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Oumar Traore

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Abstract

In this paper we study a linear population dynamics model. In this model, the birth process is described by a nonlocal term and the initial distribution is unknown. The aim of this paper is to use a controllability result of the adjoint system for the computation of the density of individuals at some time T.

Contrôlabilité à zéro et application à un problème d'assimilation de données sur un modèle linéaire de dynamique des populations.

Résumé

Dans cet article nous étudions un modèle linéaire de dynamique des populations. Dans ce modèle, le processus de naissance est défini par un terme non local et la distribution initiale des individus n'est pas connue. L'objectif ici est d'utiliser un resultat de contôlabilité du système adjoint pour la détermination de la densité des individus à un instant T.

1. Introduction

We consider a population living in a bounded open set Ω of $\mathbb{R}^N, N \geq 1$. The boundary of Ω that is Γ, is assumed to be sufficiently smooth. Let $y(t,a,x)$ be the distribution of individuals of age a at time t and location $x \in \Omega$ and let T be a positive constant. In the sequel $\mu_0(t,a,x)$ and $\beta(t,a,x)$ stand respectively for the natural death and birth rate of individuals of age a at time t and location x. We assume that the boundary Γ is inhospitable. If the flux of individuals reads $-\nabla y$ where ∇ is the

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gradient with respect to the spatial variable, then y solves the following system

$$\left\{\begin{array}{c} \frac{\partial y}{\partial t}+\frac{\partial y}{\partial a}-\triangle y+\mu_0 y=0 \text{ in } (0,T)\times(0,A)\times\Omega \\ y(t,a,x)=0 \text{ on } (0,T)\times(0,A)\times\Gamma \\ y(0,a,x)=y_0(a,x) \text{ in } (0,A)\times\Omega \\ y(t,0,x)=\int_0^A \beta y da \text{ in } (0,T)\times\Omega \end{array}\right. \tag{1.1}$$

where Δ is the laplacian with respect to the spatial variable and $y_0(a,x)$ is the initial distribution of individuals of age a at location x. When this initial distribution y_0, is known one can use an integration along charateristic lines and an orthonormal basis of eigenfunctions of the laplacian to compute $y(T,a,x)$, see [4]. In this paper we suppose that the initial distribution y_0 is unknown and we want to compute $y(T,.,.)$ using some observations on the state y during an interval of time $(0,T)$.
This is in fact a data assimilation problem. More precisely, the problem is to predict the density of individuals at some $t>T$ from the knowledge of some observations during an interval of time $(0,T)$.
The classical way to solve such problem is to compute first the initial distribution. This kind of problem is generally ill posed and requires traditionally, Tychonof regularization and minimization of a quadratic functional.[9], [7].
As soon as the initial distribution is determined, one can compute $y(t,.,.)$ in a classical way.
Here, we want to compute first $y(T,.,.)$ and afterwards, one can use it as a "‘new initial distribution"’ for the study of $y(t,.,.)$, for $t>T$.
The problem of recovering unknown data in population dynamics model was extensively studied. In [12], the author performed a technique for recovering the natural death rate in a Mc Kendrick model. The method there uses an overdeterminated data $y(T,a)=\psi(a)$ and the explicit form of the solution. In [5] the problem is also to recover the natural birth and death rates from the knowledge of the initial and final distribution. In [8], the goal is different from the previous one. More precisely in [8] the authors studied a method for determining the individual survival and reproduction function from data on population size and cumulative number of birth in a linear population model of Mc Kendrick type. These goals are quite different from the one we study here. Our method uses essentially a null controllability result of an adjoint problem. Similar problem in the framework of parabolic equation was studied earlier by JP Puel in [11].

In [14], an application of the approximate controllability property to data assimilation problems was studied by the author. The question adressed there is whether one can use an approximate controllability result for recovering the initial data for a linear population dynamics model.

A lot of papers are devoted to the study of null controllability property for population dynamics models. In [3], a null controllability result was established for a linear population dynamics. The method used a fixed point theorem and Carleman inequality for parabolic equations. Here, we will establish a new observability inequality with a weight. This result allows us to control on the whole domain $(0, A) \times \Omega$.
The remainder of this paper is as follows: in Section 2, we state assumptions and prove the null controllability result. The Section 3 is devoted to the statement of an approximation method for computing the distribution at time T.

We have also included an Appendix, where we give the proof of the Carleman inequality with the careful study of the dependence of the constants on s, λ, T and A.

2. Assumptions and null controllability results

We state first the hypotheses which will be used.

A_1**:** $\mu_0(t,a,x) = \mu_1(a) + \mu(t,a,x)$ a.e. in $(0,T) \times (0,A) \times \Omega$ with $\mu_1 \in L^1_{loc}(0,A)$; $\mu \in C([0,T] \times [0,A] \times \overline{\Omega})$. In addition we suppose that $\int_0^A \mu_1(a)\, da = \infty$; $\mu_1 \geq 0$ and $\mu \geq 0$.

A_2**:** $\beta \in C([0,A])$;$\beta \geq 0$ and there exists $0 < a_0 < a_1 < A$ such that $\beta(a) = 0$ a.e. in $((0,a_0) \cup (a_1,A)$ and $a_1 + a_0 < A$.

The following notations will be used in the sequel: $Q = (0,T) \times (0,A) \times \Omega$; $Q_T = (0,T) \times \Omega$; $Q_A = (0,A) \times \Omega$; $\Sigma = (0,T) \times (0,A) \times \Gamma$; $Q_\omega = (0,T) \times (0,A) \times \omega$ and $C(A, \beta, \ldots)$ are several positive contants depending on A, $\beta, \ldots$ Sometimes we will write dQ instead of $dt\, da\, dx$.

Remark 2.1. Assumptions A_1 and A_2 are classic in the study of population dynamics. Indeed, $\int_0^A \mu_1(a)\, da = +\infty$ means that the survival likelihood of individuals, that is $exp(-\int_0^a \mu_1(s)\, ds)$ tends towards zero as a goes to A. In other words, all individuals die before the age A.
Assumption A_2 means that the young and the old individuals are not fertile.

The main goal of this section is to prove a null controllability result. First, let us consider the system

$$\begin{cases} -\frac{\partial \tilde{p}}{\partial t} - \frac{\partial \tilde{p}}{\partial a} - \triangle \tilde{p} + \mu_0 \tilde{p} = \beta \tilde{p}(t,0,x) + v 1_\omega(x) \text{ in } \ Q \\ \tilde{p}(t,a,x) = 0 \text{ on } \ \Sigma \\ \tilde{p}(T,a,x) = \rho g(a,x) \text{ in } \ Q_A \\ \tilde{p}(t,A,x) = 0 \text{ in } \ Q_T \end{cases} \tag{2.1}$$

where $g \in L^2((0,A) \times \Omega)$, ω is a non empty open set of Ω and ρ is a function which will be precised later on.
The system (2.1) is said to be null controllable if for any $g \in L^2(Q_A)$ there exists a control $v \in L^2(Q_\omega)$ such that the corresponding solution verifies

$$\tilde{p}(0,a,x) = 0 \text{ a.e. in } \ (0,A) \times \Omega. \tag{2.2}$$

Our null controllability result is stated in the following theorem.

Theorem 2.2. *Suppose that assumptions $A_1 - A_2$ are fulfilled. For any $g \in L^2(Q_A)$ there exists a control $v \in L^2(Q_\omega)$ such that the corresponding solution of (2.1) verifies (2.2)*

Remark 2.3. In order to work with bounded coefficients we make the following change of variables: $p = exp(-\int_0^a \mu_1(s)\, ds)\tilde{p}$. Then p solves the problem:

$$\begin{cases} -\frac{\partial p}{\partial t} - \frac{\partial p}{\partial a} - \triangle p + \mu p = \tilde{\beta} p(t,0,x) + \pi v 1_\omega(x) \text{ in } \ Q \\ p(t,a,x) = 0 \text{ on } \ \Sigma \\ p(T,a,x) = \pi \rho g \text{ in } \ Q_A \\ p(t,A,x) = 0 \ \text{ in } \ Q_T \end{cases} \tag{2.3}$$

where $\pi(a) = exp(-\int_0^a \mu_1(s)\, ds)$ and $\tilde{\beta} = \pi\beta$.
The problem is reduced to find for any $g \in L^2(Q_A)$ a function $v \in L^2(Q_\omega)$ such that the corresponding solution p verifies (2.2).
On the other hand it is obvious that $\tilde{\beta}$ verifies A_2.
In this section, we will consider the previous system and we write β instead of $\tilde{\beta}$.

We want now to give a Carleman inequality from which we will derive an observability inequality.

Let us introduce the following adjoint system of (2.3) in which $z_0 \in L^2(Q_A)$

$$\begin{cases} \frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} - \triangle z + \mu z = 0 & \text{in } (0,T)\times(0,A)\times\Omega \\ z(t,a,x) = 0 & \text{on } \Sigma \\ z(0,a,x) = z_0(a,x) & \text{in } (0,A)\times\Omega \\ z(t,0,x) = \int_0^A \beta z\, da & \text{in } (0,T)\times\Omega. \end{cases} \tag{2.4}$$

We recall that for any nonempty open set $\omega_0 \subset \omega$ there exists a function denoted $\Psi \in C^2(\overline{\Omega})$ such that $\Psi(x) = 0, \forall x \in \partial\Omega$; $\nabla\Psi(x) \neq 0$, for $x \in \overline{\Omega - \omega_0}$ and $\Psi(x) > 0, \forall x \in \Omega$, see [6].
Setting $\eta(t,a,x) = \frac{e^{2\lambda\|\Psi\|_\infty} - e^{\lambda\Psi(x)}}{a(A-a)t(T-t)}$ and $\varphi(t,a,x) = \frac{e^{\lambda\Psi(x)}}{a(A-a)t(T-t)}$ we have the following result.

Proposition 2.4. *There exist positive constants $\lambda_0 > 1$ and $C(\Psi)$ such that for $\lambda > \lambda_0$ and*

$$s > s_0(\lambda) = C(\Psi)\frac{TA}{4}e^{2\lambda\|\Psi\|}s^2\left(\frac{T^2A^2}{4} + T^2A^3 + T^3A^2 + T + A\right)$$

such that for all solution z of (2.4) the following inequality holds:

$$\int_Q e^{-2s\eta}\left(s\lambda\varphi\,|\nabla z|^2 + s^3\lambda^4\varphi^3\,|z|^2\right)dQ \leq Cs^3\lambda^4\int_{Q_\omega}\varphi^3 e^{-2s\eta}z^2 dQ. \tag{2.5}$$

Remark 2.5. The proof of this Carleman inequality follows the method of [10] for parabolic equation. In [13] we have established similar Carleman inequality, but without the particular form of the constants.See also [2]. For completeness and in order to justify the particular form of the constants λ_0 and $s_0(\lambda)$ we give the entire proof in the appendix, at the end of the paper.

The goal now, is to derive from the Carleman inequality the following observability inequality which is helpful for the proof of Theorem 2.2.

Proposition 2.6. *Suppose that $A_1 - A_2$ are fulfilled. Then there exists a positive constant C depending only on a_0, A, Ψ, Ω and T such that*

$$\int_{Q_A}\rho(a)z^2(T,a,x)\,dx\,da \leq C\int_{Q\omega}z^2(t,a,x)\,dt\,dx\,da. \tag{2.6}$$

where

$$\rho(a) = exp(-2s\frac{e^{\lambda\|\Psi\|}}{a_0^2\left|a\left(a-T\right)\right|}) \tag{2.7}$$

and $s > s_0(\lambda)$.

Proof. Let us assume first that $T > A$. We will prove in this case that

$$\int_{Q_A} z^2(T,a,x)\,dx\,da \le C\int_{Q\omega} z^2(t,a,x)\,dt\,da\,dx. \tag{2.8}$$

Note that this implies inequality (2.6) since $\rho(a) \le 1$. Let $\sigma \in (T-A,T)$, we set $q(a,x) = z(\sigma+a,a,x)$, $\widehat{\mu}(a,x) = \mu(\sigma+a,a,x)$, $a \in (0,T-\sigma)$. Then, since z solves (2.4) it follows that q solves the system:

$$\begin{cases} \frac{\partial q}{\partial a} - \triangle q + \widehat{\mu}(a)q = 0 \quad in \quad (0,T-\sigma)\times\Omega \\ q(a,x) = 0 \quad on \quad (0,T-\sigma)\times\Gamma \\ q(0,x) = z(\sigma,0,x) \quad in \quad \Omega \end{cases} \tag{2.9}$$

Multiplying (2.9) by q and integrating the result over $(0,T-\sigma)\times\Omega$, we get:

$$\int_\Omega q^2(T-\sigma,x)\,dx \le \int_\Omega q^2(0,x)\,dx. \tag{2.10}$$

Since $q(0,x) = z(\sigma,0,x)$, using (2.10) and (2.4), and thanks to the Cauchy Schwarz inequality we get:

$$\int_\Omega q^2(T-\sigma,x)\,dx \le C(\beta)\int_{a_0}^{a_1}\int_\Omega z^2(\sigma,a,x)\,da\,dx. \tag{2.11}$$

Note that

$$\varphi(\sigma,a,x) \ge \frac{1}{A^2T^2} \quad in \quad Q.$$

As $a_0 < A - a_1$, one obtains:

$$e^{-2s\eta(\sigma,a,x)} \ge e^{-2s\frac{e^{2\lambda\|\Psi\|_\infty}}{a_0^2(T-A)(T-\sigma)}} \quad in \quad (T-A,T)\times(a_0,a_1)\times\Omega$$

Subsequently, setting $\theta(T-\sigma) = e^{-2s\frac{e^{2\lambda\|\Psi\|_\infty}}{a_0^2(T-A)(T-\sigma)}}$, (2.11) yields:

$$\int_\Omega \theta(T-\sigma)q^2(T-\sigma,x)\,dx \le C_0\int_{Q_A} e^{-2s\eta}\varphi^3 z^2(\sigma,a,x)\,da\,dx \tag{2.12}$$

where $C_0 = A^6T^6C(\beta,\Psi)$.
This means :

$$\int_\Omega \theta(T-\sigma)z^2(T,T-\sigma,x)\,dx \leq C_0\int_{Q_A} e^{-2s\eta}\varphi^3z^2(\sigma,a,x)\,da\,dx.$$

An integration on $(T-A,T)$ with respect to σ yields:

$$\int_{T-A}^{T}\int_\Omega \theta(T-\sigma)z^2(T,T-\sigma,x)\,d\sigma\,dx \leq C_0\int_Q e^{-2s\eta}\varphi^3z^2(\sigma,a,x)\,d\sigma\,da\,dx.$$

Therefore:

$$\int_{Q_A} \theta(a)z^2(T,a,x)\,da\,dx \leq C_0\int_Q e^{-2s\eta}\varphi^3z^2(\sigma,a,x)\,d\sigma\,da\,dx \tag{2.13}$$

Let $\epsilon < \frac{1}{2}Min\,(T-A,a_0)$. From (2.13) we get:

$$\int_\epsilon^A\int_\Omega z^2(T,a,x)\,dx \leq e^{2s\frac{e^{2\lambda\|\Psi\|_\infty}}{\epsilon^4}}C_0\int_Q e^{-2s\eta}\varphi^3z^2(t,a,x)\,dt\,da\,dx. \tag{2.14}$$

Using now (2.5), and setting $C = e^{2s\frac{e^{2\lambda\|\Psi\|_\infty}}{\epsilon^4}}C_0$ it follows

$$\int_\epsilon^A\int_\Omega z^2(T,a,x)\,dx\,da \leq C\int_{Q_\omega} e^{-2s\eta}\varphi^3z^2(t,a,x)\,dt\,da\,dx. \tag{2.15}$$

Note that (2.15) holds for all $t \leq T$ such that $t-A > \epsilon$.
Let now $\sigma \in (T-\epsilon,T)$. On the one hand we have $\sigma - A > T-A-\epsilon > \epsilon$.
On the other hand, let us consider system (2.9). Inequality (2.11) yields:

$$\int_\Omega q^2(T-\sigma,x)\,dx \leq C(\beta)\int_\epsilon^A\int_\Omega z^2(\sigma,a,x)\,dx\,da. \tag{2.16}$$

This gives

$$\int_\Omega z^2(T,T-\sigma,x)\,dx \leq C(\beta)\int_\epsilon^A\int_\Omega z^2(\sigma,a,x)\,dx\,da. \tag{2.17}$$

Combining (2.15) and (2.17) we obtain:

$$\int_\Omega z^2(T,T-\sigma,x)\,dx \leq C(\beta)C\int_{Q_\omega} e^{-2s\eta}\varphi^3z^2(t,a,x)\,dt\,da\,dx. \tag{2.18}$$

Integrating now both sides of (2.18) over $(T-\epsilon,T)$ with respect to the variable σ, one deduces after a standard change of variables:

$$\int_0^{\epsilon}\int_\Omega z^2(T,a,x)\,dx\,da \leq C\int_{Q_\omega} e^{-2s\eta}\varphi^3 z^2(t,a,x)\,dt\,da\,dx. \tag{2.19}$$

Adding (2.15) and (2.19) we get (2.8).
We suppose now that $T < A$. Let $\sigma \in (0, A-T)$. Consider the characteristic line $C = \{(t,\sigma+t); t \in (0,T)\}$ and set $q(t,x) = z(t,\sigma+t,x)$. It follows immediately that q solves the following system:

$$\begin{cases} \frac{\partial q}{\partial t} - \triangle q + \widehat{\mu} q = 0 \ \text{ in } \ (0,T)\times\Omega \\ q(t,x) = 0 \ \text{ on } \ (0,T)\times\Gamma \\ q(0,x) = z(0,\sigma,x) \ \text{ in } \ \Omega \end{cases} \tag{2.20}$$

Using now the standard observability inequality for the heat equation [6], we infer that:

$$\int_\Omega q^2(T,x)\,dx \leq C(\frac{1}{T},T,\mu)\int_0^T\int_\omega q^2(t,x)\,dx\,dt. \tag{2.21}$$

This is equivalent to:

$$\int_\Omega z^2(T,\sigma+T,x)\,dx \leq C(\frac{1}{T},T,\mu)\int_0^T\int_\omega z^2(t,\sigma+t,x)\,dx\,dt. \tag{2.22}$$

Integrating both sides over $(0, A-T)$ with respect to the variable σ gives:

$$\int_0^{A-T}\int_\Omega z^2(T,\sigma+T,x)\,dx\,d\sigma \\ \leq C(\frac{1}{T},T,\mu)\int_0^{A-T}\int_0^T\int_\omega z^2(t,\sigma+t,x)\,dx\,dt\,d\sigma. \tag{2.23}$$

Making the following change of variables: $a = \sigma + T$ in the left hand term, and $a = \sigma + t$ in the right hand term we get:

$$\int_T^A\int_\Omega z^2(T,a,x)\,dx\,da \leq C(\frac{1}{T},T,\mu)\int_0^A\int_0^T\int_\omega z^2(t,a,x)\,dx\,dt\,da. \tag{2.24}$$

Let us now take $\sigma \in (0,T)$ and consider the following characteristic line $C = \{\sigma + a, a), a \in (0, T-\sigma)\}$. Let $q(a,x) = z(\sigma+a,a,x)$. One can see that q solves the system (2.25) below:

$$\begin{cases} \frac{\partial q}{\partial a} - \triangle q + \mu q = 0 \quad in \quad (0, T-\sigma)\times\Omega \\ q(a,x) = 0 \quad on \quad (0,T-\sigma)\times\Gamma \\ q(0,x) = z(\sigma,0,x) \quad in \quad \Omega \end{cases} \tag{2.25}$$

Let us multlply (2.25) by q and integrating over $(0, T-\sigma)\times\Omega$. We obtain recalling system (2.4):

$$\int_\Omega q^2(T-\sigma,x)\,dx \leq A\int_\Omega\int_0^A \beta^2 z^2(\sigma,a,x)\,da\,dx \tag{2.26}$$

From the assumption A_2 and the boundedness of η and φ on $(a_0,a_1)\times\Omega$ we have:

$$\int_\Omega \rho(\sigma)q^2(T-\sigma,x)\,dx \leq C_1\int_\Omega\int_0^A \varphi^3(\sigma,a,x)e^{-2s\eta}z^2(\sigma,a,x)\,da\,dx$$

where $C_1 = |\beta|^2_\infty\, a^3(A-a_0)^3T_0^3$.
Then, we deduce after an integration over $(0,T)$ with respect to the variable σ:

$$\int_0^T\int_\Omega \rho(\sigma)q^2(T-\sigma,x)\,dx\,d\sigma \leq C_1\int_Q \varphi^3 e^{-2s\eta}z^2(\sigma,a,x)\,d\sigma\,da\,dx.$$

Using now the last definition of q;inequality (2.5) and setting $a = T-\sigma$, we get:

$$\int_0^T\int_\Omega \rho(a)z^2(T,a,x)\,dx\,da \leq C_1C(\Psi)\int_\omega\int_0^T\int_0^A z^2(t,a,x)\,dQ. \tag{2.27}$$

Adding now(2.24) and (2.27) and taking into a account that $\rho(a)\leq 1$ we obtain (2.6) where $C = max(C_1C(\Psi), C(\frac{1}{T},T,\mu))$. The proof is now complete. □

Let us prove now the Theorem 2.2.

Proof. We assume that $A_1 - A_2$ are satisfied. For $g\in L^2(Q_A)$, we introduce for $\alpha > 0$ the functional J_α defined on $L^2(Q_\omega)$ by:

$$J_\alpha(v) = \frac{1}{2\alpha}\int_0^A\int_\Omega p^2(0,a,x)\,da\,dx + \frac{1}{2}\int_{Q_\omega} v^2 da\,dx\,dt \tag{2.28}$$

where p solves (2.3). The functional J_α is continuous, convex and verifies:

$$\lim_{\|v\|\to\infty} J_\alpha(v) = +\infty.$$

Consequently, J_α achieves its minimum at a unique point v_α. Moreover the maximum principle gives:

$$v_\alpha = -z_\alpha 1_\omega \tag{2.29}$$

where z_α solves

$$\begin{cases} \frac{\partial z_\alpha}{\partial t} + \frac{\partial z_\alpha}{\partial a} - \triangle z_\alpha + \mu z_\alpha = 0 \quad in \quad (0,T)\times(0,A)\times\Omega \\ z_\alpha(t,a,x) = 0 \quad on \quad (0,T)\times(0,A)\times\Gamma \\ z_\alpha(0,a,x) = \frac{1}{\alpha} p_\alpha(0,a,x) \quad in \quad (0,A)\times\Omega \\ z_\alpha(t,0,x) = \int_0^A \beta z_\alpha da \quad in \quad (0,T)\times\Omega. \end{cases} \tag{2.30}$$

and p_α is the solution to

$$\begin{cases} -\frac{\partial p_\alpha}{\partial t} - \frac{\partial p_\alpha}{\partial a} - \triangle p_\alpha + \mu p_\alpha = \beta p_\alpha(t,0,x) + \pi v_\alpha 1_\omega(x) \text{ in } \ Q \\ p(t,a,x) = 0 \text{ on } \ \Sigma \\ p_\alpha(T,a,x) = \pi\rho g \text{ in } \ Q_A \\ p_\alpha(t,A,x) = 0 \ \text{ in } \ Q_T \end{cases} \tag{2.31}$$

Multiplying (2.31) by z_α and integrating over Q we obtain:

$$\int_0^A \int_\Omega z_\alpha(T,a,x)\rho\pi g(a,x)\,dx\,da - \frac{1}{\alpha}\int_0^A \int_\Omega p_\alpha^2(0,a,x)\,dx\,da \\ = \int_0^A \int_0^T \int_\omega v_\alpha^2 dx\,da\,dt.$$

Then,

$$\frac{1}{\alpha}\int_0^A \int_\Omega p_\alpha^2(0,a,x)\,dx\,da + \int_0^A \int_0^T \int_\omega v_\alpha^2 da\,dx\,dt \leq \\ \frac{1}{2}\int_0^A \int_\Omega \rho z_\alpha^2(T,a,x)\,da\,dx + \frac{1}{2}\int_0^A \int_\Omega \rho\pi^2 g^2(a,x)\,da\,dx.$$

This yields using inequality (2.6) and (2.29)

$$\frac{1}{\alpha}\int_0^A \int_\Omega p_\alpha^2(0,a,x)\,dx\,da + \frac{1}{2}\int_{Q_\omega} v_\alpha^2 dt\,da\,dx \leq 2C\,\|\rho g\|^2_{L^2((0,A)\times\Omega))}. \tag{2.32}$$

Consequently, the sequence (v_α, p_α) is bounded in $L^2(Q_\omega) \times L^2(Q)$. Then, when $\alpha \to 0$, after extraction of a subsequence, the sequence still denoted (v_α, p_α) converges weakly towards (v, p), which solves (2.3). Particularly, we have $p_\alpha(0, ., .) \rightharpoonup p(0, ., .)$ in $L^2\left((0, A) \times \Omega\right)$ weakly so that p verifies (2.2). Note that the following inequality holds too:

$$\int_{Q_\omega} v^2 dt\, da\, dx \leq 2C \int_{Q_A} g^2 da\, dx. \tag{2.33}$$

Let us prove that (v_α, p_α) converges strongly to (v, p) in $L^2(Q_\omega) \times L^2(Q)$.
Since v_α is the unique minimizer of J_α, we infer that $J_\alpha(v_\alpha) \leq J_\alpha(v)$. This gives

$$\|v_\alpha\|_{L^2(Q_\omega)} \leq \|v\|_{L^2(Q_\omega)}.$$

Consequently, the weak convergence of v_α towards v, yields that v_α converges strongly to v in $L^2(Q_\omega)$ as n goes to ∞. This implies obviously that p_α converges strongly to p in $L^2(Q)$. So, the sequence (v_α, p_α) converges strongly to (v, p) in $L^2(Q_\omega) \times L^2(Q)$. This ends the proof. $\square$

3. Recovery of the state value $y(T)$

We give here our data assimilation result. This result uses mainly the null controllability result proved above. Next, we give a possible approximation method of the null controllability problem by means of some optimal control problems.
Beforehand, we will first prove the following proposition.

Proposition 3.1. *The space $L^2\left((0, A) \times \Omega\right)$ has an orthonormal basis of the form ρg_k,with the function $g_k \in L^2\left((0, A) \times \Omega\right)$; $k = 1, 2, ...$ where ρ is defined by (2.7).*

Proof. We will prove the proposition when $A > T$, the case $A < T$ can be proved using analogous arguments.
This proof will be done in two steps:
Step 1: construction of an adapted countable and dense set.
Let $O = ((0, T) \cup (T, A)) \times \Omega$. For $m = 1, 2, \ldots$ we set

$$O_m = \left\{(a, x) \in O; a \geq \frac{1}{m}; |a - T| \geq \frac{1}{m}; dist\,(x, \partial\Omega) \geq \frac{1}{m}\right\}.$$

It follows that for all m, O_m is a bounded subset of O. Now, let P be the set of all polynomials on R^{N+1} with rational coefficients. Let $P_m = \{f1_{O_m}; f \in P\}$. The set $G = \cup_{m\geq 1} P_m$ is countable and is dense in $L^2(O)$. See [1, page 29]. Let us consider $f \in G$, there exists an integer m such that $f \in P_m$. Let $g = \rho^{-1}(a)f$. From the defintion of P_m it follows now that the function $g \in L^2(O)$. Furthermore, writting $G_0 = \{\rho^{-1}f; f \in G\}$ and $F = \{\rho g; g \in G_0\}$ it follows that F is countable and dense in $L^2(O)$.
Step 2: construction of the orthonormal basis.
Let us write $f_1, f_2, ...$ the functions of F. It suffices to extract from this sequence an infinite and dense sequence of linearly independent elements, and after to apply the orthogonalization method. For this aim, we exclude from the sequence (f_k) all function f_j which can be represented as a linear combinaison of f_i with $i < j$. We obtain thus doing, the desired sequence. The proof is complete. □

We assume that the initial distribution y_0 belongs to $L^2(Q_A)$. This assumption is natural since y stands for the density of the population. Therefore, it follows that $y(T, ., .) \in L^2(Q_A)$.
Now, let us consider an orthonormal basis of the form (ρg_k) with $g_k \in L^2(Q_A)$. Then, on the one hand, we have:

$$y(T, a, x) = \Sigma_{k=0}^{+\infty} y_k \rho(a) g_k(a, x) \;\; a.e. \;\; \in Q_A$$

where $y_k = \int_0^A \int_\Omega y(T, a, x)\rho(a)g_k(a, x)\, dx\, da$.
On the other hand, for all k, by virtue of Theorem 2.2, there exists $\tilde{v}(g_k) \in L^2(Q_\omega)$ such that the associated solution $\tilde{p}$ of (2.1) verifies (2.2). Then, multiplying (1.1) by $\tilde{p}$ and integrating the result over Q, we obtain

$$\int_{Q_A} \rho g_k(a, x) y(T, a, x)\, da\, dx = \int_{Q_\omega} \tilde{v_k}(g)(t, a, x) y(t, a, x)\, dt\, da\, dx. \quad (3.1)$$

Therefore,

$$y_k = \int_{Q_\omega} \tilde{v_k}(g)(t, a, x) y(t, a, x)\, dt\, da\, dx. \quad (3.2)$$

This equation gives the coefficients of the desired state value $y(T)$ from the measurements of the solution on the subset ω. At the same time, if we use an approximation of the exact value of y on ω, this formula describes the effect of the error on the coefficients of $y(T)$.

Using (3.1), we get thanks to the Cauchy Schwarz inequality:

$$\int_{Q_A} \rho g_k(a,x) y(T,a,x)\, da\, dx$$
$$\leq C \left(\int_{Q_A} \rho^2 g_k^2(a,x)\, da\, dx \right)^{1/2} \left(\int_{Q_\omega} y^2(t,a,x)\, dt\, da\, dx \right)^{1/2}.$$

This yields

$$\int_{Q_A} y^2(T,a,x)\, da\, dx \leq C \int_{Q_\omega} y^2(t,a,x)\, dt\, da\, dx. \tag{3.3}$$

Note that (3.3) is a stability inequality.
We now, summarize the method for retrieving the state value $y(T)$ in the following Proposition.

Proposition 3.2. *Let us consider an orthonormal basis of $L^2(Q_A)$ denoted by $(\rho(g_k)_{k\geq 1})$.*
Suppose that $A_1 - A_2$ are fulfilled.
i) For $\alpha > 0$ there exists a unique minimizer $v_{k,\alpha} \in L^2(Q_\omega)$ of J_α and $v_{\alpha,k}$ is characterized by the optimality system

$$\begin{cases} -\frac{\partial p_{\alpha,k}}{\partial t} - \frac{\partial p\alpha,k}{\partial a} - \triangle p_{\alpha,k} + \mu p_{\alpha,k} = \beta p_{\alpha,k}(t,0,x) + \pi v_{\alpha,k} 1_\omega \ in \ \ Q \\ p_{\alpha,k}(t,a,x) = 0 \ on \ \ \Sigma \\ p_{\alpha,k}(T,a,x) = \rho\pi g_k(a,x) \ in \ \ Q_A \\ p_{\alpha,k}(t,A,x) = 0 \, in \ \ Q_T \end{cases} \tag{3.4}$$

$$\begin{cases} \frac{\partial z_{\alpha,k}}{\partial t} + \frac{\partial z_{\alpha,k}}{\partial a} - \triangle z_{\alpha,k} + \mu(t,a,x) z_{\alpha,k} = 0 \ in \ \ Q \\ z_{\alpha,k}(t,a,x) = 0 \ on \ \ \Sigma \\ z_{\alpha,k}(0,a,x) = \frac{1}{\alpha} p_{\alpha,k}(0,a,x) \ in \ \ Q_A \\ z_{\alpha,k}(t,0,x) = \int_0^A \beta z_{\alpha,k} da \, in \ \ Q_T \end{cases} \tag{3.5}$$

$$v_{\alpha,k} = -z_{\alpha,k}.1_\omega \quad a.e. \ \ Q_\omega. \tag{3.6}$$

ii) When α tends towards zero, $v_{\alpha,k} \to v_k$ in $L^2(Q_\omega)$, $p_{\alpha,k} \to p_k$ in $L^2(Q)$ where (v_k, p_k) satisfies:

$$\begin{cases} -\frac{\partial p_k}{\partial t} - \frac{\partial p_k}{\partial a} - \triangle p_k + \mu p_k = \beta p_k(t,0,x) + \pi v_k 1_\omega \ in \ \ Q \\ p_k(t,a,x) = 0 \ on \ \ \Sigma \\ p_k(T,a,x) = \rho\pi g_k(a,x) \ in \ \ Q_A \\ p_k(t,A,x) = 0 \, in \ \ Q_T \end{cases} \tag{3.7}$$

and

$$p_k(0,a,x) = 0 \quad a.e. \quad in \quad Q_A \tag{3.8}$$

iii) The state at time T, $y(T,..)$ is given by

$$y(T,a,x) = \sum_{k\geq 1} y_k \rho(a) g_k(a,x) \quad a.e.\ in \quad Q_A \tag{3.9}$$

where

$$y_k = \int_{Q_\omega} v_k y dt\, da\, dx. \tag{3.10}$$

4. **Concluding Remark**

This paper adresses the essential problem of data assimilation. Here, we have shown that from the knowledge of the density of individuals on a small open set and during the interval of time $(0,T)$, one can compute $y_c := y(T,a,x)$, the density at the time T. From this, we can now compute this density at any time t such that $T < t < T'$ by means of the following system:

$$\left\{ \begin{array}{c} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - \triangle y + \mu_0 y = 0 \text{ in } \ (T,T') \times (0,A) \times \Omega \\ y(t,a,x) = 0 \text{ on } \ (T,T') \times (0,A) \times \Gamma \\ y(T,a,x) = \pi^{-1} y_c(a,x) \text{ in } \ (0,A) \times \Omega \\ y(t,0,x) = \int_0^A \beta y da \text{ in } \ (T,T') \times \Omega \end{array} \right. \tag{4.1}$$

The method we have been studying here gives a theoretical result but it could also be used for a pratical recovery of the state value $y(T)$ from measurements of the solution on a small open set. We then have to recover an approximation of $y(T)$ on a finite dimensional basis.

The choice of this basis is crucial as it has to provide a good approximation for $y(T)$ but it has to contain a small number of elements to minimize the adjoint control problems to be solved. This will be the subject of a forthcoming work and we will compare the results given by this method with classical methods using Tychonov regularization.

5. **Appendix: proof of Proposition 2.4**

Here, we suppose that the function $z \in C^2(\overline{Q})$ and verifies

$$\begin{cases} \frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} - \triangle z + \mu z = f \quad in \quad (0,T) \times (0,A) \times \Omega \\ z(t,a,x) = 0 \quad on \quad \Sigma \\ z(0,a,x) = z_0(a,x) \quad in \quad (0,A) \times \Omega \\ z(t,0,x) = \int_0^A \beta z da \quad in \quad (0,T) \times \Omega. \end{cases} \tag{5.1}$$

where $f \in L^2(Q)$ and we prove the following more general Carleman inequality

Proposition 5.1. *There exist positive constants $\lambda_0 > 1$ and $C(\Psi)$ such that for $\lambda > \lambda_0$ and*

$$s > s_0(\lambda) = C(\Psi) \frac{TA}{4} e^{2\lambda \|\Psi\|} s^2 \left(\frac{T^2 A^2}{4} + T^2 A^3 + T^3 A^2 + T + A \right)$$

and all solution z of (5.1) the following inequality hold:

$$I(s,\lambda) \leq C \int_Q e^{-2s\eta} f^2 dt\, da\, dx + C \int_{Q_\omega} s^3 \lambda^4 \varphi^3 e^{-2s\eta} z^2 da\, dx\, dt \tag{5.2}$$

where $I(s,\lambda) = \int_Q e^{-2s\eta} \left(s\lambda\varphi \left|\nabla z\right|^2 + s^3 \lambda^4 \varphi^3 \left|z\right|^2 \right) da\, dx\, dt.$

Taking $f = 0$ one obtains inequality (2.5).

Proof. We make the following change of variables $u = e^{-s\eta} z$. Then immediately it follows by using the definition of η and z that:

$$u(0,a,x) = u(T,a,x) = 0 \quad in \quad (0,A) \times \Omega; \tag{5.3}$$

$$u(t,0,x) = u(t,A,x) = 0 \quad in \quad (0,T) \times \Omega; \tag{5.4}$$

and

$$u(t,a,\sigma) = 0 \quad in \quad (0,T) \times (0,A) \times \partial\Omega. \tag{5.5}$$

Observe that:

$$\nabla \eta = -\lambda \varphi \nabla \Psi \tag{5.6}$$

and

$$\nabla \varphi = \lambda \varphi \nabla \Psi. \tag{5.7}$$

Using once again the definition of η and φ one can prove that:

$$\left| \frac{\partial \eta}{\partial a} \right| \leq e^{2\lambda \|\psi\|} \frac{TA^2}{4} \varphi^2; \quad \left| \frac{\partial \eta}{\partial t} \right| \leq e^{2\lambda \|\psi\|} \frac{T^2 A}{4} \varphi^2;$$

$$\left|\frac{\partial^2 \eta}{\partial a \partial t}\right| \leq e^{2\lambda\|\psi\|} T A \varphi^3; \quad \left|\frac{\partial^2 \eta}{\partial t^2}\right| \leq e^{2\lambda\|\psi\|} \frac{T^3 A^4}{4} \varphi^3$$

and

$$\left|\frac{\partial^2 \eta}{\partial a^2}\right| \leq e^{2\lambda\|\psi\|} \frac{T^4 A^3}{4} \varphi^3.$$

Similary we get:

$$\left|\frac{\partial \varphi}{\partial a}\right| \leq \frac{T^2 A}{4} \varphi^2; \; \left|\frac{\partial \varphi}{\partial t}\right| \leq \frac{T^2 A}{4} \varphi^2; \; \left|\frac{\partial^2 \varphi}{\partial a \partial t}\right| \leq T A \varphi^3;$$

$$\left|\frac{\partial^2 \varphi}{\partial t^2}\right| \leq \frac{T^3 A^4}{4} \varphi^3 \text{ and } \left|\frac{\partial^2 \varphi}{\partial a^2}\right| \leq \frac{T^4 A^3}{4} \varphi^3.$$

Note also that $\varphi \leq \frac{T^4 A^4}{16} \varphi^3$. All these inequalities will be used in the sequel.
We have:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -s\left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial a}\right) u + e^{-s\eta}\left(\frac{\partial z}{\partial t} + \frac{\partial z}{\partial a}\right). \tag{5.8}$$

From (5.6) and (5.7) we get:

$$\begin{aligned} \Delta u = s\lambda \Delta\Psi \varphi u + s\lambda^2 \left|\nabla\Psi\right|^2 \varphi u - s^2\lambda^2 \left|\nabla\Psi\right|^2 \varphi^2 u + \\ 2s\lambda\varphi \nabla\Psi . \nabla u + e^{-s\eta}\Delta w. \end{aligned} \tag{5.9}$$

Consequently:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} - \Delta u + \mu u = e^{-s\eta} f - s\lambda^2 u\varphi \left|\nabla\Psi\right|^2 - 2s\lambda\varphi\nabla\Psi.\nabla u + \\ s^2\lambda^2\varphi^2 \left|\nabla\Psi\right|^2 u - s\left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial a}\right) u - s\lambda\varphi u \Delta\Psi. \end{aligned} \tag{5.10}$$

This equation can be rewritten as:

$$P_1 u + P_2 u = g_s \tag{5.11}$$

where

$$P_1 u = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + 2s\lambda\varphi\nabla\Psi.\nabla u + 2s\lambda^2 u\varphi \left|\nabla\Psi\right|^2 \tag{5.12}$$

$$P_2 u = -\Delta u + s\left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial a}\right) u - s^2\lambda^2\varphi^2 \left|\nabla\Psi\right|^2 u \tag{5.13}$$

and

$$g_s = e^{-s\eta} f + s\lambda^2 u\varphi \left|\nabla\Psi\right|^2 - \mu u - s\lambda u\varphi\Delta\Psi. \tag{5.14}$$

Taking the square of (5.11) and intregrating the result over Q yield:

$$\int_Q |P_1 u|^2 \, dQ + \int_Q |P_2 u|^2 \, dQ + 2 \int_Q P_2 u P_1 u dQ = \int_Q g_s^2 dQ. \tag{5.15}$$

Let us compute $K = \int_Q P_2 u P_1 u dQ$. This computation gives twelve terms denoted $I_{i,j}, i = 1, ..., 4. j = 1, 2, 3$.
We have by integration by parts:

$$I_{1,1} = -\int_Q \frac{\partial u}{\partial t} \Delta u dQ = -\int_\Sigma \frac{\partial u}{\partial t}\frac{\partial u}{\partial \nu} dt \, da \, d\sigma + \frac{1}{2}\int_Q \frac{\partial}{\partial t} |\nabla u|^2 \, dQ.$$

Hence using (5.5)and (5.3) it follows:

$$I_{11} = 0. \tag{5.16}$$

$$I_{1,2} = s \int_Q \frac{\partial u}{\partial t}\left(\frac{\partial \eta}{\partial a} + \frac{\partial \eta}{\partial a}\right) u dQ.$$

An integration by parts leads to:

$$I_{1,2} = -\frac{s}{2}\int_Q |u|^2 \frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial a} + \frac{\partial \eta}{\partial a}\right) dQ. \tag{5.17}$$

$$I_{1,3} = -s^2\lambda^2 \int_Q \frac{\partial u}{\partial t}\varphi^2 u \, |\nabla \Psi|^2 \, dQ.$$

This gives

$$I_{1,3} = -\frac{s^2\lambda^2}{2}\int_Q \frac{\partial \, |u|^2}{\partial t}\varphi^2 \, |\nabla \Psi|^2 \, dQ.$$

Keeping in mind (5.3), an integration by parts with respect to the variable t yields:

$$I_{1,3} = s^2\lambda^2 \int_Q |u|^2 \frac{\partial \varphi}{\partial t}\varphi \, |\nabla \Psi|^2 \, dQ. \tag{5.18}$$

Likewise, one gets easily that:

$$I_{21} = 0; \tag{5.19}$$

$$I_{2,2} = -\frac{s}{2}\int_Q |u|^2 \frac{\partial}{\partial a}\left(\frac{\partial \eta}{\partial a} + \frac{\partial \eta}{\partial a}\right) dQ \tag{5.20}$$

and

$$I_{2,3} = s^2\lambda^2 \int_Q \varphi \, |u|^2 \frac{\partial \varphi}{\partial a} |\nabla \Psi|^2 \, dQ. \tag{5.21}$$

Now, we are concerned by the term $I_{3,j}$.
We have:

$$I_{3,1} = -2s\lambda \int_Q \varphi \nabla\Psi.\nabla u \Delta u dQ.$$

Then we have by an integration by parts:

$$I_{3,1} = -2s\lambda \int_\Sigma \varphi \nabla\Psi.\nabla u \frac{\partial u}{\partial \nu} dt\, da\, d\sigma + 2s\lambda \int_Q \nabla u.\nabla(\varphi \nabla\Psi.\nabla u)\, dQ.$$

From the definition of Ψ and since (5.5) is fulfilled we see that for all $\sigma \in \partial\Omega$ we have

$$\nabla u(t,a,\sigma) = (\nabla u(t,a,\sigma).\nu(\sigma))\nu(\sigma)$$

and

$$\nabla\Psi(\sigma) = (\nabla\Psi(\sigma).\nu(\sigma))\nu(\sigma).$$

As a consequence, it follows, using also (5.7) that

$$I_{3,1} = -2s\lambda \int_\Sigma \varphi(\nabla\Psi.\nu) \left|\nabla u.\nu\right|^2 dt\, da\, d\sigma + 2s\lambda^2 \int_Q \left|\nabla u.\nabla\Psi\right|^2 \varphi dQ +$$

$$2s\lambda \Sigma_{i,j=1}^N \left(\int_Q \varphi \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial \Psi}{\partial x_j} dQ + \int_Q \varphi \frac{\partial u}{\partial x_i} \frac{\partial^2 \Psi}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_j} dQ \right). \quad (5.22)$$

We have

$$2s\lambda \Sigma_{i,j=1}^N \int_Q \varphi \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial \Psi}{\partial x_j} dQ = s\lambda \int_\Sigma \varphi(\nabla\Psi.\nu) \left|\nabla u.\nu\right|^2 dt\, da\, d\sigma$$

$$-s\lambda^2 \int_Q \left|\nabla u\right|^2 \left|\nabla\Psi\right|^2 \varphi dt\, da\, dx - s\lambda \int_Q \varphi \left|\nabla u\right|^2 \Delta\Psi dt\, da\, dx.$$

Therefore,

$$I_{3,1} = -s\lambda \int_\Sigma \varphi(\nabla\Psi.\nu) \left|\nabla u.\nu\right|^2 dt\, da\, d\sigma + 2s\lambda^2 \int_Q \left|\nabla u.\nabla\Psi\right|^2 \varphi dt\, da\, dx$$

$$-s\lambda^2 \int_Q \left|\nabla u\right|^2 \left|\nabla\Psi\right|^2 \varphi dt\, da\, dx - s\lambda^2 \int_Q \left|\nabla u\right|^2 \left|\nabla\Psi\right|^2 \varphi dt\, da\, dx -$$

$$s\lambda \int_Q \varphi \left|\nabla u\right|^2 \Delta\Psi dt\, da\, dx + 2s\lambda \Sigma_{i,j=1}^N \int_Q \varphi \frac{\partial^2 \Psi}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} dt\, da\, dx \quad (5.23)$$

$$I_{3,2} = 2s^2\lambda \int_Q \varphi \nabla\Psi.\nabla u \left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial a} \right) u dt\, da\, dx.$$

Classical computations give:

$$I_{3,2} = s^2\lambda^2 \int_Q \varphi \left|\nabla\psi\right|^2 |u|^2 \left(\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial a}\right) dt\, da\, dx -$$

$$s^2\lambda \int_Q \varphi \, |u|^2 \, \nabla.\left(\nabla\Psi \left(\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial a}\right)\right) dQ. \tag{5.24}$$

$$I_{3,3} = -2s^3\lambda^3 \int_Q \varphi^3 \nabla\Psi.\nabla u \left|\nabla\Psi\right|^2 u dQ.$$

Equality (5.7) and an integration by part give:

$$I_{3,3} = 3s^3\lambda^4 \int_Q \varphi^3 u^2 \left|\nabla\Psi\right|^4 dQ + s^3\lambda^3 \int_Q \varphi^3 \, |u|^2 \, \nabla.(\nabla\Psi \left|\nabla\Psi\right|^2) \, dQ. \tag{5.25}$$

Now we compute the terms $I_{4,j}$.

$$I_{4,1} = -2s\lambda^2 \int_Q \varphi u \left|\nabla\Psi\right|^2 \Delta u dQ = 2s\lambda^2 \int_Q \nabla(\varphi u \left|\nabla\Psi\right|^2).\nabla u dQ.$$

Consequently,

$$I_{4,1} = 2s\lambda^3 \int_Q \varphi u \nabla\Psi.\nabla u \left|\nabla\Psi\right|^2 dQ + 2s\lambda^2 \int_Q \varphi \left|\nabla u\right|^2 \left|\nabla\Psi\right|^2 \, dQ +$$

$$2s\lambda^2 \int_Q \varphi u \nabla u.\nabla \left(\left|\nabla\Psi\right|^2\right) dQ. \tag{5.26}$$

Directly, we have:

$$I_{42} = 2s^2\lambda^2 \int_Q \varphi \left|\nabla\Psi\right|^2 \left(\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial a}\right) |u|^2 dQ \tag{5.27}$$

and

$$I_{43} = -2s^3\lambda^4 \int_Q \varphi^3 \left|\nabla\Psi\right|^4 u^2 dQ. \tag{5.28}$$

Grouping all the terms $I_{i,j}$ one can write:

$$2\int_Q P_1 u P_2 u dQ = X_1 + X_2 + X_3 - 2s\lambda \int_\Sigma \varphi \nabla\Psi.\nu \left|\nabla u.\nu\right|^2 dt\, da\, d\sigma +$$

$$4s\lambda^2 \int_Q \varphi \left|\nabla u.\nabla\Psi\right|^2 dQ + 2s\lambda^2 \int_Q \varphi \left|\nabla u\right|^2 \left|\nabla\Psi\right|^2 dQ +$$

$$2s^3\lambda^4 \int_Q \varphi^3 u^2 \left|\nabla\Psi\right|^4 dQ \tag{5.29}$$

where

$$X_1 = -s\lambda \int_Q \varphi \left|\nabla u\right|^2 \Delta\Psi dQ + 2s\lambda\Sigma_{i,j=1}^{N} \int_Q \varphi \frac{\partial^2\Psi}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} dQ$$

$$X_2 = -\frac{s}{2} \int_Q |u|^2 \frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial a} + \frac{\partial \eta}{\partial a}\right) dQ + s^2\lambda^2 \int_Q |u|^2 \frac{\partial \varphi}{\partial t} \varphi \left|\nabla\Psi\right|^2 dQ$$

$$-\frac{s}{2} \int_Q |u|^2 \frac{\partial}{\partial a}\left(\frac{\partial \eta}{\partial a} + \frac{\partial \eta}{\partial a}\right) dQ + s^2\lambda^2 \int_Q \varphi |u|^2 \frac{\partial \varphi}{\partial a} \left|\nabla\Psi\right|^2 dQ +$$

$$s^2\lambda \int_Q \varphi |u|^2 \nabla.\left(\nabla\Psi\left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial a}\right)\right) + 2s^2\lambda^2 \int_Q \varphi \left|\nabla\Psi\right|^2 \left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial a}\right) |u|^2 dQ$$

$$X_3 = 2s\lambda^3 \int_Q \varphi u \nabla\Psi.\nabla u \left|\nabla\Psi\right|^2 dQ + 2s\lambda^2 \int_Q \varphi u \nabla u.\nabla \left|\nabla\Psi\right|^2 \, dQ$$

verify:

$$X_1 \leq C\left(\Psi\right) s\lambda \int_Q \varphi \left|\nabla u\right|^2 dQ; \tag{5.30}$$

$$X_3 \leq C\left(\Psi\right) \lambda^2 \int_Q \varphi \left|\nabla u\right|^2 dQ + s^2\lambda^4 \frac{A^4T^4}{256} \int_Q \varphi^3 |u|^2 dQ \tag{5.31}$$

and

$$X_2 \leq C(\Psi) \frac{TA}{4} e^{2\lambda\|\Psi\|} C(s,\lambda,T,A) \int_Q \varphi^3 |u|^2 dQ \tag{5.32}$$

where

$$C(s,\lambda,T,A) = s^2\lambda^2 \left(\frac{T^2A^2}{4} + T^2A^3 + T^3A^2 + T + A\right).$$

Note that ν is the outward normal vector to $\partial\Omega$. So, using the fact that $\Psi(x) > 0$ for all $x \in \Omega$ and $\Psi(\sigma) = 0$ for all $\sigma \in \partial\Omega$ we infer that $\nabla\Psi.\nu < 0$. As a consequence, (5.29) yields:

$$2\int_Q P_1 u P_2 u dQ \geq X_1 + X_2 + X_3 + 2s\lambda^2 \int_Q \varphi \left|\nabla u\right|^2 \left|\nabla\Psi\right|^2 dQ +$$

$$2s^3\lambda^4 \int_Q \varphi^3 u^2 \left|\nabla\Psi\right|^4 dQ. \tag{5.33}$$

Note also that $\Psi \in C^2(\overline{\Omega})$ and $|\nabla\Psi| \neq 0$ in $\overline{\Omega - \tilde{\omega}}$. Consequently, there exists a positive constant δ such that $|\nabla\Psi| > \delta$ in $\overline{\Omega - \tilde{\omega}}$.
So that (5.33) gives:

$$2\int_Q P_1uP_2udQ + 2s\lambda^2\delta^2 \int_{\tilde{q}} \varphi\,|\nabla u|^2\,dQ + 2s^3\lambda^4\delta^4 \int_{\tilde{q}} \varphi^3u^2dQ \geq$$
$$X_1 + X_2 + X_3 + 2s\lambda^2\delta^2 \int_Q \varphi\,|\nabla u|^2\,dQ + 2s^3\lambda^4\delta^4 \int_Q \varphi^3u^2dQ \qquad (5.34)$$

where $\tilde{q} = (0,T) \times (0,A) \times \tilde{\omega}$.

Furthermore, we have:

$$\int_Q g_s^2 dQ \leq \int_Q e^{-2s\eta}f^2dQ + X_1 + X_2 + X_3. \qquad (5.35)$$

Then, it follows from (5.15) and (5.34):

$$\int_Q e^{-2s\eta}f^2dQ + X_1 + X_2 + X_3 + 2s^3\lambda^4\delta^4 \int_{\tilde{q}} \varphi^3\,|u|^2\,dQ +$$
$$2s\lambda^2\delta^2 \int_{\tilde{q}} \varphi\,|\nabla u|^2\,dQ \geq \int_Q |P_1u|^2\,dQ + \int_Q |P_2u|^2\,dQ +$$
$$2s\lambda^2\delta^2 \int_Q \varphi\,|\nabla u|^2\,dQ + 2s^3\lambda^4\delta^4 \int_Q \varphi^3\,|u|^2\,dQ. \qquad (5.36)$$

Recalling (5.30)-(5.31), one can choose s and λ sufficiently large so that

$$s\lambda^2\delta^2 \int_Q \varphi\,|\nabla u|^2\,dQ + s^3\lambda^4\delta^4 \int_Q \varphi^3\,|u|^2\,dQ \geq X_1 + X_2 + X_3.$$

This means more precisely that there exists positive constants $\lambda_0 > 1$ such that:

$$\lambda > \lambda_0 \Rightarrow \lambda^2 > C(\Psi)\left(1 + \frac{\lambda}{s}\right).$$

Furthermore since

$$X_2 \leq C(\Psi)\frac{TA}{4}e^{2\lambda\|\Psi\|} \times s^2\lambda^4\left(\frac{T^2A^2}{4} + T^2A^3 + T^3A^2 + T + A\right)\int_Q \varphi^3\,|u|^2\,dQ$$

let us take

$$s > C(\Psi)\frac{TA}{4}e^{2\lambda\|\Psi\|}s^2\left(\frac{T^2A^2}{4} + T^2A^3 + T^3A^2 + T + A\right).$$

It follows that

$$\int_Q e^{-2s\eta} f^2 dQ + 2s^3\lambda^4\delta^4 \int_{\tilde{q}} \varphi^3 |u|^2 dQ + 2s\lambda^2\delta^2 \int_{\tilde{q}} \varphi |\nabla u|^2 dQ \geq$$
$$\int_Q |P_1 u|^2 dQ + \int_Q |P_2 u|^2 dQ + s\lambda^2\delta^2 \int_Q \varphi |\nabla u|^2 dQ + s^3\lambda^4\delta^4 \int_Q \varphi^3 |u|^2 dQ. \tag{5.37}$$

Actually, we want now to eliminate in (5.37) the term

$$2s\lambda^2\delta^2 \int_{\tilde{q}} \varphi |\nabla u|^2 dQ.$$

For this aim, we introduce a cut-off function α such that: $\alpha \in C_0^\infty(\omega)$; $0 \leq \alpha \leq 1$; and $\alpha = 1$ on $\tilde{\omega}$.
Multiplying $P_2 u$ by $\varphi\alpha^2 u$ and integrating the result over Q leads to:

$$\int_Q \varphi\alpha^2 u P_2 u dQ = -s \int_Q \left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial a}\right) u^2 \varphi\alpha^2 dQ -$$
$$s^2\lambda^2 \int_Q u^2 \varphi^3 \alpha^2 |\Psi|^2 dQ - \int_Q u\Delta u \varphi\alpha^2 dQ. \tag{5.38}$$

Note that:

$$\int_Q u\Delta u \varphi\alpha^2 dQ = -\int_Q |\nabla u|^2 \varphi\alpha^2 dQ - \lambda \int_Q u\nabla u.\nabla\Psi \varphi\alpha^2 dQ -$$
$$2\int_Q u\nabla u.\nabla\alpha\varphi\alpha dQ. \tag{5.39}$$

Then,

$$\int_Q \varphi\alpha^2 u P_2 u dQ = -s \int_Q \left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial a}\right) u^2 \varphi\alpha^2 dQ - s^2\lambda^2 \int_Q u^2 \varphi^3 \alpha^2 |\Psi|^2 dQ +$$
$$\int_Q |\nabla u|^2 \varphi\alpha^2 dQ + \lambda \int_Q u\nabla u.\nabla\Psi\varphi\alpha^2 dQ + 2\int_Q u\nabla u.\nabla\alpha\varphi\alpha dQ. \tag{5.40}$$

This gives:

$$\int_Q |\nabla u|^2 \varphi\alpha^2 dQ = s^2\lambda^2 \int_Q u^2\varphi^3\alpha^2 |\Psi|^2 dQ + s\int_Q \left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial a}\right) u^2\varphi\alpha^2 dQ +$$
$$\int_Q \varphi\alpha^2 u P_2 u dQ - \lambda \int_Q u\nabla u.\nabla\Psi\varphi\alpha^2 dQ - 2\int_Q u\nabla u.\nabla\alpha\varphi\alpha dQ. \tag{5.41}$$

Note that:

$$-\lambda \int_Q u\nabla u.\nabla\Psi\varphi\alpha^2 dQ \leq C\lambda^2 \int_Q |u|^2 \varphi\alpha^2 dQ + \frac{1}{2}\int_Q |\nabla u|^2 \varphi\alpha^2 dQ$$

where C is a positive constant.
Now, since $\varphi \leq C\varphi^3$ with C a positive constant, and using the properties of α and Ψ we deduce:

$$\int_{\tilde{q}} |\nabla u|^2 \varphi\alpha^2 dQ \leq C\int_Q \varphi\alpha^2 u P_2 u dQ +$$

$$Cs^2\lambda^2 \int_Q u^2\varphi^3\alpha^2 dQ + C\int_Q u\varphi^{1/2} |\nabla u| \varphi^{1/2}\alpha dQ. \tag{5.42}$$

Therefore we deduce from the previous estimate that:

$$2s\lambda^2\delta^2 \int_{\tilde{q}} |\nabla u|^2 \varphi dQ \leq \frac{1}{2}\int_Q |P_2 u|^2 dQ + Cs^2\lambda^2 \int_q u^2\varphi^3 dQ \tag{5.43}$$

where C is a positive constant.
Combining (5.37) and (5.43) we get:

$$C\left(\int_Q e^{-2s\eta} f^2 dQ + s^3\lambda^4 \int_q \varphi^3 u^2 dQ\right) \geq \int_Q |P_1 u|^2 dQ + \int_Q |P_2 u|^2 dQ +$$

$$s\lambda^2 \int_Q \varphi |\nabla u|^2 dQ + s^3\lambda^4 \int_Q \varphi^3 u^2 dQ. \tag{5.44}$$

We want now to turn back to the variable z. Note that $u = e^{-s\eta}z$. Then, we have:

$$\int_Q \varphi^3 |u|^2 dQ = \int_Q e^{-2s\eta}\varphi^3 |w|^2 dQ$$

and

$$\int_q \varphi^3 |u|^2 dQ = \int_q e^{-2s\eta}\varphi^3 |w|^2 dQ.$$

As a result, one gets from (5.44)

$$s^3\lambda^4 \int_Q \varphi^3 e^{-2s\eta} z^2 dQ \leq C\int_Q e^{-2s\eta} f^2 dQ + Cs^3\lambda^4 \int_q e^{-2s\eta}\varphi^3 z^2 dQ. \tag{5.45}$$

On the other hand we have $\nabla u = s\lambda e^{-s\eta}\nabla\Psi z + e^{-s\eta}\nabla z$. Then it follows that

$$e^{-2s\eta} |\nabla z|^2 \leq C\left(s^2\lambda^2 e^{-2s\eta} |\nabla\Psi|^2 z^2 + |\nabla u|^2\right).$$

Integrating this over Q and using (5.44) and (5.45) we derive that:

$$\int_Q e^{-2s\eta} s\lambda\varphi \left|\nabla z\right|^2 dQ \leq C\left(\int_Q e^{-2s\eta} f^2 dQ + \int_{Q_\omega} s^3\lambda^4\varphi^3 e^{-2s\eta} z^2 dQ\right). \tag{5.46}$$

Adding now (5.46) and (5.45) one gets (5.2). $\square$

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Oumar Traore
Laboratoire d'Analyse Mathématique des
Equations (L.A.M.E)
Université de Ouagadougou
03 BP 7021
Ouagadougou, 03
Burkina Faso
traore.oumar@univ-ouaga.bf