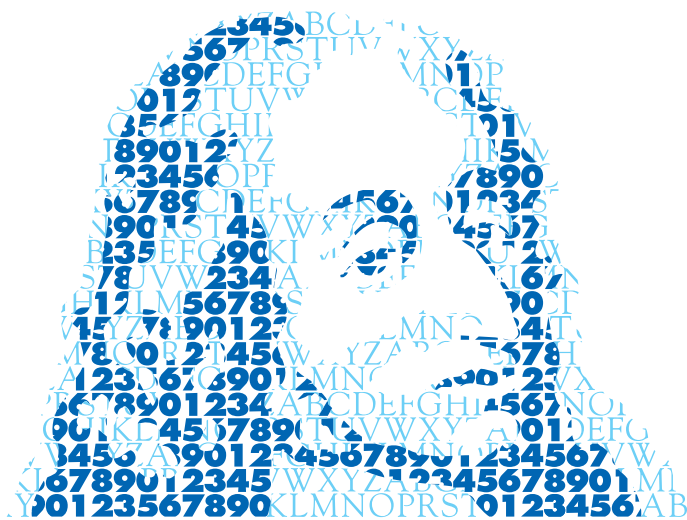# ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Some examples of harmonic maps for g-natural metrics

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Abstract

We produce new examples of harmonic maps, having as source manifold a space (M, g) of constant curvature and as target manifold its tangent bundle TM, equipped with a suitable Riemannian g-natural metric. In particular, we determine a family of Riemannian g-natural metrics G on $T\mathbb{S}^2$, with respect to which all conformal gradient vector fields define harmonic maps from $\mathbb{S}^2$ into $(T\mathbb{S}^2, G)$.

Résumé

On produit des nouveaux exemples d'applications harmoniques, ayant chacune comme espace de départ une variété (M, g) á courbure constante et comme espace d'arrivée son fibré tangent TM, muni d'une métrique g-naturelle Riemannienne appropriée. En particulier, on va déterminer une famille de métriques g-naturelles Riemanniennes G sur $T\mathbb{S}^2$, par rapport auxquelles tous les champs de vecteurs gradients conformes définissent des applications harmoniques de $\mathbb{S}^2$ dans $(T\mathbb{S}^2, G)$.

1. Introduction

In the theory of harmonic maps, a fundamental question concerns the existence of harmonic maps between two given Riemannian manifolds (M, g) and (M', g').

If (M, g) is compact and (M', g') is of non-positive sectional curvature, then there exists a harmonic map $f : (M, g) \to (M', g')$ in each homotopy class [8]. However, there is no general existence result when (M', g') admits some positive sectional curvatures. This fact makes it interesting to find examples of harmonic maps having such a target manifold. Clearly,

the non-applicability of the standard existence theory for harmonic maps, leads to the construction of examples by an *ad hoc* approach. Some examples in this direction are given in [5], where harmonic maps are defined by tangent vector fields. Moreover, given an arbitrary section σ of a Riemannian manifold (M, g), Oniciuc [12] showed how to construct a suitable metric on the tangent bundle TM, which depends on σ and makes $\sigma : (M, g) \to TM$ totally geodesic, hence harmonic.

The *Sasaki metric g^s* is by far the most investigated among all possible Riemannnian metrics on TM. Nouhaud [11] and Ishihara [9] independently proved that parallel vector fields on a compact Riemannian manifold are the only defining harmonic maps $V : (M, g) \to (TM, g^s)$. This rigid behaviour of the Sasaki metric with respect to harmonicity, and the fact that g^s and other well-known Riemannian metrics on TM are g-natural, motivated in [1] the study of the harmonicity of $V : (M, g) \to (TM, G)$, with TM equipped with an arbitrary Riemannian g-natural metric. Riemannian g-natural metrics form a large family of Riemannian metrics on TM, which depends on six independent smooth functions from $\mathbb{R}^+$ to $\mathbb{R}$. Several different behaviours have been found for the harmonic maps of (TM, G), with respect to different Riemannian g-natural metrics G. In particular, it was proved that, for a suitable choice of G, some remarkable non-parallel vector fields, such as Hopf vector fields on the unit sphere $\mathbb{S}^{2m+1}$ and, more generally, the Reeb vector field of a contact metric manifold, are harmonic maps of (TM, G).

In this paper, we give new examples of harmonic maps from a Riemannian manifold (M, g) of constant sectional curvature and its tangent bundle TM, equipped with some Riemannian g-natural metric G. More precisely, in Section 4 we show that there exists a family of Riemannian g-natural metrics G on $T\mathbb{S}^2$ (also including some metrics conformal to the Sasaki metric g^s), with respect to which all conformal gradient vector fields define harmonic maps from $\mathbb{S}^2$ into $(T\mathbb{S}^2, G)$. It is worthwhile to remark that $(T\mathbb{S}^2, G)$ necessarily admits some positive sectional curvatures, and that the metrics of Cheeger-Gromoll type $h_{p,q}$, studied by M. Benyounes, E. Loubeau and C.M. Wood ([5],[6]), fail to provide examples of harmonic maps from $(\mathbb{S}^2, g)$ to $(T\mathbb{S}^2, h_{p,q})$, for any Riemannian metric g on $\mathbb{S}^2$. In Section 5, we show that if (M, g) has constant non-positive sectional curvature, then a Riemannian g-natural metric can be found, such that all vector fields of a fixed constant length are harmonic maps. In Section 6, we suppose (M, g) flat and characterize harmonic maps for a

family of Riemannian g-natural metrics on TM. Within this family, there exist Riemannian g-natural metrics for which the target manifold TM has nonnegative sectional curvature. The definition and basic properties of g-natural metrics on TM will be recalled in Section 2, while the expression of the tension field associated to $V : (M, g) \to (TM, G)$ is in Section 3.

2. **Basic formulas on g-natural metrics on tangent bundles**

Let (M, g) be an n-dimensional Riemannian manifold and ∇ its Levi-Civita connection. At any point (x, u) of its *tangent bundle TM*, the tangent space of TM splits into the horizontal and vertical subspaces with respect to ∇: $(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$. For any vector $X \in M_x$, there exists a unique vector $X^h \in \mathcal{H}_{(x,u)}$ (the *horizontal lift* of X to $(x, u) \in TM$), such that $\pi_* X^h = X$, where $\pi : TM \to M$ is the natural projection. The *vertical lift* of a vector $X \in M_x$ to $(x, u) \in TM$ is a vector $X^v \in \mathcal{V}_{(x,u)}$ such that $X^v(df) = Xf$, for all functions f on M. Here we consider 1-forms df on M as functions on TM (i.e., $(df)(x, u) = uf$). The map $X \to X^h$ is an isomorphism between the vector spaces M_x and $\mathcal{H}_{(x,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between M_x and $\mathcal{V}_{(x,u)}$. Each tangent vector $\tilde{Z} \in (TM)_{(x,u)}$ can be written $\tilde{Z} = X^h + Y^v$, where $X, Y \in M_x$ are uniquely determined vectors. Horizontal and vertical lifts of vector fields on M can be defined in an obvious way and define vector fields on TM.

The Sasaki metric g^s has been the most investigated among all possible Riemannnian metrics on TM. However, in many different contexts such metrics showed a very "rigid" behaviour. Moreover, g^s represents only one possible choice inside a wide family of Riemannian metrics on TM, known as *Riemannian g-natural metrics*, which depend on several independent smooth functions from $\mathbb{R}^+$ to $\mathbb{R}$. As their name suggests, those metrics arise from a very "natural" construction starting from a Riemannian metric g over M. g-natural metrics arise from the description of all possible first order natural operators $D : S^2_+ T^* \rightsquigarrow (S^2 T^*)T$, transforming Riemannian metrics on manifolds into metrics on their tangent bundles, where $S^2_+ T^*$ and $S^2 T^*$ denote the bundle functors of all Riemannian metrics and all symmetric $(0, 2)$-tensors over manifolds. For more details about the concept of naturality and related notions, we refer to [10].

We shall call *g-natural metric* a metric G on TM, coming from g by a first-order natural operator $S^2_+ T^* \rightsquigarrow (S^2 T^*)T$ [2]. Given an arbitrary g-natural metric G on the tangent bundle TM of a Riemannian manifold (M, g), there are six smooth functions $\alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, such that for every $u, X, Y \in M_x$, we have

$$\begin{cases} G_{(x,u)}(X^h, Y^h) &= (\alpha_1 + \alpha_3)(r^2) g_x(X, Y) \\ &\quad + (\beta_1 + \beta_3)(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) &= \alpha_2(r^2) g_x(X, Y) + \beta_2(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^v, Y^h) &= \alpha_2(r^2) g_x(X, Y) + \beta_2(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) &= \alpha_1(r^2) g_x(X, Y) + \beta_1(r^2) g_x(X, u) g_x(Y, u), \end{cases} \quad (2.1)$$

where $r^2 = g_x(u, u)$. For $n = 1$, the same holds with $\beta_i = 0$, $i = 1, 2, 3$. Put

- $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$,

- $\alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t)$,

- $\phi(t) = \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t)$,

for all $t \in \mathbb{R}^+$. Then, a g-natural metric G on TM is Riemannian if and only if

$$\alpha_1(t) > 0, \quad \phi_1(t) > 0, \qquad \alpha(t) > 0, \quad \phi(t) > 0, \quad \forall\, t \in \mathbb{R}^+. \quad (2.2)$$

Convention a) In the sequel, all Riemannian g-natural metrics G on TM will be defined by the functions $\alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, satisfying (2.1)-(2.2).

b) Unless otherwise stated, all real functions α_i, β_i, ϕ_i, α and ϕ and their derivatives are evaluated at $r^2 := g_x(u, u)$.

c) We shall denote respectively by R and Q the *curvature tensor* and the *Ricci operator* of a Riemannian manifold (M, g). The tensor R is taken with the sign convention

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for all vector fields X, Y, Z on M.

All examples presented in this paper will concern Riemannian g-natural metrics satisfying

$$\alpha_2 = \beta_2 = 0, \quad \alpha_1 > 0, \quad \alpha_1 + \alpha_3 > 0, \quad (2.3)$$
$$\phi_1 = \alpha_1 + t\beta_1 > 0 \text{ and } \alpha_1 + \alpha_3 + t(\beta_1 + \beta_3) > 0.$$

Note that, because of (2.1), $\alpha_2 = \beta_2 = 0$ is equivalent to requiring that the horizontal and vertical distributions are mutually orthogonal with respect to G. On the other hand, inequalities in (2.3) are nothing but (2.2), rewritten when $\alpha_2 = \beta_2 = 0$.

The Levi-Civita connection $\bar{\nabla}$ of an arbitrary g-natural metric G on TM was written down in [3], to which we refer for more details. Restricting ourselves to the case when $\alpha_2 = \beta_2 = 0$, we have the following

Proposition 2.1. *Let (M, g) be a Riemannian manifold, ∇ its Levi-Civita connection and R its curvature tensor. Let G be a Riemannian g-natural metric on TM, satisfying $\alpha_2 = \beta_2 = 0$. Then, the Levi-Civita connection $\bar{\nabla}$ of (TM, G) is characterized by*

$$(i) \ \ (\bar{\nabla}_{X^h} Y^h)_{(x,u)} = (\nabla_X Y)^h_{(x,u)} + v\{B(u; X_x, Y_x)\},$$

$$(ii) \ \ (\bar{\nabla}_{X^h} Y^v)_{(x,u)} = (\nabla_X Y)^v_{(x,u)} + h\{C(u; X_x, Y_x)\},$$

$$(iii) \ \ (\bar{\nabla}_{X^v} Y^h)_{(x,u)} = h\{C(u; Y_x, X_x)\},$$

$$(iv) \ \ (\bar{\nabla}_{X^v} Y^v)_{(x,u)} = v\{F(u; X_x, Y_x)\},$$

for all vector fields X, Y on M and $(x, u) \in TM$, where B, C and F are defined, for all u, X, $Y \in M_x$, $x \in M$, by:

$$B(u; X, Y) = B_2 R(X, Y)u + B_3[g_x(Y, u)X + g_x(X, u)Y]$$
$$+ B_5 g_x(X, Y)u + B_6 g_x(X, u)g_x(Y, u)u,$$

where

$$\begin{cases} B_2 = -\frac{1}{2}, \quad B_3 = -\frac{\beta_1 + \beta_3}{2\alpha_1}, \quad B_5 = -\frac{(\alpha_1 + \alpha_3)'}{\phi_1}, \\ B_6 = \frac{-\alpha_1(\beta_1 + \beta_3)' + \beta_1(\beta_1 + \beta_3)}{\alpha_1 \phi_1}, \end{cases}$$

$$C(u; X, Y) = C_1 R(Y, u)X + C_2 g_x(X, u)Y + C_3 g_x(Y, u)X$$
$$+ C_4 g_x(R(X, u)Y, u)u + C_5 g_x(X, Y)u + C_6 g_x(X, u)g_x(Y, u)u,$$

where

$$\begin{cases} C_1 = -\frac{\alpha_1}{2(\alpha_1+\alpha_3)}, \quad C_2 = -\frac{\beta_1+\beta_3}{2(\alpha_1+\alpha_3)}, \quad C_3 = \frac{(\alpha_1+\alpha_3)'}{\alpha_1+\alpha_3}, \\ C_4 = \frac{\alpha_1(\beta_1+\beta_3)}{2(\alpha_1+\alpha_3)(\phi_1+\phi_3)}, \quad C_5 = \frac{\beta_1+\beta_3}{2(\phi_1+\phi_3)}, \\ C_6 = \frac{2\alpha_1(\alpha_1+\alpha_3)(\beta_1+\beta_3)'-\alpha_1(\beta_1+\beta_3)[2(\alpha_1+\alpha_3)'+(\beta_1+\beta_3)]}{2\alpha_1\phi_1}, \end{cases}$$

$$F(u;X,Y) = F_1[g_x(Y,u)X + g_x(X,u)Y] + F_2 g_x(X,Y)u$$

$$+F_3 g_x(X,u)g_x(Y,u)u,$$

where

$$F_1 = \frac{\alpha_1'}{\alpha_1}, \quad F_2 = \frac{\beta_1 - \alpha_1'}{\phi_1}, \quad F_3 = \frac{\alpha_1\beta_1' - 2\alpha_1'\beta_1}{\alpha_1\phi_1}.$$

3. Harmonicity of $V : (M,g) \to (TM,G)$

We briefly recall that, given $f : (M,g) \to (M',g')$ a smooth map between Riemannian manifolds, with M compact, the *energy* of f is defined as the integral

$$E(f) := \int_M e(f)dv_g$$

where $e(f) = \frac{1}{2}||f_*||^2 = \frac{1}{2}\text{tr}_g f^* g'$ is the *energy density* of f. With respect to a local orthonormal basis of vector fields $\{e_1,..,e_n\}$ on M, the energy density is $e(f) = \frac{1}{2}\sum_{i=1}^n g'(f_*e_i, f_*e_i)$. Critical points of the energy functional on $C^\infty(M,M')$ are called *harmonic maps*. They have been characterized in [8] as maps having vanishing *tension field* $\tau(f) = \text{tr}\nabla df$. When M is not compact, a map $f : (M,g) \to (M',g')$ is said to be harmonic if $\tau(f) = 0$. For more on harmonic maps, cf. [7],[15].

Now, let (M,g) be a compact Riemannian manifold of dimension n and (TM,G) its tangent bundle, equipped with a g-natural Riemannian metric. Each vector field $V \in \mathcal{X}(M)$ defines a smooth map $V : (M,g) \to (TM,G)$, $p \mapsto V_p \in M_p$. By definition, the *energy* $E(V)$ of V is the energy associated to the corresponding map $V : (M,g) \to (TM,G)$. Therefore, $E(V) = \int_M e(V)dv_g$, where the density function $e(V)$ is given by

$$e_p(V) = \frac{1}{2}||V_{*p}||^2 = \frac{1}{2}\text{tr}_g(V^* G)_p = \frac{1}{2}\sum_{i=1}^n (V^* G)_p(e_i, e_i), \qquad (3.1)$$

$\{e_1, .., e_n\}$ being any local orthonormal basis of vector fields defined in a neighborhood of p. Harmonicity of the map $V : (M, g) \to (TM, G)$ is studied in [1]. Using formulas of Proposition 2.1, we have

$$2\, e(V) = n(\alpha_1 + \alpha_3)(r^2) + (\beta_1 + \beta_3)(r^2)r^2 + \alpha_1(r^2)||\nabla V||^2$$
$$+ \tfrac{1}{4}\beta_1(r^2)||\text{grad}\, r^2||^2$$

where $r = ||V||$.

Next, considering the horizontal and vertical components of the tension field $\tau(V) = \text{tr}(\nabla dV)$ associated to the map $V : (M, g) \to (TM, G)$, harmonic maps $V : (M, g) \to (TM, G)$ where characterized in [1] for all Riemannian g-natural metrics on TM. Restricting ourselves to the case of Riemannian g-natural metrics satisfying (2.3), from the main result of [1] we have at once the following

Theorem 3.1. *Let (M, g) be a compact Riemannian manifold and G a Riemannian g-natural metric on TM, determined by functions α_i, β_i satisfying (2.3). A vector field $V \in \mathfrak{X}(M)$ defines a harmonic map $V : (M, g) \to (TM, G)$ if and only if*

$$
\begin{aligned}
\tau_h(V) \;=\; & -\frac{\alpha_1}{\alpha_1 + \alpha_3}\text{tr}[R(\nabla.V, V)\cdot] + \frac{(\alpha_1 + \alpha_3)'}{\alpha_1 + \alpha_3}\text{grad}\, r^2 - \frac{\beta_1 + \beta_3}{\alpha_1 + \alpha_3}\nabla_V V \\
& + \Big[\frac{\alpha_1(\beta_1 + \beta_3)}{(\alpha_1 + \alpha_3)(\phi_1 + \phi_3)}g(\text{tr}[R(\nabla.V, V)\cdot], V) + \frac{\beta_1 + \beta_3}{\phi_1 + \phi_3}\text{div}V \\
& + \frac{2\alpha_1(\alpha_1 + \alpha_3)(\beta_1 + \beta_3)' - 2\alpha_1(\beta_1 + \beta_3)(\alpha_1 + \alpha_3)'}{2\alpha_1\phi_1}V(r^2) \\
& - \frac{\alpha_1(\beta_1 + \beta_3)(\beta_1 + \beta_3)}{2\alpha_1\phi_1}V(r^2)\Big]V = 0, \quad\quad (3.2) \\
\tau_v(V) \;=\; & -\bar{\Delta}V + \frac{\alpha_1'}{\alpha_1}\nabla_{\text{grad}\, r^2}V + \Big[-\frac{\beta_1 + \beta_3}{\alpha_1} - n\frac{(\alpha_1 + \alpha_3)'}{\phi_1} \\
& + \frac{\beta_1 - \alpha_1'}{\phi_1}||\nabla V||^2 + \frac{\beta_1(\beta_1 + \beta_3) - \alpha_1(\beta_1 + \beta_3)'}{\alpha_1\phi_1}r^2 \quad\quad (3.3) \\
& + \frac{\alpha_1\beta_1' - 2\alpha_1'\beta_1}{4\alpha_1\phi_1}\big|\big|\text{grad}\, r^2\big|\big|^2\Big]V = 0.
\end{aligned}
$$

Remark 3.2. As usual, we can assume condition $\tau(V) = 0$ as a *definition* of harmonic maps when M is not compact, and Theorem 3.1 extends at once to the non-compact case.

Remark 3.3. In the next Sections, we shall use Riemannian g-natural metrics on the tangent bundle TM, and the characterization obtained in this Section for their harmonic maps, in order to determine new examples of harmonic maps. There are some prices to pay when the Sasaki metric is replaced by Riemannian g-natural metrics:

- With respect to these metrics, parallel vector fields are *not* always harmonic [1].

- The projection map $\pi : (TM, G) \to (M, g)$ is *not* always a Riemannian submersion [3].

4. Harmonic maps $V : \mathbb{S}^2 \to (\mathbb{S}^2, G)$

Let $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ be the unit sphere with the induced Riemannian metric g. Xin [16] proved that for any compact manifold (N, h), all non-constant harmonic maps $f : (\mathbb{S}^n, g) \to (N, h)$ are unstable, provided that $n \geq 3$. Vector fields responsible for this unstability are the conformal gradient vector fields. Given any vector $a \in \mathbb{R}^{n+1}$, $a \neq 0$, the *conformal gradient vector field* V_a corresponding to a, is defined as $V_a = \nabla \lambda_a$, where, for all $x \in \mathbb{S}^n$, $\lambda_a(x) = < x, a >$. These vector fields satisfy [16]:

$$\nabla_X V_a = -\lambda_a X, \quad \bar{\Delta} V_a = V_a. \tag{4.1}$$

Moreover, putting $c^2 = ||a||^2$, one has [5]

$$r^2 = ||V_a||^2 = c^2 - \lambda_a^2, \quad \nabla r^2 = -2\lambda_a V_a. \tag{4.2}$$

Finally, (4.1) also easily implies

$$\mathrm{div} V_a = -n\lambda_a, \qquad ||\nabla V_a||^2 = n\lambda_a^2. \tag{4.3}$$

Let now G be an arbitrary Riemannian g-natural metric on $T\mathbb{S}^n$. Using (4.1), (2.1) and (2.3), we obtain

$$V_a^* G = \{(1+\lambda_a^2)\alpha_1(r^2)+\alpha_3(r^2)\}g+\{(1+\lambda_a^2)\beta_1(r^2)+\beta_3(r^2)\}\omega_a\otimes\omega_a, \tag{4.4}$$

where ω_a is the 1-form dual to V_a.

Next, we recall that the volume of V_a is defined as

$$\text{vol}(V_a) = \int_{\mathbb{S}^n} \sqrt{\det V_a^* G}\, v_g = \int_{\mathbb{S}^n} \sqrt{\mu_1 \dots \mu_n}\, v_g,$$

where $\mu_1 \dots \mu_n$ are the eigenvalues of $V_a^* G$. In particular, if $n = 2$, then $2\sqrt{\mu_1 \mu_2} \le \mu_1 + \mu_2$ and so,

$$\text{vol}(V_a) \le E(V_a) = \frac{1}{2} \int_{\mathbb{S}^n} (\mu_1 + \mu_2)\, v_g$$

and the equality holds if and only if $\mu_1 = \mu_2$, that is, V_a is a *conformal map*. On the other hand, Sanini [14] proved that if $f : (M, g) \to (M', g')$ is a conformal map and M is a compact two-dimensional manifold, then the energy of f remains unaltered for any deformation of the metric g on M. Therefore, we have the following

Proposition 4.1. *Let G be a Riemannian g-natural metric on $T\mathbb{S}^2$, satisfying $\beta_1 = \beta_2 = \beta_3 = \alpha_2 = 0$. Then,*

$$E(V_a) = \int_{\mathbb{S}^n} ((1 + \lambda_a^2)\alpha_1(r^2) + \alpha_3(r^2))\, v_g = \text{vol}(V_a).$$

Moreover, $E(V_a)$ is stationary with respect to an arbitrary deformation of the standard metric g_0 on $\mathbb{S}^2$.

Harmonicity of conformal gradient vector fields was investigated in [5], equipping the tangent bundle with a metric of Cheeger-Gromoll type. However, in the examples found in [5], the metrics are not Riemannian, but have varying signature. Moreover, these metrics fail to provide examples of harmonic maps defined on $\mathbb{S}^2$. We will show in the sequel how to construct such examples using Riemannian g-natural metrics.

Consider a Riemannian g-natural metric G on TM of the special type described by (2.3). Moreover, we also assume $\beta_1 = \beta_3 = 0$. We then rewrite harmonicity conditions $\tau_h(V) = \tau_v(V) = 0$ for a conformal gradient vector field $V_a = \nabla \lambda_a$. Using (4.1) and (4.3), we obtain that $V_a : \mathbb{S}^n \to (T\mathbb{S}^n, G)$ is a harmonic map if and only if

$$\begin{cases} -2(\alpha_1 + \alpha_3)' + \alpha_1 - n\alpha_1 = 0, \\ -\alpha_1 - n(\alpha_1 + \alpha_3)' + (2 - n)\alpha_1' \lambda_a^2 = 0. \end{cases} \tag{4.5}$$

Calculating $(\alpha_1 + \alpha_3)'$ from the first equation in (4.5) and replacing into the second one, we see that if $n = 2$, then (4.5) reduces to its first equation, namely,

$$\alpha_1 + 2(\alpha_1 + \alpha_3)' = 0, \tag{4.6}$$

which can be integrated finding solutions compatible with the Riemannian conditions (2.2). Thus, if G is a Riemannian g-natural metric on $T\mathbb{S}^2$ determined by

$$\alpha_1 > 0, \ \alpha_1 + \alpha_3 > 0, \ \alpha_1 + 2(\alpha_1 + \alpha_3)' = 0, \ \alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0, \ (4.7)$$

then $V_a : \mathbb{S}^2 \to (T\mathbb{S}^2, G)$ is a harmonic map. For example, we can take explicitly

$$\alpha_1(t) = \mu e^{-\frac{1}{2(\mu_1+1)}t}, \quad \alpha_3(t) = \mu_1 \alpha_1(t), \quad \alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0, \quad (4.8)$$

for any real constants $\mu > 0$ and $\mu_1 \geq 0$. Note that if $\mu_1 = 0$, then the Riemannian g-natural metric G determined by (4.8) is conformal to the Sasaki metric g^s on $T\mathbb{S}^2$; moreover, if G is a Riemannian g-natural metric on $T\mathbb{S}^2$ determined by (4.7), then (by (4.4)) $V_a^* G$ is conformal to g.

Next, we recall the following decreasing property for harmonic immersions of a surface, proved by Sampson ([13], Theorem 7, p.217): if $f : (M^2, g) \to (\tilde{M}, \tilde{g})$ is a harmonic immersion and $f^* \tilde{g}$ is conformal to g, then the sectional curvatures of $(M^2, f^* \tilde{g})$ and $(\tilde{M}, \tilde{g})$ satisfy

$$K_{f^* \tilde{g}}(T_x M^2) \leq K_{\tilde{g}}(f_* T_x M^2),$$

for any $x \in M^2$. This result ensures that $(T\mathbb{S}^2, G)$ admits some positive sectional curvatures for any Riemannian g-natural metric G on $T\mathbb{S}^2$ satisfying (4.7). In fact, the Gauss-Bonnet Theorem then gives

$$\frac{1}{2\pi} \int_{\mathbb{S}^2} K_{V_a^* \tilde{g}}(T_x \mathbb{S}^2) = \chi(\mathbb{S}^2) = 2 > 0,$$

where $\chi(\mathbb{S}^2)$ denotes the Euler number of $\mathbb{S}^2$. Therefore, we proved the following

Theorem 4.2. *Let G be a Riemannian g-natural metric satisfying* (4.7). *Then,* all *conformal gradient vector fields $V_a = \nabla \lambda_a$ on $\mathbb{S}^2$ define harmonic maps $V_a : \mathbb{S}^2 \to (T\mathbb{S}^2, G)$. Moreover, $(T\mathbb{S}^2, G)$ admits some positive sectional curvatures.*

Remark 4.3. Standard calculations show that system (4.5), when $n \geq 3$, admit solutions which depend on the function $\lambda_a^2 = c^2 - r^2$ defining the conformal gradient vector field V_a. Explicitly, we find

$$\alpha_1(t) = \mu(c^2 - t)^{\frac{1-n}{2}}$$

and

$$\alpha_3(t) = \begin{cases} -\alpha_1(t) + \mu \ln(c^2 - t) + \kappa & \text{if } n = 3, \\ -\alpha_1(t) - \mu \frac{n-1}{n-3}(c^2 - t)^{\frac{3-n}{2}} + \kappa & \text{if } n > 3, \end{cases}$$

for arbitrary constants $\mu > 0$ and κ. However, in general these solutions are not compatible with the Riemannian conditions (2.2). Indeed, an easy analysis of the above expressions of α_3 leads to the following conclusions:

- The metric G is Riemannian if and only if $n > 3$, $\kappa > 0$ and $c^2 - \left[\frac{n-3}{n-1}\frac{\kappa}{\mu}\right]^{\frac{2}{3-n}} \leq 0$.

- The metric G is pseudo-Riemannian of signature (n, n) if and only if either $n = 3$ and $c^2 - e^{-\frac{\kappa}{\mu}} \leq 0$, or $n > 3$ and $\kappa \leq 0$.

- In the remaining cases, the metric G is of varying signature.

The following result has been proved in [4]: let $f : (M^2, g) \to (M', g')$ be a conformal harmonic map, with M compact. If f is volume-stable, then it is also energy-stable. Theorem 4.2 above then implies the following

Corollary 4.4. *Let G be a Riemannian g-natural metric satisfying (4.7). If the harmonic map $V_a : \mathbb{S}^2 \to (T\mathbb{S}^2, G)$ is volume-stable, then it is also energy-stable.*

5. Harmonic maps $V : (M(k), g) \to (TM, G)$ when $k \leq 0$

Let $(M(k), g)$ be an n-dimensional Riemannian manifold of constant sectional curvature $k \leq 0$, and V a vector field on M, of constant length $\sqrt{\rho}$. Consider a Riemannian g-natural metric G on TM, determined by functions $\alpha_i(t), \beta_i(t)$ satisfying

$$\alpha_2 = \beta_2 = 0, \qquad \beta_3 = -\beta_1 - k\alpha_1. \tag{5.1}$$

Using the second equation of (5.1), one can easily rewrite the connection functions given in Proposition 2.1 and then the horizontal and vertical components of the tension field $\tau(V)$. We also use the constancy of the sectional curvature and of the length of V. We conclude that $\tau_h(V) = 0$ identically, and

$$\tau_v(V) = -\bar{\Delta}V + \left[k - n\frac{(\alpha_1 + \alpha_3)'}{\phi_1} - k\rho\frac{\beta_1 - \alpha_1'}{\phi_1} - \frac{\alpha_1' - \beta_1}{\phi_1}\|\nabla V\|^2\right]V. \tag{5.2}$$

By (5.2), $\tau_v(V) = 0$ if and only if

$$\bar{\Delta} V = \left[k - n\frac{(\alpha_1 + \alpha_3)'}{\phi_1} - k\rho\frac{\beta_1 - \alpha_1'}{\phi_1} - \frac{\alpha_1' - \beta_1}{\phi_1}||\nabla V||^2\right]V. \qquad (5.3)$$

Since in this case $\bar{\Delta} V = \frac{1}{\rho}||\nabla V||^2 V$, replacing into (5.3) we find

$$(\alpha_1 + \rho\alpha_1')||\nabla V||^2 = -n\rho(\alpha_1 + \alpha_3)' + k\rho(\alpha_1 + \rho\alpha_1'). \qquad (5.4)$$

In particular, assuming $\alpha_1 + \alpha_3 = \mu$ and taking $\alpha_1(t) = ce^{-\frac{1}{\rho}t}$, for two positive constants μ, c, (5.4) is automatically satisfied. Then, we get

Theorem 5.1. *Let $(M(k), g)$ be an n-dimensional Riemannian manifold of nonpositive constant sectional curvature k, and G a Riemannian g-natural metric on TM, determined by functions $\alpha_i(t), \beta_i(t)$ which satisfy*

$$\alpha_1(t) = ce^{-\frac{1}{\rho}t}, \quad \alpha_3 = \mu - \alpha_1, \quad \alpha_2 = \beta_2 = 0, \quad \beta_3 = -\beta_1 - k\alpha_1, \quad (5.5)$$

where ρ, μ, c are positive real constants. Then, any vector field V of constant lengt ρ defines a harmonic map $V : (M(k), g) \to (TM, G)$.

Remark 5.2. Conditions (5.5) ensure that (2.2) are satisfied and so, g-natural metrics G as in Theorem 5.1 are Riemannian. Note that when $k > 0$, then (5.5) again determines g-natural metrics G for which all vector fields of constant length $\sqrt{\rho}$ are harmonic maps. However, these G are not Riemannian, because (2.2) are not satisfied. Explicit examples of non-parallel harmonic maps of (TM, G'), when (M, g) has positive constant sectional curvature and G' is a suitable Riemannian g-natural metric, have already been found in [1].

6. Harmonic maps $V : (M, g) \to (TM, G)$ when (M, g) is flat

Let (M, g) be an n-dimensional locally flat Riemannian manifold and consider a (local) orthonormal frame of coordinate vector fields $\{\frac{\partial}{\partial x_i}\}$ on M. Clearly, a vector field $V = \sum V_i \frac{\partial}{\partial x_i}$ is parallel if and only if V_i is constant for all $i = 1, \ldots, n$.

As it is well known, if the tangent bundle TM is equipped with the Sasaki metric g^s, then parallel vector fields are harmonic maps $(M, g) \to (TM, g^s)$. Moreover, when $M = \Omega$ is an open subset of $\mathbb{R}^n$, a vector field $V = \sum V_i \frac{\partial}{\partial x_i}$ defines a harmonic map from $\Omega \to (T\Omega, g^s)$ if and only if

the map

$$f : \Omega \to \mathbb{R}^n, \qquad x \mapsto (V^1(x), .., V^n(x))$$

is harmonic. To our knowledge, these are the only general results on the harmonicity of vector fields defined in a locally flat manifold.

Consider now $M = \Omega$ an open subset of $\mathbb{R}^n$, equipped with the induced flat metric, and a Riemannian g-natural metric G on TM, satisfying (2.3) and the additional conditions

$$\alpha_1 + \alpha_3 = a > 0, \qquad \beta_1 = \beta_3 = 0, \tag{6.1}$$

where a is a constant. Then, by Theorem 3.1 and taking into account $R = 0$ and (6.1), we have that (3.2) holds identically, and the map $V : (M, g) \to (TM, G)$ is harmonic if and only if

$$-\bar{\Delta}V + \frac{\alpha_1'}{\alpha_1}\nabla_{\mathrm{grad}\, r^2}V - \frac{\alpha_1'}{\alpha_1}||\nabla V||^2 V = 0 \tag{6.2}$$

Since $V = \sum V_i \frac{\partial}{\partial x_i}$, we then have $r^2 = \sum_i V_i^2$ and we easily find

$$\bar{\Delta}V = -\sum_{i,k}\frac{\partial^2 V_k}{\partial x_i^2}\frac{\partial}{\partial x_k}, \qquad \nabla_{\mathrm{grad}\, r^2}V = 2\sum_{i,j,k}V_i\frac{\partial V_i}{\partial x_j}\frac{\partial V_k}{\partial x_j}\frac{\partial}{\partial x_k}.$$

Hence, (6.2) is equivalent to the following system of partial differential equations:

$$\sum_i\frac{\partial^2 V_k}{\partial x_i^2} + 2\frac{\alpha_1'}{\alpha_1}\sum_{i,j}V_i\frac{\partial V_i}{\partial x_j}\frac{\partial V_k}{\partial x_j} - \frac{\alpha_1'}{\alpha_1}\sum_{i,j}\left(\frac{\partial V_j}{\partial x_i}\right)^2 V_k = 0, \qquad k = 1, \ldots, n. \tag{6.3}$$

Because of the regularity of α_1, general existence results for the solutions of a system of partial differential equations, ensure that system (6.3) admits solutions. Note that whenever at least one component V_k is not constant, V is not parallel.

As an extremely special case, assume the components of V are all constant except for V_k, for a fixed index k, and that V_k only depends on one of the local coordinates, say $V_k = V_k(x_i)$. Then, system (6.3) reduces to

$$\frac{\partial^2 V_k}{\partial x_i^2} = -\frac{\alpha_1'}{\alpha_1}V_k\left(\frac{\partial V_k}{\partial x_i}\right)^2. \tag{6.4}$$

Remark 6.1. It would be worthwhile to find some explicit solutions of system (6.3), also taking into account the fact that there exist some g-natural

Riemannian metrics G satisfying (2.3) and (6.1), such that (TM, G) has *non-negative sectional curvature*. For example, we have the following

Proposition 6.2. *Let (M, g) be a flat Riemannian manifold and G a Riemannian g-natural metric defined by*

$$\alpha_1(t) = \frac{1}{1+t}, \quad \alpha_3(t) = a - \frac{1}{1+t}, \quad \alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0. \quad (6.5)$$

Then, (TM, G) has non-negative sectional curvature.

Proof. Applying the formulas for the curvature tensor of a Riemannian g-natural metric on TM [2], standard calculations show that if G satisfies (6.5), then the sectional curvature $\tilde{K}$ of (TM, G) satisfies

$$\tilde{K}_{(x,u)}(X^H, Y^H) = \tilde{K}_{(x,u)}(X^H, Y^V) = 0,$$

$$\tilde{K}_{(x,u)}(X^V, Y^V) = \frac{2 + ||u||^2}{(1 + ||u||^2)^2}[g(X, u)^2 + g(Y, u)^2] + \frac{1}{1 + ||u||^2} > 0,$$

at any point $(x, u) \in TM$ and for all tangent vectors X, Y at x $\square$

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