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Maryse BOURLARD-JOSPIN, Serge NICAISE, and Juliette VENEL

Approximation of the two-dimensional Dirichlet problem by continuous and discrete problems on one-dimensional networks

APPROXIMATION OF THE TWO-DIMENSIONAL DIRICHLET PROBLEM BY CONTINUOUS AND DISCRETE PROBLEMS ON ONE-DIMENSIONAL NETWORKS

MARYSE BOURLARD-JOSPIN, SERGE NICAISE, AND JULIETTE VENEL

Abstract. We show that the solution of the two-dimensional Dirichlet problem set in a plane domain is the limit of the solutions of similar problems set on a sequence of one-dimensional networks as their size goes to zero. Roughly speaking this means that a membrane can be seen as the limit of rackets made of strings. For practical applications, we also show that the solutions of the discrete approximated problems (again on the one-dimensional networks) also converge to the solution of the two-dimensional Dirichlet problem.

1. INTRODUCTION

Approximation of multidimensional boundary value problems by discrete problems or by boundary value problems set on less dimensional ones is very important in practice. For discrete approximations, the most popular methods are the finite difference method or the finite element method, for which a lot of convergence results are proved [6, 23]. By the less dimensional approximation, we mean that a n-dimensional problem is approximated by a family of k-dimensional ones with $k < n$. For instance the approximation of boundary value problems set on objects of $\mathbb{R}^3$ with a small thickness ϵ by boundary value problems set on objects of dimension 1 or 2 was largely considered in the literature, see for instance [9, 7, 20]. In the same spirit, let us also mention homogenization techniques that analyze the limit process of problems set on n-dimensional domains of thickness ϵ to problems still set on domains of dimension n [8].

The problems studied in this paper have some common properties with the above approaches since we will approach a two-dimensional problem by a family of continuous 1-dimensional problems but as each continuous 1-dimensional problem can be approximated by a discrete one, we also examine the limit of these discrete problems. The approximation of the low frequency spectrum of such problems was performed in [13, 12] (see also [19] for the plate problem), but to our best knowledge the approximation of the boundary value problem itself was not yet performed. Hence our goal is to fill this gap and to show that indeed the solutions of the continuous and discrete one-dimensional problems converge to the solution of the two-dimensional problem. More precisely, we first prove some error estimates between the solution of the two-dimensional problem in an arbitrary domain and the solutions of the continuous one-dimensional problems. Further we propose a numerical scheme based on the resolution of discrete one-dimensional problems and obtain error estimates similar to the standard two-dimensional finite element method. Our approach can be considered as an attractive alternative to the standard ones since its associated stiffness matrix is easier to compute and keeps the same properties (symmetry, positive definiteness and sparsity). Finally it can be used for domains with curved boundaries since no triangulation is needed.

The schedule of the paper is as follows: We recall in Section 2 the Dirichlet problem in the unit square as well as its continuous counterparts on networks that approach the square as the size goes to zero. An error estimate between the solutions of these continuous problems is proved in section 3 by using the second Strang lemma. Similarly section 4 is devoted to the error analysis between the exact solution in the unit square with the finite element approximations on the networks.

In section 5 we extend some of our previous results to the Dirichlet problem set on an arbitrary domain of the plane. Finally in section 6 some numerical tests are presented that confirm our theoretical results.

Let us finish this introduction with some notation used in the remainder of the paper: On D, the $L^2(D)$-norm will be denoted by $\| \cdot \|_D$. The usual norm and seminorm of $H^s(D)$ ($s > 0$) are denoted by $\| \cdot \|_{s,D}$ and $| \cdot |_{s,D}$, respectively. Finally, the notation $a \lesssim b$ means the existence of a positive constants C, which is independent of the size h of the edges of the network (see below) and of the considered quantities a and b such that $a \leqslant Cb$.

2. The continuous problems

2.1. **The continuous two-dimensional problem.** Let $\mathcal{S}$ denote the unit square $]0;1[\times]0;1[$ and $\partial\mathcal{S}$ its boundary. On this domain, we consider the Dirichlet problem

$$\begin{cases} -\Delta u &=& f & \text{in } \mathcal{S} \\ u &=& 0 & \text{on } \partial\mathcal{S} \end{cases} \tag{2.1}$$

with $f \in \mathcal{C}(\overline{\mathcal{S}})$.

According to Lax-Milgram lemma, there exists a unique weak solution $u \in H_0^1(\mathcal{S})$ of this problem, namely $u \in H_0^1(\mathcal{S})$ is the unique solution of

$$\int_\mathcal{S} \nabla u \cdot \nabla v \, dx = \int_\mathcal{S} f \, v \, dx, \quad \forall v \in H_0^1(\mathcal{S}).$$

According to Theorem 5.1.3.5 of [11], this solution belongs to $W^{2,p}(\mathcal{S})$, for all $p > 2$, and if f belongs to $W^{1,p}(\mathcal{S})$, with $p > 2$, is such that f is zero at each corner of $\mathcal{S}$, then this solution belongs to $W^{3,p}(\mathcal{S})$, hence in particular to $H^3(\mathcal{S})$.

2.2. **The associated problem on networks.** Now we intend to consider a similar problem set on a family of networks included in $\mathcal{S}$. First we need to introduce some notation: For any $n \in \mathbb{N}$, $n \geqslant 2$, let $h = 1/n$ and introduce the network $\mathcal{R}_h$ defined by

$$\begin{aligned} \mathcal{R}_h &=& \{]kh;(k+1)h[\times\{\ell h\}; \forall k \in \{0,\dots,n-1\}, \forall\ell \in \{1,\dots,n-1\}\} \\ &\cup& \{\{kh\}\times]\ell h;(\ell+1)h[; \forall k \in \{1,\dots,n-1\}, \forall\ell \in \{0,\dots,n-1\}\}. \end{aligned}$$

The edges of $\mathcal{R}_h$ are the intervals $]kh;(k+1)h[\times\{\ell h\}$ or $\{kh\}\times]\ell h;(\ell+1)h[$ but will be quite simply denoted by e_i, in other words,

$$\mathcal{R}_h = \{e_i; i = 1,\dots,N_h\}, \text{ with } N_h = 2n(n-1).$$

We directly check that the size (or length) of each edge of the network $\mathcal{R}_h$ is h. We further write $\mathcal{N}_h$ for the set of nodes of $\mathcal{R}_h$. Moreover we need to distinguish between nodes included into $\mathcal{S}$ or into $\partial\mathcal{S}$, so we set

$$\begin{aligned} \mathcal{N}_h^{int} &=& \{(kh;\ell h); \forall k,\ell \in \{1,\dots,n-1\}\}, \\ \mathcal{N}_h^{ext} &=& \{(0;\ell h);(1;\ell h);(\ell h;0);(\ell h;1); \forall\ell \in \{0,\dots,n\}\}, \\ \mathcal{N}_h &=& \mathcal{N}_h^{int} \cup \mathcal{N}_h^{ext}. \end{aligned}$$

It remains a last notation to indicate the set of edges adjacent to a given node:

$$\forall v \in \mathcal{N}_h, \quad \mathcal{I}_v = \{i \in \{1,\dots,N_h\} \text{ such that } v \in \overline{e_i}\}.$$

Our aim is to approximate the solution u of the continuous problem (2.1) by the solution $u_h = (u_i)_{i=1,\ldots,N_h} \in \prod_{i=1}^{N_h} H^2(e_i)$ of the following problem:

$$
\begin{cases}
-u_i'' & = & \widetilde{f}_i \text{ on } e_i & \forall i = 1 \cdots N_h, \\
u_i(v) & = & 0 & \forall v \in \mathcal{N}_h^{ext}, \forall i \in \mathcal{I}_v, \\
u_i(v) & = & u_j(v) & \forall v \in \mathcal{N}_h^{int}, \forall i, j \in \mathcal{I}_v, \\
\displaystyle\sum_{i \in \mathcal{I}_v} \frac{\partial u_i}{\partial \nu_i}(v) & = & 0 & \forall v \in \mathcal{N}_h^{int},
\end{cases}
\tag{2.2}
$$

where

$$
\widetilde{f}_i = \frac{1}{2}\gamma_i f.
\tag{2.3}
$$

In the whole paper we use the abuse of notation u'' for $\dfrac{\partial^2 u}{\partial x^2}$ or $\dfrac{\partial^2 u}{\partial y^2}$ according to the kind of the edge (horizontal or vertical). A similar abuse of notation will be used for the first order derivatives. Furthermore, $\frac{\partial}{\partial \nu_i}$ and γ_i represent respectively the outer normal derivative operator and the trace operator on the edge e_i. The last equation of problem (2.2) is nothing else but Kirchoff's law. System (2.2) is a Dirichlet problem on the network $\mathcal{R}_h$ that was largely studied in the literature, see [1, 2, 4, 5, 16, 15, 17, 18, 21] and the references there.

2.3. **Variational formulation on the networks.** The variational space associated with problem (2.2) is

$$
\begin{aligned}
V_h = \{ u_h = (u_i)_{i=1,\ldots,N_h} \in \prod_{i=1}^{N_h} H^1(e_i) \text{ s.t.} & \\
u_i(v) = u_j(v) & \qquad \forall v \in \mathcal{N}_h^{int}, \forall i, j \in \mathcal{I}_v, \\
u_i(v) = 0 & \qquad \forall v \in \mathcal{N}_h^{ext}, \forall i \in \mathcal{I}_v \},
\end{aligned}
\tag{2.4}
$$

equipped with the norm:

$$
||u||_h = |u|_{1,\mathcal{R}_h} = \left[\sum_{i=1}^{N_h} \int_{e_i} (u_i'(x))^2 dx \right]^{1/2}.
\tag{2.5}
$$

Due to the Dirichlet boundary conditions, the H^1-norm and its semi-norm are equivalent on V_h.

LEMMA 2.1. — *For every $w \in V_h$, we have*

$$
||w||_{\mathcal{R}_h} \leqslant |w|_{1,\mathcal{R}_h},
\tag{2.6}
$$

as well as

$$
||w||_{\infty,\mathcal{R}_h} := \sup_{(x,y) \in \mathcal{R}_h} |w(x,y)| \leqslant |w|_{1,\mathcal{R}_h}.
\tag{2.7}
$$

Proof. — Let us denote $L_\ell = \{(x, \ell h), 0 < x < 1\}$ and $C_k = \{(kh, y), 0 < y < 1\}$. Then

$$
\mathcal{R}_h = \left(\bigcup_{\ell=1}^{n-1} L_\ell \right) \cup \left(\bigcup_{k=1}^{n-1} C_k \right).
\tag{2.8}
$$

As $w(0, \ell h) = 0$, we have for all $x \in]0; 1[$

$$
|w(x, \ell h)| = \left| \int_0^x \frac{\partial w}{\partial x}(t, \ell h) dt \right| \leqslant \left\| \frac{\partial w}{\partial x} \right\|_{L_\ell},
\tag{2.9}
$$

according to the Cauchy-Schwarz inequality. Then

$$||w||_{L_\ell}^2 = \int_0^1 |w(x,\ell h)|^2 dx \leqslant \left\|\frac{\partial w}{\partial x}\right\|_{L_\ell}^2 \leqslant |w|_{1,L_\ell}^2. \qquad (2.10)$$

In the same way, we can check that $||w||_{C_k}^2 \leqslant |w|_{1,C_k}^2$ and by summing up these two inequalities we obtain the expected estimate (2.6).

The estimate (2.7) is a direct consequence of (2.9) and its counterpart in C_k. $\qquad \square$

Now we define a bilinear form a_h on V_h by

$$a_h : V_h \times V_h \to \mathbb{R} : (u_h, w_h) \to a_h(u_h, w_h) = \sum_{i=1}^{N_h} \int_{e_i} u_i'(x) w_i'(x) dx, \qquad (2.11)$$

that is clearly continuous and coercive on V_h according to Lemma 2.1.

PROPOSITION 2.2. — *The variational formulation of problem (2.2) is to find $u_h \in V_h$ solution of*

$$\forall w_h \in V_h, a_h(u_h, w_h) = F(w_h), \qquad (2.12)$$

with

$$F(w_h) = \sum_{i=1}^{N_h} \int_{e_i} \widetilde{f}_i(x) w_i(x) dx. \qquad (2.13)$$

Proof. — The proof is quite standard (cf. Lemma 2.2.12 in [2] for instance), we give it for the sake of completeness. Let us assume that there exists a solution $u_h = (u_i)_{i=1,...,N_h} \in \prod_{i=1}^{N_h} H^2(e_i)$ of problem (2.2). Obviously u_h belongs to V_h. Moreover u_h is solution of (2.12). Indeed, let $w_h = (w_i)_{i=1,...,N_h} \in V_h$, then we have for all $i \in \{1, ..., N_h\}$,

$$-\int_{e_i} u_i''(x) w_i(x) dx = \int_{e_i} \widetilde{f}_i(x) w_i(x) dx.$$

Integrating by parts, we obtain

$$\int_{e_i} u_i'(x) w_i'(x) dx - [u_i'(v) w_i(v)]_{v=v_{i1}}^{v=v_{i2}} = \int_{e_i} \widetilde{f}_i(x) w_i(x) dx, \qquad (2.14)$$

where v_{i1} and $v_{i2} \in \mathcal{N}_h$ are such that $i \in \mathcal{I}_{v_{i1}} \cap \mathcal{I}_{v_{i2}}$.

We claim that

$$\sum_{i=1}^{N_h} [u_i'(v) w_i(v)]_{v=v_{i1}}^{v=v_{i2}} = 0. \qquad (2.15)$$

In fact, we have

$$[u_i'(v) w_i(v)]_{v=v_{i1}}^{v=v_{i2}} = \frac{\partial u_i}{\partial \nu_i}(v_{i1}) w_i(v_{i1}) + \frac{\partial u_i}{\partial \nu_i}(v_{i2}) w_i(v_{i2}), \qquad (2.16)$$

and consequently,

$$\sum_{i=1}^{N_h} [u_i'(v) w_i(v)]_{v=v_{i1}}^{v=v_{i2}} = \sum_{v \in \mathcal{N}_h^{ext}} \sum_{i \in \mathcal{I}_v} \frac{\partial u_i}{\partial \nu_i}(v) w_i(v) + \sum_{v \in \mathcal{N}_h^{int}} \sum_{i \in \mathcal{I}_v} \frac{\partial u_i}{\partial \nu_i}(v) w_i(v). \qquad (2.17)$$

If $v \in \mathcal{N}_h^{ext}$, then $w_i(v) = 0$, for all $i \in \mathcal{I}_v$ and therefore the second term of (2.17) is zero. If $v \in \mathcal{N}_h^{int}$, then

$$\sum_{i \in \mathcal{I}_v} \frac{\partial u_i}{\partial \nu_i}(v) w_i(v) = w_j(v) \sum_{i \in \mathcal{I}_v} \frac{\partial u_i}{\partial \nu_i}(v) \qquad (2.18)$$

for any $j \in \mathcal{I}_v$, since w_h is continuous at the nodes. Then, using Kirchoff's law, the right-hand side of the identity (2.18) is equal to zero and the first term of (2.17) is zero. Hence (2.15) is established and we conclude with (2.14) and (2.15). $\qquad \square$

3. AN APPROXIMATION RESULT BETWEEN THE CONTINUOUS PROBLEMS

In this section, we analyze the error between the solution u of problem (2.1) and the solutions u_h of (2.12). For that purpose, we make use of the second Strang lemma (see below). Hence we first estimate the consistency error:

THEOREM 3.1. — *Let u denote the solution of (2.1), and u_h the solution of (2.12). If $u \in H^3(\mathcal{S})$, then*

$$\sup_{w \in V_h} \frac{|a_h(u, w) - F(w)|}{||w||_h} \lesssim \sqrt{h} ||u||_{3,\mathcal{S}}. \tag{3.1}$$

Proof. — Since $u \in H^3(\mathcal{S})$, for all $i = 1, ..., N_h$, $u_i = \gamma_i u$ has a meaning and since u is also continuous, its restriction to $\mathcal{R}_h$, still denoted by u, belongs to V_h. Fix $w = (w_i)_{i=1,...,N_h} \in V_h$. It can be shown, as in the proof of Proposition 2.2, that

$$a_h(u, w) = -\sum_{i=1}^{N_h} \int_{e_i} u_i''(x) w_i(x) dx.$$

Then, thanks to (2.13),

$$a_h(u, w) - F(w) = -\sum_{i=1}^{N_h} \int_{e_i} (u_i''(x) + \widetilde{f}_i(x)) w_i(x) dx. \tag{3.2}$$

For every $v \in \mathcal{N}_h$, if (ξ, φ) are the coordinates of v, we define the rectangle

$$C_v^h = \left( ]\xi - \frac{h}{2}, \xi + \frac{h}{2}[\times]\varphi - \frac{h}{2}, \varphi + \frac{h}{2}[\right) \cap \mathcal{S}$$

and its intersection with $\mathcal{R}_h$

$$R_v^h = C_v^h \cap \mathcal{R}_h. \tag{3.3}$$

If $v \in \mathcal{N}_h^{int}$, R_v^h is a cross, while if $v \in \mathcal{N}_h^{ext}$, R_v^h is a half edge. The identity (3.2) can be rewritten as

$$a_h(u, w) - F(w) = -\sum_{v \in \mathcal{N}_h^{int}} \int_{R_v^h} (u'' + \tilde{f})(x) w(x) dx - \sum_{v \in \mathcal{N}_h^{ext}} \int_{R_v^h} (u'' + \tilde{f})(x) w(x) dx. \tag{3.4}$$

Step 1 : Case of the interior nodes
Fix $v \in \mathcal{N}_h^{int}$. We define the reference square $\widehat{C} =]-\frac{1}{2}; \frac{1}{2}[\times]-\frac{1}{2}; \frac{1}{2}[$ and the reference cross $\widehat{R} = (\{0\} \times]-\frac{1}{2}; \frac{1}{2}[) \cup (]-\frac{1}{2}; \frac{1}{2}[\times \{0\})$. We consider the change of variables

$$\phi : \widehat{C} \to C_v^h : \hat{x} \to x = \phi(\hat{x}) = v + h\hat{x}.$$

Note that $\phi(\widehat{R}) = R_v^h$ and

$$\int_{R_v^h} (u'' + \tilde{f})(x) w(x) dx = h \int_{\widehat{R}} (u''(\phi(\hat{x})) + \tilde{f}(\phi(\hat{x}))) w(\phi(\hat{x})) \, d\hat{x}.$$

Let us set $\hat{u} = u \circ \phi$, $\hat{w} = w \circ \phi$, then $\hat{u}' = h(u' \circ \phi)$. In the same way,

$$u'' \circ \phi = \frac{1}{h^2} \hat{u}'' \text{ and } (\Delta u) \circ \phi = \frac{1}{h^2} \Delta \hat{u}. \tag{3.5}$$

Owing to the definition (2.3) of $\tilde{f}$, $\tilde{f} \circ \phi = -\frac{1}{2} \frac{1}{h^2} \Delta \hat{u}$ and finally

$$\int_{R_v^h} (u'' + \tilde{f})(x) w(x) dx = \frac{1}{h} \int_{\widehat{R}} (\hat{u}'' - \frac{1}{2} \Delta \hat{u})(\hat{x}) \hat{w}(\hat{x}) d\hat{x} = h^{-1}(I_1 + I_2), \tag{3.6}$$

where

$$I_1 = \int_{\widehat{R}} (\hat{u}'' - \frac{1}{2} \Delta \hat{u})(\hat{x})(\hat{w}(\hat{x}) - \mathcal{M}\hat{w}) d\hat{x} \tag{3.7}$$

and

$$I_2 = \int_{\widehat{R}} (\hat{u}'' - \frac{1}{2} \Delta \hat{u})(\hat{x})(\mathcal{M}\hat{w}) d\hat{x}, \tag{3.8}$$

with the constant

$$\mathcal{M}\hat{w} = \int_{\widehat{R}} \hat{w}(\hat{x})d\hat{x}. \qquad (3.9)$$

Let us begin with the estimate of I_1. With (3.7) and the Cauchy-Schwarz inequality,

$$|I_1| \leqslant ||\hat{u}'' - \frac{1}{2}\Delta\hat{u}||_{\widehat{R}} ||\hat{w} - \mathcal{M}\hat{w}||_{\widehat{R}}. \qquad (3.10)$$

Moreover we have

$$||\hat{u}'' - \frac{1}{2}\Delta\hat{u}||_{\widehat{R}} \lesssim \sum_{|\alpha|=2} ||D^\alpha \hat{u}||_{\widehat{R}} \lesssim \sum_{|\alpha|=2} ||D^\alpha \hat{u}||_{1,\widehat{C}}. \qquad (3.11)$$

by using a trace theorem [11, Thm 1.5.1.2]. We recall that due to the Poincaré-Friedrichs inequality,

$$||\hat{w} - \mathcal{M}\hat{w}||_{\widehat{R}} \lesssim |\hat{w}|_{1,\widehat{R}}. \qquad (3.12)$$

Thanks to (3.10), (3.11) and (3.12), we have shown

$$|I_1| \lesssim |\hat{w}|_{1,\widehat{R}} \sum_{|\alpha|=2} ||D^\alpha \hat{u}||_{1,\widehat{C}}. \qquad (3.13)$$

Now in order to estimate I_2, we need the following lemma that can be proved by easy computations.

LEMMA 3.2. —

$$\forall \hat{p} \in \mathbb{P}_2(\widehat{R}), \quad \int_{\widehat{R}} (\hat{p}'' - \frac{1}{2}\Delta\hat{p})(\hat{x})d\hat{x} = 0, \qquad (3.14)$$

where $\mathbb{P}_2(\widehat{R})$ represents the set of polynomials of degree at most 2 on $\widehat{R}$.

Owing to (3.8) and (3.14), for all $\hat{p} \in \mathbb{P}_2(\widehat{R})$,

$$I_2 = \int_{\widehat{R}} [(\hat{u} - \hat{p})'' - \frac{1}{2}\Delta(\hat{u} - \hat{p})](\hat{x})(\mathcal{M}\hat{w})d\hat{x}.$$

According to the Cauchy-Schwarz inequality, and since $\mathcal{M}\hat{w}$ is a constant,

$$\begin{aligned} |I_2| &\lesssim |\mathcal{M}\hat{w}|||(\hat{u} - \hat{p})'' - \frac{1}{2}\Delta(\hat{u} - \hat{p})||_{\widehat{R}} \lesssim |\mathcal{M}\hat{w}| \sum_{|\alpha|=2} ||D^\alpha(\hat{u} - \hat{p})||_{\widehat{R}} \\ &\lesssim |\mathcal{M}\hat{w}| \sum_{|\alpha|=2} ||D^\alpha(\hat{u} - \hat{p})||_{1,\widehat{C}} \lesssim |\mathcal{M}\hat{w}| ||\hat{u} - \hat{p}||_{3,\widehat{C}} \end{aligned}$$

by using the same trace theorem as previously. Let $\hat{p}$ be the orthogonal projection of $\hat{u}$ on $\mathbb{P}_2(\widehat{R})$ for the $H^3(\widehat{C})$-norm, then

$$||\hat{u} - \hat{p}||_{3,\widehat{C}} \lesssim |\hat{u}|_{3,\widehat{C}}. \qquad (3.15)$$

Moreover, due to (3.9), $|\mathcal{M}\hat{w}| \lesssim ||\hat{w}||_{\widehat{R}}$, so the three last inequalities imply that

$$|I_2| \lesssim ||\hat{w}||_{\widehat{R}} |\hat{u}|_{3,\widehat{C}}. \qquad (3.16)$$

Now we recall the next lemma that specifies the change of H^m-semi-norms from a domain to a reference domain [6].

LEMMA 3.3. — Consider $m \in \mathbb{N}$ and let us denote $\hat{f} = f \circ \phi$. Then

$$\forall f \in H^m(R_v^h), |f|_{m,R_v^h} = \frac{1}{h^{m-1/2}}|\hat{f}|_{m,\widehat{R}}$$

and

$$\forall f \in H^m(C_v^h), |f|_{m,C_v^h} = \frac{1}{h^{m-1}}|\hat{f}|_{m,\widehat{C}}.$$

By (3.13),

$$|I_1| \lesssim |\hat{w}|_{1,\widehat{R}} \left[\sum_{|\alpha|=2} ||D^\alpha \hat{u}||_{\widehat{C}} + \sum_{|\alpha|=2} |D^\alpha \hat{u}|_{1,\widehat{C}} \right]. \tag{3.17}$$

It follows from Lemma 3.3 and (3.5) that

$$|I_1| \lesssim h^2 \sqrt{h} |w|_{1,R_v^h} \left[\sum_{|\alpha|=2} h^{-1} ||D^\alpha u||_{C_v^h} + \sum_{|\alpha|=2} |D^\alpha u|_{1,C_v^h} \right],$$

and finally,

$$|I_1| \lesssim h^{3/2} |w|_{1,R_v^h} ||u||_{3,C_v^h}. \tag{3.18}$$

According to Lemma 3.3, (3.16) leads to

$$|I_2| \lesssim h^{3/2} ||w||_{R_v^h} |u|_{3,C_v^h}. \tag{3.19}$$

Gathering the results (3.6), (3.18) and (3.19), we have proved that

$$\left| \int_{R_v^h} (u'' + \tilde{f})(x)w(x)dx \right| \lesssim \sqrt{h} \left(|w|_{1,R_v^h} ||u||_{3,C_v^h} + ||w||_{R_v^h} |u|_{3,C_v^h} \right). \tag{3.20}$$

Step 2 : Case of the exterior nodes
Fix $v \in \mathcal{N}_h^{ext}$ and let us denote $\widehat{R}_{1/2} = \phi^{-1}(R_v^h)$ and $\widehat{C}_{1/2} = \phi^{-1}(C_v^h)$.
We show as (3.6) that

$$\int_{R_v^h} (u'' + \tilde{f})(x)w(x)dx = \frac{1}{h} \int_{\widehat{R}_{1/2}} (\hat{u}'' - \frac{1}{2}\Delta\hat{u})(\hat{x})\hat{w}(\hat{x})d\hat{x}. \tag{3.21}$$

Using the Cauchy-Schwarz inequality, this implies

$$\left| \int_{R_v^h} (u'' + \tilde{f})(x)w(x)dx \right| \leqslant \frac{1}{h} ||\hat{u}'' - \frac{1}{2}\Delta\hat{u}||_{\widehat{R}_{1/2}} ||\hat{w}||_{\widehat{R}_{1/2}}. \tag{3.22}$$

Arguing as for (3.11), since $u \in H^3(\mathcal{S})$, we get

$$||\hat{u}'' - \frac{1}{2}\Delta\hat{u}||_{\widehat{R}_{1/2}} \lesssim \sum_{|\alpha|=2} ||D^\alpha \hat{u}||_{1,\widehat{C}_{1/2}}. \tag{3.23}$$

On the other hand, $w \in V_h$ implies that $\hat{w}(0) = 0$, so it can be proved as in Lemma 2.1 that

$$||\hat{w}||_{\widehat{R}_{1/2}} \lesssim |\hat{w}|_{1,\widehat{R}_{1/2}}. \tag{3.24}$$

Thanks to (3.22), (3.23) and (3.24), we have

$$\left| \int_{R_v^h} (u'' + \tilde{f})(x)w(x)dx \right| \lesssim \frac{1}{h} |\hat{w}|_{1,\widehat{R}_{1/2}} \left[\sum_{|\alpha|=2} ||D^\alpha \hat{u}||_{\widehat{C}_{1/2}} + \sum_{|\alpha|=2} |D^\alpha \hat{u}|_{1,\widehat{C}_{1/2}} \right]. \tag{3.25}$$

Using Lemma 3.3 and the identity (3.5), it comes

$$\left| \int_{R_v^h} (u'' + \tilde{f})(x)w(x)dx \right| \lesssim \frac{1}{h} \sqrt{h} |w|_{1,R_v^h} h^2 \left[\sum_{|\alpha|=2} h^{-1} ||D^\alpha u||_{C_v^h} + \sum_{|\alpha|=2} |D^\alpha u|_{1,C_v^h} \right]$$

$$\lesssim \sqrt{h} |w|_{1,R_v^h} ||u||_{3,C_v^h}. \tag{3.26}$$

Step 3 : Conclusion
The identity (3.4) leads to

$$|a_h(u,w) - F(w)| \leqslant \sum_{v \in \mathcal{N}_h^{int}} \left| \int_{R_v^h} (u'' + \tilde{f})(x)w(x)dx \right| + \sum_{v \in \mathcal{N}_h^{ext}} \left| \int_{R_v^h} (u'' + \tilde{f})(x)w(x)dx \right|. \tag{3.27}$$

Summing (3.20) for all $v \in \mathcal{N}_h^{int}$ and (3.26) for all $v \in \mathcal{N}_h^{ext}$, we deduce from the previous inequality

$$
\begin{aligned}
|a_h(u,w) - F(w)| &\lesssim \sqrt{h} \sum_{v \in \mathcal{N}_h^{int}} \left(|w|_{1,R_v^h} ||u||_{3,C_v^h} + ||w||_{R_v^h} |u|_{3,C_v^h} \right) \\
&+ \sqrt{h} \sum_{v \in \mathcal{N}_h^{ext}} |w|_{1,R_v^h} ||u||_{3,C_v^h} \\
&\lesssim \sqrt{h} \sum_{v \in \mathcal{N}_h^{int}} ||w||_{R_v^h} |u|_{3,C_v^h} + \sqrt{h} \sum_{v \in \mathcal{N}_h} |w|_{1,R_v^h} ||u||_{3,C_v^h}.
\end{aligned}
$$

By the discrete Cauchy-Schwarz inequality, we obtain

$$
|a_h(u,w) - F(w)| \lesssim \sqrt{h} \left(|w|_{1,\mathcal{R}_h} ||u||_{3,\mathcal{S}} + ||w||_{\mathcal{R}_h} |u|_{3,\mathcal{S}} \right). \qquad (3.28)
$$

We conclude the proof thanks to Lemma 2.1 and inequality (3.28). $\qquad \square$

Now we recall the following result which is a consequence of the second Strang Lemma and can be found for example in [6, Thm 4.2.2].

LEMMA 3.4. — *Let u denote the solution of (2.1) supposed to belong to V_h, and let u_h be the solution of (2.12). Then*

$$
||u - u_h||_h \lesssim \sup_{w_h \in V_h} \frac{|a_h(u,w_h) - F(w_h)|}{||w_h||_h}. \qquad (3.29)
$$

Remark 3.5. — Note that the upper bound in the second Strang Lemma contains another term, namely $\inf_{v_h \in V_h} ||u - v_h||_h$, called the "interpolation error". Here only the "consistency error" term appears as we have assumed that $u \in V_h$, the interpolation error being obviously equal to zero.

COROLLARY 3.6. — *Let u denote the solution of (2.1), and let u_h be the solution of (2.12). If $u \in H^3(\mathcal{S})$, then*

$$
||u - u_h||_{1,\mathcal{R}_h} \lesssim \sqrt{h} ||u||_{3,\mathcal{S}}, \qquad (3.30)
$$

and

$$
||u - u_h||_{\infty,\mathcal{R}_h} \lesssim \sqrt{h} ||u||_{3,\mathcal{S}}. \qquad (3.31)
$$

Proof. — We deduce from Theorem 3.1 and Lemma 3.4 that

$$
|u - u_h|_{1,\mathcal{R}_h} \lesssim \sqrt{h} ||u||_{3,\mathcal{S}}. \qquad (3.32)
$$

Now the estimates (3.30) and (3.31) are a direct consequence of Lemma 2.1 since $u - u_h \in V_h$. $\qquad \square$

4. THE FINITE ELEMENT METHOD ON THE NETWORKS

In the previous section, we have checked that u_h is a good approximation of u. However, problem (2.2) is still set in an infinite dimensional space and except for some specific right-hand sides $\tilde{f}$, its solution u_h cannot be computed analytically. Hence in practice problem (2.2) has to be discretized. Here we choose the finite element method and propose to deal with two different cases according to the regularity $H^3(\mathcal{S})$ or $\mathcal{C}^3(\overline{\mathcal{S}})$ of the solution u.

4.1. **A less regular solution.** Here we assume that the solution of the continuous problem (2.1) u belongs to $H^3(\mathcal{S})$ and f is a continuous function in $\overline{\mathcal{S}}$. Let $\mathbb{P}_1(\overline{e_i})$ denote the set of polynomials of degree at most 1 on $\overline{e_i}$, for all $i \in \{1, \ldots, N_h\}$. We define the discrete variational space

$$
W_h = \{ w_h = (w_i)_{i=1,\ldots,N_h} \in V_h \text{ s.t. } w_i \in \mathbb{P}_1(\overline{e_i}), \forall i = 1, \ldots, N_h \}. \qquad (4.1)
$$

Let $U_h \in W_h$ be the solution of the finite element problem

$$
a_h(U_h, w_h) = F(w_h), \forall w_h \in W_h. \qquad (4.2)
$$

In order to compare u and U_h in $\mathcal{S}$, we will use an interpolant $I_h u$ of u and a lifting $R_h U_h$ of U_h defined as follows: Let us denote $K_{k,\ell}^h =]kh, (k+1)h[\times]\ell h, (\ell+1)h[$, for each $k, \ell \in \{0, \ldots, n-1\}$. Observe that

$$\mathcal{S} = \left(\bigcup_{k=0}^{n-1} \bigcup_{\ell=0}^{n-1} K_{k,\ell}^h \right) \cup \mathcal{R}_h, \tag{4.3}$$

and therefore the set of $\bar{K}_{k,\ell}^h$ is a triangulation of $\mathcal{S}$. Hence let $I_h u$ denote the Lagrange interpolation of u related to this triangulation, namely $I_h u$ is the function such that its restriction to $K_{k,\ell}^h$ belongs to $Q_1(K_{k,\ell}^h)$ (where Q_1 is the space of polynomials in (x, y) of degree at most 1 in each variable x and y) and that coincides with u at each node $v \in \mathcal{N}_h$. As a consequence $I_h u$ is continuous on $\mathcal{S}$. Finally we define $R_h U_h = I_h U_h$ in the sense that its restriction to $K_{k,\ell}^h$ fulfils $R_h U_h \in Q_1(K_{k,\ell}^h)$ and

$$R_h U_h(v) = U_h(v), \forall v \in \overline{K_{k,\ell}^h} \cap \mathcal{N}_h. \tag{4.4}$$

Thus $R_h U_h$ coincides with U_h on $\mathcal{R}_h$ and is continuous on $\mathcal{S}$.

Now we aim at approximating u by U_h. The estimate of the error is made with the help of the following three lemmas.

LEMMA 4.1. —
$$|I_h u - R_h U_h|_{1,\mathcal{S}} \lesssim \sqrt{h}|I_h u - U_h|_{1,\mathcal{R}_h}. \tag{4.5}$$

Proof. — According to Lemma 3.3, with $\tilde{\Phi} : \hat{C} \to K_{k,\ell}^h : \hat{x} \mapsto (kh, \ell h) + h(\hat{x} + \frac{1}{2})$, we have

$$|I_h u - R_h U_h|_{1,K_{k,\ell}^h} \lesssim |\widehat{I_h u} - \widehat{R_h U_h}|_{1,\hat{C}}. \tag{4.6}$$

Let us denote $Q_1^0(\hat{C}) = \{q \in Q_1(\hat{C}), \int_{\partial \hat{C}} q = 0\}$.

Take $q \in Q_1(\hat{C})$, then

$$|q|_{1,\hat{C}} = |\pi q|_{1,\hat{C}} \leqslant \|\pi q\|_{1,\hat{C}},$$

where $\pi q = q - \frac{1}{4} \int_{\partial \hat{C}} q \in Q_1^0(\hat{C})$. Note that $Q_1^0(\hat{C})$ is a finite dimensional space and $|\cdot|_{1,\partial \hat{C}}$ is a norm on this space. So

$$\|\pi q\|_{1,\hat{C}} \lesssim |\pi q|_{1,\partial \hat{C}} = |q|_{1,\partial \hat{C}}.$$

We have thus proved that

$$\forall q \in Q_1(\hat{C}), |q|_{1,\hat{C}} \lesssim |q|_{1,\partial \hat{C}}. \tag{4.7}$$

As $\widehat{I_h u} - \widehat{R_h U_h} \in Q_1(\hat{C})$, thanks to (4.6) and (4.7), we have

$$|I_h u - R_h U_h|_{1,K_{k,\ell}^h} \lesssim |\widehat{I_h u} - \widehat{R_h U_h}|_{1,\partial \hat{C}},$$

and owing to Lemma 3.3 again,

$$|I_h u - R_h U_h|_{1,K_{k,\ell}^h} \lesssim \sqrt{h}|I_h u - U_h|_{1,\partial K_{k,\ell}^h}. \tag{4.8}$$

Collecting the pieces, we obtain

$$\begin{aligned} |I_h u - R_h U_h|_{1,\mathcal{S}}^2 &= \sum_{k,\ell} |I_h u - R_h U_h|_{1,K_{k,\ell}^h}^2 \\ &\lesssim h \sum_{k,\ell} |I_h u - U_h|_{1,\partial K_{k,\ell}^h}^2 \\ &\lesssim h|I_h u - U_h|_{1,\mathcal{R}_h}^2, \end{aligned}$$

the last inequality following from the fact that each edge of $\mathcal{R}_h$ is in the boundary of two domains $K_{k,\ell}^h$. $\qquad \square$

LEMMA 4.2. — If $u \in H^2(\mathcal{S})$, then

$$|I_h u - u|_{1,\mathcal{R}_h} \lesssim \sqrt{h}|u|_{2,\mathcal{S}}. \tag{4.9}$$

Proof. — Using Lemma 3.3,

$$|I_h u - u|_{1,\partial K_{k,\ell}^h} = h^{-1/2}|\widehat{I_h u} - \widehat{u}|_{1,\partial\widehat{C}} \leqslant h^{-1/2}||\widehat{I_h u} - \widehat{u}||_{1,\partial\widehat{C}}.$$

Thanks to a trace theorem (see Theorem 1.5.2.1 in [11] for instance), this leads to

$$|I_h u - u|_{1,\partial K_{k,\ell}^h} \lesssim h^{-1/2}||\widehat{I_h u} - \widehat{u}||_{2,\widehat{C}}. \tag{4.10}$$

By the classical interpolation error estimate (see for instance Theorem 3.1.6 in [6]), we have

$$||\widehat{I_h u} - \widehat{u}||_{2,\widehat{C}} \lesssim |\widehat{u}|_{2,\widehat{C}}.$$

Owing to Lemma 3.3 again,

$$||\widehat{I_h u} - \widehat{u}||_{2,\widehat{C}} \lesssim h|u|_{2,K_{k,\ell}^h}. \tag{4.11}$$

Then (4.10) and (4.11) imply

$$|I_h u - u|_{1,\partial K_{k,\ell}^h} \lesssim \sqrt{h}|u|_{2,K_{k,\ell}^h}.$$

We conclude the proof by squaring this inequality and summing up for $k, \ell \in \{0, \dots, n-1\}$. □

LEMMA 4.3. — *Let us assume that $f \in \mathcal{C}(\overline{\mathcal{S}})$. Then*

$$||f||_{\mathcal{R}_h} \leqslant \sqrt{2}\, h^{-1/2}||f||_{\infty,\mathcal{S}}. \tag{4.12}$$

Proof. — Let us use the notation of the proof of Lemma 2.1. Then

$$||f||_{L_\ell}^2 = \int_0^1 |f(x,\ell h)|^2 dx \leqslant ||f||_{\infty,\mathcal{S}}^2. \tag{4.13}$$

Obviously we have the same estimate for $||f||_{C_k}^2$. This leads to

$$||f||_{\mathcal{R}_h}^2 = \sum_{\ell=1}^{n-1}||f||_{L_\ell}^2 + \sum_{k=1}^{n-1}||f||_{C_k}^2 \leqslant 2n||f||_{\infty,\mathcal{S}}^2.$$

Since $h = 1/n$, we obtain the expected result. □

PROPOSITION 4.4. — *Let $u_h \in V_h$ denote the solution of (2.12) and $U_h \in W_h$ the solution of (4.2). Let us assume that the datum f belongs to $\mathcal{C}(\overline{\mathcal{S}})$. Then*

$$||u_h - U_h||_{1,\mathcal{R}_h} \lesssim \sqrt{h}||f||_{\infty,\mathcal{S}}. \tag{4.14}$$

Proof. — It can be proven (see for example Theorem 3.1.6 in [6]) that

$$||u_h - U_h||_{1,\mathcal{R}_h} \lesssim h|u_h|_{2,\mathcal{R}_h}. \tag{4.15}$$

But u_h is a solution of (2.2), so

$$|u_h|_{2,\mathcal{R}_h} = |\tilde{f}|_{\mathcal{R}_h} = \frac{1}{2}||f||_{\mathcal{R}_h}. \tag{4.16}$$

Due to Lemma 4.3,

$$|u_h|_{2,\mathcal{R}_h} \lesssim h^{-1/2}||f||_{\infty,\mathcal{S}}. \tag{4.17}$$

The aim then follows from (4.15) and (4.17). □

THEOREM 4.5. — *Let $U_h \in W_h$ denote the solution of (4.2), and $R_h U_h$ be defined by (4.4). Let us assume that the solution u of the continuous problem (2.1) belongs to $H^3(\mathcal{S})$, and the datum f belongs to $\mathcal{C}(\overline{\mathcal{S}})$. Then*

$$||u - R_h U_h||_{1,\mathcal{S}} \lesssim h\left(||u||_{3,\mathcal{S}} + ||f||_{\infty,\mathcal{S}}\right). \tag{4.18}$$

Proof. — As the trace of $I_h u - R_h U_h$ is equal to 0 on $\partial\mathcal{S}$, we have

$$||I_h u - R_h U_h||_{1,\mathcal{S}} \lesssim |I_h u - R_h U_h|_{1,\mathcal{S}}. \tag{4.19}$$

Lemma 4.1 leads to

$$
\begin{aligned}
||I_h u - R_h U_h||_{1,\mathcal{S}} &\lesssim \sqrt{h}|I_h u - U_h|_{1,\mathcal{R}_h} \\
&\lesssim \sqrt{h}\left(|I_h u - u|_{1,\mathcal{R}_h} + |u - u_h|_{1,\mathcal{R}_h} + |u_h - U_h|_{1,\mathcal{R}_h}\right) \tag{4.20}
\end{aligned}
$$

We deduce from Lemma 4.2, Corollary 3.6 and Proposition 4.4 that

$$||I_h u - R_h U_h||_{1,\mathcal{S}} \lesssim h\left(||u||_{3,\mathcal{S}} + ||f||_{\infty,\mathcal{S}}\right). \tag{4.21}$$

On the other hand, thanks to Theorem 3.1.6 of [6],

$$||u - I_h u||_{1,\mathcal{S}} \lesssim h|u|_{2,\mathcal{S}}. \tag{4.22}$$

We conclude with (4.21) and (4.22). $\qquad\square$

4.2. **A more regular solution.** For more regular solutions, we will exploit the analogy with a finite difference scheme to get a pointwise convergence result.
For every $v \in \mathcal{N}_h^{int}$, we define $\lambda_v \in W_h$ such that $\lambda_v(v) = 1$ and $\lambda_v(v') = 0$, for all $v' \in \mathcal{N}_h$ such that $v' \neq v$. Remark that the support of λ_v is included in $\{\bar{e}_i; i \in \mathcal{I}_v\}$ and that the set $\{\lambda_v, v \in \mathcal{N}_h^{int}\}$ forms a basis of the space W_h. The stiffness matrix M_h of problem (4.2) is easily computed. More precisely, we enumerate the interior nodes $v \in \mathcal{N}_h^{int}$ line by line, namely let us denote $v_1 = (h, h)$, $v_2 = (2h, h)$, ..., $v_{n-1} = ((n-1)h, h)$, $v_n = (h, 2h)$, $v_{n+1} = (2h, 2h)$, ..., $v_{2n-2} = ((n-1)h, 2h)$, ... $v_{(n-1)^2} = ((n-1)h, (n-1)h)$. Let M_h denote the stiffness matrix such that

$$(M_h)_{i,j} = a_h(\lambda_{v_i}, \lambda_{v_j}), \forall i, j \in \{1, \ldots, (n-1)^2\}.$$

Then M_h is a symmetric matrix that can be written

$$M_h = \frac{1}{h}\tilde{A}_h \tag{4.23}$$

where

$$
\tilde{A}_h = \begin{pmatrix}
A_{1,1} & A_{1,2} & 0 & \ldots & & 0 \\
A_{2,1} & A_{2,2} & A_{2,3} & \ddots & & 0 \\
0 & A_{3,2} & A_{3,3} & \ddots & & 0 \\
\vdots & \ddots & \ddots & \ddots & & \vdots \\
0 & \ldots & 0 & & A_{n-1,n-2} & A_{n-1,n-1}
\end{pmatrix}. \tag{4.24}
$$

The blocks $A_{k,l}$ are symmetric matrices of dimension $(n-1)$ and satisfy for all $k \in \{1, \ldots, n-1\}$, $A_{k,k-1} = A_{k-1,k} = -I_{n-1}$ (I_{n-1} is the identity matrix of dimension $(n-1)$), and

$$
A_{k,k} = \begin{pmatrix}
4 & -1 & \ldots & 0 \\
-1 & 4 & \ddots & \vdots \\
\vdots & \ddots & \ddots & -1 \\
0 & \ldots & -1 & 4
\end{pmatrix}. \tag{4.25}
$$

Set $m = (n-1)^2$ (for shortness we skip the dependence of m on h), as U_h belongs to W_h, it can be expressed in the basis $(\lambda_{v_k})_{k=1,\ldots,m}$ as follows:

$$U_h = \sum_{k=1}^{m} U_h(v_k)\lambda_{v_k}.$$

As usual, U_h is the solution of problem (4.2) if and only if

$$M_h \widetilde{U_h} = \widetilde{F_h} \tag{4.26}$$

where $\widetilde{U_h} = (U_h(v_1), \ldots, U_h(v_m))^\top$ and $\widetilde{F_h} = (F(\lambda_{v_1}), \ldots, F(\lambda_{v_m}))^\top$.

Now we want to check that the values of U_h at the nodes are a good approximation of the values of u. To this end, we observe that M_h is closely related to the matrix obtained by using the finite difference method to approximate the continuous problem (2.1). Indeed if D_h denote the approximation of the solution u of (2.1) with the finite difference method, then D_h is solution of the linear system [14]

$$A_h D_h = F_h, \tag{4.27}$$

where

$$A_h = \frac{1}{h} M_h = \frac{1}{h^2} \tilde{A}_h \tag{4.28}$$

with $\tilde{A}_h$ defined by (4.24) and $F_h = (f(v_1), ..., f(v_m))^\top$.

For further purposes, we state the following two results (see Lemma 6.2 of [3] for the proof of the first result, the second one being proved in a fully similar way, see also Property 1.20 of [22]).

PROPOSITION 4.6. — *Let $A \in \mathbb{R}^{m \times m}$ satisfying the following conditions*
(1) $\forall i \neq j$, $a_{ij} \leqslant 0$, and
(2) $\forall i = 1, \ldots, m$, $\sum_{j=1}^m a_{ij} > 0$,
then A is a monotone matrix, i. e., if $X = (x_i)_{i=1,\ldots,m} \in \mathbb{R}^m$ is such that $AX \geqslant 0$ (in the sense that $(Ax)_i \geqslant 0$, for all $i = 1, \ldots, m$), then $X \geqslant 0$.

Remark 4.7. — The result of Proposition 4.6 still holds if the assumption (2) is replaced by
(2') A is a regular matrix and for all $i = 1, \ldots, m$, $\sum_{j=1}^m a_{ij} \geqslant 0$.

COROLLARY 4.8. — *$\tilde{A}_h$ and A_h given by (4.28) are monotone matrices.*

Proof. — Since $\tilde{A}_h$ is symmetric and positive definite, $\tilde{A}_h$ is a regular matrix. Moreover, $\tilde{A}_h$ fulfils condition (1) of Proposition 4.6 and condition (2') of Remark 4.7. □

PROPOSITION 4.9. — *Consider u the solution of Problem (2.1) and suppose that $u \in \mathcal{C}^3(\overline{\mathcal{S}})$. Set $U = (u(v_1), ..., u(v_m))^\top$ and let D_h be the solution of equation (4.27). Then*

$$A_h(U - D_h) = \overline{\eta}(u),$$

with $\overline{\eta}(u) = (\eta(u)(v_1), ..., \eta(u)(v_m))^\top$, where

$$\eta(u)(x_i, y_i) = -\frac{h}{6} \left[\frac{\partial^3 u}{\partial x^3}(x_i + \theta_{i,1}h, y_i) - \frac{\partial^3 u}{\partial x^3}(x_i - \theta_{i,2}h, y_i) \right.$$
$$\left. + \frac{\partial^3 u}{\partial y^3}(x_i, y_i + \theta_{i,3}h) - \frac{\partial^3 u}{\partial y^3}(x_i, y_i - \theta_{i,4}h) \right]$$

with some $\theta_{i,j} \in]0; 1[$ and (x_i, y_i) being the coordinates of v_i. Moreover, one has

$$||\overline{\eta}(u)||_\infty = \max_{i=1,\ldots,m} |\eta(u)(x_i, y_i)| \lesssim h M_3,$$

where $M_3 = ||D^3 u||_\infty = \max_{\substack{(x, y) \in \overline{\mathcal{S}} \\ |\alpha| = 3}} |D^\alpha u(x, y)|$.

Proof. — This result is just a consequence of Taylor's formula. We refer the reader to [14] for the details. □

LEMMA 4.10. — *Let $W = (w_1, \ldots, w_m)^\top, G = (g_1, \ldots, g_m)^\top \in \mathbb{R}^m$ be such that $A_h W = G$. Then, for all $i = 1, \ldots, m$,*

$$|w_i| \leqslant \frac{1}{4}[x_i(1 - x_i) + y_i(1 - y_i)] \, ||G||_\infty \tag{4.29}$$

where (x_i, y_i) are the coordinates of v_i and $||G||_\infty = \max_{i=1,\ldots,m} |g_i|$.

Proof. — Let us consider $\tilde{w}$ defined by $\tilde{w}(x, y) = \frac{1}{4}(x(1 - x) + y(1 - y))\tilde{h}$, with $\tilde{h} = ||G||_\infty$. We notice that $\frac{\partial^2 \tilde{w}}{\partial x^2} = \frac{\partial^2 \tilde{w}}{\partial y^2} = -\frac{1}{2}\tilde{h}$, and thus $\tilde{w} \in \mathcal{C}^4(\overline{\mathcal{S}})$ is solution of

$$\begin{cases} -\Delta \tilde{w} = \tilde{h} & \text{in } \mathcal{S} \\ \tilde{w} = 0 & \text{on } \partial \mathcal{S}. \end{cases}$$

We write D_h^w for the solution of the following finite difference problem:

$$A_h D_h^w = \tilde{H}$$

where $\tilde{H} = \tilde{h}(1, \cdots, 1)^\top$. Owing to Proposition 4.9 and noticing that $\eta(\tilde{w}) = 0$, we get

$$A_h(\tilde{W} - D_h^w) = 0$$

where $\tilde{W} = (\widetilde{w_1}, ..., \widetilde{w_m})^\top$ with $\widetilde{w_i} = \tilde{w}(v_i)$, for every $i \in \{1, \ldots, m\}$. Comparing the two last identities, we obtain

$$A_h \tilde{W} = \tilde{H}. \tag{4.30}$$

Since for all $i \in \{1, \ldots, m\}$, $\tilde{h} = ||G||_\infty \geqslant |g_i|$, we deduce from (4.30) that $(A_h\tilde{W})_i \geqslant |(A_hW)_i|$. This implies that $A_h(\tilde{W} - W) \geqslant 0$ and $A_h(\tilde{W} + W) \geqslant 0$. As A_h is a monotone matrix, this leads to $\tilde{W} - W \geqslant 0$ and $\tilde{W} + W \geqslant 0$. In other words, for every $i = 1, ..., m$, $\widetilde{w_i} \geqslant |w_i|$ and thanks to the definition of $\tilde{w}$, we finally get (4.29). □

PROPOSITION 4.11. — *The finite difference problem (4.27) admits a unique solution D_h.*
Assume that $u \in \mathcal{C}^3(\overline{\mathcal{S}})$ and set $D_h = (D_h(v_1), ..., D_h(v_m))^\top$, then for every $i = 1, \ldots, m$, we have

$$|u(v_i) - D_h(v_i)| \lesssim M_3 h[x_i(1 - x_i) + y_i(1 - y_i)], \tag{4.31}$$

with $M_3 = ||D^3u||_\infty$, (x_i, y_i) denotes the coordinates of v_i, and the numerical constant appearing here (and below) is independent of u, h and i.

Proof. — Due to Corollary 4.8, A_h is a monotone matrix and consequently A_h is regular. This implies that there exists a unique solution D_h of (4.27).
Owing to Proposition 4.9,

$$A_h(U - D_h) = \overline{\eta}(u).$$

Let us apply Lemma 4.10 with $W = U - D_h$ and $G = \overline{\eta}(u)$. Then

$$|(U - D_h)(v_i)| \leqslant \frac{1}{4}[x_i(1 - x_i) + y_i(1 - y_i)]||\overline{\eta}(u)||_\infty.$$

The conclusion follows directly from the estimates of $||\overline{\eta}(u)||_\infty$ given in Proposition 4.9. □

PROPOSITION 4.12. — *If $f \in \mathcal{C}^1(\overline{\mathcal{S}})$, then*

$$\left| \frac{F(\lambda_v)}{h} - f(v) \right| \lesssim h||\nabla f||_{\infty, \overline{\mathcal{S}}}, \forall v \in \mathcal{N}_h^{int}.$$

Proof. — Let $v \in \mathcal{N}_h^{int}$, it is easy to prove that for all $i \in \mathcal{I}_v$,

$$\int_{e_i} \lambda_v(x)dx = \frac{h}{2}. \tag{4.32}$$

Thus we get successively

$$
\begin{aligned}
\left| \frac{F(\lambda_v)}{h} - f(v) \right| &= \frac{1}{h} \left| \left(\sum_{i \in \mathcal{I}_v} \int_{e_i} \widetilde{f}_i(x) \lambda_v(x) dx \right) - h f(v) \right| \\
&\leqslant \frac{1}{h} \sum_{i \in \mathcal{I}_v} \left| \int_{e_i} \widetilde{f}_i(x) \lambda_v(x) dx - \frac{h}{4} f(v) \right| \\
&\leqslant \frac{1}{h} \sum_{i \in \mathcal{I}_v} \left| \int_{e_i} \widetilde{f}_i(x) \lambda_v(x) dx - \frac{f(v)}{2} \int_{e_i} \lambda_v(x) dx \right| \\
&\leqslant \frac{1}{h} \sum_{i \in \mathcal{I}_v} \max_{x \in e_i} \left| \widetilde{f}_i(x) - \frac{f(v)}{2} \right| \int_{e_i} \lambda_v(x) dx \\
&\leqslant \frac{1}{h} \sum_{i \in \mathcal{I}_v} \max_{x \in e_i} \left| \widetilde{f}_i(x) - \frac{f(v)}{2} \right| \frac{h}{2}.
\end{aligned}
\tag{4.33}
$$

Since $\widetilde{f}_i = \frac{1}{2} \gamma_i f$, we have

$$
\max_{x \in e_i} \left| \widetilde{f}_i(x) - \frac{f(v)}{2} \right| \leqslant \sup_{z \in B(v,h)} \left| \frac{f(z)}{2} - \frac{f(v)}{2} \right|
\tag{4.34}
$$

where $B(v,h) = \{ z \in \mathcal{S} \text{ s.t. } \|z - v\|_2 < h \}$. And since $card(\mathcal{I}_v) = 4$, (4.33) and (4.34) imply

$$
\left| \frac{F(\lambda_v)}{h} - f(v) \right| \leqslant 2 \sup_{z \in B(v,h)} \left| \frac{f(z)}{2} - \frac{f(v)}{2} \right|.
\tag{4.35}
$$

As $f \in \mathcal{C}^1(\overline{\mathcal{S}})$, we have

$$
\forall z \in B(v,h), |f(z) - f(v)| \leqslant h \max_{\xi \in \overline{\mathcal{S}}} |\nabla f(\xi)|.
\tag{4.36}
$$

The aim follows from (4.35) and (4.36). □

THEOREM 4.13. — *If $u \in \mathcal{C}^3(\overline{\mathcal{S}})$, then*

$$
|(u - \widetilde{U_h})(v_i)| \lesssim h \|u\|_{\mathcal{C}^3(\overline{\mathcal{S}})}, \forall i = 1, \cdots, m.
$$

Proof. — Equalities (4.26) and (4.28) imply

$$
A_h \widetilde{U_h} = \frac{\widetilde{F_h}}{h}.
$$

Owing to (4.27), this leads to

$$
A_h (D_h - \widetilde{U_h}) = F_h - \frac{\widetilde{F_h}}{h}.
$$

Thanks to Lemma 4.10, this implies

$$
\begin{aligned}
|(D_h - \widetilde{U_h})(v_i)| &\leqslant \frac{1}{4}[x_i(1 - x_i) + y_i(1 - y_i)] \max_i \left| \left(F_h - \frac{\widetilde{F_h}}{h} \right)_i \right| \\
&\leqslant \frac{1}{4}[x_i(1 - x_i) + y_i(1 - y_i)] \max_{v \in \mathcal{N}_h^{int}} \left| f(v) - \frac{1}{h} F(\lambda_v) \right|.
\end{aligned}
$$

Owing to Proposition 4.12 (since $f \in \mathcal{C}^1(\overline{\mathcal{S}})$), we get

$$
|(D_h - \widetilde{U_h})(v_i)| \lesssim h[x_i(1 - x_i) + y_i(1 - y_i)] \|u\|_{\mathcal{C}^3(\overline{\mathcal{S}})}.
$$

Combining this estimate with (4.31) we obtain the expected estimate. □

5. Some results for an arbitrary domain

Our goal is to extend some of the previous results to an arbitrary domain of the plane. Let us start with some notation. Let $\Omega \subset \mathbb{R}^2$ denote a bounded open domain with a smooth boundary. Without loss of generality we can assume that $\overline{\Omega} \subset \mathcal{S}$, where $\mathcal{S}$ denotes the square $]0;1[\times]0;1[$. We here consider the Dirichlet problem in Ω:

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{5.1}$$

with $f \in \mathcal{C}(\overline{\Omega})$.

To approximate this problem by similar ones on a family of networks, we cut the square $\mathcal{S}$ as previously and use the same notation as before. Let us further denote

$$\mathcal{N}_{h,\Omega}^{int} = \mathcal{N}_h^{int} \cap \Omega, \, \mathcal{N}_{h,\Omega}^{ext} = \partial\Omega \cap \overline{\mathcal{R}_h}, \, \mathcal{N}_{h,\Omega} = \mathcal{N}_{h,\Omega}^{int} \cup \mathcal{N}_{h,\Omega}^{ext}, \, \mathcal{R}_{h,\Omega} = \mathcal{R}_h \cap \Omega$$

and

$$\mathcal{R}_{h,\Omega} = \{e_i^\Omega; i = 1, \ldots, N_{h,\Omega}\}, \text{ where } e_i^\Omega = \Omega \cap e_i, \forall i = 1, ..., N_{h,\Omega}.$$

We define the variational space

$$V_h^\Omega = \{u_h = (u_i)_{i=1,\ldots,N_{h,\Omega}} \in \prod_{i=1}^{N_{h,\Omega}} H^1(e_i^\Omega) \text{ s.t.}$$
$$u_i(v) = u_j(v) \qquad \forall v \in \mathcal{N}_{h,\Omega}^{int}, \forall i, j \in \mathcal{I}_v,$$
$$u_i(v) = 0 \qquad \forall v \in \mathcal{N}_{h,\Omega}^{ext}, \forall i \in \mathcal{I}_v\}, \tag{5.2}$$

equipped with the norm

$$||u||_{h,\Omega} = |u|_{1,\mathcal{R}_{h,\Omega}} = \left[\sum_{i=1}^{N_{h,\Omega}} \int_{e_i^\Omega} (u_i'(x))^2 dx\right]^{1/2}. \tag{5.3}$$

Introducing the bilinear and linear forms on V_h^Ω

$$a_h^\Omega : V_h^\Omega \times V_h^\Omega \to \mathbb{R} : (u_h, w_h) \to a_h^\Omega(u_h, w_h) = \sum_{i=1}^{N_{h,\Omega}} \int_{e_i^\Omega} u_i'(x)w_i'(x)dx, \tag{5.4}$$

$$F^\Omega(w_h) = \sum_{i=1}^{N_{h,\Omega}} \int_{e_i^\Omega} \widetilde{f_i}(x)w_i(x)dx, \forall w_h \in V_h^\Omega, \tag{5.5}$$

we can consider the solution $u_h^\Omega \in V_h^\Omega$ of (compare with (2.12))

$$\forall w_h \in V_h^\Omega, a_h^\Omega(u_h^\Omega, w_h) = F^\Omega(w_h). \tag{5.6}$$

5.1. **An approximation result between the continuous problems.** Lemma 3.4 still holds : for all $u \in V_h^\Omega$, one has

$$||u - u_h^\Omega||_{h,\Omega} \lesssim \sup_{w_h \in V_h^\Omega} \frac{|a_h^\Omega(u, w_h) - F^\Omega(w_h)|}{||w_h||_{h,\Omega}}. \tag{5.7}$$

Therefore we only need to estimate the consistency error.

THEOREM 5.1. — *Let u denote the solution of (5.1) supposed to belong to* $H^3(\Omega)$, *then*

$$\sup_{w_h \in V_h^\Omega} \frac{|a_h^\Omega(u, w_h) - F^\Omega(w_h)|}{||w_h||_{h,\Omega}} \lesssim \sqrt{h}||u||_{3,\Omega}. \tag{5.8}$$

Proof. — As we have assumed $\overline{\Omega} \subset \mathcal{S}$, there exists $h_0 > 0$ small enough such that $\overline{\Omega} \subset]\frac{h_0}{2}, 1 - \frac{h_0}{2}[\times]\frac{h_0}{2}, 1 - \frac{h_0}{2}[$. From now on, we suppose that $h \in]0, h_0[$. For $w \in V_h^{\Omega}$, we denote $\tilde{w}$, the extension of w by 0 outside Ω. Then $\tilde{w} \in V_h$. Owing to Theorem 1.4.3.1 of [11], there exists an extension $Eu \in H_0^1(\mathcal{S}) \cap H^3(\mathcal{S})$ of u such that

$$||Eu||_{3,\mathcal{S}} \leqslant c||u||_{3,\Omega}, \tag{5.9}$$

where c is a positive constant independent of u and that depends only on Ω and most importantly, Eu coincides with u on Ω. Then for $w \in V_h^{\Omega}$,

$$
\begin{aligned}
a_h^{\Omega}(u,w) - F^{\Omega}(w) &= \sum_{i=1}^{N_{h,\Omega}} \int_{e_i^{\Omega}} u_i'(x) w_i'(x) dx - \sum_{i=1}^{N_{h,\Omega}} \int_{e_i^{\Omega}} \tilde{f}_i(x) w_i(x) dx \\
&= -\sum_{i=1}^{N_{h,\Omega}} \int_{e_i^{\Omega}} u_i''(x) w_i(x) dx - \sum_{i=1}^{N_{h,\Omega}} \int_{e_i^{\Omega}} \tilde{f}_i(x) w_i(x) dx
\end{aligned}
$$

where we have used Kirchoff's law satisfied by u at each interior nodes. Consequently,

$$
\begin{aligned}
a_h^{\Omega}(u,w) - F^{\Omega}(w) &= -\sum_{i=1}^{N_{h,\Omega}} \int_{e_i^{\Omega}} (u_i'' + \widetilde{f}_i)(x) w_i(x) dx \\
&= -\sum_{i=1}^{N_h} \int_{e_i} ((Eu)_i'' + \widetilde{f}_i)(x) \widetilde{w}_i(x) dx
\end{aligned}
$$

since $\tilde{w} = 0$ in $\mathcal{S} \setminus \Omega$ and $Eu'' = u''$ in Ω. As $h < h_0$, $\tilde{w} = 0$ in R_v^h defined by (3.3) for $v \in \mathcal{N}_h^{ext}$ and hence

$$a_h^{\Omega}(u,w) - F^{\Omega}(w) = -\sum_{v \in \mathcal{N}_h^{int}} \int_{R_v^h} ((Eu)'' + \tilde{f})(x) \tilde{w}(x) dx.$$

Thanks to (3.20),

$$
\begin{aligned}
\left| a_h^{\Omega}(u,w) - F^{\Omega}(w) \right| &\leqslant \sum_{v \in \mathcal{N}_h^{int}} \left| \int_{R_v^h} ((Eu)'' + \tilde{f})(x) \tilde{w}(x) dx \right| \\
&\lesssim \sqrt{h} \sum_{v \in \mathcal{N}_h^{int}} \left(|\tilde{w}|_{1,R_v^h} ||Eu||_{3,C_v^h} + ||\tilde{w}||_{R_v^h} |Eu|_{3,C_v^h} \right).
\end{aligned}
$$

The Cauchy-Schwarz inequality leads to

$$\left| a_h^{\Omega}(u,w) - F^{\Omega}(w) \right| \lesssim \sqrt{h} \left(|\tilde{w}|_{1,\mathcal{R}_h} ||Eu||_{3,\mathcal{S}} + ||\tilde{w}||_{\mathcal{R}_h} |Eu|_{3,\mathcal{S}} \right).$$

Applying Lemma 2.1 to $\tilde{w} \in V_h$, we get

$$\left| a_h^{\Omega}(u,w) - F^{\Omega}(w) \right| \lesssim \sqrt{h} |\tilde{w}|_{1,\mathcal{R}_h} ||Eu||_{3,\mathcal{S}}.$$

As $|\tilde{w}|_{1,\mathcal{R}_h} = |w|_{1,\mathcal{R}_h^{\Omega}}$, and thanks to (5.9), we arrive at (5.8). $\qquad\square$

The estimates (5.7) and (5.8) directly lead to the

COROLLARY 5.2. — *Let u denote the solution of (5.1), and let u_h^{Ω} be the solution of (5.6). If $u \in H^3(\mathcal{S})$, then*

$$||u - u_h^{\Omega}||_{h,\Omega} \lesssim \sqrt{h} ||u||_{3,\Omega}. \tag{5.10}$$

5.2. **The finite element method on the networks.** Let us define the discrete variational space

$$W_h^\Omega = \{w_h = (w_i)_{i=1,\ldots,N_{h,\Omega}} \in V_h^\Omega \text{ s.t. } w_i \in \mathbb{P}_1(\overline{e_i^\Omega}), \forall i = 1,\ldots,N_{h,\Omega}\}. \quad (5.11)$$

We write $U_h^\Omega \in W_h^\Omega$ for the unique solution of the finite element problem

$$a_h^\Omega(U_h^\Omega, w_h) = F^\Omega(w_h), \forall w_h \in W_h^\Omega. \quad (5.12)$$

PROPOSITION 5.3. — *Let* $u_h^\Omega \in V_h^\Omega$ *denote the solution of (5.6) and let* $U_h^\Omega \in W_h^\Omega$ *be the solution of (5.12). Let us assume that the datum* f *belongs to* $\mathcal{C}(\overline{\Omega})$. *Then*

$$||u_h^\Omega - U_h^\Omega||_{h,\Omega} \lesssim \sqrt{h}||f||_{\infty,\Omega}. \quad (5.13)$$

Proof. — As in the proof of Lemma 4.3, we can show that

$$||f||_{L_\ell \cap \Omega}^2 \leqslant ||f||_{\infty,\Omega}^2. \quad (5.14)$$

So the equivalent of Lemma 4.3 holds:

$$||f||_{\mathcal{R}_{h,\Omega}} \leqslant \sqrt{2}\, h^{-1/2}||f||_{\infty,\Omega}. \quad (5.15)$$

Then we argue exactly as in the proof of Proposition 4.4, replacing Lemma 4.3 with inequality (5.15). $\square$

Note that under the assumptions $u \in H^3(\Omega)$ and $f \in \mathcal{C}(\overline{\Omega})$, the estimates (5.10) and (5.13) yield

$$||u - U_h^\Omega||_{h,\Omega} \lesssim \sqrt{h}(||u||_{3,\Omega} + ||f||_{\infty,\Omega}),$$

which shows the convergence of U_h^Ω to u. Note further that a similar estimate in the H^1-norm of Ω (i.e. an estimate like (4.18)) seems difficult to obtain since the estimate of $\tilde{u} - I_h\tilde{u}$ (where $\tilde{u}$ is the extension of u by zero outside Ω) is problematic near the boundary of Ω. Nevertheless such an estimate holds far from the boundary, namely if we set

$$\Omega_h = \cup_{k,\ell: \bar{K}_{k,\ell} \subset \Omega} K_{k,\ell},$$

then, with the same assumptions as before, as in Theorem 4.5 we can prove that

$$|u - R_h \tilde{U}_h^\Omega|_{1,\Omega_h} \lesssim h\left(||u||_{3,\Omega} + ||f||_{\infty,\Omega}\right), \quad (5.16)$$

where $\tilde{U}_h^\Omega$ is the extension by zero of U_h^Ω outside Ω.

Remark 5.4. —   The error estimates (4.18) and (5.16) show the same order of convergence than the standard finite element method but require a higher regularity on the solution and on the data. Nevertheless, our method can be considered as an attractive alternative to the standard ones for the three following reasons:
1. the cartesian networks $\mathcal{R}_h^\Omega$ are easily built,
2. the associated stiffness matrix is easier to compute and is still symmetric, positive definite and sparse,
3. as no two-dimensional mesh is necessary, our method is easy to implement for arbitrary domains.

6. NUMERICAL RESULTS

To illustrate our theoretical results we propose some numerical tests. First we take as exact solution:

$$u(x, y) = (x(1 - x)y(1 - y))^\alpha,$$

with a parameter $\alpha > 1.5$. Note that this solution belongs to $H^3(\mathcal{S})$ whenever $\alpha > 2.5$.
First of all, we want to compare the solution u_h of problem (2.2) and the approximation of u (solution of problem (2.1)) by the P1-finite element method in $\mathcal{S}$, called

u_{FE}. On the one hand, we easily compute u_h since for example, on a horizontal edge $y = y_0$

$$u_h(x, y_0) = \int \int \tilde{f}(\cdot, y_0) + Q(x)$$

where Q is a linear polynomial. By imposing Dirichlet boundary conditions, the continuity at the nodes and Kirchoff's law, we obtain that the coefficients of those polynomials are solutions of a linear system. On the other hand, u_{FE} is computed with the help of the FreeFem++ software [10] using a triangular mesh with as many nodes as there are in the network. First for $\alpha = 3$, we observe in Figure 6.1 that the contour lines of u_h and of u_{FE} at the same level of resolution are very similar.
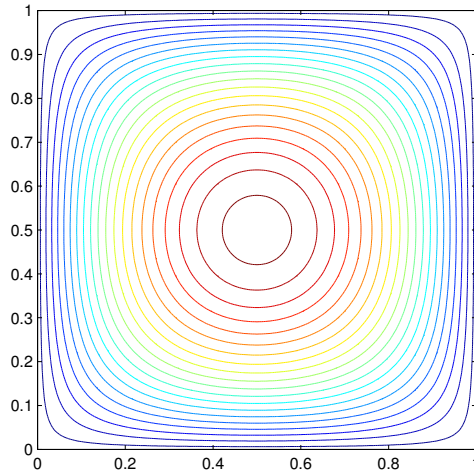
(a) Contour lines of u_h for $\alpha = 3$
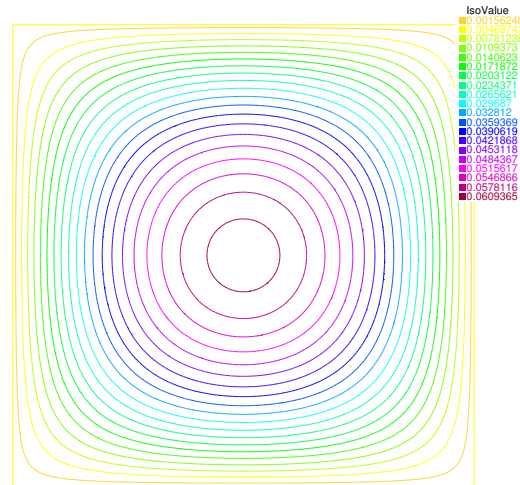
(b) Contour lines of u_{FE} for $\alpha = 3$

FIGURE 6.1. Comparison between u_h and u_{FE}

In Figures 6.2 and 6.3, we have plotted the L^∞-error and the H^1-error between the exact solution u and the solution u_h (defined on the network) on a log-log scale for different values of α. As expected, straight lines are obtained with different slopes

specified in Table 6.1. We recover the expected rate of convergence $1/2$ for the H^1 error whenever $\alpha > 2.5$ whereas the results regarding the L^∞ norm are better than those stated by Corollary 3.6. It is even better than the case of a regular solution treated by Theorem 4.13 that predicted a converence rate of one. Actually for the chosen solution this improvement is caused by the small size of the term estimated in Proposition 4.12.
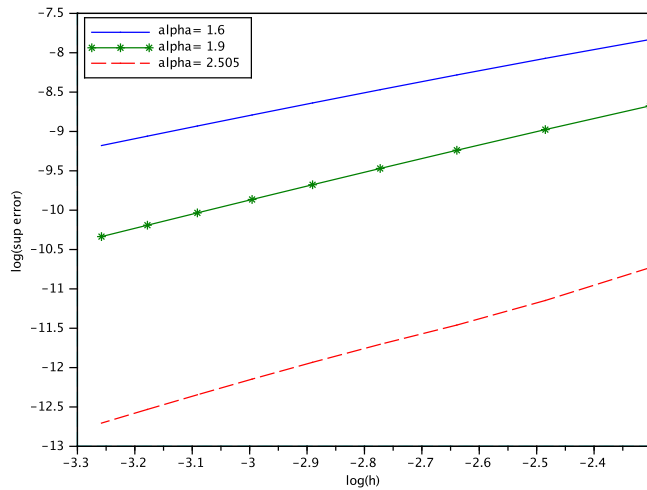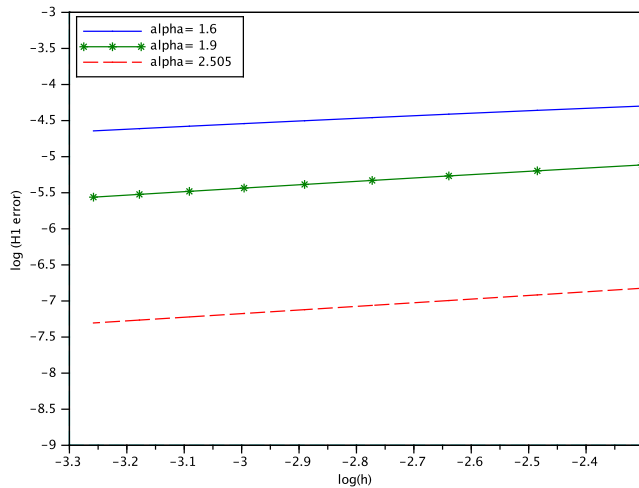

FIGURE 6.2. L^∞ error for some values α

FIGURE 6.3. H^1 error for some values of α

That is why a second example is considered where the exact solution is defined by $u(x, y) = \sin(10\pi x)\sin(10\pi y)$. In Figure 6.4, we see that the experimental rate of convergence of the L^∞-norm is 1, as asserted in Theorem 4.13.

α	L^∞ error	H^1 error
2.505	1.988	0.502
1.9	1.767	0.474
1.6	1.443	0.371

TABLE 6.1. Convergence rates of L^∞ and H^1 errors
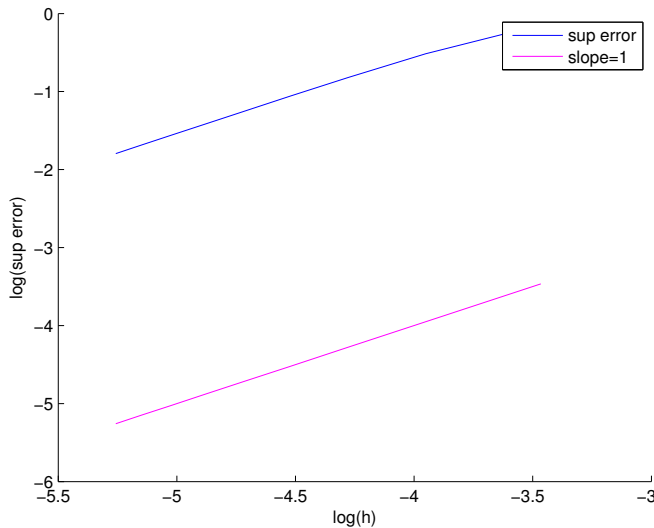
FIGURE 6.4. L^∞ error for the second example

REFERENCES

[1] F. Ali Mehmeti. A characterisation of generalized c^∞ notion on nets. *Integral Eq. and Operator Theory*, 9:753–766, 1986.

[2] F. Ali Mehmeti. *Nonlinear wave in networks*, volume 80 of *Math. Res.* Akademie Verlag, 1994.

[3] O. Axelsson. *Iterative solution methods.* Cambridge University Press, Cambridge, 1994.

[4] J. von Below. A characteristic equation associated to an eigenvalue problem on c^2-networks. *Linear Algebra and Appl.*, 71:309–325, 1985.

[5] J. von Below. Parabolic network equations. Technical report, Eberhard-Karls-Universität Tübingen, 1993. Habilitationsschrift.

[6] P. G. Ciarlet. *The finite element method for elliptic problems.* North-Holland, Amsterdam, 1978.

[7] P. G. Ciarlet. *Plates and junctions in elastic multi-structures*, volume 14 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1990. An asymptotic analysis.

[8] D. Cioranescu and J. Saint Jean Paulin. *Homogenization of reticulated structures*, volume 136 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1999.

[9] P. Destuynder and M. Salaun. *Mathematical analysis of thin plate models*, volume 24 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 1996.

[10] FreeFEM++ finite element programming environment. `http://www.freefem.org/ff++/`.

[11] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics.* Pitman, Boston–London–Melbourne, 1985.

[12] A. V. Komarov and O. M. Penkin. On the spectrum of a nonperiodic woven membrane. *Sovrem. Mat. Prilozh.*, (16, Differ. Uravn. Chast. Proizvod.):3–21, 2004.

[13] A. V. Komarov, O. M. Penkin, and Y. V. Pokornyǐ. On the spectrum of a uniform network of strings. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (4):23–27, 2000.

[14] P. Lascaux and R. Théodor. *Analyse numérique matricielle appliquée à l'art de l'ingénieur. Tome 1.* Masson, Paris, 1986.

[15] G. Lumer. Connecting of local operators and evolution equations on networks. In *Potential theory, Copenhagen 1979 (Proc. Colloq., Copenhagen, 1979)*, volume 787 of *Lecture Notes in Math.*, pages 219–234. Springer, Berlin, 1980.

[16] G. Lumer. Espaces ramifiés, et diffusions sur les réseaux topologiques. *C. R. Acad. Sci. Paris Sér. A-B*, 291(12):A627–A630, 1980.

[17] S. Nicaise. *Diffusion sur les espaces ramifiés*. PhD thesis, U. Mons (Belgium), 1986.

[18] S. Nicaise. Spectre des réseaux topologiques finis. *Bull. Sc. Math., 2ème série*, 111:401–413, 1987.

[19] S. Nicaise and O. Penkin. Relationship between the lower frequency spectrum of plates and networks of beams. *Math. Methods Appl. Sci.*, 23(16):1389–1399, 2000.

[20] G. Panasenko. *Multi-scale modelling for structures and composites*. Springer, Dordrecht, 2005.

[21] O. Penkin. *Some qualitative properties of the boundary values problems on graphs*. PhD thesis, U. Voronezh (Russia), 1988.

[22] A. Quarteroni, R. Sacco, and F. Saleri. *Numerical mathematics*, volume 37 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 2000.

[23] A. Quarteroni and A. Valli. *Numerical approximation of partial differential equations*, volume 23 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1994.

Maryse BOURLARD-JOSPIN, Serge NICAISE & Juliette VENEL
Université de Valenciennes et du Hainaut Cambrésis, LAMAV, FR CNRS 2956, Institut des Sciences et Techniques de Valenciennes, F-59313 Valenciennes Cedex 9, France
maryse.jospin@univ-valenciennes.fr
serge.nicaise@univ-valenciennes.fr (Corresponding author)
juliette.venel@univ-valenciennes.fr