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On the range of the Fourier transform connected with Riemann-Liouville operator

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Abstract

We characterize the range of some spaces of functions by the Fourier transform associated with the Riemann-Liouville operator $\mathscr{R}_\alpha,\ \alpha \geqslant 0$ and we give a new description of the Schwartz spaces. Next, we prove a Paley-Wiener and a Paley-Wiener-Schwartz theorems.

1. **Introduction**

In [3], the first author with the others consider the so-called Riemann-Liouville transform $\mathscr{R}_\alpha;\ \alpha \geqslant 0$, defined on the space $\mathscr{C}_*(\mathbb{R}^2)$ (the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable) by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f\big(rs\sqrt{1-t^2}, x+rt\big) \\ \qquad\qquad \times\big(1-t^2\big)^{\alpha-\frac{1}{2}}\big(1-s^2\big)^{\alpha-1} dt\ ds, & \text{if } \alpha > 0; \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f\big(r\sqrt{1-t^2}, x+rt\big)\dfrac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha = 0. \end{cases}$$

The mapping $\mathscr{R}_\alpha$ generalizes the mean operator $\mathscr{R}_0$ defined by

$$\mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f\big(r\sin\theta, x + r\cos\theta\big)\ d\theta.$$

The dual operator ${}^t\mathscr{R}_0$ of $\mathscr{R}_0$ is defined by

$${}^t\mathscr{R}_0(g)(r,x) = \frac{1}{\pi}\int_{\mathbb{R}} g\big(\sqrt{r^2+(x-y)^2}, y\big)dy.$$

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The mean operator $\mathscr{R}_0$ and its dual ${}^t\mathscr{R}_0$ play an important role and have many applications; for example, in image processing of the so-called synthetic aperture radar (SAR) data [9, 10] or in the linearized inverse scattering problem in acoustics [8]. The operators $\mathscr{R}_0$ and ${}^t\mathscr{R}_0$ have been studied by many authors and from many points of view [2, 13, 14]. In [3]; the authors associated to the Riemann-Liouville operator the Fourier transform $\mathscr{F}_\alpha$ defined by

$$\mathscr{F}_\alpha(f)(\mu,\lambda) = \frac{1}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\int_{\mathbb{R}}\int_0^{+\infty} f(r,x)\ j_\alpha(r\sqrt{\mu^2+\lambda^2})e^{-i\lambda x}r^{2\alpha+1}\ dr\ dx$$

where, j_α is a modified Bessel function. They have constructed the harmonic analysis related to the Fourier transform $\mathscr{F}_\alpha$ (inversion formula, Plancherel formula, Paley-Wiener theorem, Plancherel theorem ...).

Our investigation in the present work consists to characterize the range of some spaces of functions by the Fourier transform $\mathscr{F}_\alpha$ and to establish a real Paley-Wiener theorem and a Paley-Wiener-Schwartz theorem for this transform. More precisely, in the second section of this paper, we characterize the range of some subspace of $L^2([0,+\infty[\times\mathbb{R}; r^{2\alpha+1}\ dr\otimes dx)$ (the space of square integrable functions on $[0,+\infty[\times\mathbb{R}$ with respect to the measure $r^{2\alpha+1}\ dr\otimes dx$). In the third section; we give a new characterization of the Schwartz's space $S_*(\mathbb{R}^2)$ (the space of infinitely differentiable functions on $\mathbb{R}^2$; even with respect to the first variable, rapidly decreasing together with all their derivatives)[15, 16, 18]. Using this; we give a nice description of the space $S_*(\Gamma)$ (the space of infinitely differentiable functions on $\Gamma = \mathbb{R}^2 \cup \{(it,x);\ (t,x)\in\mathbb{R}^2,\ |t|\leqslant|x|\}$; even with respect to the first variable, rapidly decreasing with all their derivatives). In the last section, using the idea of [4]; we establish a real Paley-Wiener theorem and a Paley-Wiener-Schwartz theorem.

We recall that in [21]; the author obtains similar results for the Hankel transform and the generalized Hankel transform on the half line.

2. **Fourier transform associated with Riemann-Liouville operator.**

In this section, we recall some properties of the Fourier transform associated with the Riemann-Liouville operator.

For all $(\mu,\lambda)\in\mathbb{C}^2$; we put

$$\varphi_{\mu,\lambda}(r,x)=\mathscr{R}_\alpha\big(\cos(\mu.)exp(-i\lambda.)\big)(r,x),$$

where $\mathscr{R}_\alpha$ is the Riemann-Liouville transform defined in the introduction. Then, the function $\varphi_{\mu,\lambda}$ is given by

$$\varphi_{\mu,\lambda}(r,x)=j_\alpha(r\sqrt{\mu^2+\lambda^2})e^{-i\lambda x}, \tag{2.1}$$

where j_α is the modified Bessel function defined by

$$\begin{aligned} j_\alpha(s)=2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(s)}{s^\alpha}&=\Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{s}{2}\right)^{2k}\\ &=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}e^{-its}dt; \end{aligned} \tag{2.2}$$

and J_α is the Bessel function of first kind and index α [6, 7, 12, 22].

Moreover,

- For all $(\mu,\lambda)\in\Gamma$, we have
$$\sup_{(r,x)\in\mathbb{R}^2}\big|\varphi_{\mu,\lambda}(r,x)\big|=1$$
where Γ is the set given by
$$\Gamma=\mathbb{R}^2\cup\Big\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}^2,\ |\mu|\leqslant|\lambda|\Big\}. \tag{2.3}$$

- For all $(\mu,\lambda)\in\mathbb{C}^2$; the function $\varphi_{\mu,\lambda}$ is the unique solution of the system
$$\begin{cases}\Delta_1u(r,x)=-i\lambda u(r,x),\\ \Delta_2u(r,x)=-\mu^2u(r,x),\\ u(0,0)=1,\dfrac{\partial u}{\partial r}(0,x)=0;\ \forall x\in\mathbb{R};\end{cases}$$
where
$$\Delta_1=\frac{\partial}{\partial x},$$
$$\Delta_2=\frac{\partial^2}{\partial r^2}+\frac{2\alpha+1}{r}\frac{\partial}{\partial r}-\frac{\partial^2}{\partial x^2};\quad (r,x)\in]0,+\infty[\times\mathbb{R},\ \alpha\geqslant 0.$$

In the following, we shall define the Fourier transform associated with the Riemann-Liouville operator and we give some properties that we need in the next section.

We denote by

- $d\nu_\alpha(r,x)$ the measure defined on $[0,+\infty[\times\mathbb{R}$, by

$$d\nu_\alpha(r,x)=\frac{1}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\, r^{2\alpha+1}dr\otimes dx.$$

- $L^p(d\nu_\alpha)$, $p\in[1,+\infty]$, the space of measurable functions f on $[0,+\infty[\times\mathbb{R}$, satisfying

$$\|f\|_{p,\nu_\alpha}=\begin{cases}\left(\int_0^{+\infty}\int_{\mathbb{R}}|f(r,x)|^p d\nu_\alpha(r,x)\right)^{\frac{1}{p}}<+\infty, & 1\leqslant p<+\infty;\\[2ex] \operatorname*{ess\ sup}_{(r,x)\in[0,+\infty[\times\mathbb{R}}|f(r,x)|<+\infty, & p=+\infty.\end{cases}$$

- Γ_+ the subset of Γ given by

$$\Gamma_+=[0,+\infty[\times\mathbb{R}\cup\big\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}^2,\ 0\leqslant\mu\leqslant|\lambda|\big\}.$$

- $\mathscr{B}_{\Gamma_+}$ the $\sigma-$algebra on Γ_+;

$$\mathscr{B}_{\Gamma_+}=\theta^{-1}\big(\mathscr{B}_{[0,+\infty[\times\mathbb{R}}\big),$$

where θ is the bijective function defined on Γ_+ by

$$\theta(\mu,\lambda)=\big(\sqrt{\mu^2+\lambda^2},\lambda\big). \tag{2.4}$$

- $d\gamma_\alpha(\mu,\lambda)$ the measure defined on Γ_+ by

$$\gamma_\alpha(A)=\nu_\alpha\big(\theta(A)\big);\ A\in\mathscr{B}_{\Gamma_+}$$

- $L^p(d\gamma_\alpha)$, $p\in[1,+\infty]$, the space of measurable functions f on Γ_+, satisfying

$$\|f\|_{p,\gamma_\alpha}<+\infty.$$

- $dm_n(x)$ the measure defined on $\mathbb{R}^n$, by

$$dm_n(x)=\frac{1}{(2\pi)^{\frac{n}{2}}}\ dx.$$

- $L^p(dm_n)$, $p\in[1,+\infty]$, the space of measurable functions f on $\mathbb{R}^n$, satisfying

$$\|f\|_{p,m_n}<+\infty.$$

Proposition 2.1. *i. For all non negative measurable function f on Γ_+ (respectively integrable on Γ_+ with respect to the measure $d\gamma_\alpha$), we have*

$$\int\int_{\Gamma_+} f(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) =$$
$$\frac{\int_{\mathbb{R}}\int_0^{+\infty} f(\mu,\lambda)\big(\mu^2+\lambda^2\big)^\alpha \mu d\mu d\lambda + \int_{\mathbb{R}}\int_0^{|\lambda|} f(i\mu,\lambda)\big(\lambda^2-\mu^2\big)^\alpha \mu d\mu d\lambda}{\sqrt{2\pi}\ 2^\alpha\Gamma(\alpha+1)}$$

ii. For all non negative measurable function g on $[0,+\infty[\times\mathbb{R}$ (respectively integrable on $[0,+\infty[\times\mathbb{R}$ with respect to the measure $d\nu_\alpha$), we have

$$\int_{\mathbb{R}}\int_0^{+\infty} g(r,x)d\nu_\alpha(r,x) = \int\int_{\Gamma_+} g\circ\theta(\mu,\lambda)d\gamma_\alpha(\mu,\lambda). \tag{2.5}$$

Definition 2.2. The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(d\nu_\alpha)$ by

$$\forall(\mu,\lambda)\in\Gamma;\quad \mathscr{F}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x)\varphi_{\mu,\lambda}(r,x)\ d\nu_\alpha(r,x),$$

where Γ is the set defined by the relation (2.3) and $\varphi_{\mu,\lambda}$ is the eigenfunction given by (2.1).

We have the following properties

- For every $f\in L^1(d\nu_\alpha)$ and $(\mu,\lambda)\in\Gamma$, we have

$$\mathscr{F}_\alpha(f)(\mu,\lambda) = \big(B\circ\widetilde{\mathscr{F}_\alpha}\big)(f)(\mu,\lambda) \tag{2.6}$$

where,

$$\forall(\mu,\lambda)\in\mathbb{R}^2;\ \widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda) = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x)\ j_\alpha(r\mu)\ e^{-i\lambda x}d\nu_\alpha(r,x),$$

and

$$\forall(\mu,\lambda)\in\Gamma,\quad B(f)(\mu,\lambda) = f\big(\sqrt{\mu^2+\lambda^2},\lambda\big) = f\circ\theta(\mu,\lambda). \tag{2.7}$$

- For $f\in L^1(d\nu_\alpha)$, the function $\mathscr{F}_\alpha(f)$ is continuous on Γ and

$$\lim_{\substack{\mu^2+2\lambda^2\longrightarrow+\infty\\(\mu,\lambda)\in\Gamma}} \mathscr{F}_\alpha(f)(\mu,\lambda) = 0.$$

- For $f \in L^1(d\nu_\alpha)$ such that $\mathscr{F}_\alpha(f) \in L^1(d\gamma_\alpha)$, we have the inversion formula for $\mathscr{F}_\alpha$; for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}$,

$$f(r,x) = \int\int_{\Gamma_+} \mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda).$$

- For all $p \in [1,+\infty]$ and $f \in L^p(d\nu_\alpha)$,

$$B(f) \in L^p(d\gamma_\alpha) \quad \text{and} \quad \|B(f)\|_{p,\gamma_\alpha} = \|f\|_{p,\nu_\alpha}. \tag{2.8}$$

In particular, from the relations (2.5), (2.7) and the fact that the function θ defined by (2.4), is bijective from Γ_+ onto $[0,+\infty[\times\mathbb{R}$; we deduce that the mapping B is an isometric isomorphism from $L^2(d\nu_\alpha)$ onto $L^2(d\gamma_\alpha)$.

It's well known [19, 20], that the transform $\widetilde{\mathscr{F}_\alpha}$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself. Then, using the relations (2.5), (2.6) and (2.7), we have the following result

Theorem 2.3. *(Plancherel theorem) The transform $\mathscr{F}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto $L^2(d\gamma_\alpha)$. In particular, we have the Parseval's equality; for all $f,\ g \in L^2(d\nu_\alpha)$*

$$\int_{\mathbb{R}}\int_0^{+\infty} f(r,x)\overline{g(r,x)}d\nu_\alpha(r,x) = \int\int_{\Gamma_+} \mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\mathscr{F}_\alpha(g)(\mu,\lambda)}d\gamma_\alpha(\mu,\lambda).$$

3. Fourier transform of $L^2(d\nu_\alpha)$- rapidly decreasing functions.

This section consists to characterize, by the Fourier transform associated with the Riemann-Liouville operator, a space of functions having only some integral conditions at infinity. This permits in the coming section, to give an other description of the Schwartz's space on the set Γ.

We denote by [3, 13]

- $S(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives, and $S_*(\mathbb{R}^2)$ its subset consisting of even functions with respect to the first variable.

- $S_*(\Gamma)$ the space of infinitely differentiable functions on Γ, even with respect to the first variable, rapidly decreasing together with

all their derivatives, which means
$\forall (k_1, k_2) \in \mathbb{N}^2,\ \forall \alpha \in \mathbb{N}$,

$$\sup \big\{ \big(1 + |\mu|^2 + |\lambda|^2\big)^{\alpha} |(\frac{\partial}{\partial \mu})^{k_1} (\frac{\partial}{\partial \lambda})^{k_2} f(\mu, \lambda)|;\ (\mu, \lambda) \in \Gamma \big\} < +\infty,$$

where

$$\frac{\partial f}{\partial \mu}(\mu, \lambda) = \begin{cases} \dfrac{\partial}{\partial r}(f(r, \lambda)), & if\ \mu = r \in \mathbb{R}; \\ \dfrac{1}{i} \dfrac{\partial}{\partial t}(f(it, \lambda)), & if\ \mu = it, |t| \leqslant |\lambda|. \end{cases}$$

To prove the main result of this section, we need the following lemma.

Lemma 3.1. *Let $a_0,\ a_1,\ b_0,\ b_1$ be real numbers such that $a_i < b_i$; $i \in \{0, 1\}$; and let*

$$\psi :\ \mathbb{R}^2 \times [a_0,\ b_0] \times [a_1,\ b_1] \longrightarrow \mathbb{C}$$

be a bounded function such that

i. *For all $(\mu,\ \lambda)\ \in \mathbb{R}^2$; the function*

$$(r, x) \longmapsto \psi\big((\mu, \lambda); (r, x)\big)$$

belongs to $L^1\big([a_0, b_0] \times [a_1, b_1];\ dm_2(r, x)\big)$.

ii.

$$\lim_{\mu^2+\lambda^2 \longrightarrow +\infty} \int_{\alpha_0}^{\beta_0} \int_{\alpha_1}^{\beta_1} \psi\big((\mu, \lambda); (r, x)\big)\ dm_2(r, x)\ = 0$$

uniformly with respect to $\alpha_i,\ \beta_i;\ 0 \leqslant i \leqslant 1$ and $a_i \leqslant \alpha_i \leqslant \beta_i \leqslant b_i$.

Then, for all $f \in L^1\big([a_0, b_0] \times [a_1, b_1];\ dm_2(r, x)\big)$;

$$\lim_{\mu^2+\lambda^2 \longrightarrow +\infty} \int_{a_0}^{b_0} \int_{a_1}^{b_1} \psi\big((\mu, \lambda); (r, x)\big) f(r, x)\ dm_2(r, x)\ = 0.$$

Proof. • Suppose firstly that $f \in S(\mathbb{R}^2)$. By integration by parts; we have

$$\int_{a_0}^{b_0}\int_{a_1}^{b_1} f(r,x)\psi((\mu,\lambda);(r,x))\ dm_2(r,x)$$
$$= f(b_0,\ b_1)\int_{a_0}^{b_0}\int_{a_1}^{b_1}\psi((\mu,\lambda);(r,x))\ dm_2(r,x)$$
$$-\int_{a_1}^{b_1}\frac{\partial f}{\partial x}(b_0,x)\Big[\int_{a_1}^{x}\int_{a_0}^{b_0}\psi((\mu,\lambda);(t,y))\ dm_2(t,y)\Big]dx$$
$$-\ \int_{a_0}^{b_0}\frac{\partial f}{\partial r}(r,b_1)\Big[\int_{a_1}^{b_1}\int_{a_0}^{r}\psi((\mu,\lambda);(t,y))\ dm_2(t,y)\Big]dr$$
$$+\int_{a_0}^{b_0}\int_{a_1}^{b_1}\frac{\partial^2 f}{\partial r\partial x}(r,x)\Big[\int_{a_0}^{r}\int_{a_1}^{x}\psi((\mu,\lambda);(t,y))\ dm_2(t,y)\Big]dr\ dx.$$

Then, the result follows from the hypothesis *ii*) and the fact that f and all its derivatives are bounded on $\mathbb{R}^2$.

• If f is any function in $L^1\big([a_0,b_0]\times[a_1,b_1];\ dm_2(r,x)\big)$; then for all $\varepsilon>0$, there exists $g\in S(\mathbb{R}^2)$ such that

$$\int_{a_0}^{b_0}\int_{a_1}^{b_1}|f(r,x)-g(r,x)|\ dm_2(r,x)\leqslant\ \frac{\varepsilon}{2}\frac{1}{\|\psi\|_\infty}.$$

Consequently;

$$\Big|\int_{a_0}^{b_0}\int_{a_1}^{b_1} f(r,x)\psi((\mu,\lambda);(r,x))\ dm_2(r,x)\Big|$$
$$\leqslant\ \frac{\varepsilon}{2}+\Big|\int_{a_0}^{b_0}\int_{a_1}^{b_1} g(r,x)\psi((\mu,\lambda);(r,x))\ dm_2(r,x)\Big|$$

and the required result follows from the first case. □

Example 3.2. Let a be a positive real number and let

$$\psi:\ \mathbb{R}^2\times[0,a]\times[-a,a]\longrightarrow\mathbb{C}$$

defined by

$$\psi((\mu,\lambda);(r,x)) = (r\mu)^{\alpha+\frac{1}{2}}j_\alpha(r\mu)\ e^{-i\lambda x}\mathbf{1}_{[0,+\infty[}(\mu).$$

From the asymptotic expansion of the function j_α [12, 22]; it follows that the functions

$$r\longmapsto r^{\alpha+\frac{1}{2}}j_\alpha(r)$$

and

$$g(r) = \int_0^r s^{\alpha+\frac{1}{2}} j_\alpha(s) ds$$

are bounded on $[0, +\infty[$. On the other hand, for all $(\mu, \lambda) \in \mathbb{R}^2$;

- $$\begin{aligned} \int_0^a \int_{-a}^a \Big|\psi((\mu,\lambda);(r,x))\Big| dm_2(r,x) &\leqslant \frac{a}{\pi}\int_0^a |(r\mu)^{\alpha+\frac{1}{2}} j_\alpha(r\mu)| dr \\ &\leqslant \frac{a^2}{\pi}\left\|s^{\alpha+\frac{1}{2}} j_\alpha\right\|_\infty . \end{aligned}$$

- For all $[\alpha_0, \beta_0] \subset [0, a]$ and $[\alpha_1, \beta_1] \subset [-a, a]$;

$$\begin{aligned} &\int_{\alpha_0}^{\beta_0}\int_{\alpha_1}^{\beta_1} \psi((\mu,\lambda);(r,x))\ dm_2(r,x) \\ &\qquad = \frac{1}{2\pi}\frac{e^{-i\alpha_1\lambda} - e^{-i\beta_1\lambda}}{i\lambda} \times \frac{g(\beta_0\mu) - g(\alpha_0\mu)}{\mu}. \end{aligned}$$

  Thus,

$$\lim_{\mu^2+\lambda^2 \longrightarrow +\infty} \int_{\alpha_0}^{\beta_0}\int_{\alpha_1}^{\beta_1} \psi((\mu,\lambda);\ (r,x))\ dm_2(r,x) = 0$$

  uniformly for $[\alpha_0, \beta_0] \subset [0, a]$ and $[\alpha_1, \beta_1] \subset [-a, a]$.

Consequently; from lemma 3.1, we deduce that
$\forall\ f \in L^1([0, +\infty[\times\mathbb{R}, dm_2(r,x))$;

$$\lim_{\mu^2+\lambda^2 \longrightarrow +\infty} \int_0^a \int_{-a}^a f(r,x)(r\mu)^{\alpha+\frac{1}{2}} j_\alpha(r\mu) e^{-i\lambda x}\ dm_2(r,x) = 0.$$

In the following, to give a nice description of rapidly decreasing functions; we need the following notations

- $\dfrac{\partial}{\partial\mu^2} = \dfrac{1}{\mu}\dfrac{\partial}{\partial\mu}$
- $C = \dfrac{\partial}{\partial\lambda} - \lambda\dfrac{\partial}{\partial\mu^2}$
- $l_\alpha = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r}$

- $L_\alpha = l_\alpha + \dfrac{\partial^2}{\partial x^2}$

- $K_\alpha = (\mu^2+\lambda^2)\big(\dfrac{\partial}{\partial \mu^2}\big)^2 + (2\alpha+2)\dfrac{\partial}{\partial \mu^2}$

- $A_\alpha = K_\alpha + \big(\dfrac{\partial}{\partial \lambda} - \lambda\dfrac{\partial}{\partial \mu^2}\big)^2 = K_\alpha + C^2.$

Then, for all $f \in S_*(\mathbb{R}^2)$; we have the following properties

- $$B\Big(\frac{\partial}{\partial \mu^2} f\Big) = \frac{\partial}{\partial \mu^2} B(f). \tag{3.1}$$

- For all $(k_1,\ k_2) \in \mathbb{N}^2$;
$$B\Big(l_\alpha^{k_1}\big(\frac{\partial}{\partial \lambda}\big)^{k_2} f\Big) = K_\alpha^{k_1} C^{k_2} B(f). \tag{3.2}$$

- For all $k \in \mathbb{N}$;
$$B(L_\alpha^k f) = A_\alpha^k B(f). \tag{3.3}$$

Where B is the mapping given by the relation (2.7).

Now, we are able to prove the main result of this section.

Theorem 3.3. *Let $f \in L^2(d\nu_\alpha)$. Then, the following assumptions are equivalent*
1. For all $(k_1,\ k_2) \in \mathbb{N}^2$; the function
$$(r,x) \longmapsto r^{k_1} x^{k_2} f(r,x)$$
belongs to the space $L^2(d\nu_\alpha)$.
2. The Fourier transform $\mathscr{F}_\alpha(f)$ of f satisfies the following properties

i. *The function $\mathscr{F}_\alpha(f)$ is infinitely differentiable on Γ, even with respect to the first variable.*

ii. *For all $(k_1, k_2) \in \mathbb{N}^2$ the function $K_\alpha^{k_1} C^{k_2} \mathscr{F}_\alpha(f) \in L^2(d\gamma_\alpha)$.*

iii. *For all $(k_1,\ k_2) \in \mathbb{N}^2$;*
$$\lim_{\substack{\mu^2+2\lambda^2 \longrightarrow +\infty \\ (\mu,\lambda)\in\Gamma}} \Big(1 + (\mu^2+\lambda^2)^{\frac{2\alpha+1}{4}}\Big) K_\alpha^{k_1} C^{k_2} \mathscr{F}_\alpha(f)(\mu,\lambda) = 0.$$

iv. *For all* $(k_1,\ k_2)\in\mathbb{N}^2$*;*

$$\lim_{\substack{\mu^2+2\lambda^2\to+\infty\\(\mu,\lambda)\in\Gamma}}(\mu^2+\lambda^2)^{\frac{2\alpha+3}{4}}\frac{\partial}{\partial\mu^2}K_\alpha^{k_1}C^{k_2}\mathscr{F}_\alpha(f)(\mu,\lambda)=0.$$

Proof. • Suppose that for all $(k_1,\ k_2)\in\mathbb{N}^2$; the function

$$(r,x)\longmapsto r^{k_1}x^{k_2}f(r,x)$$

belongs to the space $L^2(d\nu_\alpha)$. Then, for all $(l_1,\ l_2)\in\mathbb{N}^2$; the function

$$(r,x)\longmapsto r^{l_1}x^{l_2}f(r,x)$$

belongs to $L^1(d\nu_\alpha)$.

i. From the relation (2.2), we deduce that for all $k\in\mathbb{N}$ and $s\in\mathbb{R}$;

$$\big|j_\alpha^{(k)}(s)\big|\leqslant 1; \tag{3.4}$$

then, by derivative's theorem, it follows that the function

$$\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)=\int_0^\infty\int_{\mathbb{R}}f(r,x)j_\alpha(r\mu)e^{-i\lambda x}d\nu_\alpha(r,x)$$

is infinitely differentiable on $\mathbb{R}^2$, even with respect to the first variable. Hence, from the relation (2.6), the function $\mathscr{F}_\alpha(f)$ is infinitely differentiable on Γ, even with respect to the first variable.

ii. For all $(k_1,\ k_2)\in\mathbb{N}^2$ and using the relations (2.6) and (3.2), we get

$$\begin{aligned}K_\alpha^{k_1}C^{k_2}\mathscr{F}_\alpha(f) &= K_\alpha^{k_1}C^{k_2}\big(B(\widetilde{\mathscr{F}_\alpha}(f))\big)\\ &= B\Big(l_\alpha^{k_1}(\frac{\partial}{\partial\lambda})^{k_2}\widetilde{\mathscr{F}_\alpha}(f)\Big)\\ &= B\Big(\widetilde{\mathscr{F}_\alpha}\big((-r^2)^{k_1}(-ix)^{k_2}f\big)\Big)\\ &= \mathscr{F}_\alpha\big((-r^2)^{k_1}(-ix)^{k_2}f\big).\end{aligned}$$

Since, the function

$$(r,x)\longmapsto r^{2k_1}x^{k_2}f(r,x)$$

belongs to the space $L^2(d\nu_\alpha)$; by Plancherel theorem's; the function

$$K_\alpha^{k_1}C^{k_2}\mathscr{F}_\alpha(f)=\mathscr{F}_\alpha\big((-r^2)^{k_1}(-ix)^{k_2}f\big)$$

belongs to $L^2(d\gamma_\alpha)$.

iii. For all $f\in L^1(d\nu_\alpha)$; the function $\widetilde{\mathscr{F}_\alpha}(f)$ belongs to the space $\mathscr{C}_{*,0}(\mathbb{R}^2)$

(the space of continuous functions g on $\mathbb{R}^2$; even with respect to the first variable and such that $\lim\limits_{\mu^2+\lambda^2\longrightarrow+\infty} g(\mu,\lambda)=0$). Then,

$$\begin{aligned}
&\lim_{\substack{\mu^2+2\lambda^2\longrightarrow+\infty\\(\mu,\lambda)\in\Gamma}} K_\alpha^{k_1}C^{k_2}\mathscr{F}_\alpha(f)(\mu,\lambda)=\\
&\lim_{\substack{\mu^2+2\lambda^2\longrightarrow+\infty\\(\mu,\lambda)\in\Gamma}} \widetilde{\mathscr{F}_\alpha}\big((-r^2)^{k_1}(-ix)^{k_2}f\big)\big(\sqrt{\mu^2+\lambda^2},\lambda\big)=0.
\end{aligned}\tag{3.5}$$

On the other hand; for all $(\mu,\lambda)\in[0,+\infty[\times\mathbb{R}$, we have

$$\begin{aligned}
&\mu^{\frac{2\alpha+1}{2}}\widetilde{\mathscr{F}_\alpha}\Big((-r^2)^{k_1}(-ix)^{k_2}f\Big)(\mu,\lambda)\\
=&\frac{\int_0^a\int_{-a}^a(-r^2)^{k_1}(-ix)^{k_2}f(r,x)\mu^{\alpha+\frac12}j_\alpha(r\mu)e^{-i\lambda x}r^{2\alpha+1}drdx}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\\
+&\frac{\int\int_{[0,+\infty[\times\mathbb{R}\backslash I_a}(-r^2)^{k_1}(-ix)^{k_2}f(r,x)\mu^{\alpha+\frac12}j_\alpha(r\mu)e^{-i\lambda x}r^{2\alpha+1}drdx}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}
\end{aligned}$$

where $a>0$ and $I_a=[0,a]\times[-a,a]$. Let

$$C_\alpha=\sup_{s\geqslant0}\big|s^{\alpha+\frac12}j_\alpha(s)\big|,$$

and $l\in\mathbb{N}$ such that

$$\int_{\mathbb{R}}\int_0^{+\infty}\frac{dr\ dx}{(1+r^2+x^2)^{2l}}<+\infty,$$

we have;

$$\left|\frac{\int\int_{[0,+\infty[\times\mathbb{R}\setminus I_a}(-r^2)^{k_1}(-ix)^{k_2}f(r,x)\mu^{\alpha+\frac{1}{2}}j_\alpha(r\mu)e^{-i\lambda x}r^{2\alpha+1}drdx}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\right|$$
$$\leqslant \frac{C_\alpha}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\int\int_{[0,+\infty[\times\mathbb{R}\setminus I_a} r^{2k_1}|x|^{k_2}|f(r,x)|r^{\alpha+\frac{1}{2}}dr\ dx$$
$$\leqslant \frac{C_\alpha}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\Big(\int_0^{+\infty}\int_{\mathbb{R}}\frac{drdx}{(1+r^2+x^2)^{2l}}\Big)^{\frac{1}{2}}\times$$
$$\Big(\int\int_{[0,+\infty[\times\mathbb{R}\setminus I_a}(1+r^2+x^2)^{2l}|r|^{4k_1}|x|^{2k_2}|f(r,x)|^2r^{2\alpha+1}drdx\Big)^{\frac{1}{2}}$$
$$= \frac{C_\alpha}{(2^\alpha\Gamma(\alpha+1)\sqrt{2\pi})^{\frac{1}{2}}}\Big(\int_0^{+\infty}\int_{\mathbb{R}}\frac{drdx}{(1+r^2+x^2)^{2l}}\Big)^{\frac{1}{2}}\times$$
$$\Big(\int\int_{[0,+\infty[\times\mathbb{R}\setminus I_a}(1+r^2+x^2)^{2l}|r|^{4k_1}|x|^{2k_2}|f(r,x)|^2d\nu_\alpha(r,x)\Big)^{\frac{1}{2}}. \tag{3.6}$$

Let $\varepsilon>0$. Since,

$$\int_0^{+\infty}\int_{\mathbb{R}}(1+r^2+x^2)^{2l}r^{4k_1}|x|^{2k_2}|f(r,x)|^2d\nu_\alpha(r,x)<+\infty;$$

by (3.6); there exists $a>1$ such that

$$\Big|\int\int_{[0,+\infty[\times\mathbb{R}\setminus I_a}(-r^2)^{k_1}(-ix)^{k_2}f(r,x)\mu^{\alpha+\frac{1}{2}}j_\alpha(r\mu)e^{-i\lambda x}d\nu_\alpha(r,x)\Big|\leqslant\frac{\varepsilon}{2}$$

Let ψ be the function defined in example 3.2 by

$$\psi\big((\mu,\lambda),(r,x)\big)=\mu^{\alpha+\frac{1}{2}}\ j_\alpha(r\mu)\ e^{-i\lambda x}\ r^{\alpha+\frac{1}{2}}\mathbf{1}_{[0,+\infty[}(\mu)$$

and

$$g(r,x)=(-1)^{k_1}r^{2k_1+\alpha+\frac{1}{2}}(-ix)^{k_2}f(r,x).$$

By Hölder's inequality, we have

$$\int_0^{+\infty}\int_{\mathbb{R}}|g(r,x)|dm_2(r,x)=\int_0^{+\infty}\int_{\mathbb{R}}r^{2k_1}|x|^{k_2}|f(r,x)|r^{\alpha+\frac{1}{2}}dm_2(r,x)$$
$$\leqslant\Big(\frac{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}{2\pi}\Big)^{\frac{1}{2}}\Big(\int_0^{+\infty}\int_{\mathbb{R}}\frac{dm_2(r,x)}{(1+r^2+x^2)^{2l}}\Big)^{\frac{1}{2}}$$
$$\times\Big(\int_0^{+\infty}\int_{\mathbb{R}}(1+r^2+x^2)^{2l}r^{4k_1}x^{2k_2}|f(r,x)|^2\ d\nu_\alpha(r,x)\Big)^{\frac{1}{2}}<+\infty.$$

Applying the result of example 3.2; we deduce that
$$\int_0^a\int_{-a}^a \mu^{\alpha+\frac{1}{2}}(-1)^{k_1}r^{2k_1+2\alpha+1}(-ix)^{k_2}f(r,x)j_\alpha(r\mu)e^{-i\lambda x}drdx \overset{\mu^2+\lambda^2\to+\infty}{\longrightarrow} 0.$$
This shows that
$$\lim_{\mu^2+\lambda^2\to+\infty}\mu^{\alpha+\frac{1}{2}}\widetilde{\mathscr{F}}_\alpha\big((-r^2)^{k_1}(-ix)^{k_2}f\big)(\mu,\lambda)=0$$
and consequently;
$$\lim_{\substack{\mu^2+2\lambda^2\longrightarrow+\infty\\(\mu,\lambda)\in\Gamma}}\big(\mu^2+\lambda^2\big)^{\frac{2\alpha+1}{4}}\widetilde{\mathscr{F}}_\alpha\big((-r^2)^{k_1}(-ix)^{k_2}f\big)\big(\sqrt{\mu^2+\lambda^2},\lambda\big)=0. \tag{3.7}$$
Combining the relations (2.6), (3.2), (3.5) and (3.7), we get
$$\lim_{\substack{\mu^2+2\lambda^2\longrightarrow+\infty\\(\mu,\lambda)\in\Gamma}}\Big(1+\big(\mu^2+\lambda^2\big)^{\frac{2\alpha+1}{4}}\Big)K_\alpha^{k_1}\ C^{k_2}\ \mathscr{F}_\alpha(f)(\mu,\lambda)=0.$$
iv. From the relation
$$\frac{\partial}{\partial\mu}\big(j_\alpha(r\mu)\big)=-\frac{r^2\mu}{2(\alpha+1)}\ j_{\alpha+1}(r\mu), \tag{3.8}$$
and from the derivative's theorem, We have
$$\mu^{\frac{2\alpha+3}{2}}\frac{\partial}{\partial\mu^2}\widetilde{\mathscr{F}}_\alpha\big((-r^2)^{k_1}(-ix)^{k_2}f\big)(\mu,\lambda)=\frac{1}{2(\alpha+1)}$$
$$\int_0^{+\infty}\int_{\mathbb{R}}(-r^2)^{k_1+1}(-ix)^{k_2}f(r,x)\mu^{\alpha+\frac{3}{2}}j_{\alpha+1}(r\mu)e^{-i\lambda x}d\nu_\alpha(r,x).$$
Using the same argument as in iii) and the example 3.2, with
$$\widetilde{\psi}\big((\mu,\lambda),(r,x)\big)=(r\mu)^{\alpha+\frac{3}{2}}j_{\alpha+1}(r\mu)e^{-i\lambda x}\mathbf{1}_{[0,+\infty[}(\mu),$$
and
$$\widetilde{g}(r,x)=(-1)^{k_1+1}r^{2k_1+\alpha+\frac{1}{2}}(-ix)^{k_2}f(r,x)$$
we deduce that
$$\lim_{\mu^2+\lambda^2\longrightarrow+\infty}\mu^{\frac{2\alpha+3}{2}}\frac{\partial}{\partial\mu^2}\widetilde{\mathscr{F}}_\alpha\big((-r^2)^{k_1}(-ix)^{k_2}f\big)(\mu,\lambda)=0,$$
and therefore
$$\lim_{\substack{\mu^2+2\lambda^2\longrightarrow+\infty\\(\mu,\lambda)\in\ \Gamma}}B\Big(\mu^{\frac{2\alpha+3}{2}}\frac{\partial}{\partial\mu^2}\widetilde{\mathscr{F}}_\alpha\big((-r^2)^{k_1}(-ix)^{k_2}f\big)\Big)(\mu,\lambda)=0.$$

Wich means that

$$\lim_{\substack{\mu^2+2\lambda^2\longrightarrow+\infty\\(\mu,\lambda)\in\ \Gamma}} (\mu^2+\lambda^2)^{\frac{2\alpha+3}{4}} \frac{\partial}{\partial\mu^2}\ K_\alpha^{k_1}\ C^{k_2}\mathscr{F}_\alpha(f)(\mu,\lambda)=0.$$

• Conversely; suppose that $f\in L^2(d\nu_\alpha)$ and $\mathscr{F}_\alpha(f)$ satisfies the assertion 2) of theorem. In particular; for every $(k_1,k_2)\in\mathbb{N}^2$, the function $K_\alpha^{k_1}\ C^{k_2}\mathscr{F}_\alpha(f)$ belongs to $L^2(d\gamma_\alpha)$. In virtue of the relations (2.5) and (3.2), we deduce that for all $(k_1,k_2)\in\mathbb{N}^2$; the function $l_\alpha^{k_1}(\frac{\partial}{\partial\lambda})^{k_2}\widetilde{\mathscr{F}_\alpha}(f)$ belongs to $L^2(d\nu_\alpha)$.

Let's denote by Λ_n; $n\in\mathbb{N}^*$, the usual Fourier transform defined on $L^1(dm_n)$ by

$$\Lambda_n(f)(\lambda)=\int_{\mathbb{R}^n} f(x)\ e^{-i\langle\lambda/x\rangle}dm_n(x)$$

and F_α the Fourier Bessel transform defined on the space

$$L^1([0,+\infty[,\ \frac{1}{2^\alpha\Gamma(\alpha+1)}r^{2\alpha+1}\ dr)$$

by

$$F_\alpha(f)(\mu)=\frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^{+\infty} f(r)\ j_\alpha(r\mu)\ r^{2\alpha+1}dr.$$

Let $k\in\mathbb{N}$. Since

$$\int_0^{+\infty}\int_{\mathbb{R}}\Big|(\frac{\partial}{\partial\lambda})^k\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)\Big|^2 d\nu_\alpha(\mu,\lambda)<+\infty;$$

then, there exists a null set $N_1\subset[0,+\infty[$; such that for all $\mu\in\ N_1^c$;

$$\int_{\mathbb{R}}\Big|(\frac{\partial}{\partial\lambda})^k\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)\Big|^2 d\lambda<+\infty. \tag{3.9}$$

For $\mu\in\ N_1^c$; we put

$$f_{k,\mu}(t)=(\frac{\partial}{\partial t})^k\widetilde{\mathscr{F}_\alpha}(f)(\mu,t)$$

and

$$g_{k,\mu}^n(y)=\int_{-n}^n f_{k,\mu}(t)e^{ity}dm_1(t);\ n\in\mathbb{N}.$$

By (3.9); the function $f_{k,\mu}$ belongs to $L^2(dm_1)$ and

$$\lim_{n\longrightarrow+\infty} g_{k,\mu}^n=\Lambda_1^{-1}(f_{k,\mu})\quad\text{in } L^2(dm_1). \tag{3.10}$$

However; by integration by parts; we have

$$g_{k,\mu}^{n}(y)=\frac{1}{\sqrt{2\pi}}\big[e^{ity}f_{k-1,\mu}(t)\big]_{-n}^{n}-\int_{-n}^{n}f_{k-1,\mu}(t)iye^{ity}dm_1(t). \tag{3.11}$$

On the other hand, from the hypothesis $iii)$ and by writing

$$\begin{aligned}(1+\mu^{\frac{2\alpha+1}{2}})l_{\alpha}^{k_1}(\frac{\partial}{\partial\lambda})^{k_2}\widetilde{\mathscr{F}}_{\alpha}(f)(\mu,\lambda)&\\ =\left[1+(\lambda^2+(\mu^2-\lambda^2))^{\frac{2\alpha+1}{4}}\right]K_{\alpha}^{k_1}C^{k_2}\mathscr{F}_{\alpha}(f)(\sqrt{\mu^2-\lambda^2},\lambda),&\end{aligned}$$

if $\mu\geqslant|\lambda|$ and

$$\begin{aligned}&(1+\mu^{\frac{2\alpha+1}{2}})l_{\alpha}^{k_1}(\frac{\partial}{\partial\lambda})^{k_2}\widetilde{\mathscr{F}}_{\alpha}(f)(\mu,\lambda)\\ &=\left[1+(\lambda^2+(i\sqrt{\lambda^2-\mu^2})^2)^{\frac{2\alpha+1}{4}}\right]K_{\alpha}^{k_1}C^{k_2}\mathscr{F}_{\alpha}(f)(i\sqrt{\lambda^2-\mu^2},\lambda),\end{aligned} \tag{3.12}$$

if $\mu<|\lambda|$. We deduce that for all $(k_1,k_2)\in\mathbb{N}^2$;

$$\lim_{\mu^2+\lambda^2\longrightarrow+\infty}(1+\mu^{\frac{2\alpha+1}{2}})l_{\alpha}^{k_1}(\frac{\partial}{\partial\lambda})^{k_2}\widetilde{\mathscr{F}}_{\alpha}(f)(\mu,\lambda)=0.$$

In particular; for all $\mu\in[0,+\infty[$;

$$\lim_{|\lambda|\longrightarrow+\infty}(\frac{\partial}{\partial\lambda})^{k-1}\widetilde{\mathscr{F}}_{\alpha}(f)(\mu,\lambda)=0.$$

Consequently; for all $\mu\in\ N_1^c$;

$$\lim_{n\longrightarrow+\infty}\left[e^{ity}f_{k-1,\mu}(t)\right]_{-n}^{n}=0. \tag{3.13}$$

Combining the relations (3.11) and (3.13), we get

$$\lim_{n\longrightarrow+\infty}g_{k,\mu}^{n}(y)=\lim_{n\longrightarrow+\infty}(-iy)\int_{-n}^{n}f_{k-1,\mu}(t)\ e^{ity}dm_1(t),$$

and by iteration, we deduce that

$$\lim_{n\longrightarrow+\infty}g_{k,\mu}^{n}(y)=(-iy)^k\lim_{n\longrightarrow+\infty}\int_{-n}^{n}f_{0,\mu}(t)\ e^{ity}dm_1(t).$$

Using the relation (3.10), we obtain

$$\Lambda_1^{-1}(f_{k,\mu})=(-iy)^k\ \Lambda_1^{-1}(f_{0,\mu}). \tag{3.14}$$

Since the usual Fourier transform Λ_1 is an isometric isomorphism from $L^2(dm_1)$ onto itself, the relation (3.14) involves that

$$\int_{\mathbb{R}} |f_{k,\mu}(\lambda)|^2 dm_1(\lambda) = \int_{\mathbb{R}} \lambda^{2k} |\Lambda_1^{-1}(f_{0,\mu})(\lambda)|^2 dm_1(\lambda)$$

or

$$\int_{\mathbb{R}} \Big|(\frac{\partial}{\partial\lambda})^k \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)\Big|^2 dm_1(\lambda) = \int_{\mathbb{R}} \lambda^{2k} \Big|F_\alpha(f(.,\lambda))(\mu)\Big|^2 dm_1(\lambda).$$

Integrating over $[0,+\infty[$ with respect to the measure $\dfrac{r^{2\alpha+1}\ dr}{2^\alpha\Gamma(\alpha+1)}$ and using the fact that the Fourier-Bessel transform F_α is an isometric isomorphism from $L^2([0,+\infty[, \dfrac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}dr)$ onto itself, we deduce that

$$\int_0^{+\infty}\int_{\mathbb{R}} \lambda^{2k}|f(\mu,\lambda)|^2 d\nu_\alpha(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} \Big|(\frac{\partial}{\partial\lambda})^k \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)\Big|^2 d\nu_\alpha(\mu,\lambda)$$
$$< +\infty$$

which shows that for all $k \in \mathbb{N}$;

$$\int_0^{+\infty}\int_{\mathbb{R}} |x^k\ f(r,x)|^2 d\nu_\alpha(r,x) < +\infty. \tag{3.15}$$

By the same way, and using the fact that for all $k \in \mathbb{N}$;

$$\int_0^{+\infty}\int_{\mathbb{R}} |l_\alpha^k\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|^2 d\nu_\alpha(\mu,\lambda) < +\infty,$$

we deduce that there exists a null set $N_2 \subset \mathbb{R}$ such that for all $\lambda \in\ N_2^c$;

$$\int_0^{+\infty} |l_\alpha^k\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|^2 \mu^{2\alpha+1} d\mu < +\infty.$$

Let

$$h_{k,\lambda}^n(r) = \frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^n\ l_\alpha^k\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)\ j_\alpha(r\mu)\mu^{2\alpha+1}d\mu$$

then;

$$\lim_{n\longrightarrow+\infty} h_{k,\lambda}^n(r) = F_\alpha(l_\alpha^k\ \widetilde{\mathscr{F}}_\alpha(f)(.,\lambda))(r) \tag{3.16}$$

in $L^2([0,+\infty[,\ \dfrac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}dr)$. Now; integrating by parts; we have

$$h^n_{k,\lambda}(r)=\frac{1}{2^\alpha\Gamma(\alpha+1)}\Big\{\Big[j_\alpha(r\mu)\mu^{2\alpha+1}\frac{\partial}{\partial\mu}(l_\alpha^{k-1}\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda))\Big]_0^n - \Big[\mu^{2\alpha+1}\frac{\partial}{\partial\mu}(j_\alpha(r\mu))l_\alpha^{k-1}\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)\Big]_0^n\Big\}-r^2\ h^n_{k-1,\lambda}(r). \tag{3.17}$$

On the other hand, from the hypothesis $iii)$ and by the relation (3.12), we deduce that for all $k\in\mathbb{N}$;

$$\lim_{\mu^2+\lambda^2\longrightarrow+\infty}(1+\mu^{\frac{2\alpha+1}{2}})l_\alpha^k\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)=0.$$

In particular, for all $\lambda\in\mathbb{R}$;

$$\lim_{\mu\longrightarrow+\infty}\mu^{\frac{2\alpha+1}{2}}\ l_\alpha^k\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)=0. \tag{3.18}$$

However, from the relation (3.8) we have,

$$\Big|\mu^{2\alpha+1}\frac{\partial}{\partial\mu}(j_\alpha(r\mu))l_\alpha^{k-1}\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)\Big| \leqslant \frac{C_{\alpha+1}}{2(\alpha+1)}r^{-\alpha+\frac12}\mu^{\alpha+\frac12}\big|l_\alpha^{k-1}\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)\big|$$

and by the relation (3.18), we deduce that for all $\lambda\in\mathbb{R}$

$$\lim_{\mu\longrightarrow+\infty}\mu^{2\alpha+1}\frac{\partial}{\partial\mu}(j_\alpha(r\mu))l_\alpha^{k-1}\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)=0.$$

By the same way, we have

$$\Big|j_\alpha(r\mu)\ \mu^{2\alpha+1}\frac{\partial}{\partial\mu}(l_\alpha^{k-1}\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda))\Big| \leqslant C_\alpha\ r^{-\alpha-\frac12}\Big|\mu^{\alpha+\frac32}\ \frac{\partial}{\partial\mu^2}(l_\alpha^{k-1}\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda))\Big|,$$

using the relation (3.1) and (3.2), we get

$$\Big|j_\alpha(r\mu)\ \mu^{2\alpha+1}\frac{\partial}{\partial\mu}\big(l_\alpha^{k-1}\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)\big)\Big| \leqslant C_\alpha\ r^{-\alpha-\frac{1}{2}}\times$$

$$\begin{cases} \big(\lambda^2+(\mu^2-\lambda^2)\big)^{\frac{2\alpha+3}{4}}\dfrac{\partial}{\partial\mu^2}\ K_\alpha^k\ \mathscr{F}_\alpha(f)(\sqrt{\mu^2-\lambda^2},\lambda), & \text{if } \mu\geqslant|\lambda|; \\[2ex] \big(\lambda^2+(i\sqrt{\lambda^2-\mu^2})^2\big)^{\frac{2\alpha+3}{4}}\dfrac{\partial}{\partial\mu^2}\ K_\alpha^k\ \mathscr{F}_\alpha(f)(i\sqrt{\lambda^2-\mu^2},\lambda), & \text{if } \mu<|\lambda|. \end{cases}$$

By the hypothesis $iv)$, it follows that for all $\lambda\in\mathbb{R}$;

$$\lim_{\mu\longrightarrow+\infty} j_\alpha(r\mu)\ \mu^{2\alpha+1}\frac{\partial}{\partial\mu}\big(l_\alpha^{k-1}\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)\big)=0. \tag{3.19}$$

Combining the relations (3.16), (3.17), (3.18) and (3.19), we deduce that for all $\lambda\in N_2^c$; the function

$$r\longmapsto(-r^2)\ F_\alpha\big(l_\alpha^{k-1}\widetilde{\mathscr{F}}_\alpha(f)(.,\lambda)\big)(r)$$

belongs to $L^2([0,+\infty[,\ \dfrac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}dr)$ and

$$F_\alpha\big(l_\alpha^{k}\ \widetilde{\mathscr{F}}_\alpha(f)(.,\lambda)\big)(r)=(-r^2)\ F_\alpha\big(l_\alpha^{k-1}\ \widetilde{\mathscr{F}}_\alpha(f)(.,\lambda)\big)(r).$$

By iteration, for all $\lambda\in N_2^c$, the function

$$r\longmapsto(-r^2)^k\ F_\alpha\big(\widetilde{\mathscr{F}}_\alpha(f)(.,\lambda)\big)(r)$$

belongs to $L^2([0,+\infty[,\ \dfrac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}dr)$ and we have

$$\begin{aligned} F_\alpha\big(l_\alpha^{k}\ \widetilde{\mathscr{F}}_\alpha(f)(.,\lambda)\big)(r) &= (-r^2)^k\ F_\alpha\big(\widetilde{\mathscr{F}}_\alpha(f)(.,\lambda)\big)(r) \\ &= (-r^2)^k\ \Lambda_1\big(f(r,.)\big)(\lambda). \end{aligned} \tag{3.20}$$

Integrating over $[0,+\infty[\times\mathbb{R}$, with respect to the measure $d\nu_\alpha(r,\lambda)$ and using the Fubini's theorem and Plancherel theorem's, respectively for F_α and Λ_1; the relation (3.20) leads to

$$\begin{aligned} &\int_0^{+\infty}\int_{\mathbb{R}}\big|r^{2k}f(r,x)\big|^2\ d\nu_\alpha(r,x) \\ &\qquad\qquad=\int_0^{+\infty}\int_{\mathbb{R}}\big|l_\alpha^k\widetilde{\mathscr{F}}_\alpha(f)(r,\lambda)\big|^2\ d\nu_\alpha(r,\lambda)<+\infty. \end{aligned}$$

This shows that for all $k \in \mathbb{N}$;

$$\int_0^{+\infty}\int_{\mathbb{R}} \left|r^k\ f(r,x)\right|^2 d\nu_\alpha(r,x) < +\infty. \tag{3.21}$$

Thus, by the relations (3.15), (3.21) and the Cauchy-Shwartz inequality, we deduce that for all $(k_1,k_2) \in \mathbb{N}^2$, the function

$$(r,x) \longmapsto r^{k_1} x^{k_2} f(r,x)$$

belongs to $L^2(d\nu_\alpha)$. This completes the proof of theorem 3.3. □

4. Best charcterizations of the spaces $S_*(\mathbb{R}^2)$and $S_*(\Gamma)$.

In this section, using the theorem 3.3, we give new characterizations of the Schwartz's spaces $S_*(\mathbb{R}^2)$ and $S_*(\Gamma)$. For this, we need the following important result

Proposition 4.1. *Let f be a continuous function on $\mathbb{R}^2$, even with respect to the first variable. Then, the following assumptions are equivalent.*

i. *For all $(k_1,k_2) \in \mathbb{N}^2$; the functions*

$$(r,x) \longmapsto r^{k_1}\ x^{k_2}\ f(r,x) \ \textit{and} \quad (\mu,\lambda) \longmapsto \mu^{k_1}\ \lambda^{k_2}\ \widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)$$

are bounded on $[0,+\infty[\times\mathbb{R}$.

ii. *For all $(k_1,k_2) \in \mathbb{N}^2$; the functions*

$$(r,x) \longmapsto r^{k_1}\ x^{k_2}\ f(r,x) \ \textit{and} \quad (\mu,\lambda) \longmapsto \mu^{k_1}\ \lambda^{k_2}\ \widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)$$

belong to $L^2(d\nu_\alpha)$.

Proof. • It's clear that, if for all $(k_1,k_2) \in \mathbb{N}^2$; the functions

$$(r,x) \longmapsto r^{k_1}\ x^{k_2}\ f(r,x) \text{ and } \quad (\mu,\lambda) \longmapsto \mu^{k_1}\ \lambda^{k_2}\ \widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)$$

are bounded on $[0,+\infty[\times\mathbb{R}$, then for all $(l_1,l_2) \in \mathbb{N}^2$; the functions

$$(r,x) \longmapsto r^{l_1}\ x^{l_2}\ f(r,x) \text{ and } \quad (\mu,\lambda) \longmapsto \mu^{l_1}\ \lambda^{l_2}\ \widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)$$

belong to $L^2(d\nu_\alpha)$.

• Conversely, suppose that for all $(k_1,k_2) \in \mathbb{N}^2$; the functions

$$(r,x) \longmapsto r^{k_1}\ x^{k_2}\ f(r,x) \text{ and } \quad (\mu,\lambda) \longmapsto \mu^{k_1}\ \lambda^{k_2}\ \widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)$$

belong to $L^2(d\nu_\alpha)$. Then by Hölder's inequality, we deduce that for all $(l_1,l_2)\in\mathbb{N}^2$; the functions

$$(r,x)\longmapsto r^{l_1}\ x^{l_2}\ f(r,x) \text{ and } \quad (\mu,\lambda)\longmapsto \mu^{l_1}\ \lambda^{l_2}\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)$$

belong to $L^1(d\nu_\alpha)$, and by derivative's theorem, the relation (3.4) and the inversion formula for the transform $\widetilde{\mathscr{F}}_\alpha$, that is

$$f(r,x)=\int_0^{+\infty}\int_{\mathbb{R}}\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)\ j_\alpha(r\mu)\ e^{i\lambda x}\ d\nu_\alpha(\mu,\lambda);$$

we deduce that the functions f and $\widetilde{\mathscr{F}}_\alpha(f)$ are infinitely differentiable on $\mathbb{R}^2$, even with respect to the first variable. Moreover, for all $(k_1,k_2)\in\mathbb{N}^2$;

$$\lim_{r^2+x^2\longrightarrow+\infty}\Big(\frac{\partial}{\partial r}\Big)^{k_1}\Big(\frac{\partial}{\partial x}\Big)^{k_2}f(r,x)=0 \tag{4.1}$$

and

$$\lim_{\mu^2+\lambda^2\longrightarrow+\infty}\Big(\frac{\partial}{\partial \mu}\Big)^{k_1}\Big(\frac{\partial}{\partial \lambda}\Big)^{k_2}\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)=0 \tag{4.2}$$

1. For all $(k_1,k_2)\in\mathbb{N}^2$; such that $k_1\geqslant 2\alpha+1$; the function

$$(r,x)\longmapsto r^{k_1}\ x^{k_2}\ f(r,x)$$

belongs to $L^1\big([0,+\infty[\times\mathbb{R},\ dm_2(r,x)\big)$. Indeed

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}}\big|r^{k_1}\ x^{k_2}\ f(r,x)\big|dm_2(r,x)\\
&\qquad=\int_0^{1}\int_{\mathbb{R}}\big|r^{k_1}\ x^{k_2}\ f(r,x)\big|dm_2(r,x)\\
&\qquad\quad+\int_1^{+\infty}\int_{\mathbb{R}}\big|r^{k_1}\ x^{k_2}\ f(r,x)\big|dm_2(r,x)\\
&\qquad\leqslant\frac{2^\alpha\Gamma(\alpha+1)}{\sqrt{2\pi}}\Big\{\int_0^{1}\int_{\mathbb{R}}\big|x^{k_2}\ f(r,x)\big|d\nu_\alpha(r,x)\\
&\qquad\qquad\qquad+\int_1^{+\infty}\int_{\mathbb{R}}\big|r^{k_1}x^{k_2}f(r,x)\big|d\nu_\alpha(r,x)\Big\}\\
&\qquad\leqslant\frac{2^\alpha\Gamma(\alpha+1)}{\sqrt{2\pi}}\Big\{\int_0^{+\infty}\int_{\mathbb{R}}\big|x^{k_2}\ f(r,x)\big|d\nu_\alpha(r,x)\\
&\qquad\qquad\qquad+\int_0^{+\infty}\int_{\mathbb{R}}\big|r^{k_1}x^{k_2}f(r,x)\big|d\nu_\alpha(r,x)\Big\}\\
&\qquad<+\infty.
\end{aligned}$$

2. For all $(k_1,k_2)\in\mathbb{N}^2$ and $a\in\mathbb{R};\ a>0$, the function

$$(r,x)\longmapsto r^{k_1+a}\ x^{k_2}\ f(r,x)$$

is bounded on $[0,+\infty[\times\mathbb{R}$.

In fact; let $m\in\mathbb{N};\ m\geqslant 3$ and $m\geqslant\dfrac{2(\alpha+1)}{a}$. By a simple calculus and using the fact that f and all its derivatives are bounded on $[0,+\infty[\times\mathbb{R}$; we deduce that for all $(k_1,k_2)\in\mathbb{N}^2$; there exists $C_{k_1,k_2,m,a}>0$ such that

$$\begin{aligned}&\Big|\frac{\partial}{\partial r}\frac{\partial}{\partial x}\Big[\big(r^{k_1+a}\ x^{k_2}\ f(r,x)\big)^m\Big]\Big|\\ &\leqslant C_{k_1,k_2,m,a}\times\Big\{\big|r^{m(k_1+a)-1}\ x^{mk_2-1}\ f(r,x)\big|+\big|r^{m(k_1+a)-1}\ x^{mk_2}\ f(r,x)\big|\\ &+\big|r^{m(k_1+a)}\ x^{mk_2-1}\ f(r,x)\big|+2\ \big|r^{m(k_1+a)}\ x^{mk_2}\ f(r,x)\big|\Big\},\end{aligned}$$

and by 1) of this proof, we deduce that the function

$$(r,x)\longmapsto\frac{\partial}{\partial r}\frac{\partial}{\partial x}\Big[\big(r^{k_1+a}\ x^{k_2}\ f(r,x)\big)^m\Big]$$

is integrable on $[0,+\infty[\times\mathbb{R}$ with respect to the measure $dm_2(r,x)$ and by (4.1), we have

$$\begin{aligned}&\big(r^{k_1+a}\ x^{k_2}\ f(r,x)\big)^m\\ &=\begin{cases}\displaystyle\int_0^r\int_0^x\frac{\partial}{\partial t}\frac{\partial}{\partial y}\Big[\big(t^{k_1+a}\ y^{k_2}\ f(t,y)\big)^m\Big]dt\ dy, & \text{if } k_2\geqslant 1;\\ \displaystyle\int_0^r\int_{-\infty}^x\frac{\partial}{\partial t}\frac{\partial}{\partial y}\Big[\big(t^{k_1+a}\ f(t,y)\big)^m\Big]dt\ dy, & \text{if } k_2=0.\end{cases}\end{aligned}$$

This shows that the function

$$(r,x)\longmapsto r^{k_1+a}\ x^{k_2}\ f(r,x)$$

is bounded on $[0,+\infty[\times\mathbb{R}$ and for all $(r,x)\in[0,+\infty[\times\mathbb{R}$;

$$\big|r^{k_1+a}\ x^{k_2}\ f(r,x)\big|\leqslant\Big(2\pi\Big\|\frac{\partial}{\partial r}\frac{\partial}{\partial x}\big(r^{k_1+a}x^{k_2}f\big)\Big\|_{1,m_2}\Big)^{\frac{1}{m}}.$$

3. For all $(k_1,k_2)\in\mathbb{N}^2$; the function

$$(r,x)\longmapsto r^{k_1}\ x^{k_2}\ f(r,x)$$

belongs to $L^1\big([0,+\infty[\times\mathbb{R},\ dm_2(r,x)\big)$. Indeed

$$\int_0^{+\infty}\int_{\mathbb{R}}|r^{k_1}\ x^{k_2}\ f(r,x)|dm_2(r,x)=\int_0^1\int_{\mathbb{R}}|r^{k_1}\ x^{k_2}\ f(r,x)|dm_2(r,x)$$
$$+\int_1^{+\infty}\int_{\mathbb{R}}|r^{k_1}\ x^{k_2}\ f(r,x)|dm_2(r,x).$$

From 2) there exists $C_{k_1,k_2}>0$ such that

$$\forall\ (r,x)\in[0,+\infty[\times\mathbb{R};\quad \big|r^{k_1}\ x^{k_2}\ f(r,x)\big|\leqslant\frac{C_{k_1,k_2}}{\sqrt{r}(1+x^2)},$$

thus;

$$\begin{aligned}\int_0^1\int_{\mathbb{R}}|r^{k_1}\ x^{k_2}\ f(r,x)|dm_2(r,x)\ &\leqslant\ \frac{1}{2\pi}C_{k_1,k_2}\int_0^1\frac{dr}{\sqrt{r}}\int_{\mathbb{R}}\frac{1}{(1+x^2)}dx\\ &=\ C_{k_1,k_2}.\end{aligned}$$

On the other hand;

$$\int_1^{+\infty}\int_{\mathbb{R}}|r^{k_1}\ x^{k_2}\ f(r,x)|dm_2(r,x)\leqslant\frac{2^\alpha\ \Gamma(\alpha+1)}{\sqrt{2\pi}}\ \big\|r^{k_1}x^{k_2}f\big\|_{1,\nu_\alpha},$$

which proves that for all $(k_1,k_2)\in\mathbb{N}^2$;

$$\int_0^{+\infty}\int_{\mathbb{R}}|r^{k_1}\ x^{k_2}\ f(r,x)|dm_2(r,x)<+\infty.$$

4. For all $(k_1,k_2)\in\mathbb{N}^2$; the function

$$(r,x)\longmapsto r^{k_1}\ x^{k_2}\ f(r,x)$$

is bounded on $[0,+\infty[\times\mathbb{R}$. Indeed; for $k_1\geqslant 1$, the result follows from 2) Let's prove that for all $k\in\mathbb{N};\ k\geqslant 1$; the function

$$(r,x)\longmapsto x^k\ f(r,x)$$

is bounded on $[0,+\infty[\times\mathbb{R}$. From the fact that f and all its derivatives are bounded, we deduce that there exists $C_k>0$ such that;

$\forall\ (r,x)\in[0,+\infty[\times\mathbb{R}$;

$$\Big|\frac{\partial}{\partial r}\frac{\partial}{\partial x}\big[\big(x^k\ f(r,x)\big)^3\big]\Big|\leqslant C_k\Big\{\big|x^{3k-1}f(r,x)\big|+\big|x^{3k}f(r,x)\big|\Big\},$$

and by 3) we deduce that the function

$$(r,x)\longmapsto\frac{\partial}{\partial r}\frac{\partial}{\partial x}\Big[\big(x^k\ f(r,x)\big)^3\Big]$$

belongs to $L^1([0,+\infty[\times\mathbb{R},\ dm_2(r,x))$, and by (4.1) we have;

$$\big(x^k\ f(r,x)\big)^3 = \int_{-\infty}^{r}\int_0^x \frac{\partial}{\partial t}\ \frac{\partial}{\partial y}\Big[\big(y^k\ f(t,y)\big)^3\Big]dt\ dy.$$

Consequently, for all $(r,x)\in[0,+\infty[\times\mathbb{R}$;

$$|x^k\ f(r,x)| \leqslant \Big(2\pi\|\frac{\partial}{\partial r}\ \frac{\partial}{\partial x}[(x^k\ f)^3]\|_{1,m_2}\Big)^{\frac{1}{3}}.$$

By the same method and using the relation (4.2), we prove that for all $(k_1,k_2)\in\mathbb{N}^2$; the function

$$(\mu,\lambda)\longmapsto \mu^{k_1}\lambda^{k_2}\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)$$

is bounded on $[0,+\infty[\times\mathbb{R}$.
This achieves the proof of proposition 4.1. □

In the sequel; we give a new description of the Schwartz's space $S_*(\mathbb{R}^2)$. Namely, we have

Theorem 4.2. *Let f be a continuous function on $\mathbb{R}^2$, even with respect to the first variable. Then, the following properties are equivalent.*

i. *For all $(k_1,k_2)\in\mathbb{N}^2$; the functions*

$$(r,x)\longmapsto r^{k_1}\ x^{k_2}\ f(r,x)\ \textit{and}\quad (\mu,\lambda)\longmapsto \mu^{k_1}\ \lambda^{k_2}\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)$$

are bounded on $[0,+\infty[\times\mathbb{R}$.

ii. *The function f is infinitely differentiable on $\mathbb{R}^2$, even with respect to the first variable, bounded together with all its derivatives on $[0,+\infty[\times\mathbb{R}$ and for all $(k_1,k_2)\in\mathbb{N}^2$; the function*

$$(r,x)\longmapsto r^{k_1}\ x^{k_2}\ f(r,x)$$

is bounded on $[0,+\infty[\times\mathbb{R}$.

iii. *The function f belongs to the space $S_*(\mathbb{R}^2)$.*

iv. *For all $(k_1,k_2)\in\mathbb{N}^2$; the functions*

$$(r,x)\longmapsto r^{k_1}\ x^{k_2}\ f(r,x)\ \textit{and}\quad (\mu,\lambda)\longmapsto \mu^{k_1}\ \lambda^{k_2}\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)$$

belong to $L^2(d\nu_\alpha)$.

Proof. • From the proof of proposition 4.1, we deduce that ii) holds if i) is satisfied.
• Suppose that f satisfies ii). Then, for all $(k_1,k_2)\in\mathbb{N}^2$; we have

$$\int_0^r t^{2k_1}\ x^{2k_2}\Big|\frac{\partial f}{\partial t}(t,x)\Big|^2dt=\int_0^r t^{2k_1}\ x^{2k_2}\frac{\partial f}{\partial t}(t,x)(\overline{\frac{\partial f}{\partial t}})(t,x)dt$$
$$=\Big[t^{2k_1}\ x^{2k_2}f(t,x)(\overline{\frac{\partial f}{\partial t}})(t,x)\Big]_0^r-\int_0^r x^{2k_2}f(t,x)2k_1t^{2k_1-1}(\overline{\frac{\partial f}{\partial t}})(t,x)dt$$
$$-\int_0^r x^{2k_2}f(t,x)t^{2k_1}(\overline{\frac{\partial^2 f}{\partial t^2}})(t,x)dt$$
$$=r^{2k_1}\ x^{2k_2}f(r,x)(\overline{\frac{\partial f}{\partial r}})(r,x)-2k_1\int_0^r x^{2k_2}t^{2k_1-1}f(t,x)(\overline{\frac{\partial f}{\partial t}})(t,x)dt$$
$$-\int_0^r t^{2k_1}x^{2k_2}f(t,x)(\overline{\frac{\partial^2 f}{\partial t^2}})(t,x)dt.$$

And by hypothesis, we deduce that for all $(k_1,k_2)\in\mathbb{N}^2$; the function

$$(r,x)\longmapsto\int_0^r t^{2k_1}\ x^{2k_2}\Big|\frac{\partial f}{\partial t}(t,x)\Big|^2dt \tag{4.3}$$

is bounded on $[0,+\infty[\times\mathbb{R}$.
By the same way, for all $(k_1,k_2)\in\mathbb{N}^2$; the function

$$(r,x)\longmapsto\int_0^x r^{2k_1}\ y^{2k_2}\Big|\frac{\partial f}{\partial y}(r,y)\Big|^2dy \tag{4.4}$$

is bounded on $[0,+\infty[\times\mathbb{R}$.
On the other hand, for all $(k_1,k_2)\in\mathbb{N}^2$;

$$\frac{\partial}{\partial r}\Big(r^{3k_1}\ x^{3k_2}(\frac{\partial f}{\partial r}(r,x))^3\Big)=3k_1\ r^{3k_1-1}\ x^{3k_2}(\frac{\partial f}{\partial r}(r,x))^3$$
$$+3r^{3k_1}\ x^{3k_2}(\frac{\partial f}{\partial r}(r,x))^2\frac{\partial^2 f}{\partial r^2}(r,x).$$

Consequently,

$$(r^{k_1}\ x^{k_2}\ \frac{\partial f}{\partial r}(r,x))^3=3k_1\int_0^r t^{3k_1-1}\ x^{3k_2}(\frac{\partial f}{\partial t}(t,x))^2(\frac{\partial f}{\partial t}(t,x))dt$$
$$+3\int_0^r t^{3k_1}\ x^{3k_2}(\frac{\partial f}{\partial t}(t,x))^2\frac{\partial^2 f}{\partial t^2}(t,x)\ dt.$$

From (4.3), we deduce that for all $(k_1, k_2) \in \mathbb{N}^2$; the function

$$(r, x) \longmapsto r^{k_1}\ x^{k_2}\ \frac{\partial f}{\partial r}(r, x)$$

is bounded on $[0, +\infty[\times\mathbb{R}$. By the same way, and using (4.4) it follows that the function

$$(r, x) \longmapsto r^{k_1}\ x^{k_2}\ \frac{\partial f}{\partial x}(r, x)$$

is bounded on $[0, +\infty[\times\mathbb{R}$.
Thus, the functions $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial x}$ satisfy the same hypothesis as the function f. By iteration, we deduce that for all $(l_1, l_2) \in \mathbb{N}^2$; the function

$$(r, x) \longmapsto r^{k_1} x^{k_2}\ (\frac{\partial}{\partial r})^{l_1} (\frac{\partial}{\partial x})^{l_2} f(r, x)$$

is bounded on $[0, +\infty[\times\mathbb{R}$.
Which means that the function f lies in $S_*(\mathbb{R}^2)$.
• It's clear that if f belongs to $S_*(\mathbb{R}^2)$, then for all $(k_1, k_2) \in \mathbb{N}^2$; the functions

$$(r, x) \longmapsto r^{k_1}\ x^{k_2}\ f(r, x) \quad \text{and} \quad (\mu, \lambda) \longmapsto \mu^{k_1}\ \lambda^{k_2}\ \widetilde{\mathscr{F}}_\alpha(f)(\mu, \lambda)$$

belong to $L^2(d\nu_\alpha)$, because the transform $\widetilde{\mathscr{F}}_\alpha$ is an isomorphism from $S_*(\mathbb{R}^2)$ onto itself.
• Lastly, if the hypothesis iv) is satisfied, then by proposition 4.1 we deduce that i) holds. □

Corollary 4.3. *Let f be a continuous function on Γ, even with respect to the first variable. Then the following assertions are equivalent.*

i. *For all $(k_1, k_2) \in \mathbb{N}^2$;*

$$\sup_{(\mu,\lambda)\in\Gamma_+} \left|(\mu^2 + \lambda^2)^{\frac{k_1}{2}} \lambda^{k_2} f(\mu, \lambda)\right| < +\infty$$

and

$$\sup_{(r,x)\in\mathbb{R}_+\times\mathbb{R}} \left|r^{k_1}\ x^{k_2}\ \mathscr{F}_\alpha^{-1}(f)(r, x)\right| < +\infty.$$

ii. *The function f is infinitely differentiable on Γ, bounded together with all its derivatives on Γ_+, and for all $(k_1, k_2) \in \mathbb{N}^2$; the function*

$$(\mu, \lambda) \longmapsto (\mu^2 + \lambda^2)^{\frac{k_1}{2}} \lambda^{k_2} f(\mu, \lambda)$$

is bounded on Γ_+.

iii. *The function* f *belongs to* $S_*(\Gamma)$.

iv. *For all* $(k_1, k_2) \in \mathbb{N}^2$*; the functions*

$$(\mu, \lambda) \longmapsto (\mu^2 + \lambda^2)^{\frac{k_1}{2}} \lambda^{k_2} f(\mu, \lambda)$$

respectively

$$(r, x) \longmapsto r^{k_1} \ x^{k_2} \ \mathscr{F}_\alpha^{-1}(f)(r, x)$$

belong in $L^2(d\gamma_\alpha)$*, respectively* $L^2(d\nu_\alpha)$.

Proof. let f be a continuous function on Γ, even with respect to the first variable. We consider the function g defined on $[0, +\infty[\times\mathbb{R}$ by

$$g(r, x) = \begin{cases} f(\sqrt{r^2 - x^2}, x), & \text{if } r \geqslant |x|; \\ \\ f(i\sqrt{x^2 - r^2}, x), & \text{if } r < |x|. \end{cases}$$

Then,

- For all $(\mu, \lambda) \in \Gamma$;
$$B(g)(\mu, \lambda) = g \circ \theta(\mu, \lambda) = f(\mu, \lambda).$$

- $$\sup_{(r,x)\in\mathbb{R}_+\times\mathbb{R}} \left|r^{k_1} \ x^{k_2} \ g(r, x)\right| = \sup_{(\mu,\lambda)\in\Gamma_+} \left|(\mu^2 + \lambda^2)^{\frac{k_1}{2}} \lambda^{k_2} f(\mu, \lambda)\right|.$$

- For every $(r, x) \in [0, +\infty[\times\mathbb{R}$;
$$\widetilde{\mathscr{F}_\alpha}(g)(r, x) = \mathscr{F}_\alpha^{-1}(f)(r, -x).$$

So, if the function f satisfies the assertion $i)$ of this corollary; then for all $(k_1, k_2) \in \mathbb{N}^2$; the functions

$$(r, x) \longmapsto r^{k_1} \ x^{k_2} \ g(r, x)$$

and

$$(\mu, \lambda) \longmapsto \mu^{k_1} \lambda^{k_2} \widetilde{\mathscr{F}_\alpha}(g)(\mu, \lambda)$$

are bounded on $[0, +\infty[\times\mathbb{R}$. Consequently, the result follows from theorem 4.2 and the fact that for all $g \in S_*(\mathbb{R}^2)$; the function $f = g \circ \theta$ belongs to $S_*(\Gamma)$. $\square$

5. **Fourier transform of functions with bounded supports.**

In this section, we characterize some spaces of functions by their Fourier transforms. More precisely, we establish a real Paley-Wiener theorem and a Paley-Wiener-Schwartz theorem for the Fourier transform connected with the Riemann-Liouville operator.

Theorem 5.1. *(Paley-Wiener) Let f be a function in $L^2(d\gamma_\alpha)$ and $g = \mathscr{F}_\alpha^{-1}(f)$.*

i. *If g has a compact support, then f satisfies the assertion 2) of theorem 3.3. Moreover, the sequence $\left(\|A_\alpha^n \mathscr{F}_\alpha(g)\|_{2,\gamma_\alpha}^{\frac{1}{2n}}\right)_n$ converges to σ_g, where*

$$\sigma_g = \sup\{|(r,x)|;\ (r,x) \in supp\ g\}; \quad |(r,x)| = \sqrt{r^2+x^2}.$$

ii. *Conversely, let $g \in L^2(d\nu_\alpha)$ such that $\mathscr{F}_\alpha(g)$ satisfies the assertion 2) of theorem 3.3 and the sequence $\left(\|A_\alpha^n \mathscr{F}_\alpha(g)\|_{2,\gamma_\alpha}^{\frac{1}{2n}}\right)_n$ has a finite limit σ, then g has a compact support and $\sigma = \sigma_g$.*

Proof. i. Suppose that g has a compact support, then for all $(k_1,k_2) \in \mathbb{N}^2$; the function

$$(r,x) \longmapsto r^{k_1}\ x^{k_2}\ g(r,x)$$

belongs to $L^2(d\nu_\alpha)$. By theorem 3.3, we deduce that the function $f = \mathscr{F}_\alpha(g)$ satisfies the assertion 2) of theorem 3.3. From the relation (3.3), we have;

$$\forall\ n \in \mathbb{N}; \quad A_\alpha^n\ \mathscr{F}_\alpha(g) = B\big(L_\alpha^n \widetilde{\mathscr{F}_\alpha}(g)\big).$$

Then, by (2.8), we get

$$\begin{aligned}\|A_\alpha^n \mathscr{F}_\alpha(g)\|_{2,\gamma_\alpha} &= \|L_\alpha^n \widetilde{\mathscr{F}_\alpha}(g)\|_{2,\nu_\alpha}\\ &= \|\widetilde{\mathscr{F}_\alpha}\big(-(r^2+x^2)^n g\big)\|_{2,\nu_\alpha}.\end{aligned}$$

Applying Plancherel theorem for the transform $\widetilde{\mathscr{F}_\alpha}$, it follows that for all $n \in \mathbb{N}$;

$$\|A_\alpha^n \mathscr{F}_\alpha(g)\|_{2,\gamma_\alpha}^{\frac{1}{2n}} = \|(r^2+x^2)^n g\|_{2,\nu_\alpha}^{\frac{1}{2n}}. \tag{5.1}$$

Thus, for every $n \in \mathbb{N}$;

$$\|A_\alpha^n \mathscr{F}_\alpha(g)\|_{2,\gamma_\alpha}^{\frac{1}{2n}} \leqslant \sigma_g \ \|g\|_{2,\nu_\alpha}^{\frac{1}{2n}}$$

and consequently;

$$\limsup_{n \longrightarrow +\infty} \|A_\alpha^n \mathscr{F}_\alpha(g)\|_{2,\gamma_\alpha}^{\frac{1}{2n}} \leqslant \sigma_g. \tag{5.2}$$

On the other hand, from (5.1), for all $\varepsilon > 0$ and $n \in \mathbb{N}$; we have

$$\begin{aligned} \|A_\alpha^n \mathscr{F}_\alpha(g)\|_{2,\gamma_\alpha}^{\frac{1}{2n}} &\geqslant \Big(\int\int_{(r^2+x^2) \geqslant (\sigma_g - \varepsilon)^2} (r^2+x^2)^{2n} |g(r,x)|^2 d\nu_\alpha(r,x) \Big)^{\frac{1}{4n}} \\ &\geqslant (\sigma_g - \varepsilon) \Big(\int\int_{(r^2+x^2) \geqslant (\sigma_g - \varepsilon)^2} |g(r,x)|^2 d\nu_\alpha(r,x) \Big)^{\frac{1}{4n}}. \end{aligned}$$

where,

$$\int\int_{r^2+x^2 \geqslant (\sigma_g - \varepsilon)^2} \big|g(r,x)\big|^2 d\nu_\alpha(r,x) > 0.$$

Hence, for all $\varepsilon > 0$;

$$\liminf_{n \longrightarrow +\infty} \|A_\alpha^n \mathscr{F}_\alpha(g)\|_{2,\gamma_\alpha}^{\frac{1}{2n}} \geqslant \sigma_g - \varepsilon,$$

which implies that

$$\liminf_{n \longrightarrow +\infty} \|A_\alpha^n \mathscr{F}_\alpha(g)\|_{2,\gamma_\alpha}^{\frac{1}{2n}} \geqslant \sigma_g. \tag{5.3}$$

>From (5.2) and (5.3), we deduce that the sequence $\Big(\|A_\alpha^n \mathscr{F}_\alpha(g)\|_{2,\gamma_\alpha}^{\frac{1}{2n}}\Big)_n$ is convergent and

$$\lim_{n \longrightarrow +\infty} \|A_\alpha^n \mathscr{F}_\alpha(g)\|_{2,\gamma_\alpha}^{\frac{1}{2n}} = \sigma_g.$$

ii. Let $g \in L^2(d\nu_\alpha)$ such that $\mathscr{F}_\alpha(g)$ satisfies the assertion 2) of theorem 3.3 and the sequence $\Big(\|A_\alpha^n \mathscr{F}_\alpha(g)\|_{2,\gamma_\alpha}^{\frac{1}{2n}}\Big)_n$ has a finite limit σ.
Suppose that there exists $\varepsilon > 0$ such that the set

$$\Big\{(r,x) \in \mathbb{R}_+ \times \mathbb{R};\ \sqrt{r^2+x^2} > \sigma + \varepsilon; \quad g(r,x) \neq 0\Big\}$$

has a positive measure. Then

$$\begin{aligned}
\|A_\alpha^n \mathscr{F}_\alpha(g)\|_{2,\gamma_\alpha}^{\frac{1}{2n}} &= \|(r^2+x^2)^n g\|_{2,\nu_\alpha}^{\frac{1}{2n}} \\
&= \Big(\int_0^{+\infty}\int_{\mathbb{R}} \big(r^2+x^2\big)^{2n}\big|g(r,x)\big|^2 d\nu_\alpha(r,x)\Big)^{\frac{1}{4n}} \\
&\geqslant \Big(\int\int_{r^2+x^2>(\sigma+\varepsilon)^2} \big(r^2+x^2\big)^{2n}\big|g(r,x)\big|^2 d\nu_\alpha(r,x)\Big)^{\frac{1}{4n}} \\
&\geqslant (\sigma+\varepsilon)\Big(\int\int_{r^2+x^2>(\sigma+\varepsilon)^2} \big|g(r,x)\big|^2 d\nu_\alpha(r,x)\Big)^{\frac{1}{4n}},
\end{aligned}$$

and by hypothesis, we get;

$$\sigma \geqslant \sigma+\varepsilon$$

which is impossible. This shows that g has a bounded support and by the proof of i) we can show that $\sigma=\sigma_g$. □

In the following, we shall give a new characterization of infinitely differentiable functions with bounded supports, by means of their Fourier transforms. For this, let $(\sigma_1,\sigma_2)\in(\mathbb{R}_+^*)^2$; we denote by

- $\mathcal{H}^{(\sigma_1,\sigma_2)}(\mathbb{C}^2)$; the space of entire functions g on $\mathbb{C}^2$, slowly increasing of exponential type, i.e, there exists an integer k such that

$$\sup_{(\mu,\lambda)\in\mathbb{C}^2}\frac{\big|g(\mu,\lambda)\big|e^{-\sigma_1|\Im m\mu|-\sigma_2|\Im m\lambda|}}{\big(1+|\mu|^2+|\lambda|^2\big)^k}<+\infty.$$

- $\mathbb{H}^{(\sigma_1,\sigma_2)}(\mathbb{C}^2)$; the space of entire functions f on $\mathbb{C}^2$, rapidly decreasing of exponential type, i.e for all $k\in\mathbb{N}$;

$$\sup_{(\mu,\lambda)\in\mathbb{C}^2}\big|f(\mu,\lambda)\big|\big(1+|\mu|^2+|\lambda|^2\big)^k e^{-\sigma_1|\Im m\mu|-\sigma_2|\Im m\lambda|}<+\infty.$$

  and $\mathbb{H}_*^{(\sigma_1,\sigma_2)}(\mathbb{C}^2)$, its subset consisting of even functions with respect to the first variable.

- $\mathbb{H}_*(\mathbb{C}^2)=\displaystyle\bigcup_{(\sigma_1,\sigma_2)\in(\mathbb{R}_+^*)^2}\mathbb{H}_*^{(\sigma_1,\sigma_2)}(\mathbb{C}^2)$.

- $\mathscr{E}(\mathbb{R}^2)$, the space of infinitely differentiable functions on $\mathbb{R}^2$.

- $\mathscr{E}'_{(\sigma_1,\sigma_2)}(\mathbb{R}^2)$; the space of distributions on $\mathbb{R}^2$, with support in $[-\sigma_1,\sigma_1]\times[-\sigma_2,\sigma_2]$.

- $S'(\mathbb{R}^2)$, the space of tempered distributions on $\mathbb{R}^2$.

- $\mathscr{D}_*^{(\sigma_1,\sigma_2)}(\mathbb{R}^2)$, the space of infinitely differentiable functions, even with respect to the first variable and with support in $[-\sigma_1,\sigma_1]\times[-\sigma_2,\sigma_2]$.

- $\mathscr{D}_*(\mathbb{R}^2) = \displaystyle\bigcup_{(\sigma_1,\sigma_2)\in(\mathbb{R}_+^*)^2} \mathscr{D}_*^{(\sigma_1,\sigma_2)}(\mathbb{R}^2)$

- For all $f\in\mathcal{H}^{(\sigma_1,\sigma_2)}(\mathbb{C}^2)$;

  $\sigma_{f,i} = \sup\{|P_i(r,x)|;\ (r,x)\in \operatorname{supp}\Lambda_2^{-1}(T_f)\};\quad i\in\{0,1\}$,

  with $P_0(r,x)=r$ and $P_1(r,x)=x$; $(r,x)\in\mathbb{R}^2$, and T_f the tempered distribution given by the function f.

The following result is a consequence of Bernstein's inequality and the theorem of Kolmogoroff [1, 5, 17].

Proposition 5.2. *Let $\sigma=(\sigma_1,\sigma_2)\in(\mathbb{R}_+^*)^2$. For all $f\in\mathcal{H}^\sigma(\mathbb{C}^2)\cap L^p(dm_2)$; $p\in[1,+\infty]$, the functions $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial x}$ belong to $\mathcal{H}^\sigma(\mathbb{C}^2)\cap L^p(dm_2)$; and we have*

i.
$$\|\frac{\partial}{\partial r}f\|_{p,m_2}\leqslant \sigma_1\ \|f\|_{p,m_2}.$$

ii.
$$\|\frac{\partial}{\partial x}f\|_{p,m_2}\leqslant \sigma_2\ \|f\|_{p,m_2}.$$

Proposition 5.3. *Let $p\in[1,+\infty]$ and $f\in\mathscr{E}(\mathbb{R}^2)$ such that, for all $(l_1,l_2)\in\mathbb{N}^2$; the function*

$$(r,x)\longmapsto(\frac{\partial}{\partial r})^{l_1}(\frac{\partial}{\partial x})^{l_2}f(r,x)$$

belongs to $L^p(dm_2)$. Then, for all $n\in\mathbb{N}^*$ and $k\in\mathbb{N}$; $0<k<n$, we have

i.
$$\Big\|(\frac{\partial}{\partial r})^k f\Big\|^n_{p,m_2} \leqslant (\frac{\pi}{2})^n \ \|f\|^{n-k}_{p,m_2} \ \|(\frac{\partial}{\partial r})^n f\|^k_{p,m_2}.$$

ii..
$$\Big\|(\frac{\partial}{\partial x})^k f\Big\|^n_{p,m_2} \leqslant (\frac{\pi}{2})^n \ \|f\|^{n-k}_{p,m_2} \ \|(\frac{\partial}{\partial x})^n f\|^k_{p,m_2}.$$

Proof. • In the case $p = +\infty$, the proof can be found in [17].

• Suppose that $p \in [1, +\infty[$ and let
$$h_1(r,x) = \frac{\overline{(\frac{\partial}{\partial r})^k f(r,x)}}{|(\frac{\partial}{\partial r})^k f(r,x)|} \frac{|(\frac{\partial}{\partial r})^k f(r,x)|^{p-1}}{\||(\frac{\partial}{\partial r})^k f|^{p-1}\|_{p',m_2}}$$
where p' is the conjugate exponent of p. Then
$$\|h_1\|_{p',m_2} = 1 \tag{5.4}$$
and
$$\int\int_{\mathbb{R}^2} h_1(r,x)(\frac{\partial}{\partial r})^k f(r,x) dm_2(r,x) = \Big\|(\frac{\partial}{\partial r})^k f\Big\|_{p,m_2}. \tag{5.5}$$
Let
$$F(r) = \int\int_{\mathbb{R}^2} h_1(t,x) f(r+t,x) dm_2(t,x).$$
Applying lemma 8 of [17] and using the hypothesis, we deduce that the function F is infinitely differentiable on $\mathbb{R}$, and we have
$$F^{(k)}(r) = \int\int_{\mathbb{R}^2} h_1(t,x)(\frac{\partial}{\partial r})^k f(r+t,x) dm_2(t,x); \quad 0 < k < n.$$
Then, by Hölder's inequality, we get
$$|F^{(k)}(r)| \leqslant \|h_1\|_{p',m_2} \Big\|(\frac{\partial}{\partial r})^k f\Big\|_{p,m_2},$$
and by (5.4), we deduce that for all $k \in \mathbb{N};\ 0 < k < n$
$$\|F^{(k)}\|_{\infty,m_2} \leqslant \Big\|(\frac{\partial}{\partial r})^k f\Big\|_{p,m_2}. \tag{5.6}$$
On the other hand, using the relation (5.5) we have
$$|F^{(k)}(0)| = \Big\|(\frac{\partial}{\partial r})^k f\Big\|_{p,m_2}. \tag{5.7}$$

However, applying the theorem of Kolmogoroff to F [11, 17], we obtain

$$\|F^{(k)}\|^n_{\infty,m_2} \leqslant (\frac{\pi}{2})^n \|F\|^{n-k}_{\infty,m_2} \|F^{(n)}\|^k_{\infty,m_2}. \tag{5.8}$$

Combining the relations (5.6), (5.7) and (5.8) we obtain

$$\Big\|(\frac{\partial}{\partial r})^k f\Big\|^n_{p,m_2} \leqslant (\frac{\pi}{2})^n \|f\|^{n-k}_{p,m_2} \Big\|(\frac{\partial}{\partial r})^n f\Big\|^k_{p,m_2}.$$

- We obtain the result by the same way and using the function

$$G(x) = \int\int_{\mathbb{R}^2} h_2(t,y)\ f(t,x+y)\ dm_2(t,y)$$

where

$$h_2(r,x) = \frac{\overline{(\frac{\partial}{\partial x})^k f(r,x)}}{|(\frac{\partial}{\partial x})^k f(r,x)|} \frac{|(\frac{\partial}{\partial x})^k f(r,x)|^{p-1}}{\||(\frac{\partial}{\partial x})^k f|^{p-1}\|_{p',m_2}}.$$

□

Theorem 5.4. *Let $p \in [1,+\infty]$ and let f be a function satisfying the hypothesis of proposition 5.3.*

1. *If $\sigma_{f,0} + \sigma_{f,1} < +\infty$, then the sequences $\Big(\|(\frac{\partial}{\partial r})^k f\|^{\frac{1}{k}}_{p,m_2}\Big)_k$ and $\Big(\|(\frac{\partial}{\partial x})^k f\|^{\frac{1}{k}}_{p,m_2}\Big)_k$ converge respectively to $\sigma_{f,0}$ and $\sigma_{f,1}$.*

2. *If there exist $(M_1,M_2) \in (\mathbb{R}^*_+)^2$ such that for all $(l_1,l_2) \in \mathbb{N}^2$*
$$\|(\frac{\partial}{\partial r})^{l_1}(\frac{\partial}{\partial x})^{l_2} f\|_{p,m_2} \leqslant M_1^{l_1}\ M_2^{l_2}\ \|f\|_{p,m_2}$$
then, $\sigma_{f,0} < +\infty$ and $\sigma_{f,1} < +\infty$. Moreover, the sequences $\Big(\|(\frac{\partial}{\partial r})^k f\|^{\frac{1}{k}}_{p,m_2}\Big)_k$ and $\Big(\|(\frac{\partial}{\partial x})^k f\|^{\frac{1}{k}}_{p,m_2}\Big)_k$ converge respectively to $\sigma_{f,0}$ and $\sigma_{f,1}$.

Proof. 1. If f satisfies the hypothesis of Proposition 5.3, then T_f and $\Lambda_2^{-1}(T_f)$ belong to $S'(\mathbb{R}^2)$. Suppose that $\sigma_{f,0} + \sigma_{f,1} < +\infty$.

Since the Fourier transform Λ_2 is an isomorphism from $\mathscr{E}'_{(\sigma_{f,0},\sigma_{f,1})}(\mathbb{R}^2)$ onto $\mathcal{H}^{(\sigma_{f,0},\sigma_{f,1})}(\mathbb{C}^2)$, the function f lies in $\mathcal{H}^{(\sigma_{f,0},\sigma_{f,1})}(\mathbb{C}^2)$.

On the other hand, by proposition 5.3, for all $n \in \mathbb{N}^*$ and $k \in \mathbb{N}$; $0 < k < n$; we have

$$\Big\|(\frac{\partial}{\partial r})^k f\Big\|_{p,m_2}^{\frac{1}{k}} \leqslant (\frac{\pi}{2})^{\frac{1}{k}} \ \|f\|_{p,m_2}^{\frac{1}{k}-\frac{1}{n}} \ \Big\|(\frac{\partial}{\partial r})^n f\Big\|_{p,m_2}^{\frac{1}{n}}. \tag{5.9}$$

Applying the proposition 5.2, we get

$$\Big\|(\frac{\partial}{\partial r})^k f\Big\|_{p,m_2}^{\frac{1}{k}} \leqslant \sigma_{f,0} \ (\frac{\pi}{2})^{\frac{1}{k}} \ \|f\|_{p,m_2}^{\frac{1}{k}},$$

then,

$$\liminf_{k \longrightarrow +\infty} \Big\|(\frac{\partial}{\partial r})^k f\Big\|_{p,m_2}^{\frac{1}{k}} \leqslant \ \sigma_{f,0}. \tag{5.10}$$

Now, from the inequality (5.9), we deduce that for all $k \in \mathbb{N}^*$;

$$\Big\|(\frac{\partial}{\partial r})^k f\Big\|_{p,m_2}^{\frac{1}{k}} \ (\frac{\pi}{2})^{-\frac{1}{k}} \ \|f\|_{p,m_2}^{-\frac{1}{k}} \leqslant \ \liminf_{n \longrightarrow +\infty} \|(\frac{\partial}{\partial r})^n f\|_{p,m_2}^{\frac{1}{n}}$$

then,

$$\limsup_{k \longrightarrow +\infty} \|(\frac{\partial}{\partial r})^k f\|_{p,m_2}^{\frac{1}{k}} \ \leqslant \ \liminf_{k \longrightarrow +\infty} \|(\frac{\partial}{\partial r})^k f\|_{p,m_2}^{\frac{1}{k}}.$$

This shows that the sequence $\Big(\|(\frac{\partial}{\partial r})^k f\|_{p,m_2}^{\frac{1}{k}}\Big)_k$ converges and by (5.10)

$$\lim_{k \longrightarrow +\infty} \|(\frac{\partial}{\partial r})^k f\|_{p,m_2}^{\frac{1}{k}} = \sigma_0 \leqslant \sigma_{f,0}.$$

Let's prove that $\sigma_0 = \sigma_{f,0}$. Indeed, suppose that $\sigma_0 < \sigma_{f,0}$

- The case $p = +\infty$.
  let $\varepsilon > 0$ such that

  $$\sigma_0 + 2\varepsilon < \sigma_{f,0} \tag{5.11}$$

  then, there exists $M > 0$ such that

  $$\forall \ k \in \mathbb{N}; \quad \|(\frac{\partial}{\partial r})^k f\|_{\infty,m_2} \leqslant M \ (\sigma_0 + \varepsilon)^k. \tag{5.12}$$

  From proposition 5.2 and the relation (5.12), we deduce that, for all $(l_1, l_2) \in \mathbb{N}$;

  $$\|(\frac{\partial}{\partial r})^{l_1} (\frac{\partial}{\partial x})^{l_2} f\|_{\infty,m_2} \leqslant M \ (\sigma_0 + \varepsilon)^{l_1} \ \sigma_{f,1}^{l_2}.$$

Let $(\mu, \lambda) \in \mathbb{C}^2$, $\mu = x_1 + iy_1$ and $\lambda = x_2 + iy_2$, we have

$$\begin{aligned}
&\sum_{l_1,l_2=0}^{+\infty} \frac{|(\frac{\partial}{\partial r})^{l_1}(\frac{\partial}{\partial x})^{l_2} f(x_1,x_2)(iy_1)^{l_1}(iy_2)^{l_2}|}{l_1!\ l_2!} \\
&\qquad \leqslant M\ \Big(\sum_{l_1=0}^{+\infty} \frac{(\sigma_0+\varepsilon)^{l_1}|y_1|^{l_1}}{l_1!}\Big)\Big(\sum_{l_2=0}^{+\infty} \frac{\sigma_{f,1}^{l_2}|y_2|^{l_2}}{l_2!}\Big) \\
&\qquad = M\ \exp\big((\sigma_0+\varepsilon)|\Im m\mu| + \sigma_{f,1}|\Im m\lambda|\big).
\end{aligned}$$

This shows that f belongs to the space $\mathcal{H}^{(\sigma_0+\varepsilon,\sigma_{f,1})}(\mathbb{C}^2)$. Again, by Paley-Wiener Theorem's it follows that

$$\text{supp } \Lambda_2^{-1}(T_f) \subset [-\sigma_0 - \varepsilon, \sigma_0 + \varepsilon] \times [-\sigma_{f,1}, \sigma_{f,1}].$$

Consequently;

$$\sigma_{f,0} \leqslant \sigma_0 + \varepsilon,$$

which contradicts (5.11).

- The case $p \in [1, +\infty[$.

Let $\varphi \in \mathscr{D}_*(\mathbb{R}^2)$; $0 \leqslant \varphi \leqslant 1$ such that

$$\int\int_{\mathbb{R}^2} \varphi(r,x)\ dm_2(r,x) = 1.$$

We put;

$$\varphi_n(r,x) = n^2\varphi(nr,nx); \quad n \in \mathbb{N}^*$$

and

$$F_n(r,x) = \int\int_{\mathbb{R}^2} f(r+t, x+y)\ \varphi_n(t,y)\ dm_2(t,y). \tag{5.13}$$

By applying lemma 8 of [17] and using the hypothesis, we deduce that for all $n \in \mathbb{N}^*$; the function F_n is infinitely differentiable on $\mathbb{R}^2$ and for all $k \in \mathbb{N}$; we have

$$(\frac{\partial}{\partial r})^k F_n(r,x) = \int\int_{\mathbb{R}^2} (\frac{\partial}{\partial r})^k f(r+t,x+y)\varphi_n(t,y)\ dm_2(t,y).$$

By Hölder's inequality, we get

$$\begin{aligned}
\|(\frac{\partial}{\partial r})^k F_n\|_{\infty,m_2} &\leqslant \|(\frac{\partial}{\partial r})^k f\|_{p,m_2}\|\varphi_n\|_{p',m_2} \\
&\leqslant\ n^{\frac{1}{p}}\|(\frac{\partial}{\partial r})^k f\|_{p,m_2}
\end{aligned} \tag{5.14}$$

where p' is the conjugate exponent of p, then,

$$\|(\frac{\partial}{\partial r})^k F_n\|_{\infty,m_2}^{\frac{1}{k}} \leqslant\ n^{\frac{1}{kp}}\|(\frac{\partial}{\partial r})^k f\|_{p,m_2}^{\frac{1}{k}}. \tag{5.15}$$

>From the relation (5.13), we deduce that the function F_n can be written in the form

$$F_n(r,x) = f * \varphi_n(r,x),$$

where $*$ is the usual convolution product in $\mathbb{R}^2$.
So,

$$\Lambda_2^{-1}(T_{F_n}) = \Lambda_2(\varphi_n)\ \Lambda_2^{-1}(T_f).$$

In particular,

$$\sigma_{F_n,0} + \sigma_{F_n,1} < +\infty.$$

Using the case $p = +\infty$ and the relation (5.15), we deduce that

$$\forall n \in \mathbb{N}^*; \quad \sigma_{F_n,0} \leqslant\ \sigma_0 \leqslant\ \sigma_{f,0}. \tag{5.16}$$

Consequently,

$$\liminf_{n\longrightarrow+\infty} \sigma_{F_n,0} \leqslant \sigma_{f,0}.$$

Suppose that

$$\liminf_{n\longrightarrow+\infty}\ \sigma_{F_n,0} < \sigma_{f,0},$$

then, there exists $r \in P_0\big(\text{supp } \Lambda_2^{-1}(T_f)\big)$ such that

$$\big|r\big| > \liminf_{n\longrightarrow+\infty}\ \sigma_{F_n,0} = a.$$

We assume that $r \geqslant 0$ (the same proof holds if $r < 0$).
Let $\varepsilon > 0$ such that $a < r - 3\varepsilon$. There exists a subsequence $\big(\sigma_{F_{\theta(n)},0}\big)_n$ satisfying,

$$\forall n \in \mathbb{N}^*; \quad \sigma_{F_{\theta(n)},0} < r - 2\varepsilon. \tag{5.17}$$

Now, since the sequence $(\varphi_n)_n$ is an approximate identity and using the relation (5.13), we deduce that

$$\lim_{n\longrightarrow+\infty} \big\|F_{\theta(n)} - f\big\|_{p,m_2} = 0$$

and consequently,

$$\lim_{n\longrightarrow+\infty} \Lambda_2^{-1}(T_{F_{\theta(n)}}) = \Lambda_2^{-1}(T_f) \tag{5.18}$$

in $S'(\mathbb{R}^2)$.
Let $\psi \in \mathscr{D}_*(\mathbb{R}^2)$ such that

$$P_0\big(\mathrm{supp}(\psi)\big) \subset \big[r-\varepsilon, r+\varepsilon\big]$$

and

$$< \Lambda_2^{-1}(T_f), \psi > \neq\ 0.$$

However, by (5.17) for all $n \in \mathbb{N}$;

$$< \Lambda_2^{-1}(T_{F_{\theta(n)}}),\ \psi >= 0$$

and by (5.18)

$$< \Lambda_2^{-1}(T_f),\ \psi > =\ 0.$$

Which gives a contradiction. Hence,

$$\liminf_{n\longrightarrow+\infty} \sigma_{F_n,0}\ =\ \sigma_{f,0}.$$

Using, the relation (5.16), we deduce that

$$\sigma_0\ =\ \sigma_{f,0}$$

which means that

$$\lim_{k\longrightarrow+\infty} \|(\frac{\partial}{\partial r})^k f\|_{p,m_2}^{\frac{1}{k}}\ =\ \sigma_{f,0}.$$

By the same way, we prove that

$$\lim_{k\longrightarrow+\infty} \|(\frac{\partial}{\partial x})^k f\|_{p,m_2}^{\frac{1}{k}}\ =\ \sigma_{f,1}.$$

2. Suppose that there exists $(M_1, M_2) \in (\mathbb{R}_+^*)^2$ such that

$$\forall (k_1,k_2) \in \mathbb{N}^2;\quad \Big\|(\frac{\partial}{\partial r})^{k_1}(\frac{\partial}{\partial x})^{k_2} f\Big\|_{p,m_2} \leqslant\ M_1^{k_1} M_2^{k_2}\ \|f\|_{p,m_2}$$

- The case $p = +\infty$.
  Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$; we have

$$\begin{aligned}\sum_{(k_1,k_2)\in\mathbb{N}^2} &\Big|\frac{(\frac{\partial}{\partial z_1})^{k_1}(\frac{\partial}{\partial z_2})^{k_2} f(x_1,x_2)\ (iy_1)^{k_1}(iy_2)^{k_2}}{k_1!\ k_2!}\Big| \\ &\leqslant\ \|f\|_{\infty,m_2} \sum_{k_1=0}^{\infty} \frac{\big(M_1|y_1|\big)^{k_1}}{k_1!} \sum_{k_2=0}^{\infty} \frac{\big(M_2|y_2|\big)^{k_2}}{k_2!} \\ &=\ \|f\|_{\infty,m_2}\ e^{M_1|\Im m z_1| + M_2|\Im m z_2|}.\end{aligned}$$

This shows that the function f is entire on $\mathbb{C}^2$, slowly increasing of exponential type and by Paley-Wiener theorem's for the distributions, we deduce that

$$\operatorname{supp} \Lambda_2^{-1}(T_f) \subset \ [-M_1, M_1] \times [-M_2, M_2].$$

In particular, $\sigma_{f,0} + \sigma_{f,1}$ is finite and from the first assumption of this theorem, the sequences

$$\Big(\|(\frac{\partial}{\partial r})^k f\|_{p,m_2}^{\frac{1}{k}}\Big)_k \text{ and } \Big(\|(\frac{\partial}{\partial x})^k f\|_{p,m_2}^{\frac{1}{k}}\Big)_k$$

converge respectively to $\sigma_{f,0}$ and $\sigma_{f,1}$.

- The case $p \in [1, +\infty[$.
  Let $\big(F_n\big)_n$ be the sequence defined by

  $$F_n(r,x) = \int\int_{\mathbb{R}^2} f(r+t, x+y)\ \varphi_n(t,y)\ dm_2(t,y).$$

  By the relation (5.14); for all $(k_1,k_2) \in \mathbb{N}^2$;

  $$\|(\frac{\partial}{\partial r})^{k_1}(\frac{\partial}{\partial x})^{k_2} F_n\|_{\infty,m_2} \leqslant \ \|f\|_{p,m_2}\ n^{\frac{1}{p}}\ M_1^{k_1} M_2^{k_2}.$$

  >From the case $p = +\infty$; we deduce that for all $n \in \mathbb{N}$, the function F_n is entire on $\mathbb{C}^2$, and for all $(z_1,z_2) \in \mathbb{C}^2$;

  $$\big|F_n(z_1,z_2)\big| \leqslant \ n^{\frac{1}{p}}\ \|f\|_{\infty,m_2}\ e^{M_1|\Im m z_1| + M_2|\Im m z_2|},$$

  which implies that for all $n \in \mathbb{N}^*$;

  $$\operatorname{supp} \Lambda_2^{-1}(T_{F_n}) \subset \ [-M_1, M_1] \times [-M_2, M_2].$$

  Since, $\big(\Lambda_2^{-1}(T_{F_n})\big)_n$ converges to $\Lambda_2^{-1}(T_f)$ in $S'(\mathbb{R}^2)$, we deduce that;

  $$\operatorname{supp} \Lambda_2^{-1}(T_f) \subset \ [-M_1, M_1] \times [-M_2, M_2].$$

  This achieves the proof.

$\square$

We denote by

- $\widetilde{\gamma}_\alpha$ the measure defined on Γ_+ by

  $$d\widetilde{\gamma}_\alpha(\mu,\lambda) = \frac{2^\alpha \Gamma(\alpha+1)}{\sqrt{2\pi}\ \big(\mu^2+\lambda^2\big)^{\alpha+\frac{1}{2}}}\ d\gamma_\alpha(\mu,\lambda).$$

- $L^p(d\widetilde{\gamma}_\alpha)$; $1 \leqslant p \leqslant +\infty$ the space of measurable functions on Γ_+ satisfying

$$\|f\|_{p,\widetilde{\gamma}_\alpha} = \begin{cases} \left(\displaystyle\int\int_{\Gamma_+} |f(\mu,\lambda)|^p d\widetilde{\gamma}_\alpha(\mu,\lambda)\right)^{\frac{1}{p}} < +\infty, & \text{if } 1 \leqslant p < +\infty; \\ \operatorname*{ess\ sup}_{(\mu,\lambda)\in\Gamma_+} |f(\mu,\lambda)| < +\infty, & \text{if } p = +\infty. \end{cases}$$

Lemma 5.5. *The mapping W_α defined on $\mathscr{D}_*(\mathbb{R}^2)$ by*

$$W_\alpha(g)(r,x) = \frac{1}{2^{\alpha+\frac{1}{2}}\ \Gamma(\alpha+\frac{1}{2})} \int_r^{+\infty} (t^2-r^2)^{\alpha-\frac{1}{2}} g(t,x)\ 2t\ dt$$

is a topological isomorphism from $\mathscr{D}_*(\mathbb{R}^2)$ onto itself.
The inverse isomorphism is given by

$$W_\alpha^{-1}(f) = (-1)^{[\alpha]+1}\ W_{[\alpha]+1-\alpha}\Big(\big(\frac{\partial}{\partial r^2}\big)^{[\alpha]+1}(f)\Big).$$

Moreover, for all $g \in \mathscr{D}_*(\mathbb{R}^2)$;

$$\sup\{|P_i(r,x)|;\ (r,x) \in\ supp\ W_\alpha(g)\} = \sup\{|P_i(r,x)|;\ (r,x) \in\ supp\ g\} \tag{5.19}$$

The proof of this lemma can be found in [19, 20]

Proposition 5.6. *Let f be a function in $S_*(\mathbb{R}^2)$. Then, the function $\widetilde{\mathscr{F}}_\alpha^{-1}(f)$ belongs to the space $\mathscr{D}_*(\mathbb{R}^2)$ if, and only if for all $p \in [1,+\infty]$, there exist $(M_1,M_2) \in (\mathbb{R}_+^*)^2$ such that*

$$\forall (k_1,k_2) \in \mathbb{N}^2; \quad \Big\|\big(\frac{\partial}{\partial r}\big)^{k_1}\big(\frac{\partial}{\partial x}\big)^{k_2} f\Big\|_{p,m_2} \leqslant\ M_1^{k_1} M_2^{k_2}\ \|f\|_{p,m_2}.$$

Moreover, the sequences $\Big(\|\big(\frac{\partial}{\partial r}\big)^k f\|_{p,m_2}^{\frac{1}{k}}\Big)_k$ and $\Big(\|\big(\frac{\partial}{\partial x}\big)^k f\|_{p,m_2}^{\frac{1}{k}}\Big)_k$ converge respectively to $\sigma_{f,0}$ and $\sigma_{f,1}$.

Proof. • Suppose that $\widetilde{\mathscr{F}}_\alpha^{-1}(f)$ belongs to the space $\mathscr{D}_*(\mathbb{R}^2)$.

Since, the transform $\widetilde{\mathscr{F}}_\alpha$ is an isomorphism from $\mathscr{D}_*(\mathbb{R}^2)$ onto $\mathbb{H}_*(\mathbb{C}^2)$, then there exist $(\sigma_1,\sigma_2) \in (\mathbb{R}_+^*)^2$ such that

$$f \in \mathbb{H}^{(\sigma_1,\sigma_2)}(\mathbb{C}^2) \subset \mathcal{H}^{(\sigma_1,\sigma_2)}(\mathbb{C}^2),$$

and from Proposition 5.2, we have

$$\|(\frac{\partial}{\partial r})^{k_1} f\|_{p,m_2} \leqslant \sigma_1^{k_1} \; \|f\|_{p,m_2}$$

and

$$\|(\frac{\partial}{\partial x})^{k_2} f\|_{p,m_2} \leqslant \sigma_2^{k_2} \; \|f\|_{p,m_2}.$$

Then, for all $(k_1, k_2) \in \mathbb{N}^2$;

$$\Big\|(\frac{\partial}{\partial r})^{k_1} (\frac{\partial}{\partial x})^{k_2} f\Big\|_{p,m_2} \quad \leqslant \quad \sigma_1^{k_1} \; \sigma_2^{k_2} \; \|f\|_{p,m_2}$$

and by assertion 2) of theorem 5.4, we deduce that the sequences $\Big(\|(\frac{\partial}{\partial r})^k f\|_{p,m_2}^{\frac{1}{k}}\Big)_k$ and $\Big(\|(\frac{\partial}{\partial x})^k f\|_{p,m_2}^{\frac{1}{k}}\Big)_k$ converge respectively to $\sigma_{f,0}$ and $\sigma_{f,1}$.

- Conversely, suppose that there exists $(M_1, M_2) \in (\mathbb{R}_+^*)^2$ such that

$$\forall (k_1, k_2) \in \mathbb{N}^2; \quad \Big\|(\frac{\partial}{\partial r})^{k_1} (\frac{\partial}{\partial x})^{k_2} f\Big\|_{p,m_2} \leqslant \; M_1^{k_1} M_2^{k_2} \; \|f\|_{p,m_2}.$$

Again, From the second assertion of theorem 5.4, we deduce that the distribution $\Lambda_2^{-1}(T_f)$ has a bounded support. Since, the mapping Λ_2 is a topological isomorphism from $S_*(\mathbb{R}^2)$ onto itself, then $\Lambda_2^{-1}(f)$ lies in $\mathscr{D}_*(\mathbb{R}^2)$. Now, from the relation

$$\widetilde{\mathscr{F}}_\alpha^{-1} = W_\alpha^{-1} \circ \Lambda_2^{-1}$$

and by lemma 5.5, it follows that $\widetilde{\mathscr{F}}_\alpha^{-1}(f)$ belongs to $\mathscr{D}_*(\mathbb{R}^2)$.

□

Remark 5.7. For every $f \in S_*(\mathbb{R}^2)$ and $(k_1, k_2) \in \mathbb{N}^2$, we have

$$E^{k_1} \; C^{k_2} \; B(f) = B\Big((\frac{\partial}{\partial r})^{k_1} (\frac{\partial}{\partial x})^{k_2} f\Big)$$

where

$$E = (\mu^2 + \lambda^2)^{\frac{1}{2}} \frac{\partial}{\partial \mu^2},$$

B and C are defined as above. Then, by the relation (2.8), we deduce that

$$\|E^{k_1} \; C^{k_2} \; B(f)\|_{p,\widetilde{\gamma}_\alpha} = \Big\|(\frac{\partial}{\partial r})^{k_1} (\frac{\partial}{\partial x})^{k_2} f\Big\|_{p,m_2}. \tag{5.20}$$

Theorem 5.8. *(Paley-Wiener-Schwartz) Let f be a function in $S_*(\Gamma)$. Then, the function $\mathscr{F}_\alpha^{-1}(f)$ belongs to the space $\mathscr{D}_*(\mathbb{R}^2)$ if, and only if for all $p \in [1,+\infty]$, there exist $(M_1, M_2) \in (\mathbb{R}_+^*)^2$ such that*

$$\forall (k_1,k_2) \in \mathbb{N}^2; \quad \Big\| E^{k_1}\ C^{k_2}\ (f) \Big\|_{p,\widetilde{\gamma}_\alpha} \leqslant\ M_1^{k_1} M_2^{k_2}\ \|f\|_{p,\widetilde{\gamma}_\alpha}.$$

Moreover, the sequences $\Big(\|E^k(f)\|_{p,\widetilde{\gamma}_\alpha}^{\frac{1}{k}}\Big)_k$ and $\Big(\|C^k(f)\|_{p,\widetilde{\gamma}_\alpha}^{\frac{1}{k}}\Big)_k$ converge respectively to $\delta_{f,0}$ and $\delta_{f,1}$; where

$$\delta_{f,i} = \sup\Big\{|P_i(r,x)|;\ (r,x) \in\ supp\ \mathscr{F}_\alpha^{-1}(f)\Big\}; \quad i \in \{0,1\}.$$

Proof. We know that the Fourier transform $\mathscr{F}_\alpha$ is a topological isomorphism from $S_*(\mathbb{R}^2)$ onto $S_*(\Gamma)$, where the isomorphism inverse is given by

$$\mathscr{F}_\alpha^{-1}(f)(r,x) = \int\int_{\Gamma_+} f(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda).$$

Also; the Fourier-Bessel transform $\widetilde{\mathscr{F}_\alpha}$ is a topological isomorphism from $S_*(\mathbb{R}^2)$ onto itself. Then, from the relation (2.6), we deduce that the mapping B defined by the relation (2.7) is an isomorphism from $S_*(\mathbb{R}^2)$ onto $S_*(\Gamma)$.
Let $f \in S_*(\Gamma)$ and $g = B^{-1}(f)$, we have;

$$\mathscr{F}_\alpha^{-1}(f) = \widetilde{\mathscr{F}}_\alpha^{-1}(g).$$

>From proposition 5.6, $\widetilde{\mathscr{F}}_\alpha^{-1}(g)$ belongs to $\mathscr{D}_*(\mathbb{R}^2)$ if, and only if for all $p \in [1,+\infty]$, there exists $(M_1, M_2) \in (\mathbb{R}_+^*)^2$ such that

$$\forall (k_1,k_2) \in \mathbb{N}^2; \quad \Big\|(\frac{\partial}{\partial r})^{k_1}(\frac{\partial}{\partial x})^{k_2} g\Big\|_{p,m_2} \leqslant\ M_1^{k_1} M_2^{k_2}\ \|g\|_{p,m_2}. \tag{5.21}$$

Using the relation (5.20), when applied to the function g and the fact that

$$\|f\|_{p,\widetilde{\gamma}_\alpha} = \|g\|_{p,m_2},$$

we deduce that, the function $\mathscr{F}_\alpha^{-1}(f)$ belongs to $\mathscr{D}_*(\mathbb{R}^2)$ if, and only if, for all $p \in [1,+\infty]$, there exists $(M_1, M_2) \in (\mathbb{R}_+^*)^2$ such that

$$\forall (k_1,k_2) \in \mathbb{N}^2; \quad \Big\| E^{k_1}\ C^{k_2}\ f \Big\|_{p,\widetilde{\gamma}_\alpha} \leqslant\ M_1^{k_1} M_2^{k_2}\ \|f\|_{p,\widetilde{\gamma}_\alpha}.$$

From the relation (5.21) and proposition 5.6, the sequences

$$\Big(\|(\frac{\partial}{\partial r})^k g\|_{p,m_2}^{\frac{1}{k}}\Big)_k \text{ and } \Big(\|(\frac{\partial}{\partial x})^k g\|_{p,m_2}^{\frac{1}{k}}\Big)_k$$

converge respectively to $\sigma_{g,0}$ and $\sigma_{g,1}$. However,

$$\forall i \in \{0,1\}; \quad \sigma_{g,i} = \sup\left\{|P_i(r,x)|;\ (r,x) \in \text{ supp } \Lambda_2^{-1}(g)\right\}$$

and by the relation (5.19);

$$\begin{aligned} \sigma_{g,i} &= \sup\left\{|P_i(r,x)|;\ (r,x) \in \text{ supp } W_\alpha^{-1}(\Lambda_2^{-1}(g))\right\} \\ &= \sup\left\{|P_i(r,x)|;\ (r,x) \in \text{supp } \mathscr{F}_\alpha^{-1}(f)\right\} \\ &= \delta_{f,i}. \end{aligned}$$

□

References

[1] N. Akhiezer, *Vorlesungen über Approximations Theorie*, Akademieverlag, Berlin, 1953.

[2] L. E. Andersson, *On the determination of a function from spherical averages*, SIAM. J. Math Anal **19** (1988), 214–234.

[3] C. Baccar, N. B. Hamadi, and L. T. Rachdi, *Inversion formulas for the Riemann-Liouville transform and its dual associated with singular partial differential operators*, Internat. J. Math. Math. Sci. 2006, Article ID 86238 (2006), 1–26.

[4] H. H. Bang, *A property of infinitely differentiable functions*, Proc. Amer. Math. Soc **108** (1990), 73–76.

[5] R. P. Boas, *Entire Functions*, Academic Press, New-York, 1954.

[6] A. Erdely and all, *Higher Transcendental Functions*, vol. I, Mc Graw-Hill Book Compagny, New-York, 1953.

[7] ———, *Tables of Integral Transforms*, vol. II, Mc Graw-Hill Book Compagny, New-York, 1954.

[8] J. A. Fawcett, *Inversion of n-dimensional spherical means*, SIAM. J. Appl. Math. **45** (1985), 336–341.

[9] H. Helesten and L. E. Anderson, *An inverse method for the processing of synthetic aperture radar data*, Inv. Prob. **3** (1987), 111–124.

[10] M. Herberthson, *A numerical Implementation of an inverse formula for CARABAS raw data*, National Defense Research Institute, Internal Report D 30430-3.2, Linköping, Sweden, 1986.

[11] A. N. Kolmogoroff, *On inequalities between upper bounds of the successive derivatives of an arbitrary function on an infinite interval*, vol. 4, Amer. Math. Soc. Translation, 1949.

[12] N.N. Lebedev, *Special Functions and Their Applications*, Dover publications, Inc., New-York, 1972.

[13] M. M. Nessibi, L. T. Rachdi, and K. Trimèche, *Ranges and inversion formulas for spherical mean operator and its dual*, J. Math. Anal. Appl. **196** (1995), 861–884.

[14] L. T. Rachdi and K. Trimèche, *Weyl transforms associated with the spherical mean operator*, Anal. Appl. **1** (2003), 141–164, No. 2.

[15] L. Schwartz, *Theory of Distributions*, I, Hermann, Paris, 1957.

[16] ———, *Theorie des distributions*, Hermann, Paris, 1978.

[17] E. M. Stein, *Functions of exponential type*, Ann. of Math. **65, No 2** (1957), 582–592.

[18] CH. Swartz, *Convergence of convolution operators*, Studia.Math. **42** (1972), 249–257.

[19] K. Trimèche, *Transformation intégrale de Weyl et théorème de Paley-Wiener associés à un opérateur différentiel singulier sur* $(0,+\infty)$, J. Math. Pures Appl. **60** (1981), 51–98.

[20] ———, *Inversion of the Lions translation operator using generalized wavelets*, Appl. Comput. Harmonic Anal. **4** (1997), 97–112.

[21] Vu Kim Tuan, *On the range of the Hankel and extended Hankel transforms*, J. Math. Anal. Appl. **209** (1997), 460–478.

[22] G.N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed. Cambridge Univ. Press., London/New-York, 1966.

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