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MAN-HO HO

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# Remarks on flat and differential $K$ -theory

MAN-HO HO

## Abstract

In this note we prove some results in flat and differential  $K$ -theory. The first one is a proof of the compatibility of the differential topological index and the flat topological index by a direct computation. The second one is the explicit isomorphisms between Bunke-Schick differential  $K$ -theory and Freed-Lott differential  $K$ -theory.

*Remarques sur les  $K$ -théories plate et différentielle*

## Résumé

Dans cette note, nous prouvons certains résultats en  $K$ -théories plate et différentielle. Le premier est une preuve de la compatibilité de l'indice topologique différentiel et de l'indice topologique plat par un calcul direct. Le second est un isomorphisme explicite entre les  $K$ -théories différentielles de Bunke-Schick et de Freed-Lott.

## 1. Introduction

In this note we prove some results in flat and differential  $K$ -theory. While some of these results are known to the experts, the proofs given here have not appeared in the literature. We first prove the compatibility of the flat topological index  $\text{ind}_L^t$  and the differential topological index  $\text{ind}_{\text{FL}}^t$  by a direct computation, i.e., the following diagram commutes ([7, Proposition 8.10])

$$\begin{array}{ccc}
 K_L^{-1}(X; \mathbb{R}/\mathbb{Z}) & \xrightarrow{i} & \widehat{K}_{\text{FL}}(X) \\
 \text{ind}^t \downarrow & & \downarrow \text{ind}_{\text{FL}}^t \\
 K_L^{-1}(B; \mathbb{R}/\mathbb{Z}) & \xrightarrow[i]{} & \widehat{K}_{\text{FL}}(B)
 \end{array} \tag{1.1}$$

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where  $i$  is the canonical inclusion,  $K_L^{-1}(X; \mathbb{R}/\mathbb{Z})$  is the geometric model of  $K$ -theory with  $\mathbb{R}/\mathbb{Z}$  coefficients and  $\widehat{K}_{\text{FL}}(X)$  is Freed-Lott differential  $K$ -theory. The commutativity of (1.1) is a consequence of the compatibility of the differential analytic index  $\text{ind}_{\text{FL}}^{\text{a}}$  and the flat analytic index  $\text{ind}_L^{\text{a}}$  together with the differential family index theorem [7, Theorem 7.35]. The differential topological index  $\text{ind}_{\text{FL}}^{\text{t}}$  is defined to be the composition of an embedding pushforward and a projection pushforward. When defining the embedding pushforward, currential  $K$ -theory [7, §2.28] is used instead of differential  $K$ -theory due to the Bismut-Zhang current [2, Definition 1.3]. It is not clear whether currential  $K$ -theory should be regarded as a differential cohomology or a “differential homology” (see [6, §4.5] for a detailed discussion), so it may be clearer by looking at the direct computation.

Second we construct the unique natural isomorphisms between Bunke-Schick differential  $K$ -theory [4] and Freed-Lott differential  $K$ -theory by writing down the explicit formulas, which are inspired by [4, Corollary 5.5]. The uniqueness follows from [5, Theorem 3.10]. Together with [10, Theorem 4.34] and [8, Theorem 1] all the explicit isomorphisms between all the existing differential  $K$ -groups [9], [4], [7], [12] are known.

The paper is organized as follows: Section 2 contains all the necessary background material, including the Freed-Lott differential  $K$ -theory, the differential topological index, the pairing between flat  $K$ -theory and  $K$ -homology, and Bunke-Schick differential  $K$ -theory. In Section 3 we prove the main results.

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## 2. Background material

### 2.1. Freed-Lott differential $K$ -theory and the differential topological index

In this section we review Freed-Lott differential  $K$ -theory and the construction of the differential topological index [7, §4, 5]. We refer the readers to [7] for the details.

REMARKS ON FLAT AND DIFFERENTIAL  $K$ -THEORY

The Freed–Lott differential  $K$ -group  $\widehat{K}_{\text{FL}}(X)$  is the abelian group generated by quadruples  $\mathcal{E} = (E, h, \nabla, \phi)$ , where  $(E, h, \nabla) \rightarrow X$  is a complex vector bundle with a Hermitian metric  $h$  and a unitary connection  $\nabla$ , and  $\phi \in \frac{\Omega^{\text{odd}}(X)}{\text{Im}(d)}$ . The only relation is  $\mathcal{E}_1 = \mathcal{E}_2$  if and only if there exists a generator  $(F, h^F, \nabla^F, \phi^F)$  of  $\widehat{K}_{\text{FL}}(X)$  such that  $E_1 \oplus F \cong E_2 \oplus F$  and  $\phi_1 - \phi_2 = \text{CS}(\nabla^{E_2} \oplus \nabla^F, \nabla^{E_1} \oplus \nabla^F)$ .

There is an exact sequence [7, (2.20)]

$$0 \longrightarrow K_{\mathbb{L}}^{-1}(X; \mathbb{R}/\mathbb{Z}) \xrightarrow{i} \widehat{K}_{\text{FL}}(X) \xrightarrow{\text{ch}_{\widehat{K}}} \Omega_{\text{BU}}^{\text{even}}(X) \longrightarrow 0 \quad (2.1)$$

where  $K_{\mathbb{L}}^{-1}(X; \mathbb{R}/\mathbb{Z})$  is the geometric model of  $\mathbb{R}/\mathbb{Z}$   $K$ -theory [11],  $i$  is the canonical inclusion map,

$$\Omega_{\text{BU}}^{\text{even}}(X) = \{\omega \in \Omega_{d=0}^{\text{even}}(X) \mid [\omega] \in \text{Im}(r \circ \text{ch} : K^0(X) \rightarrow H^{\text{even}}(X; \mathbb{R}))\},$$

and  $\text{ch}_{\widehat{K}_{\text{FL}}}(E, h, \nabla, \phi) := \text{ch}(\nabla) + d\phi$ . Elements in  $K_{\mathbb{L}}^{-1}(X; \mathbb{R}/\mathbb{Z})$  are required to have virtual rank zero. The canonical inclusion map  $i$  in (2.1) is defined by  $i(E, h, \nabla, \phi) = (E, h, \nabla, \phi)$ .

Let  $X \rightarrow B$  and  $Y \rightarrow B$  be fiber bundles of smooth manifolds with  $X$  compact. Let  $g^{T^V X}$  and  $g^{T^V Y}$  be metrics on the vertical bundles  $T^V X \rightarrow X$  and  $T^V Y \rightarrow Y$  respectively, and assume there are horizontal distributions  $T^H X$  and  $T^H Y$ . Let  $\mathcal{E} = (E, h^E, \nabla^E, \phi) \in \widehat{K}_{\text{FL}}(X)$  and  $\iota : X \hookrightarrow Y$  be an embedding of manifolds. We assume the codimension of  $X$  in  $Y$  is even, and the normal bundle  $\nu \rightarrow X$  of  $X$  in  $Y$  carries a  $\text{spin}^c$  structure. As in [7, §5] we assume for each  $b \in B$ , the map  $\iota_b : X_b \rightarrow Y_b$  is an isometric embedding and is compatible with projections to  $B$ . Denote by  $\mathcal{S}(\nu) \rightarrow X$  the spinor bundle associated to the  $\text{spin}^c$ -structure of  $\nu \rightarrow X$ . We can locally choose a  $\text{spin}$  structure for  $\nu \rightarrow X$  with spinor bundle  $\mathcal{S}^{\text{spin}}(\nu)$ . Then there exists a locally defined Hermitian line bundle  $L^{\frac{1}{2}}(\nu)$  such that  $\mathcal{S}(\nu) \cong \mathcal{S}^{\text{spin}}(\nu) \otimes L^{\frac{1}{2}}(\nu)$ . Note that the tensor product on the right is globally defined, and so is the Hermitian line bundle  $L(\nu) \rightarrow X$  defined by  $L(\nu) := (L^{\frac{1}{2}}(\nu))^2$ . Let  $\nabla^\nu$  be a metric compatible connection on  $\nu \rightarrow X$ . It has a unique lift to a connection on  $\mathcal{S}^{\text{spin}}(\nu)$ , still denoted by  $\nabla^\nu$ . Choose a unitary connection  $\nabla^{L(\nu)}$  on  $L(\nu) \rightarrow X$ , which induces a connection on  $L^{\frac{1}{2}}(\nu)$ . The tensor product of  $\nabla^\nu$  and the induced connection on  $L^{\frac{1}{2}}(\nu)$  is

a connection on  $\mathcal{S}(\nu) \rightarrow X$ , denoted by  $\widehat{\nabla}^\nu$ . Define

$$\text{Todd}(\widehat{\nabla}^\nu) := \widehat{A}(\nabla^\nu) \wedge e^{\frac{1}{2}c_1(\nabla^L(\nu))}.$$

The embedding pushforward  $\widehat{\iota}_* : \widehat{K}_{\text{FL}}(X) \rightarrow {}_\delta\widehat{K}_{\text{FL}}(Y)$  [7, Definition 4.14] is defined to be

$$\widehat{\iota}_*(\mathcal{E}) = \left( F, h^F, \nabla^F, \frac{\phi^E}{\text{Todd}(\widehat{\nabla}^\nu)} \wedge \delta_X - \gamma \right),$$

where  ${}_\delta\widehat{K}_{\text{FL}}(Y)$  is the currential  $K$ -group,  $\delta_X$  is the current of integration over  $X$  and  $\gamma$  is the Bismut-Zhang current.  $(F, h^F, \nabla^F)$  is a Hermitian bundle with a Hermitian metric and a unitary connection chosen as in [7, Lemma 4.4]. Note that  $\gamma$  satisfies the following transgression formula [2, Theorem 1.4]

$$d\gamma = \text{ch}(\nabla^F) - \frac{\text{ch}(\nabla^E)}{\text{Todd}(\widehat{\nabla}^\nu)} \wedge \delta_X.$$

As noted in [7, p.926] the horizontal distributions of the fiber bundles  $X \rightarrow B$  and  $Y \rightarrow B$  need not be compatible. An odd form  $\widetilde{C} \in \frac{\Omega^{\text{odd}}(X)}{\text{Im}(d)}$  is defined to correct this non-compatibility, and it satisfies the following transgression formula [7, (5.6)]

$$d\widetilde{C} = \iota^* \text{Todd}(\widehat{\nabla}^{T^V Y}) - \text{Todd}(\widehat{\nabla}^{T^V X}) \wedge \text{Todd}(\widehat{\nabla}^\nu).$$

The modified embedding pushforward  $\widehat{\iota}_*^{\text{mod}} : \widehat{K}_{\text{FL}}(X) \rightarrow {}_{\text{WF}}\widehat{K}_{\text{FL}}(Y)$  [7, Definition 5.8] is defined to be

$$\widehat{\iota}_*^{\text{mod}}(\mathcal{E}) := \widehat{\iota}_*(\mathcal{E}) - j \left( \frac{\widetilde{C}}{\iota^* \text{Todd}(\widehat{\nabla}^{T^V Y}) \wedge \text{Todd}(\widehat{\nabla}^\nu)} \wedge \text{ch}_{\widehat{K}_{\text{FL}}}(\mathcal{E}) \wedge \delta_X \right). \quad (2.2)$$

See [7, §3.1] for the definition of  ${}_{\text{WF}}\widehat{K}_{\text{FL}}(X)$ .

The differential topological index  $\text{ind}_{\text{FL}}^t : \widehat{K}_{\text{FL}}(X) \rightarrow \widehat{K}_{\text{FL}}(B)$  [7, Definition 5.34] is defined by taking  $Y = \mathbb{S}^N \times B$  for some even  $N$  and composing the embedding pushforward with the submersion pushforward  $\widehat{\pi}_*^{\text{prod}}$  defined in [7, Lemma 5.13], i.e.,  $\text{ind}_{\text{FL}}^t := \widehat{\pi}_*^{\text{prod}} \circ \widehat{\iota}_*^{\text{mod}}$ .

## 2.2. Pairing between flat $K$ -theory and topological $K$ -homology

Let  $X$  be an odd-dimensional closed  $\text{spin}^c$  manifold,  $\mathcal{E} = (E, h^E, \nabla^E, \phi) \in \widehat{K}_{\text{FL}}(X)$ , and  $D^{X,E}$  be the twisted Dirac operator on  $\mathcal{S}(X) \otimes E \rightarrow X$ .

A modified reduced eta-invariant  $\bar{\eta}(X, \mathcal{E}) \in \mathbb{R}/\mathbb{Z}$  [7, Definition 2.33] is defined by

$$\bar{\eta}(X, \mathcal{E}) := \bar{\eta}(\mathbb{D}^{X,E}) + \int_X \text{Todd}(\widehat{\nabla}^{TX}) \wedge \phi \pmod{\mathbb{Z}}.$$

$\bar{\eta} : {}_\delta\widehat{K}_{\text{FL}}(X) \rightarrow \mathbb{R}/\mathbb{Z}$  is a well defined homomorphism [7, Prop 2.25]. If  $\mathcal{E}$  is a generator of  $K_{\mathbb{L}}^{-1}(X; \mathbb{R}/\mathbb{Z})$ , by [7, (2.37)] we have

$$\bar{\eta}(X, i(\mathcal{E})) = \langle [X], \mathcal{E} \rangle, \quad (2.3)$$

where  $[X] \in K_{-1}(X)$  is the fundamental  $K$ -homology class. Here  $\langle [X], \mathcal{E} \rangle$  is the perfect pairing between flat  $K$ -theory and topological  $K$ -homology [11, Prop 3]

$$K_{\mathbb{L}}^{-1}(X; \mathbb{R}/\mathbb{Z}) \times K_{-1}(X) \rightarrow \mathbb{R}/\mathbb{Z}. \quad (2.4)$$

### 2.3. Bunke-Schick differential $K$ -theory

In this subsection we briefly recall Bunke-Schick differential  $K$ -theory  $\widehat{K}_{\text{BS}}$ , and refer to [4] for the details.

A generator of  $\widehat{K}_{\text{BS}}(B)$  is of the form  $(\mathcal{E}, \phi)$ , where  $\mathcal{E}$  is an even-dimensional geometric family [4, Definition 2.2] over a compact manifold  $B$  and  $\phi \in \frac{\Omega^{\text{odd}}(B)}{\text{Im}(d)}$ . Roughly speaking a geometric family over  $B$  is the geometric data needed to construct the index bundle. There is a well defined notion of isomorphic and sum of generators [4, Definition 2.5, 2.6]. Two geometric families  $(\mathcal{E}_0, \phi_0)$  and  $(\mathcal{E}_1, \phi_1)$  are equivalent if there exists a geometric family  $(\mathcal{E}', \phi')$  such that  $(\mathcal{E}_0, \rho_0) + (\mathcal{E}', \phi')$  is paired with  $(\mathcal{E}_1, \rho_1) + (\mathcal{E}', \phi')$  [4, Definition 2.10, Lemma 2.13]. Two generators  $(\mathcal{E}_0, \phi_0)$  and  $(\mathcal{E}_1, \phi_1)$  are paired if

$$\rho_1 - \rho_0 = \eta^{\text{B}}((\mathcal{E}_0 \sqcup_B (\mathcal{E}_1)^{\text{op}})_t),^1$$

where  $(\mathcal{E} \sqcup_B (\mathcal{E}')^{\text{op}})_t$  is a certain tamed geometric family [4, Definition 2.7], and  $\eta^{\text{B}}$  is the Bunke eta form [3].

As noted in [4, 2.14] and [3, 4.2.1], a complex vector bundle  $E \rightarrow B$  with a Hermitian metric  $h^E$  and a unitary connection  $\nabla^E$  can be naturally considered as a zero-dimensional geometric family over  $B$ , denoted by  $\mathbb{E}$ .

---

<sup>1</sup>It differs by a sign in [4].

### 3. Main results

#### 3.1. Compatibility of the topological indices

Note that every element  $\mathcal{E} - \mathcal{F} \in \widehat{K}_{\text{FL}}(X)$  can be written in the form

$$\widetilde{\mathcal{E}} - [n].$$

Here  $\widetilde{\mathcal{E}} = (E \oplus G, h^E \oplus h^G, \nabla^E \oplus \nabla^G, \phi^E + \phi^G)$ , where  $(G, h^G, \nabla^G, \phi^G)$  is a generator of  $\widehat{K}_{\text{FL}}(X)$  such that

$$(F \oplus G, h^F \oplus h^G, \nabla^F \oplus \nabla^G, \phi^F + \phi^G) = (\mathbb{C}^n, h, d, 0) =: [n].$$

The existence of the connection  $\nabla^G$  such that  $\text{CS}(\nabla^F \oplus \nabla^G, d) = 0$ , where  $d$  is the trivial connection on the trivial bundle  $X \times \mathbb{C}^n \rightarrow X$ , follows from [12, Theorem 1.8]. Here  $\phi^G := -\phi^F$ . Henceforth we assume an element of  $\widehat{K}_{\text{FL}}(X; \mathbb{R}/\mathbb{Z})$  is of the form  $\mathcal{E} - [n]$ . These arguments also apply to elements in  $K_{\text{L}}^{-1}(X; \mathbb{R}/\mathbb{Z})$ .

**Proposition 3.1.** *Let  $\pi : X \rightarrow B$  be a fiber bundle with  $X$  compact and such that the fibers are of even dimension. The following diagram commutes.*

$$\begin{array}{ccc} K_{\text{L}}^{-1}(X; \mathbb{R}/\mathbb{Z}) & \xrightarrow{i} & \widehat{K}_{\text{FL}}(X) \\ \text{ind}^{\text{t}} \downarrow & & \downarrow \text{ind}_{\text{FL}}^{\text{t}} \\ K_{\text{L}}^{-1}(B; \mathbb{R}/\mathbb{Z}) & \xrightarrow{i} & \widehat{K}_{\text{FL}}(B) \end{array}$$

*Proof.* Let  $\mathcal{E}' - [n]' \in K_{\text{L}}^{-1}(X)$  and write  $\mathcal{E} - [n] = i(\mathcal{E}' - [n]')$ , where  $i$  is given in (2.1). Consider the difference

$$h := \text{ind}_{\text{FL}}^{\text{t}}(\mathcal{E} - [n]) - i(\text{ind}^{\text{t}}(\mathcal{E}' - [n]')).$$

We prove that  $h = 0$ . By [7, Lemma 5.36] and the fact that  $\text{ch}_{\widehat{K}_{\text{FL}}} \circ i = 0$  (see (2.1)), we have

$$\begin{aligned} & \text{ch}_{\widehat{K}_{\text{FL}}}(\text{ind}_{\text{FL}}^{\text{t}}(\mathcal{E} - [n])) - \text{ch}_{\widehat{K}_{\text{FL}}}(i(\text{ind}^{\text{t}}(\mathcal{E}' - [n]'))) \\ &= \text{ch}_{\widehat{K}_{\text{FL}}}(\text{ind}_{\text{FL}}^{\text{t}}(\mathcal{E} - [n])) \\ &= \int_{X/B} \text{Todd}(\widehat{\nabla}^{T^V X}) \wedge (\text{ch}(\nabla^E) - \text{rank}(E) + d\phi^E) \\ &= 0. \end{aligned}$$

By (2.1), there exists an element  $a \in K^{-1}(B; \mathbb{R}/\mathbb{Z})$  such that  $i(a) = h$ . To prove  $a = 0 \in K_{\mathbb{L}}^{-1}(B; \mathbb{R}/\mathbb{Z})$ , it follows from (2.4) that it is sufficient to show that for all  $\alpha \in K_{-1}(B; \mathbb{Z})$ ,

$$\langle \alpha, a \rangle = 0 \in \mathbb{R}/\mathbb{Z}. \quad (3.1)$$

Using the geometric picture of  $K$ -homology [1], we may, without loss of generality, let  $\alpha = f_*[M]$  for some smooth map  $f : M \rightarrow B$ , where  $M$  is a closed odd-dimensional  $\text{spin}^c$  manifold, and  $[M]$  is the fundamental  $K$ -homology in  $K_{-1}(M)$ . Since  $\langle \alpha, a \rangle = \langle [M], f^*a \rangle$ , we pull everything back to  $M$  and we may assume  $B$  is an arbitrary closed odd-dimensional  $\text{spin}^c$  manifold. Thus proving (3.1) is equivalent to proving

$$\langle [B], a \rangle = 0 \in \mathbb{R}/\mathbb{Z}. \quad (3.2)$$

Since

$$\langle [B], a \rangle = \bar{\eta}(B, \text{ind}_{\text{FL}}^{\dagger}(\mathcal{E} - [n])) - \bar{\eta}(B, i(\text{ind}^{\dagger}(\mathcal{E}' - [n]'))) \pmod{\mathbb{Z}},$$

proving (3.2) is equivalent to proving

$$\bar{\eta}(B, \text{ind}_{\text{FL}}^{\dagger}(\mathcal{E} - [n])) = \bar{\eta}(B, i(\text{ind}^{\dagger}(\mathcal{E}' - [n]'))) \pmod{\mathbb{Z}}. \quad (3.3)$$

In the following, we write  $a \equiv b$  as  $a = b \pmod{\mathbb{Z}}$ . By [7, (6.7)], we have

$$\begin{aligned} \bar{\eta}(B, \text{ind}_{\text{FL}}^{\dagger}(\mathcal{E} - [n])) &\equiv \bar{\eta}(\mathbb{D}^{X, E-n}) + \int_X \frac{\iota^* \text{Todd}(\widehat{\nabla}^{T(\mathbb{S}^N \times B)})}{\text{Todd}(\widehat{\nabla}^{\nu})} \wedge \phi^E \\ &\quad - \int_X \frac{\pi^* \text{Todd}(\widehat{\nabla}^{TB})}{\text{Todd}(\widehat{\nabla}^{\nu})} \wedge \tilde{C} \wedge \text{ch}_{\widehat{K}_{\text{FL}}}(\mathcal{E} - [n]) \\ &\equiv \bar{\eta}(\mathbb{D}^{X, E-n}) + \int_X \frac{\iota^* \text{Todd}(\widehat{\nabla}^{T(\mathbb{S}^N \times B)})}{\text{Todd}(\widehat{\nabla}^{\nu})} \wedge \phi^E \end{aligned} \quad (3.4)$$

as  $\text{ch}_{\widehat{K}_{\text{FL}}}(\mathcal{E} - [n]) = \text{ch}_{\widehat{K}_{\text{FL}}}(i(\mathcal{E}' - [n]')) = 0$ . On the other hand, by [11, (49)], we have

$$\begin{aligned} \bar{\eta}(B, i(\text{ind}^{\dagger}(\mathcal{E}' - [n]'))) &\equiv \langle [B], \text{ind}^{\dagger}(\mathcal{E}' - [n]') \rangle \\ &= \langle \pi^! [B], \mathcal{E} - [n] \rangle = \langle [X], \mathcal{E} - [n] \rangle = \bar{\eta}(X, \mathcal{E} - [n]) \\ &\equiv \bar{\eta}(\mathbb{D}^{X, E-n}) + \int_X \text{Todd}(\widehat{\nabla}^{TX}) \wedge \phi^E. \end{aligned} \quad (3.5)$$



From (3.4) and (3.5) we have

$$\begin{aligned}
 & \bar{\eta}(B, \text{ind}_{\text{FL}}^t(\mathcal{E} - [n])) - \bar{\eta}(B, i(\text{ind}^t(\mathcal{E}' - [n]'))) \\
 & \equiv \int_X \left( \frac{\iota^* \text{Todd}(\widehat{\nabla}^{T(\mathbb{S}^N \times B)})}{\text{Todd}(\widehat{\nabla}^\nu)} - \text{Todd}(\widehat{\nabla}^{TX}) \right) \wedge \phi^E \\
 & \equiv \int_X \left( \frac{\iota^* \text{Todd}(\widehat{\nabla}^{T(\mathbb{S}^N \times B)}) - \text{Todd}(\widehat{\nabla}^{TX}) \wedge \text{Todd}(\widehat{\nabla}^\nu)}{\text{Todd}(\widehat{\nabla}^\nu)} \right) \wedge \phi^E.
 \end{aligned} \tag{3.6}$$

Since  $\text{ch}_{\widehat{K}_{\text{FL}}}(\mathcal{E} - [n]) = 0$ , it follows from (2.2) that

$$\widehat{\iota}_*^{\text{mod}}(\mathcal{E} - [n]) = \widehat{\iota}_*(\mathcal{E} - [n]). \tag{3.7}$$

Recall that the purpose of the modified embedding pushforward  $\widehat{\iota}_*^{\text{mod}}$  is to correct the non-compatibility of the horizontal distributions  $T^H(\mathbb{S}^N \times B)$  and  $T^H X$ . By (3.7) we may assume that the horizontal distributions  $T^H(\mathbb{S}^N \times B)$  and  $T^H X$  are compatible by changing the one for  $X$  to be the restriction of the one for  $\mathbb{S}^N \times B$ . Thus

$$\iota^* \text{Todd}(\widehat{\nabla}^{T(\mathbb{S}^N \times B)}) = \text{Todd}(\widehat{\nabla}^{TX}) \wedge \text{Todd}(\widehat{\nabla}^\nu),$$

which implies that (3.6) is zero, and therefore  $h = 0$ .  $\square$

### 3.2. Explicit isomorphisms between $\widehat{K}_{\text{BS}}$ and $\widehat{K}_{\text{FL}}$

In this subsection we construct the explicit isomorphisms between Bunke-Schick differential  $K$ -group and the Freed-Lott differential  $K$ -group.

**Proposition 3.2.** *Let  $B$  be a compact manifold. We define two maps,  $f : \widehat{K}_{\text{FL}}(B) \rightarrow \widehat{K}_{\text{BS}}(B)$  and  $g : \widehat{K}_{\text{BS}}(B) \rightarrow \widehat{K}_{\text{FL}}(B)$ , by*

$$\begin{aligned}
 f(E, h, \nabla, \phi) &= [\mathbb{E}, \phi], \\
 g([\mathcal{E}, \phi]) &= (\text{ind}^a(\mathcal{E}), h^{\text{ind}^a(\mathcal{E})}, \nabla^{\text{ind}^a(\mathcal{E})}, \phi),
 \end{aligned}$$

where, in the definition of  $f$ ,  $\mathbb{E}$  is the zero-dimensional geometric family associated to  $(E, h, \nabla)$ . Then  $f$  and  $g$  are well defined ring isomorphisms and are inverses to each other.

*Proof.* Note that it suffices to prove the statement under the assumption that  $\text{ind}^a(\mathcal{E}) \rightarrow B$  is actually given by a kernel bundle  $\ker(D^E) \rightarrow B$  in the definition of  $g$ . Indeed, by a standard perturbation argument every class in  $\widehat{K}_{\text{BS}}$  has a representative where this is satisfied.

First of all we prove that  $f$  is well defined. Suppose

$$(E, h^E, \nabla^E, \phi^E) = (F, h^F, \nabla^F, \phi^F) \in \widehat{K}_{\text{FL}}(B).$$

Then there exists a generator  $(G, h^G, \nabla^G, \phi^G)$  of  $\widehat{K}_{\text{FL}}(B)$  such that

$$\begin{aligned} E \oplus G &\cong F \oplus G, \\ \phi^F - \phi^E &= \text{CS}(\nabla^E \oplus \nabla^G, \nabla^F \oplus \nabla^G). \end{aligned} \quad (3.8)$$

Denote by  $\mathbb{F}$  and  $\mathbb{G}$  the zero-dimensional geometric families associated to  $(F, h^F, \nabla^F)$  and  $(G, h^G, \nabla^G)$ , respectively. We prove that  $[\mathbb{E}, \phi^E] = [\mathbb{F}, \phi^F] \in \widehat{K}_{\text{BS}}(B)$ . Indeed, we prove that  $(\mathbb{E} + \mathbb{G}, \phi^E + \phi^G)$  is paired with  $(\mathbb{F} + \mathbb{G}, \phi^F + \phi^G)$ . We need to show  $\mathbb{E} \sqcup_B \mathbb{G} \cong \mathbb{F} \sqcup_B \mathbb{G}$  and

$$(\phi^F + \phi^G) - (\phi^E + \phi^G) = \eta^{\text{B}}(((\mathbb{E} \sqcup_B \mathbb{G}) \sqcup_B (\mathbb{F} \sqcup_B \mathbb{G})^{\text{op}})_t) \quad (3.9)$$

if such a taming exists. In the case of zero-dimensional geometric family,  $\mathbb{E} \sqcup_B \mathbb{G} \cong E \oplus G$  as vector bundles over  $B$ . Thus the first equality (3.8) implies  $\mathbb{E} \sqcup_B \mathbb{G} \cong \mathbb{F} \sqcup_B \mathbb{G}$ . Since the underlying proper submersion of the trivial geometric family is the identity map, the corresponding kernel bundle is just  $E \rightarrow B$  by the remark of [6, Definition 4.7]. Thus the taming in (3.9) exists and the definition of  $\eta^{\text{B}}$  shows that

$$\eta^{\text{B}}(((\mathbb{E} \sqcup_B \mathbb{G}) \sqcup_B (\mathbb{F} \sqcup_B \mathbb{G})^{\text{op}})_t) = \text{CS}(\nabla^E \oplus \nabla^G, \nabla^F \oplus \nabla^G). \quad (3.10)$$

From (3.8) and (3.9) we see that  $(\mathbb{E} + \mathbb{G}, \phi^E + \phi^G)$  is paired with  $(\mathbb{F} + \mathbb{G}, \phi^F + \phi^G)$ . Thus  $f$  is well defined.

For the map  $g$ , note that under our assumption we have  $[\mathcal{E}, 0] = [\mathbb{K}, \tilde{\eta}(\mathcal{E})]$  by [4, Corollary 5.5], where  $\mathbb{K}$  is the trivial geometric family associated to  $(\ker(\mathbb{D}^E), h^{\ker(\mathbb{D}^E)}, \nabla^{\ker(\mathbb{D}^E)})$  and  $\tilde{\eta}(\mathcal{E})$  is the associated Bismut-Cheeger eta form. Since  $[\mathcal{E}, \phi] = [\mathbb{K}, \tilde{\eta}(\mathcal{E}) + \phi]$ ,  $g$  can be written as

$$g([\mathcal{E}, \phi]) = g([\mathbb{K}, \tilde{\eta}(\mathcal{E}) + \phi]) = (\ker(\mathbb{D}^E), h^{\ker(\mathbb{D}^E)}, \nabla^{\ker(\mathbb{D}^E)}, \tilde{\eta}(\mathcal{E}) + \phi).$$

We prove that  $g$  is well defined. Suppose  $[\mathcal{E}_1, \phi^1] = [\mathcal{E}_2, \phi^2] \in \widehat{K}_{\text{BS}}(B)$ . Since  $[\mathcal{E}_i, \phi^i] = [\mathbb{K}_i, \tilde{\eta}(\mathcal{E}_i) + \phi^i]$  for  $i = 1, 2$ , to prove  $g([\mathcal{E}_1, \phi^1]) = g([\mathcal{E}_2, \phi^2])$  it suffices to show

$$\begin{aligned} &(\ker(\mathbb{D}^{E^1}), h^{\ker(\mathbb{D}^{E^1})}, \nabla^{\ker(\mathbb{D}^{E^1})}, \tilde{\eta}(\mathcal{E}_1) + \phi^1) \\ &= (\ker(\mathbb{D}^{E^2}), h^{\ker(\mathbb{D}^{E^2})}, \nabla^{\ker(\mathbb{D}^{E^2})}, \tilde{\eta}(\mathcal{E}_2) + \phi^2). \end{aligned} \quad (3.11)$$

Since  $[\mathcal{E}_1, \phi^1] = [\mathcal{E}_2, \phi^2]$ , there exists a taming  $(\mathcal{E}_1 \sqcup_B (\mathcal{E}_2)^{\text{op}})_t$ , and therefore  $\ker(\mathbb{D}^{E^1}) = \ker(\mathbb{D}^{E^2}) \in K(B)$ . Thus it suffices to show

$$\text{CS}(\nabla^{\ker(\mathbb{D}^{E^2})}, \nabla^{\ker(\mathbb{D}^{E^1})}) = \tilde{\eta}(\mathcal{E}_1) - \tilde{\eta}(\mathcal{E}_2) + \phi^2 - \phi^1 \in \frac{\Omega^{\text{odd}}(B)}{\Omega_{\text{BU}}^{\text{odd}}(B)} \quad (3.12)$$

by the exactness of [7, (2.21)]. Since

$$[\mathbb{K}_1, \tilde{\eta}(\mathcal{E}_1) + \phi^1] = [\mathcal{E}_1, \phi^1] = [\mathcal{E}_2, \phi^2] = [\mathbb{K}_2, \tilde{\eta}(\mathcal{E}_2) + \phi^2],$$

it follows that there exists a taming  $(\mathbb{K}_2 \sqcup_B (\mathbb{K}_1)^{\text{op}})_t$  such that

$$\tilde{\eta}(\mathcal{E}_1) - \tilde{\eta}(\mathcal{E}_2) + \phi^1 - \phi^2 = \eta^{\text{B}}((\mathbb{K}_2 \sqcup_B (\mathbb{K}_1)^{\text{op}})_t). \quad (3.13)$$

By the same reason as in (3.10) we have

$$\eta^{\text{B}}((\mathbb{K}_2 \sqcup_B (\mathbb{K}_1)^{\text{op}})_t) = \text{CS}(\nabla^{\ker(\mathbb{D}^{E^2})}, \nabla^{\ker(\mathbb{D}^{E^1})}). \quad (3.14)$$

(3.12) follows by comparing (3.13) and (3.14). Thus  $g$  is well defined.

We prove that  $f$  and  $g$  are inverses to each other. Let  $(E, h, \nabla, \phi)$  be a generator of  $\widehat{K}_{\text{FL}}(B)$ . Then

$$(g \circ f)(E, h, \nabla, \phi) = g[(\mathbb{E}, \phi)] = (E, h, \nabla, \phi).$$

On the other hand, for a generator  $(\mathcal{E}, \phi)$  of  $\widehat{K}_{\text{BS}}(B)$ ,

$$\begin{aligned} (f \circ g)((\mathcal{E}, \phi)) &= f(\ker(\mathbb{D}^E), h^{\ker(\mathbb{D}^E)}, \nabla^{\ker(\mathbb{D}^E)}, \tilde{\eta}(\mathcal{E}) + \phi) \\ &= [\mathbb{K}, \tilde{\eta}(\mathcal{E}) + \phi] \\ &= [\mathcal{E}, \phi] \end{aligned}$$

by [4, Corollary 5.5] again.

Since  $f$  is a ring homomorphism, the same is true for  $g$ . Thus  $f$  and  $g$  are ring isomorphisms and are inverses to each other.  $\square$

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MAN-HO HO  
 Department of Mathematics  
 Hong Kong Baptist University  
 Kowloon Tong, Kowloon  
 Hong Kong  
 homanho@hkbu.edu.hk