R}^3) \,:\, \det(DU_0) > \lambda\,\|\phi\|^3_{H^{1/2}(\partial\Omega)} \text{ in } B_\rho(x_0)\Big\}, \tag{2.3}$$

where U_0 is the harmonic extension of ϕ given by (2.1). The set $A(\phi, x_0, \rho, \lambda)$ contains all boundary data whose harmonic extensions in Ω satisfy the stated lower determinant bound in $B_\rho(x_0)$. We write $|X|$ the Lebesgue measure of the set X.

The main result is the following.

Theorem 1. — *Given $\rho > 0$, $x_0 \in \Omega'$ such that $B_\rho(x_0) \subset \Omega'$, and $\lambda > 0$, let $A(\phi, x_0, \rho, \lambda)$ be defined by (2.3). There exist n, depending on ρ, Ω, Ω' and λ only, a universal constant $\tau > 0$ and two open subsets B_+ and B_- of $B_\rho(x_0)$ such that*

$$|B_+| > \tau\,|B_\rho(x_0)| \quad \text{and} \quad |B_-| > \tau\,|B_\rho(x_0)|\,,$$

and for any $\phi \in A(\phi, x_0, \rho, \lambda)$,

$$\det\left(DU_n\right)(x) < -\tau\lambda\,\|\phi\|^3_{H^{1/2}(\partial\Omega)} \quad \text{on } B_-,$$

and

$$\det\left(DU_n\right)(x) > \tau\lambda\,\|\phi\|^3_{H^{1/2}(\partial\Omega)} \quad \text{on } B_+,$$

where U_n is the $\gamma\,(n\cdot)$-harmonic extension of ϕ given by (2.2).

In other words, there is no a priori choice of boundary data which can ensure a quantitative lower bound of the Jacobian determinant for piecewise constant scalar conductivities without additional a priori information, as any given boundary condition would fail either for harmonic maps or the two-phase composite $\gamma(n\cdot)$ at a fixed scale n. This result is an application of two existing results in the literature. The first key result is a part of Theorem 3 in [11].

Lemma 2 (see Theorem 3 in [11]). — *There exist $\delta_0 > 0$, $\tau > 0$, Y_+ and Y_- open subsets of Y both of positive measure 2τ such that the periodic corrector matrix $P = D\zeta$, where ζ is the solution of*

$$\begin{aligned}\operatorname{div}\left(\gamma D\zeta\right) &= 0 \text{ in } \mathbb{R}^3,\\ \zeta(y) - y &\in H^1_\#(Y),\end{aligned}$$

satisfies

$$\det\left(P\right)(y) > 2\tau \text{ in } Y_+ \quad \text{and} \quad \det\left(P\right)(y) < -2\tau \text{ in } Y_-.$$

The second key result is a regularity result. Because the conductivity γ is piecewise constant (and therefore piecewise smooth), and because the set Q has C^∞ smooth boundaries (and therefore $C^{1,\alpha}$ smooth boundaries), the regularity results given in [20] and [19] show that U_n is also piecewise $C^{1,\beta}$ for some $\beta > 0$, up to the boundary of the set Q in Ω'. In fact, this provides uniform $W^{1,\infty}$ estimates for U_n, independently of n, see [19]. This result has been successfully expanded to provide error estimates for DU_n see [8, 21].

Lemma 3 (See Theorem 3.4 in [19], Theorem 3.6 in [8] or Theorem 4.2 in [21])
There exists a constant C depending on Ω, Ω', δ and Q only such that

$$\begin{aligned}\|DU_n\|_{L^\infty(\Omega')} &\leqslant C\|\phi\|_{H^{1/2}(\partial\Omega,\mathbb{R}^3)},\\ \|P\,(nx)\,\|_{L^\infty(\Omega')} &\leqslant C,\end{aligned}$$

and

$$\|DU_n - P\,(nx)\,DU_0\|_{L^\infty(\Omega')} \leqslant \frac{C}{n^{1/3}}\|\phi\|_{H^{1/2}(\partial\Omega,\mathbb{R}^3)}.$$

Another variant of this result (also based on [20, 19]) is given in [4]. Combining these two ingredients, we obtain our result.

Proof of Theorem 1. — In Ω', we have

$$\begin{aligned}\det(DU_n) &= \det(P(nx)\,DU_0) + R_n,\\ &= \det(P(nx))\det(DU_0) + R_n\end{aligned}$$

with

$$\|R_n\|_{L^\infty(\Omega')} \leqslant C\|DU_n - P(nx)\,DU_0\|_{L^\infty(\Omega')} \cdot \max\Big(\|DU_n\|^2_{L^\infty(\Omega')}, \|P(nx)\,DU_0\|^2_{L^\infty(\Omega')}\Big),$$

as for any two real $d\times d$ matrices,

$$|\det(A) - \det(B)| \leqslant C(d)\|A-B\|_\infty \max\Big(\|A\|_\infty^{d-1}, \|B\|_\infty^{d-1}\Big).$$

Thanks to Lemma 3,

$$R_n \leqslant \frac{C}{n^{1/3}}\,\|\phi\|^3_{H^{1/2}(\partial\Omega,\mathbb{R}^3)}.$$

In the ball $B(x_0,\rho)$, let

$$B_\pm = \big\{x\in B(x_0,\rho)\,:\exists\, p\in\mathbb{Z}^3,\ nx-p\in Y_\pm\big\}.$$

For n large enough, $|B_\pm| > \tau\,|B(x_0,\rho)|$.

In $B_\pm$, we have thanks to Lemma 2, $\pm\det(P(nx)) > 2\tau$, therefore

$$\pm\det(DU_n) > 2\tau\det(DU_0) - \frac{C}{n^{1/3}}\,\|\phi\|^3_{H^{1/2}(\partial\Omega,\mathbb{R}^3)}.$$

With $\phi\in A(\phi,x_0,\rho,\lambda)$, this implies

$$\pm\det(DU_n) > \Big(2\tau\lambda - \frac{C}{n^{1/3}}\Big)\|\phi\|^3_{H^{1/2}(\partial\Omega,\mathbb{R}^3)} > \tau\lambda\|\phi\|^3_{H^{1/2}(\partial\Omega,\mathbb{R}^3)},$$

for $n^{1/3}\tau\lambda \geqslant C$. □

Remark. — Any larger n would lead to the same conclusion, with the same bound. Note that the above argument shows that for a given radius ρ, and λ sufficiently small, one can choose $n = C\lambda^{-3}$, with a constant C depending Ω and Ω' only.

The fact that the conductivity coefficient γ is not smooth is not required for Theorem 1 to hold. This choice was made in [11] as it corresponds to realisable composites. Consider as before the periodic function γ with

$$\gamma(y) = 1 + (\delta-1)\,\chi_Q(y) \text{ for all } y\in Y,$$

with δ chosen via Lemma 2. Introducing the standard mollifier $\eta\in C^\infty(\mathbb{R})$ given by $\eta(x) = c_\eta\exp(-1/(1-|x|^2))$ for $|x|<1$ and $\eta(x)=0$ otherwise, where c_η is chosen so that $\|\eta\|_{L^1(\mathbb{R})} = 1$, write for some $M>0$ to be chosen later $\eta_M = \eta(M\cdot)$. The function $\widetilde{\gamma} = \eta_M \star \gamma$ is smooth, Y-periodic, and enjoys the same symmetries as γ because η is radial. Note that the open subsets $Y_\pm$ given in Lemma 2 are located in parts of the periodic cell where P is smooth, that is, away from δQ (see Theorem 3 in [11]). Thus, as solutions of elliptic boundary value problems depend smoothly on their coefficients, we can set M large enough so that Lemma 2 applies to $\widetilde{\gamma}$ (for another universal constant τ).

Write $\widetilde{U}_n = (\widetilde{u}_n^1, \widetilde{u}_n^2, \widetilde{u}_n^3)$ where for $i = 1, 2, 3$, $\widetilde{u}_n^i \in H^1(\Omega; \mathbb{R}^3)$ is the solution of

$$\begin{aligned} \operatorname{div}\left(\widetilde{\gamma}\,(nx)\, D\widetilde{u}_n^i\right) &= 0 \quad \text{in } \Omega, \\ \widetilde{u}_n^i &= \phi^i \quad \text{on } \partial\Omega. \end{aligned} \tag{2.4}$$

The proof of Theorem 1 is then easily adapted. Since $\widetilde{\gamma}$ is smooth the error estimates corresponding to Lemma 3 are classical and the rate of decay of the error is then n^{-1}, see [9]. We obtain the following corollary.

Corollary 4. — *Given $\rho > 0$, $x_0 \in \Omega'$ such that $B_\rho(x_0) \subset \Omega'$, and $\lambda > 0$, let $A(\phi, x_0, \rho, \lambda)$ be defined by* (2.3). *There exist n, depending on ρ, Ω, Ω' and λ only, a universal constant $\tau > 0$ and two open subsets B_+ and B_- of $B_\rho(x_0)$ such that*

$$|B_+| > \tau\, |B_\rho(x_0)| \quad \text{and} \quad |B_-| > \tau\, |B_\rho(x_0)|\,,$$

and for any $\phi \in A(\phi, x_0, \rho, \lambda)$,

$$\det(D\widetilde{U}_n)(x) < -\tau\lambda\, \|\phi\|^3_{H^{1/2}(\partial\Omega)} \quad \text{on } B_-,$$

and

$$\det(D\widetilde{U}_n)(x) > \tau\lambda\, \|\phi\|^3_{H^{1/2}(\partial\Omega)} \quad \text{on } B_+,$$

where $\widetilde{U}_n$ is the $\widetilde{\gamma}\,(n\cdot)$-harmonic extension of ϕ given by (2.4).

3. Positive Jacobian bounds using more than d boundary data in dimension $d \geqslant 3$

Given a sufficiently smooth conductivity γ defined in $\mathbb{R}^d$, with $d \geqslant 3$ and satisfying, for some $s \geqslant 0$, and $\beta > 1$,

$$\gamma \in C^s\left(\mathbb{R}^d\right) \quad \text{such that} \quad 1 \leqslant \gamma(x) \leqslant \beta \text{ in } \Omega, \tag{3.1}$$

one can ask whether using $N \geqslant d$ boundary conditions would provide N-tuples such that the Jacobian matrix of the corresponding solutions has maximal rank.

A construction of boundary conditions $F_N = \left(\phi^1, \ldots, \phi^N\right) \in H^{1/2}(\partial\Omega)$ ensuring that the solutions $U = (u^1, \ldots, u^N)$ of

$$-\operatorname{div}\left(\gamma D u^i\right) = 0 \text{ in } \Omega, \quad u^i = \phi^i \text{ on } \partial\Omega, \tag{3.2}$$

are such that everywhere on the domain at least one d-tuples of such solutions satisfy a positive Jacobian constraint is provided in [5, 22]. Their approach relies on Complex Geometric Optics solutions [12, 23], adapted for hybrid inverse problems in [6].

Proposition 5 (See Lemma 3.3 in [5], Lemma 2.1 in [22]). — *Given $\gamma \in H^{\frac{d}{2}+3+\varepsilon}\left(\mathbb{R}^3\right)$ satisfying* (3.1). *Let $N = 2\left\lfloor \frac{d+1}{2} \right\rfloor$. There exists a non-empty open set $\mathbf{F}_N \subset (H^{1/2}(\partial\Omega))^N$ of N-tuples of illuminations such that for any $\left(\phi^1, \ldots, \phi^N\right) \in \mathbf{F}_N$ there exists a constant $c_0 > 0$ so that $\left(u^1, \ldots, u^N\right)$ given by* (3.2) *satisfy*

If d is even, $\det\left(Du^1, \ldots, Du^d\right) \geqslant c_0$ *in Ω.*

If d is odd, on an open cover of Ω of the form $\{\Omega_{2i-1}, \Omega_{2i}\}_{1\leqslant i\leqslant M}$,

$$\left|\det\left(Du^1,\ldots,Du^{d-1},Du^{d+1}\right)\right| \geqslant c_0 \quad \textit{on } \Omega_{2i-1}$$

and

$$\left|\det\left(Du^1,Du^{d-1},Du^{d}\right)\right| \geqslant c_0 \quad \textit{on } \Omega_{2i}.$$

When this note was submitted, one of the referee suggested an earlier result [15, 14]. It is shown in these articles that on any connected non compact Riemannian manifold of dimension d, there exists $2d+1$ harmonic functions which taken together give a proper embedding of the manifold in the Euclidean space $\mathbb{R}^{2d+1}$. Translated in the language and context of this paper, it shows that given $\gamma \in C^\infty\left(\mathbb{R}^d\right)$ satisfying (3.1), one can choose $N = 2d+1$ to satisfy the positivity constraint. The main property used to obtain this result is the unique continuation property of second order linear elliptic PDE. As this property holds for Lipschitz conductivities [17], this result can be extended from C^∞ to $C^{0,1}$, and is in that sense more general than Proposition 5, at the cost of a slightly larger number of boundary conditions.

Both results apply to $\widetilde{\gamma}\,(n\cdot)$ for each $n \geqslant 1$. Corollary 4 shows that the Dirichlet data must changes with n and that the lower bound on the determinant tends to nought as n grows if the $H^{1/2}(\partial\Omega)$ norm of the Dirichlet data is bounded a priori. The decay of the lower bound is exponential in n in the proof of Proposition 5.

In dimension 3, as the set of possible septuplet given in [15, 14] is very large, it is possible that there exists a good choice of boundary conditions for all conductivities satisfying a priori Lipschitz bounds. Considering $G_{\beta,L}$ given by

$$G_{\beta,L} = \left\{\gamma \in C^{0,1}\left(\mathbb{R}^3\right) \,:\, 0<1\leqslant\gamma\leqslant\beta \text{ and } \frac{|\gamma(x)-\gamma(y)|}{|x-y|} < L \text{ in } \Omega\right\},$$

there could exist $N \in \mathbb{N}$ depending on β, L and $(\phi^1,\ldots,\phi^N)$ such that for any $\gamma \in G_{\beta,L}$,

$$\operatorname{rank}\left[Du^1,\ldots,Du^N\right] = 3 \text{ on } \Omega.$$

What Theorem 1 and Corollary 4 indicate is that an a priori constraint on the oscillations of γ is unavoidable.

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