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BLAISE PASCAL

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Volume 17, n° 1 (2010), p. 183-197.

http://ambp.cedram.org/item?id=AMBP_2010__17_1_183_0

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*Publication éditée par le laboratoire de mathématiques
de l'université Blaise-Pascal, UMR 6620 du CNRS
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Successive Approximation of Neutral Functional Stochastic Differential Equations in Hilbert Spaces

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Abstract

By using successive approximation, we prove existence and uniqueness result for a class of neutral functional stochastic differential equations in Hilbert spaces with non-Lipschitzian coefficients

Approximations successives pour les équations fonctionnelles stochastiques de type neutre dans un espace de Hilbert.

Résumé

En utilisant la méthode des approximations successives, nous allons montrer un résultat d'existence et d'unicité, sous des conditions non Lipschitziennes, pour une classe d'équations fonctionnelles stochastiques de type neutre dans un espace de Hilbert.

1. Introduction

The purpose of this paper is to prove the existence and uniqueness of mild solutions for a class of neutral functional stochastic differential equations (FSDEs) described in the form

$$\begin{aligned}d[x(t) + g(t, x_t)] &= [Ax(t) + f(t, x_t)]dt + \sigma(t, x_t)dW(t), \quad 0 \leq t \leq T, \\x(t) &= \varphi(t), \quad -r \leq t \leq 0.\end{aligned}\tag{1.1}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(T(t))_{t \geq 0}$, in a Hilbert space H ; $x_t \in \mathcal{C}_r = \mathcal{C}([-r, 0], H)$ and $f : [0, T] \times \mathcal{C}_r \rightarrow H$, $g : [0, T] \times \mathcal{C}_r \rightarrow H$, $\sigma : [0, T] \times \mathcal{C}_r \rightarrow \mathcal{L}_2(Q^{1/2}E, H)$

Keywords: Semigroup of bounded linear operator, Fractional powers of closed operators, Successive approximation, Mild solution, Cylindrical Q -Wiener process.

Math. classification: 60H20, 34F05, 34G20.

are appropriate functions. Here $\mathcal{L}_2(Q^{1/2}E, H)$ denotes the space of all Q -Hilbert-Schmidt operators from $Q^{1/2}E$ into H (see section 2 below).

Neutral FSDEs arises in many areas of applied mathematics and such equations have received much attention in recent years. The theory of neutral FSDEs in finite dimensional spaces has been extensively studied in the literature; see Kolmanovskii and Nosov [8], Mao [13]-[12], Kolmanovskii *et al.* [7], and Liu and Xia [10].

However, in the infinite-dimensional Hilbert space, only a few results have been obtained in this field despite the importance and interest of the model (1.1). In this respect, it is worth mentioning that this kind of neutral equation arises from problems related to coupled oscillators in a noisy environment, or in problems of viscoelastic materials under random or stochastic influences (see [15] for a description of these problems in the deterministic case). To the best of our knowledge, there exist only few papers already published in this field. To be more precise, a version of (1.1), in the particular case where the delays are constant, is considered in [9] and some stability properties of the mild solutions are analyzed in a similar way as Dakto proved in [4] in the deterministic case, while in [6] the existence and uniqueness of mild solutions to model (1.1) is studied, as well as some results on the stability of the null solution. In [2] Caraballo *et al* studied the problem in a variational point of view. So far little is known about the neutral FSDEs in Hilbert spaces.

Our idea is inspired by a paper of Mahmudov [11] in which the author study the existence and uniqueness of equation (1.1) without delay.

The paper is organized as follows, In Section 2 we give a brief review and preliminaries needed to establish our results. Section 3 is devoted to the study of existence and uniqueness by using a Picard type iteration.

2. Preliminaries

In this section, we introduce notations, definitions and preliminary results which we require to establish the existence and uniqueness of a solution of equation (1.1).

Let E and H be two real separable Hilbert spaces. Denote by $L(E, H)$ the family of bounded linear operators from E to H . Fix a non-negative and symmetric operator $Q \in L(E, E)$. Let W be a cylindrical Q -Wiener

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process in E (cf. [3]) defined on some complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \geq 0})$. Let $\mathcal{L}_2^0 := \mathcal{L}_2(Q^{1/2}E, H)$ be the space of all Hilbert-Schmidt operators from Hilbert space $Q^{1/2}E$ to H with the inner product

$$\langle \phi, \psi \rangle_{\mathcal{L}_2^0} = \text{tr}[\phi Q \psi^*].$$

Let us consider two fixed real numbers $T > 0$ and $r > 0$, we denote by $\mathcal{C}_r = \mathcal{C}([-r, 0]; H)$ the space of all continuous functions from $[-r, 0]$ to H equipped with the norm

$$\|Z\|_{\mathcal{C}_r} = \sup_{-r \leq s \leq 0} \|Z(s)\|$$

If we consider a function $x \in \mathcal{C}([-r, T], H)$, for each $t \in [0, T]$ we will denote by $x_t \in \mathcal{C}_r$ the function defined by $x_t(s) = x(t + s) \forall s \in [-r, 0]$. We denote by $L_{\mathcal{F}}^p([0, T], H)$ the Hilbert space of all square integrable and \mathcal{F}_t adapted processes with values in H . with $\mathcal{F}_t = \mathcal{F}_0$ for $-r \leq t \leq 0$, let us denote by B_T the Banach space of all H - valued \mathcal{F}_t adapted process $x(t, \omega) : [-r, T] \times \Omega \rightarrow H$, which are continuous in t for a.e. fixed $\omega \in \Omega$ and satisfy

$$\|x\|_{B_T}^p = \mathbb{E} \sup_{-r \leq t \leq T} \|x(t, \omega)\|^p < \infty, \quad p > 2.$$

Let $A : D(A) \rightarrow H$ be the infinitesimal generator of an analytic semigroup, $(T(t))_{t \geq 0}$, of bounded linear operators on H . For the theory of strongly continuous semigroup, we refer to Pazy [14] and Goldstein [5]. We will point out here some notations and properties that will be used in this work. It is well known that there exist $\tilde{M} \geq 1$ and $\lambda \in \mathbb{R}$ such that $\|T(t)\| \leq \tilde{M}e^{\lambda t}$ for every $t \geq 0$. If $(T(t))_{t \geq 0}$ is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A , then it is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in H , and the expression

$$\|h\|_\alpha = \|(-A)^\alpha h\|$$

defines a norm in $D(-A)^\alpha$. If H_α represents the space $D(-A)^\alpha$ endowed with the norm $\|\cdot\|_\alpha$, then the following properties are well known (cf. [14], p. 74).

Lemma 2.1. *Suppose that the preceding conditions are satisfied.*

(1) *Let $0 < \alpha \leq 1$. Then H_α is a Banach space.*

(2) If $0 < \beta \leq \alpha$ then the injection $H_\alpha \hookrightarrow H_\beta$ is continuous.

(3) For every $0 < \alpha \leq 1$ there exists $C_\alpha > 0$ such that

$$\|(-A)^\alpha T(t)\| \leq \frac{C_\alpha}{t^\alpha}, \quad 0 < t \leq T.$$

We now recall the following Bihari's inequality [1].

Lemma 2.2. *Let $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous and non-decreasing function and let g, h, λ be non-negative functions on \mathbb{R}^+ such that*

$$g(t) \leq h(t) + \int_0^t \lambda(s)\rho(g(s))ds, \quad t \geq 0,$$

then

$$g(t) \leq G^{-1} \left(G(h^*(t)) + \int_0^t \lambda(s)ds \right),$$

where $G(x) := \int_{x_0}^x \frac{1}{\rho(y)}dy$ is well defined for some $x_0 > 0$, G^{-1} is the inverse function of G and $h^*(t) := \sup_{s \leq t} h(s)$. In particular, we have the Gronwall-Bellman lemma:

If

$$g(t) \leq h(t) + \int_0^t \lambda(s)g(s)ds,$$

then

$$g(t) \leq h^*(t) \exp \left(\int_0^t \lambda(s)ds \right).$$

Finally, we remark that for the proof of our theorem we shall make use of the following Lemma (see [3], p. 184)

Lemma 2.3. *Suppose that $\phi(t), t \geq 0$ is a \mathcal{L}_2^0 -valued predictable process and let $W_A^\phi = \int_0^t T(t-s)\phi(s)dW(s)$, $t \in [0, T]$. Then, for any arbitrary $p > 2$ there exists a constant $c(p, T) > 0$ such that:*

$$\mathbb{E} \sup_{t \leq T} |W_A^\phi|^p \leq c(p; T) \sup_{t \leq T} \|T(t)\|^p \mathbb{E} \int_0^T \|\phi(s)\|^p ds$$

Moreover, if $\mathbb{E} \int_0^T \|\phi(s)\|^p ds < \infty$, then there exists a continuous version of the process $\{W_A^\phi; t \geq 0\}$

3. The main result

In this section we study the existence and uniqueness of mild solution of equation (1.1). Henceforth we will assume that A is the infinitesimal generator of an analytic semigroup, $(T(t))_{t \geq 0}$, of bounded linear operators on H . Further, to avoid unnecessary notations, we suppose that $0 \in \rho(A)$ and that, see Lemma 2.1,

$$\|T(t)\| \leq \tilde{M} \quad \text{and} \quad \|(-A)^{1-\beta}T(t)\| \leq \frac{C_{1-\beta}}{t^{1-\beta}}$$

for some constants $\tilde{M}, C_{1-\beta}$ and every $t \in [0, T]$.

Definition 3.1. A continuous stochastic process $x : [-r, T] \rightarrow H$ is a mild solution of equation (1.1) on $[-r, T]$ if

- i)* $x(t)$ is measurable and \mathcal{F}_t adapted, for all $-r \leq t \leq T$,
- ii)* $\int_{-r}^T \|x(s)\|^p ds < \infty$, a.s., $p > 2$.
- iii)* $x(t) = T(t)(\varphi(0) + g(0, \varphi)) - g(t, x_t) - \int_0^t AT(t-s)g(s, x_s)ds$
 $+ \int_0^t T(t-s)f(s, x_s)ds + \int_0^t T(t-s)\sigma(s, x_s)ds$ if $t \in [0, T]$,
- iv)* $x(t) = \varphi(t)$, $-r \leq t \leq 0$.

In order to show the existence and the uniqueness of the equation (1.1), we are going to make the following hypotheses

($\mathcal{H}.1$) The function $(f, \sigma) : [0, T] \times \mathcal{C}_r \rightarrow H \times \mathcal{L}_2^0$ is measurable, continuous in ξ for each fixed $t \in [0, T]$ and there exists a function $K : [0, T] \times [0, \infty) \rightarrow [0, \infty)$ such that

(1a) $\forall t \in [0, T]$, $K(t, \cdot)$ is continuous non-decreasing and for each fixed $x \in \mathbb{R}_+$, $\int_0^T K(s, x)ds < +\infty$.

(1b) For any fixed $t \in [0, T]$ and $\xi \in L^p(\Omega; \mathcal{C}_r)$

$$\mathbb{E}(\|f(t, \xi)\|^p + \|\sigma(t, \xi)\|^p) \leq K(t, \mathbb{E}\|\xi\|^p).$$

(1c) For any constant $\alpha > 0, u_0 \geq 0$, the integral equation

$$u(t) = u_0 + \alpha \int_0^t K(s, u(s))ds \tag{3.1}$$

has a global solution on $[0, T]$.

($\mathcal{H}.2$) There exists a function $G : [0, T] \times [0, +\infty) \rightarrow [0, \infty)$ such that

(2a) $\forall t \in [0, T]$, $G(t, \cdot)$ is continuous non-decreasing with $G(t, 0) = 0$ and for each fixed $x \in \mathbb{R}_+$, $\int_0^T G(s, x) ds < +\infty$.

(2b) For any fixed $t \in [0, T]$ and $\xi, \eta \in L^p(\Omega; \mathcal{C}_r)$

$$\begin{aligned} \mathbb{E} (\|f(t, \xi) - f(t, \eta)\|^p + \|\sigma(t, \xi) - \sigma(t, \eta)\|^p) \\ \leq G(t, \mathbb{E}\|\xi - \eta\|^p), \end{aligned}$$

(2c) For any constant $D > 0$; if a non negative function $z(t), t \in [0, T]$ satisfies $z(0) = 0$ and $z(t) \leq D \int_0^t G(s, z(s)) ds$, then $z(t) = 0$ for all $t \in [0, T]$.

($\mathcal{H}.3$) There exist constants $\frac{1}{p} < \beta < 1$, l_1, l_2, M_g such that the function g is H_β -valued, $(-A)^\beta g : [0, T] \times \mathcal{C}_r \rightarrow H$ is continuous and satisfies

(3a) For all $t \in [0, T]$ and $\xi \in \mathcal{C}_r$,

$$\|(-A)^\beta g(t, \xi)\|^p \leq l_1 \|\xi\|_{\mathcal{C}_r}^p + l_2.$$

(3b) For all $t \in [0, T]$ and $\xi, \eta \in \mathcal{C}_r$

$$\|(-A)^\beta g(t, \xi) - (-A)^\beta g(t, \eta)\| \leq M_g \|x - y\|_{\mathcal{C}_r}.$$

(3c) The constants M_g, l_1 and β satisfy the following inequalities

$$4^{p-1} \|(-A)^{-\beta}\|^p M_g^p < 1, \quad 5^{p-1} \|(-A)^{-\beta}\|^p l_1 < 1.$$

Moreover, we assume that $\varphi \in L^p(\Omega, \mathcal{C}_r)$ is an \mathcal{F}_0 -measurable random variable and $p > 2$.

The main result of this paper is given in the next theorem.

Theorem 3.2. *Assume ($\mathcal{H}.1$)-($\mathcal{H}.3$) holds, then the equation (1.1) has a unique mild solution $x \in B_T$.*

For the proof, we will need the following lemmas.

Lemma 3.3. *Let $\tilde{f} \in L^p_{\mathcal{F}}([0, T], H)$, $\tilde{\sigma} \in L^p_{\mathcal{F}}([0, T], \mathcal{L}_2^0)$, and consider the equation*

$$\begin{aligned} d[x(t) + g(t, x_t)] &= [Ax(t) + \tilde{f}(t)]dt + \tilde{\sigma}(t)dW(t), \quad 0 \leq t \leq T, \\ x_0 &= \varphi. \end{aligned} \tag{3.2}$$

Under condition ($\mathcal{H}.3$), Equation (3.2) has a unique mild solution $x \in B_T$.

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Proof. Let us consider the set

$$S_T = \{x \in B_T : x(s) = \varphi(s), \text{ for } s \in [-r, 0]\}.$$

S_T is a closed subset of B_T provided with the norm $\|\cdot\|_{B_T}$.

Let ψ be the function defined on S_T by

$$\psi(x)(t) = \begin{cases} \varphi(t) & \text{if } t \in [-r, 0] \\ T(t)(\varphi(0) + g(0, \varphi)) - g(t, x_t) - \int_0^t AT(t-s)g(s, x_s)ds \\ \quad + \int_0^t T(t-s)\tilde{f}(s)ds + \int_0^t T(t-s)\tilde{\sigma}(s)dW(s), & \text{if } t \in [0, T] \end{cases}$$

We will first prove that the function ψ is well defined.

Let $x \in S_T$ and $t \in [0, T]$, we have

$$\begin{aligned} \psi(x)(t) &= T(t)(\varphi(0) + g(0, \varphi)) - g(t, x_t) - \int_0^t AT(t-s)g(s, x_s)ds \\ &\quad + \int_0^t T(t-s)\tilde{f}(s)ds + \int_0^t T(t-s)\tilde{\sigma}(s)dW(s) \\ &= \sum_{1 \leq i \leq 5} I_i(t) \end{aligned}$$

We are going to show that each function $t \rightarrow I_i(t)$ is continuous on $[0, T]$. The continuity of I_1 follows directly from the continuity of $t \rightarrow T(t)h$. By $(\mathcal{H}.3)$, the function $(-A)^\beta g$ is continuous and since the operator $(-A)^{-\beta}$ is bounded then $t \rightarrow g(t, x_t)$ is continuous on $[0, T]$

For the third term $I_3(t) = \int_0^t AT(t-s)g(s, x_s)ds$, we have

$$\begin{aligned} |I_3(t+h) - I_3(t)| &\leq \left| \int_0^t (T(h) - I)(-A)^{1-\beta}T(t-s)(-A)^\beta g(s, x_s)ds \right| \\ &\quad + \left| \int_t^{t+h} (-A)^{1-\beta}T(t+h-s)(-A)^\beta g(s, x_s)ds \right| \\ &\leq I_{31}(h) + I_{32}(h). \end{aligned}$$

By the strong continuity of $T(t)$, we have for each $s \in [0, T]$,

$$\lim_{h \rightarrow 0} (T(h) - I)(-A)^{1-\beta}T(t-s)(-A)^\beta g(s, x_s) = 0$$

and since

$$\begin{aligned} & \| (T(h) - I)(-A)^{1-\beta} T(t-s)(-A)^\beta g(s, x_s) \| \\ & \leq (\tilde{M} + 1) \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| (-A)^\beta g(s, x_s) \| \in \mathbb{L}^1([0, t], ds), \end{aligned}$$

we conclude by the Lebesgue dominated theorem that $\lim_{h \rightarrow 0} I_{31}(h) = 0$. On the other hand,

$$|I_{32}(h)|^p \leq C h^{\frac{p}{q} - p(1-\beta)} \int_0^T (l_1 \|x_s\|^p + l_2) ds;$$

then

$$\lim_{h \rightarrow 0} I_3(t+h) - I_3(t) = 0$$

Similar computations can be used to show the continuity of I_4 . The continuity of the last term follows from the Lemma 2.3. Hence, we conclude that the function $t \rightarrow \psi(x)(t)$ is continuous on $[0, T]$ a.s.

Next, to see that $\psi(S_T) \subset S_T$, let $x \in S_T$ and $t \in [0, T]$. By using condition (3a) and Hölder's inequality, with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} \|\psi(x)(t)\|^p & \leq 5^{p-1} \tilde{M}^p \|\varphi(0) + g(0, \varphi)\|^p + 5^{p-1} \|(-A)^{-\beta}\|^p [l_1 \|x_t\|^p + l_2] \\ & \quad + 5^{p-1} \left(\int_0^t \|(-A)^{1-\beta} T(t-s)\|^q ds \right)^{\frac{p}{q}} \int_0^t (l_1 \|x_s\|^p + l_2) ds \\ & \quad + 5^{p-1} \tilde{M}^p t^{\frac{p}{q}} \int_0^t \|\tilde{f}(s)\|^p ds + 5^{p-1} \left\| \int_0^t T(t-s) \tilde{\sigma}(s) dW(s) \right\|^p. \end{aligned}$$

By using Lemma 2.1 and Lemma 2.3, we obtain

$$\mathbb{E} \sup_{0 \leq s \leq T} \|\psi(x)(s)\|^p \leq C + C \mathbb{E} \sup_{-r \leq s \leq T} \|x(s)\|^p, \text{ for some constant } C > 0.$$

Since $\psi(x) = \varphi$ on $[-r, 0]$, it follows that

$$\mathbb{E} \sup_{-r \leq s \leq T} \|\psi(x)(s)\|^p < \infty.$$

The \mathcal{F}_t measurability is easily verified, so we conclude that ψ is well defined.

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Now, we are going to show that ψ is a contraction mapping in S_{T_1} with some $T_1 \leq T$ to be specified later. Let $x, y \in S_T$ and $t \in [0, T]$, we have

$$\begin{aligned} & \|\psi(x)(t) - \psi(y)(t)\|^p \\ & \leq 2^{p-1} \|g(t, x_t) - g(t, y_t)\|^p + 2^{p-1} \left\| \int_0^t AT(t-s)(g(s, x_s) - g(s, y_s)) ds \right\|^p \\ & \leq 2^{p-1} \|(-A)^{-\beta}\|^p \|(-A)^\beta g(t, x_t) - (-A)^\beta g(t, y_t)\|^p \\ & \quad + 2^{p-1} \left\| \int_0^t (-A)^{1-\beta} T(t-s) (-A)^\beta (g(s, x_s) - g(s, y_s)) ds \right\|^p. \end{aligned}$$

By condition (3b), Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned} \|\psi(x)(t) - \psi(y)(t)\|^p & \leq 2^{p-1} \|(-A)^{-\beta}\|^p M_g^p \|x_t - y_t\|^p \\ & \quad + 2^{p-1} M_g^p C_{1-\beta}^p \left(\frac{t^{1-(1-\beta)q}}{1-(1-\beta)q} \right)^{\frac{p}{q}} \int_0^t \|x_s - y_s\|^p ds. \end{aligned}$$

Hence

$$\mathbb{E} \sup_{s \in [-r, t]} \|\psi(x)(s) - \psi(y)(s)\|^p \leq \gamma(t) \mathbb{E} \sup_{s \in [-r, t]} \|x(s) - y(s)\|^p.$$

where

$$\gamma(t) = 2^{p-1} M_g^p \left\{ \|(-A)^{-\beta}\|^p + C_{1-\beta}^p \left(\frac{t^{1-(1-\beta)q}}{1-(1-\beta)q} \right)^{\frac{p}{q}} t \right\}.$$

By condition (3c), we have $\gamma(0) = 2^{p-1} \|(-A)^{-\beta}\|^p M_g^p < 1$. Then there exists $0 < T_1 \leq T$ such that $0 < \gamma(T_1) < 1$ and ψ is a contraction mapping on S_{T_1} and therefore has a unique fixed point, which is a mild solution of equation (3.2) on $[0, T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[-r, T]$ in finitely many steps. \square

We now construct a successive approximation sequence using a Picard type iteration with the help of Lemma 3.3. Let x^0 be a solution of equation (3.2) with $\tilde{f} = 0, \tilde{\sigma} = 0$. For $n \geq 0$, let x^{n+1} be the solution of equation

(3.2) on $[-r, T]$ with $\tilde{f}(t) = f(t, x_t^n)$, and $\tilde{\sigma}(t) = \sigma(t, x_t^n)$ i.e.

$$\left\{ \begin{array}{l} x^{n+1} \in B_T \\ x^{n+1}(t) = \varphi(t) \quad \text{if } t \in [-r, 0] \\ x^{n+1}(t) = T(t)(\varphi(0) + g(0, \varphi)) - g(t, x_t^{n+1}) - \int_0^t AT(t-s)g(s, x_s^{n+1})ds \\ \quad + \int_0^t T(t-s)f(s, x_s^n)ds + \int_0^t T(t-s)\sigma(s, x_s^n)dW(s), \text{ if } t \in [0, T] \end{array} \right. \quad (3.3)$$

Lemma 3.4. *Under conditions $(\mathcal{H}.1) - (\mathcal{H}.3)$, the sequence $\{x^n, n \geq 0\}$ is well defined and there exist positive constants M, D_0, D_1 such that for all $m, n \in \mathbb{N}$ and $t \in [0, T]$*

(1)

$$\mathbb{E} \sup_{-r \leq s \leq t} \|x^{m+1}(s) - x^{n+1}(s)\|^p \leq M \int_0^t G(s, \mathbb{E} \sup_{-r \leq \theta \leq s} \|x^m(\theta) - x^n(\theta)\|^p) ds \quad (3.4)$$

(2)

$$\mathbb{E} \sup_{-r \leq s \leq t} \|x^{n+1}(s)\|^p \leq D_0 + D_1 \int_0^t K(s, \mathbb{E} \sup_{-r \leq \theta \leq s} \|x^n(\theta)\|^p) ds. \quad (3.5)$$

Proof. • 1: For $m, n \in \mathbb{N}$ and $t \in [0, T]$ we have

$$\|x^{m+1}(t) - x^{n+1}(t)\|^p \leq 4^{p-1} (I_1(t) + I_2(t) + I_3(t))$$

where

$$I_1(t) := \|g(t, x_t^{m+1}) - g(t, x_t^{n+1})\|^p,$$

$$I_2(t) := \left\| \int_0^t AT(t-s)(g(t, x_s^{m+1}) - g(t, x_s^{n+1}))ds \right\|^p,$$

$$\begin{aligned} I_3(t) &:= \left\| \int_0^t T(t-s)(f(s, x_s^m) - f(s, x_s^n))ds \right\|^p \\ &\quad + \left\| \int_0^t T(t-s)(\sigma(s, x_s^m) - \sigma(s, x_s^n))dW(s) \right\|^p. \end{aligned}$$

By using condition (3b) for the terms I_1 and I_2 , we obtain

$$\begin{aligned} I_1(t) &\leq \|(-A)^{-\beta}\|^p M_g^p \|x_t^{m+1} - x_t^{n+1}\|^p, \\ I_2(t) &\leq \left(\int_0^t \left(\frac{C_{1-\beta}}{(t-s)^{1-\beta}} \right)^q ds \right)^{\frac{p}{q}} \int_0^t M_g^p \|x_s^{m+1} - x_s^{n+1}\|^p ds. \end{aligned}$$

By using Lemma 2.3 and condition (2b) for the term I_3 , we obtain

$$\mathbb{E} \sup_{0 \leq s \leq t} I_3(s) \leq C \int_0^t G(s, \mathbb{E} \sup_{-r \leq \theta \leq s} \|x^m(\theta) - x^n(\theta)\|^p) ds.$$

Using the fact that $4^{p-1} \|(-A)^{-\beta}\|^p M_g^p < 1$ and the above inequalities, we obtain that:

$$\begin{aligned} \mathbb{E} \sup_{-r \leq s \leq t} \|x^{m+1}(s) - x^{n+1}(s)\|^p &\leq C \int_0^t \mathbb{E} \sup_{-r \leq \theta \leq s} \|x^{m+1}(\theta) - x^{n+1}(\theta)\|^p ds \\ &\quad + C \int_0^t G(s, \mathbb{E} \sup_{-r \leq \theta \leq s} \|x^m(\theta) - x^n(\theta)\|^p) ds. \end{aligned}$$

By Lemma 2.2, we obtain

$$\mathbb{E} \sup_{-r \leq s \leq t} \|x^{m+1}(s) - x^{n+1}(s)\|^p \leq C \int_0^t G(s, \mathbb{E} \sup_{-r \leq \theta \leq s} \|x^m(\theta) - x^n(\theta)\|^p) ds.$$

• 2: By the same method as in the proof of assertion (1), we obtain that

$$\begin{aligned} \mathbb{E} \sup_{-r \leq s \leq t} \|x^{m+1}(s)\|^p &\leq C + C \int_0^t \mathbb{E} \sup_{-r \leq \theta \leq s} \|x^{m+1}(\theta)\|^p ds \\ &\quad + C \int_0^t K(s, \mathbb{E} \sup_{-r \leq \theta \leq s} \|x^m(\theta)\|^p) ds. \end{aligned}$$

By Lemma 2.2, we obtain

$$\mathbb{E} \sup_{-r \leq s \leq t} \|x^{m+1}(s)\|^p \leq C + C \int_0^t K(s, \mathbb{E} \sup_{-r \leq \theta \leq s} \|x^m(\theta)\|^p) ds.$$

□

Lemma 3.5. *Under conditions (H.1) – (H.3), there exists an $u(t)$ satisfying*

$$u(t) = u_0 + D \int_0^t K(s, u(s)) ds$$

for some $u_0 \geq 0$, $D > 0$ and the sequence $\{x^n, n \geq 0\}$ satisfies, for all $n \in \mathbb{N}$ and $t \in [0, T]$

$$\mathbb{E} \sup_{-r \leq s \leq t} \|x^n(s)\|^p \leq u(t) \quad (3.6)$$

Proof. Let $u : [0, T] \rightarrow \mathbb{R}$ be a global solution of the integral equation (3.1) with an initial condition $u_0 = D_0 \vee \mathbb{E} \sup_{-r \leq t \leq T} \|x^0(t)\|^p$ and with $\alpha = D_1$, where D_0, D_1 are the same constants as in Lemma 3.3. We prove inequality (3.6) by the mathematical induction.

For $n = 0$, the inequality (3.6) holds by the definition of u_0 .

Let us assume that $\mathbb{E} \sup_{-r \leq s \leq t} \|x^n(t)\|^p \leq u(t)$. Then, by (3.5), we obtain

$$\begin{aligned} \mathbb{E} \sup_{-r \leq s \leq t} \|x^{n+1}(s)\|^p &\leq D_0 + D_1 \int_0^t K(s, \mathbb{E} \sup_{-r \leq \theta \leq s} \|x^n(\theta)\|^p) ds \\ &\leq u_0 + D_1 \int_0^t K(s, u(s)) ds = u(t). \end{aligned}$$

This completes the proof. □

Proof of Theorem 3.2.

• Proof of the existence: For $t \in [0, T]$, let

$$z(t) = \lim_{m, n \rightarrow +\infty} \sup_{-r \leq s \leq t} (\mathbb{E} \sup_{-r \leq s \leq t} \|x^m(s) - x^n(s)\|^p)$$

By (3.4), (3.6) and Fatou's lemma, we see that

$$z(t) \leq M \int_0^t G(s, z(s)) ds$$

By condition (2c), we get $z(t) = 0$, which implies that

$$\lim_{m, n \rightarrow +\infty} \mathbb{E} \sup_{-r \leq s \leq T} \|x^m(s) - x^n(s)\|^p = 0$$

This implies that there exists $x \in B_T$ such that

$$\lim_{n \rightarrow +\infty} \mathbb{E} \sup_{-r \leq s \leq T} \|x^n(s) - x(s)\|^p = 0$$

Letting $n \rightarrow +\infty$ in (3.3); it is seen that x is a mild solution to equation (1.1) on $[-r, T]$.

• Proof of the uniqueness: Let x and y be two mild solutions of equation (1.1) on $[-r, T]$, then by the same procedure as for Lemma 3.4, we obtain

$$\mathbb{E} \sup_{-r \leq s \leq t} \|x(s) - y(s)\|^p \leq M \int_0^t G(s, \mathbb{E} \sup_{-r \leq \theta \leq s} \|x(\theta) - y(\theta)\|^p) ds$$

By condition (2c), we get $\mathbb{E} \sup_{-r \leq s \leq T} \|x(s) - y(s)\|^p = 0$, which implies the uniqueness. The proof of theorem is complete. \square

Remark 3.6. Let $G(t, x) = \lambda(t)\rho(x)$, where λ is a non-negative function such that $\int_0^T \lambda(s) ds < +\infty$, and $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous non-decreasing function such that $\rho(0) = 0$ and $\int_{0+} \frac{1}{\rho(x)} = +\infty$. Thanks to Bihari’s inequality (Lemma 2.2), we know that (2c) holds. Moreover if we further assume that $\rho(x)$ is a concave function, f and σ satisfy $f(\cdot, 0) \in L^p_{\mathcal{F}}([0, T], H)$, $\sigma(\cdot, 0) \in L^p_{\mathcal{F}}([0, T], \mathcal{L}^0_2)$ and for all $t \in [0, T]$, $\xi, \eta \in \mathcal{C}_r$

$$\|f(t, \xi) - f(t, \eta)\|^p + \|\sigma(t, \xi) - \sigma(t, \eta)\|^p \leq \lambda(t)\rho(\|\xi - \eta\|^p)$$

then f, σ satisfy conditions (H.1) and (H.2), where

$$K(t, x) = 2^{p-1} \left(G(t, x) + \mathbb{E}\|f(t, 0)\|^p + \mathbb{E}\|\sigma(t, 0)\|^p \right).$$

Remark 3.7. The typical concave continuous non-decreasing functions satisfying $\rho(0) = 0$ and $\int_{0+} \frac{1}{\rho(x)} = +\infty$ are given by $\rho_k(x), k = 1, 2, \dots$,

$$\rho_k(x) := \begin{cases} c_0 \cdot x \cdot \prod_{j=1}^k \log^j x^{-1}, & x \leq \eta \\ c_0 \cdot \eta \cdot \prod_{j=1}^k \log^j \eta^{-1} + c_0 \cdot \rho'_k(\eta-) \cdot (x - \eta), & x > \eta, \end{cases}$$

where $\log^j x^{-1} := \log \log \dots \log x^{-1}$ and $c_0 > 0, 0 < \eta < \frac{1}{e^k}$.

Acknowledgment: This work was supported by the Hassan II Academy of Sciences and Technology.

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