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Octave Moutsinga

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Convex hulls, Sticky particle dynamics and Pressure-less gas system

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Abstract

We introduce a new condition which extends the definition of sticky particle dynamics to the case of discontinuous initial velocities u_0 with *negative jumps*. We show the existence of a stochastic process and a forward flow ϕ satisfying $X_{s+t} = \phi(X_s, t, P_s, u_s)$ and $\mathrm{d}X_t = \mathrm{E}[u_0(X_0)/X_t]\mathrm{d}t$, where $P_s = PX_s^{-1}$ is the law of X_s and $u_s(x) = \mathrm{E}[u_0(X_0)/X_s = x]$ is the velocity of particle x at time $s \geq 0$. Results on the flow characterization and Lipschitz continuity are also given.

Moreover, the map $(x,t) \mapsto M(x,t) := P(X_t \leq x)$ is the entropy solution of a scalar conservation law $\partial_t M + \partial_x(A(M)) = 0$ where the flux A represents the particles momentum, and $\big(P_t,\ u_t,\ t > 0\big)$ is a weak solution of the pressure-less gas system of equations of initial datum P_0, u_0.

1. Introduction

Our purpose is to give the most natural assumptions which allow to define the sticky particles model, and to study its main properties. It is well known that this model is connected with the pressure-less gas system of equations

$$\begin{cases} \partial_t(\rho) + \partial_x(u\rho) & = 0 \\ \partial_t(u\rho) + \partial_x(u^2\rho) & = 0 \\ \rho(\mathrm{d}x,t) \to P_0\,, \quad u(x,t)\rho(\mathrm{d}x,t) \to u_0(x)P_0(\mathrm{d}x) & \text{weakly as } t \to 0^+ \end{cases} \tag{1.1}$$

when P_0 is a Radon measure and u_0 is a continuous function.

Definition 1.1. Let $(u(\cdot,t) : t \geq 0)$ be a family of real functions and $(\rho(\cdot,t) : t \geq 0)$ be a family of real measures, weakly continuous with respect to t. The family (ρ, u) is a weak solution of the above pressure-less

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gas system (1.1) if, for any $f \in C_c^1(\mathbb{R})$, the space of C^1-functions on $\mathbb{R}$ with compact support, and $0 < s < t$:

$$\int f(x)\rho(\mathrm{d}x,t) - \int f(x)\rho(\mathrm{d}x,s) = \int_s^t \int f'(x)u(x,r)\rho(\mathrm{d}x,r)\mathrm{d}r$$

$$\int f(x)u(x,r)\rho(\mathrm{d}x,t) - \int f(x)u(x,r)\rho(\mathrm{d}x,s) = \int_s^t \int f'(x)u(x,r)^2\rho(\mathrm{d}x,r)\mathrm{d}r.$$

The last line of system (1.1) defines the Cauchy problem of the initial datum P_0, u_0, and is related to the weak convergence of measures.

In the discrete case (P_0 with discrete support), Zeldovich [8] solved this system from the sticky particle dynamics. Its solution is given by the mass distribution $\rho(\cdot,t)$ and the velocity function $u(\cdot,t)$ of the particles at time t.

For P_0 with a continuous support with no vacuum, and u_0 continuous, E, Sinai, Rikov [5] constructed the sticky particle dynamics from a generalized variational principle and by two functions $(x,t) \mapsto \phi_t(x)$, $u(\phi_t(x),t)$ which define respectively the position and the velocity at time t of the initial particle x. Using discretization of P_0, the authors showed that $(P_0\phi_t^{-1}\,,\ u(\cdot,t)\,,\ t \geq 0)$ is a weak solution of (1.1).

Independently and as in [5], Brenier and Grenier [1] solved the system (1.1) by discretization of P_0 which has a bounded support and continuous u_0. Using discrete sticky particle dynamics, they obtained a limit cumulative distribution function (c.d.f.) $M(\cdot,t)$ which is the unique entropy solution of the scalar conservation law

$$\partial_t M + \partial_x(A(M)) = 0 \quad \text{such that} \quad M(x,0) = F_0(x) := P_0((-\infty,x]) \tag{1.2}$$

with the flux $A(m) = \int_0^m u_0\big(F_0^{-1}(z)\big)\mathrm{d}z$, $\forall m \in (0,1)$. The measure $\partial_x A(M)$ is absolutely continuous with respect to $\partial_x M =: \rho$, and a weak solution of (1.1) is given by ρ and the Radon-Nycodm derivative $u(\cdot,t)$. Although this solution can be interpreted by sticky particles, the authors do not obtain the sticky particles trajectories.

In [4], Dermoune and Moutsinga defined the sticky particles model from convex hulls, for any probability P_0 and any continuous bounded u_0, thus giving new proofs and completing the results of [1] and [5].

A first probabilistic interpretation (of [1, 5]) was made by Dermoune [3] who deduced a weak solution $\rho(\mathrm{d}x,t) = PX_t^{-1}$, $u(x,t) = \mathrm{E}[u_0(X_0)/X_t = x]$

of the pressure-less gas system (1.1) from the ordinary stochastic differential equation

$$\mathrm{d}X_t = \mathrm{E}[u_0(X_0)/X_t]\mathrm{d}t \quad \text{a.s.}, \quad PX_0^{-1} = P_0 \,. \tag{1.3}$$

In this paper, we introduce a more general condition on the initial velocity : u_0 is allowed to be discontinuous, but must have *negative jumps*. Indeed, this condition appears (in proposition 3.3) to be natural in order to have a sticky particles behavior.

We state several properties of the *forward flow* related to sticky particles, such as its characterization or its Lipschitz continuity. To complete [3], we show that (1.2) can be solved from (1.3), by $M(x,t) = P(X_t \leq x)$, as soon as X has a *forward flow property.*

We study in particular the sticky dynamics of the clusters formed before or at time $t_0 > 0$, and starting with the new velocity u_{t_0}. Although this dynamics (starting from $t_0 > 0$) is often evoked in the literature, its exact definition is new and is made possible because of the *propagation of negative jumps* (along the time).

Most of the details on the present work can be seen in [7].

2. **Main results**

For any probability P, let its support $\mathcal{S}(P) = \{x \in \mathbb{R} : \ P(x-\varepsilon, x+\varepsilon) > 0, \forall \varepsilon > 0\}$, and the subsets $\mathcal{S}_-(P) = \{x \in \mathbb{R} : \ P(x-\varepsilon, x) > 0, \forall \varepsilon > 0\}$, $\mathcal{S}_+(P) = \{x \in \mathbb{R} : \ P(x, x+\varepsilon) > 0, \forall \varepsilon > 0\}$, $\mathcal{S}_0(P) = \{x \in \mathbb{R} : \ P(\{x\}) > 0\}$. The notations $\mathcal{S}$, $\mathcal{S}_-$, $\mathcal{S}_+$, $\mathcal{S}_0$ are preferred when there is no ambiguity. For all real function u, we define the functions u^- on $\mathcal{S}_-$, and u^+ on $\mathcal{S}_+$ by

$$u^-(x) = \limsup_{\varepsilon \to 0+} \frac{\int_{[x-\varepsilon, x)} u(\eta) P(\mathrm{d}\eta)}{P\big([x-\varepsilon, x)\big)} \,, \; u^+(x) = \liminf_{\varepsilon \to 0+} \frac{\int_{(x, \varepsilon+x]} u(\eta) P(\mathrm{d}\eta)}{P\big((x, \varepsilon+x]\big)} \,.$$

As the function u is related to (1.1), it is *a priori* a Radon-Nycodim derivative (of uP w.r.t. P); thus, it is not uniquely defined on $\mathcal{S}\backslash\mathcal{S}_0$. We set $u(x) := u^-(x)$ for $x \in \mathcal{S}_-\backslash(\mathcal{S}_+ \cup \mathcal{S}_0)$, and $u(x) := u^+(x)$ for $x \in \mathcal{S}_+\backslash(\mathcal{S}_- \cup \mathcal{S}_0)$. Moreover, our version of u is required to satisfy the *negative jump condition* on $\mathcal{S}$:

$$u^-(x) \geq u(x) \, \forall \, x \in \mathcal{S}_- \,, \quad u(x) \geq u^+(x) \, \forall \, x \in \mathcal{S}_+ \,. \tag{2.1}$$

This condition is automatically fulfilled if u is continuous (on $\mathcal{S}$).

Theorem 2.1. *Suppose that a probability P_0 and a bounded real function u_0 satisfy (2.1). There exists a stochastic process (X_t) solution of (1.3), such that*

1) *The trajectories of this process define the sticky particle dynamics whose initial mass distribution is P_0 and whose initial velocity function is u_0. More precisely, at time $t \geq 0$, each $x := X_t(\omega)$ is the position of the particle whose mass and velocity are respectively $P_t(\{x\}) := P(X_t = x)$ and $u_t(x) := \mathrm{E}[u_0(X_0)/X_t = x]$.*

2) **Propagation of negative jumps :** *For all t, u_t and the law $P_t = PX_t^{-1}$ of X_t satisfy (2.1).*

3) **Forward flow property :** $\forall\,(s,t),\ X_{s+t} = \phi(X_s, t, P_s, u_s),$ *where $\phi(\cdot,\cdot,P_s,u_s)$ is a continuous forward flow defined on $\mathcal{S}(P_s)\times\mathbb{R}_+$, non-decreasing in the first variable.*

Then, as in [3], the pressure-less gas system (1.1) is solved from (1.3), by $\rho(\mathrm{d}x,t) := P_t$ and $u(x,t) := u_t(x)$. We can also get (1.2) from (1.3) and the forward flow property.

Proposition 2.2. *Let X be a solution of (1.3) which satisfies assertion 3) of theorem (2.1). A weak solution of the scalar conservation law (1.2) is given by the c.d.f. $M(x,t) = P(X_t \leq x)$.*

This result is an other proof of the one of [4], without the entropy condition (defined in [2]) which requires properties of the discontinuity lines of M. One could easily get this entropy condition, if one could show that these discontinuity lines coincide with the trajectories of sticky particles. We make the conjecture that it is indeed the case in section 4, where we give a description of these trajectories (**Fig. 1** and **Fig. 2**).

In fact, M is the entropy solution because it is related to Hamilton Jacobi equation. Indeed, one can see in next section that the process X of theorem 2.1 and its c.d.f. M are given by convex hulls. Then, Hopf's formula leads to the following, as in [4].

Proposition 2.3. *The function $(x,t) \mapsto \Psi(x,t) = \int_0^x M(y,t)\mathrm{d}y$ is the viscosity solution of the Hamilton-Jacobi equation $\partial_t\Psi + A(\partial_x\Psi) = 0$, $\Psi(x,0) = \int_0^x F_0(y)\mathrm{d}y$. Hence, M is the entropy solution of the conservation law (1.2).*

Proof of Proposition 2.2. For all (x,t), let $F_t(x) = M(x,t)$. As a weak solution of (1.2), M must satisfy
$\int\int [M(x,t)\partial_t h(x,t) + A(M(x,t))\partial_x h(x,t)]\,\mathrm{d}x\mathrm{d}t = 0 \quad \forall\, h \in \mathrm{C}^1_c(\mathbb{R}\times\mathbb{R}^*_+)$.

Let λ be the Lebesgue measure and define the random vector (F_0^{-1}, F_t^{-1}) on the probability space $((0,1), \mathcal{B}, \lambda)$. One has $F_t^{-1} = \phi(F_0^{-1}, t, P_s, u_s)$, so $(F_0^{-1},\, F_t^{-1})$ has the same law as $(X_0,\, X_t)$.

Hence, the Heaviside function $H = 1_{[0,+\infty)}$ gives $M(x,t) = \mathrm{E}H(x - X_t)$ and $A(M(x,t)) = \mathrm{E}\big(u_0(X_0)H(x - X_t)\big)$.

Then, using the Dirac measure δ_{X_t} of atom X_t, one gets

$$\begin{aligned}
\iint \partial_t h(x,t) M(x,t)\mathrm{d}x\mathrm{d}t &= \mathrm{E}\iint \partial_t h(x,t) H(x - X_t)\mathrm{d}x\mathrm{d}t \\
&= \mathrm{E}\iint h(x,t)\delta_{X_t}(\mathrm{d}x)\mathrm{d}X_t = -\mathrm{E}\iint \partial_x h(x,t)H(x-X_t)\mathrm{d}X_t\mathrm{d}x \\
&= -\iint \partial_x h(x,t)\mathrm{E}(u_0(X_0)H(x - X_t))\mathrm{d}x\mathrm{d}t \\
&= -\iint \partial_x h(x,t) A(M(x,t))\mathrm{d}x\mathrm{d}t\ .
\end{aligned}$$

□

Theorem 2.4. (Flow characterization) *The following properties characterize the forward flow of the sticky particles with initial mass distribution P and initial velocity function u (which are supposed to satisfy (2.1)).*
1) $\forall\, x\,, \quad \phi(x, 0, P, u) = x$.
2) For all t, $x \mapsto \phi(x,t,P,u)$ *is continuous and non-decreasing.*
3) For all (x,t) *,* $E(x,t) := \{y\ :\ \phi(y,t,P,u) = \phi(x,t,P,u)\}$ *satisfies*

$$\begin{aligned}
\phi(x,t,P,u) &= \frac{\int_{E(x,t)}[\eta + tu(\eta)]P(\mathrm{d}\eta)}{P(E(x,t))} && \textit{if} \quad P(E(x,t)) > 0\ , \\
\phi(x,t,P,u) &= x + tu^-(x) && \textit{if} \quad P(E(x,t)) = 0\ , \quad x \in \mathcal{S}_-\ , \\
\phi(x,t,P,u) &= x + tu^+(x) && \textit{if} \quad P(E(x,t)) = 0\ , \quad x \in \mathcal{S}_+\ .
\end{aligned}$$

4) If $E(x,t) = [\alpha,\beta]\cap\mathcal{S}(P)$ *then* $\forall\, y \in [\alpha,\beta]$ *s.t.* $P([\alpha,y))P((y,\beta]) > 0$ *:*

$$\begin{aligned}
\frac{\int_{[\alpha,y)}[\eta + tu(\eta)]P(\mathrm{d}\eta)}{P([\alpha,y))} &\geq \frac{\int_{E(x,t)}[\eta + tu(\eta)]P(\mathrm{d}\eta)}{P(E(x,t))} \\
&\geq \frac{\int_{(y,\beta]}[\eta + tu(\eta)]P(\mathrm{d}\eta)}{P((y,\beta])}\ .
\end{aligned}$$

Moreover, one or all the intervals can be closed in y.

Proposition 2.5. (Lipschitz continuity) *Let* $(P_t, u_t, t \geq 0)$ *be defined as in theorem 2.1, and let* $C \geq ||u_0||_\infty$. $\forall\, t \geq 0$, $\forall\, s, s'$, $\forall\, x \leq y$ *in* $\mathcal{S}(P_t)$,

$$\begin{aligned} \big|\phi(x, s', P_t, u_t) - \phi(y, s, P_t, u_t)\big| \leq\ & C|s' - s| + |x - y| \\ & + s \sup_{x \leq \beta \leq \alpha \leq y} \big(u_t(\alpha) - u_t(\beta)\big)\ . \end{aligned} \tag{2.2}$$

$\forall\, t > 0$, $||u_t||_\infty \leq ||u_0||_\infty$ *and one has the Œlenick type entropy condition*

$$u_t(x_2) - u_t(x_1) \leq t^{-1}(x_2 - x_1) \qquad \forall\, x_1 < x_2 \text{ in } \mathcal{S}(P_t)\ . \tag{2.3}$$

Then, $\forall\, t > 0$, $\forall\, s, s'$, $\forall\, x, y \in \mathcal{S}(P_t)$,

$$\big|\phi(x, s', P_t, u_t) - \phi(y, s, P_t, u_t)\big| \leq C|s' - s| + |x - y|(1 + st^{-1})\ . \tag{2.4}$$

In next sections 3 and 4, some results are similar to [4]; but all the proofs must hold account of the discontinuity of u_0, and are obtained by the convex hulls properties.

3. The sticky particle dynamics

Let P_0 be any probability measure with cumulative distribution function F_0, and u_0 be a real function with *negative jumps* (2.1) on the support $\mathcal{S}$ of P_0.

In our construction, each particle, of initial position x, is indexed by the total mass $F_0(x)$ of itself and all the initial particles situated on its left; and each mass m indexes two initial particles $F_d^{-1}(m), F_g^{-1}(m)$ which can be confused or separated by a vacuum. We recall that the two inverse functions of F_0 are defined for all $m \in (0,1)$ by

$$F_g^{-1}(m) = \inf\{x : F_0(x) \geq m\}\ , \quad F_d^{-1}(m) = \sup\{x : F_0(x) \leq m\}\ .$$

The function F_d^{-1} is *càdlàg* and its left-hand limit is F_g^{-1}. In the sequel, F_0^{-1} will stand independently for F_g^{-1} or F_d^{-1} (which are equal almost everywhere). For all $t \geq 0$, let $H(\cdot, t)$ be the lower convex hull of any primitive of $F_0^{-1}(\cdot) + tu_0(F_0^{-1}(\cdot))$:

$$m \in (0,1) \mapsto \int_a^m \Big[F_0^{-1}(z) + tu_0(F_0^{-1}(z))\Big]\,\mathrm{d}z =: \varphi(m, t) \tag{3.1}$$

(with $a \in (0,1)$), *i.e.* the greatest convex lower bound for $\varphi(\cdot, t)$.

We define the sticky particle dynamics with initial c.d.f. F_0 and initial velocities given by u_0, as follows.

Definition 3.1. (Sticky dynamics) The initial state of the particles is defined by the set of images $m \in (0,1) \mapsto F_g^{-1}(m),\ F_d^{-1}(m)$.

While there is no collision, particles move with constant velocities defined initially by u_0; the state of the particles at time t is given by $m \in (0,1) \mapsto F_g^{-1}(m) + tu_0\big(F_g^{-1}(m)\big),\ F_d^{-1}(m) + tu_0\big(F_d^{-1}(m)\big)$.

After the first collision, the state of the particles at time t is defined by $m \in (0,1) \mapsto \partial_m^- H(m,t) =: x_g(m,t),\ \partial_m^+ H(m,t) =: x_d(m,t)$.

Note that in abscence of collision, the definition seems to depend on the choice of a version of u_0, which is arbitrary for $x := F_g^{-1}(m) = F_d^{-1}(m)$ if $u_0^-(x) > u_0^+(x)$. In fact, it is not the case since we show that such a particle of initial position x is collided immediately after time zero (see proposition 3.2). This fact is due to the negative jump condition (2.1) which is also necessary here to guarantee the coherence of our definition with a sticky dynamics, *i.e.* particles do not disintegrate.

The idea of using such a condition (2.1) comes naturally from the following inequalities obtained from the convex hull. For all t, let $\mathcal{E}_t$ be the set of abscissas of extremal points of $H(\cdot,t)$. $\forall\, m \in \mathcal{E}_t$:

$$\begin{aligned} F_g^{-1}(m) + tu_0^-\big(F_g^{-1}(m)\big) &\le x_g(m,t) \\ &\le x_d(m,t) \le F_d^{-1}(m) + tu_0^+\big(F_d^{-1}(m)\big) \end{aligned}$$

if $F_g^{-1}(m) \notin \mathcal{S}_0$ and $F_d^{-1}(m) \notin \mathcal{S}_0$; u_0^- (resp. u_0^+) is replaced by u_0 when $F_g^{-1}(m) \in \mathcal{S}_0$ (resp. when $F_d^{-1}(m) \in \mathcal{S}_0$). This is simply given as limits of rates of increase of $\varphi(\cdot,t)$, and by definition of u_0^-, u_0^+. Using the negative jump condition (2.1), these inequalities become

$$F_g^{-1}(m) + tu_0\big(F_g^{-1}(m)\big) \le x_g(m,t) \le x_d(m,t) \le F_d^{-1}(m) + tu_0\big(F_d^{-1}(m)\big) \tag{3.2}$$

that we call *fundamental property* of convex hulls.

Proposition 3.2. (Coherence property) *We have*

$$x_g(m,t) < x_d(m,t) \implies F_g^{-1}(m) < F_d^{-1}(m)\ . \tag{3.3}$$

Moreover, the state of particles at time t is determined by $\mathcal{E}_t$, and one has $\mathcal{E}_t \subset \mathcal{E}_s\ \forall\, s < t$.

Proof. (3.3) comes from (3.2). For $m \notin \mathcal{E}_t$, $\exists m_1, m_2$ s.t. $m_1 < m < m_2$, $(m_1,m_2) \cap \mathcal{E}_t = \emptyset$. Hence $x_g(m,t) = x_d(m,t) = x_d(m_1,t) = x_g(m_2,t)$.

This shows also that the state of particles at time t is given by elements of $\mathcal{E}_t$.

Last assertion comes from the definition of $m \in \mathcal{E}_t : \forall\, m_1 < m < m_2$,

$$\frac{\int_{m_1}^{m}\left[F_0^{-1}(z) + tu_0(F_0^{-1}(z))\right]\mathrm{d}z}{m - m_1} < \frac{\int_{m}^{m_2}\left[F_0^{-1}(z) + tu_0(F_0^{-1}(z))\right]\mathrm{d}z}{m_2 - m}\,.$$

Seeing the fractions as functions of t (straight lines) which start respectively from $\int_{m_1}^{m} F_0^{-1}(z)\mathrm{d}z/(m-m_1) < \int_{m}^{m_2} F_0^{-1}(z)\mathrm{d}z/(m_2-m)$, it follows that $m \in \mathcal{E}_s$, $\forall\, s < t$. □

Assertion (3.3) implies that initial particles do not disintegrate. The second assertion suggests the fact (confirmed in proposition 4.2) that the particles have a sticky behavior, because their "number" decreases.

4. **Forward flow**

For all t, let us define $\mathcal{E}_t^- = \{m : (z,\, m) \cap \mathcal{E}_t \neq \emptyset\ ,\ \forall\, z < m\}$, $\mathcal{E}_t^+ = \{m : (m,\, z) \cap \mathcal{E}_t \neq \emptyset\ ,\ \forall\, z > m\}$. From (2.1) and the simple limits of rates of increase of the function $\varphi(\cdot, t)$ (given in (3.1) and which coincides with $H(\cdot, t)$ on $\mathcal{E}_t$), we obtain the exact expression of the positions. For all m, t and consecutive $m_1, m_2 \in \mathcal{E}_t$:

$$x_g(m,t) = \begin{cases} F_g^{-1}(m) + tu_0(F_g^{-1}(m)) & \text{if } m \in \mathcal{E}_t^- \\ \dfrac{\int_{m_1}^{m_2}\left[F_0^{-1}(m) + tu_0(F_0^{-1}(m))\right]\mathrm{d}m}{m_2 - m_1} & \text{if } m \in (m_1, m_2] \end{cases} \tag{4.1}$$

$$x_d(m,t) = \begin{cases} F_d^{-1}(m) + tu_0(F_d^{-1}(m)) & \text{if } m \in \mathcal{E}_t^+ \\ \dfrac{\int_{m_1}^{m_2}\left[F_0^{-1}(m) + tu_0(F_0^{-1}(m))\right]\mathrm{d}m}{m_2 - m_1} & \text{if } m \in [m_1, m_2)\,. \end{cases} \tag{4.2}$$

Let $\varepsilon \in \{g, d\}$. In order to study the properties of trajectories, we define for all x, t :

$$M_*(x,t) = \sup\{m : x_\varepsilon(m,t) < x\}\ , \quad M^*(x,t) = \inf\{m : x_\varepsilon(m,t) > x\}\ . \tag{4.3}$$

These functions do not depend on ε. $M_*(x,t)$ is the mass of all clusters whose positions at time t is less than x; the mass of the cluster of position x is $M^*(x,t) - M_*(x,t)$.

Theorem 4.1. *Let $C \geq ||u_0||_\infty$. We have $\forall\, m, t, s$,*

$$|x_\varepsilon(m,t) - x_\varepsilon(m,s)| \leq C|t-s| \;,$$

$$\lim_{h\to 0^+} \frac{x_\varepsilon(m,t+h) - x_\varepsilon(m,t)}{h} = u(x_\varepsilon(m,t),t) \;,$$

except if $x_g(m,t) = x_d(m,t) =: x$, $F_g^{-1}(m) < F_d^{-1}(m)$ and $M_*(x,t) = M^*(x,t)$, with

$$u(x,t) = \left\{ \begin{cases} u_0(F_\varepsilon^{-1}(m)) & \text{if } M_*(x,t) = M^*(x,t) = m, \\ \dfrac{\int_{m_1}^{m_2} u_0(F_0^{-1}(z))\mathrm{d}z}{m_2 - m_1} & \text{if } m_1 = M_*(x,t) < M^*(x,t) = m_2. \end{cases} \right. \tag{4.4}$$

Furthermore, $t \mapsto u(x_\varepsilon(m,t),t)$ is càdlàg.

Proof. The proof is due to (4.1) and (4.2), to the following four possibilities and to following propositions 4.2 and 4.3. Let $m_*(t) = M_*(x_\varepsilon(m,t),t)$, $m^*(t) = M^*(x_\varepsilon(m,t),t)\ \forall\,(m,t)$, and let $s > t$.
1) If $m_*(s) < m^*(t)$ and $m_*(t) < m^*(s)$, then the fact that $m_*(s), m^*(s)$ are consecutive in $\mathcal{E}_s$, with $m_*(t), m^*(t) \in [m_*(s), m^*(s)]$, implies that

$$\frac{\varphi(m_*(t),s) - \varphi(m^*(s),s)}{m_*(t) - m^*(s)} \leq x_\varepsilon(m,s) \leq \frac{\varphi(m_*(s),s) - \varphi(m^*(t),s)}{m_*(s) - m^*(t)}$$

(with φ given in (3.1)). As $m_*(t), m^*(t)$ are also consecutive in $\mathcal{E}_t$, one has

$$\frac{\varphi(m_*(t),t) - \varphi(m^*(s),t)}{m_*(t) - m^*(s)} \geq x_\varepsilon(m,t) \geq \frac{\varphi(m_*(s),t) - \varphi(m^*(t),t)}{m_*(s) - m^*(t)} \;.$$

Using the fact that $\varphi(m',r) = \varphi(m',0) + rA(m')\ \forall(m',r)$, one gets

$$\frac{A(m_*(t)) - A(m^*(s))}{m_*(t) - m^*(s)} \leq \frac{x_\varepsilon(m,s) - x_\varepsilon(m,t)}{s-t} \leq \frac{A(m_*(s)) - A(m^*(t))}{m_*(s) - m^*(t)} \;.$$

2) If $m = m_*(s) = m^*(t) < m^*(s)$, then

$$x_g(m,s) - x_g(m,t) = (s-t)u_0\left(F_g^{-1}(m)\right) \;,$$

$$\frac{A(m^*(s)) - A(m)}{m^*(s) - m} \leq \frac{x_d(m,s) - x_d(m,t)}{s-t} \leq u_0\left(F_d^{-1}(m)\right) \;.$$

Indeed, for the first result (with $\varepsilon = g$), the fact that $m = m_*(t) = m_*(s)$ implies that $m \in \mathcal{E}_t^- \cap \mathcal{E}_s^-$, and one concludes with (4.1). In the result with $\varepsilon = d$, the left-hand side is obtained as in 1). For the right-hand side, the equation $m = m^*(t)$ implies that $m \in \mathcal{E}_t^+$, then $x_d(m,t) =$

$F_d^{-1}(m)+tu_0\left(F_d^{-1}(m)\right)$. From (3.2), $x_d(m,s) \leq F_d^{-1}(m)+su_0\left(F_d^{-1}(m)\right)$.

3) If $m_*(s) < m_*(t) = m^*(s) = m$, then (as previously)

$$x_d(m,s) - x_d(m,t) = (s-t)u_0\left(F_d^{-1}(m)\right) \ ,$$

$$u_0\left(F_g^{-1}(m)\right) \leq \frac{x_g(m,s) - x_g(m,t)}{s-t} \leq \frac{A(m_*(s)) - A(m)}{m_*(s) - m} \ .$$

4) If $m_*(s) = m^*(s) = m$, then $m \in \mathcal{E}_t^- \cap \mathcal{E}_t^+ \cap \mathcal{E}_s^- \cap \mathcal{E}_s^+$, so $x_\varepsilon(m,s) - x_\varepsilon(m,t) = (s-t)u_0\left(F_\varepsilon^{-1}(m)\right)$.

The Lipschitz continuity is then immediate. The derivative of $t \mapsto x_\varepsilon(m,t)$ has thus the form (4.4), provided by the fact (shown in proposition 4.2) that the functions $t \mapsto m_*(t), m^*(t)$ are *càdlàg*. This fact implies also the *càdlàg* property of $t \mapsto u(x_\varepsilon(m,t),t)$. □

The velocity is not defined when $x_g(m,t) = x_d(m,t) =: x,\ F_g^{-1}(m) < F_d^{-1}(m),\ m = m_*(t) = m^*(t)$. In this case, $m \in \mathcal{E}_s^- \cap \mathcal{E}_s^+ \ \forall\, s \leq t$, and $x_\varepsilon(m,s) = F_\varepsilon^{-1}(m)+su_0(F_\varepsilon^{-1}(m))$. Thus, $u_0(F_g^{-1}(m)) > u_0(F_d^{-1}(m))$ and one can define $u(x,t)$ as any value of $\left[u_0(F_d^{-1}(m)), u_0(F_g^{-1}(m))\right]$. Remark that for all $s \neq t$, the velocity is well defined in the theorem. Indeed, as functions of s which coincide at t, one has $x_g(m,s) < x_d(m,s)\ \forall\, s < t$. For $s > t$, one gets $F_g^{-1}(m) + su_0(F_g^{-1}(m)) > F_d^{-1}(m) + su_0(F_d^{-1}(m))$, so $m \notin \mathcal{E}_s$ (from (3.2)), and $m_*(s) < m < m^*(s),\ x_g(m,s) = x_d(m,s)$.

The following results show that clusters grow up.

Proposition 4.2. *1) The functions $M_*(\cdot,t)$ and $M^*(\cdot,t)$ describe the whole set $\mathcal{E}_t$.*
2) $\forall\, \varepsilon \in \{g,d\}$, $\forall\, m$, $\forall\, s > t$:

$$M_*\left(x_\varepsilon(m,s),s\right) \leq M_*\left(x_\varepsilon(m,t),t\right) \leq M^*\left(x_\varepsilon(m,t),t\right) \leq M^*\left(x_\varepsilon(m,s),s\right).$$

3) The functions $t \mapsto M_*(x_\varepsilon(m,t),t),\ M^*(x_\varepsilon(m,t),t)$ are càdlàg.

Proof. 1) One can see that $\forall\,(x,t),\ M_*(x,t), M^*(x,t) \in \mathcal{E}_t$. Moreover, $\forall\, m \in \mathcal{E}_t$, one has

$$\forall\, z_1 < m < z_2, \qquad \frac{\varphi(m,t) - \varphi(z_1,t)}{m - z_1} < \frac{\varphi(m,t) - \varphi(z_2,t)}{m - z_2} \ .$$

Let us define

$$x := \sup_{z<m} \frac{\varphi(m,t)-\varphi(z,t)}{m-z} = \sup_{z<m} \frac{H(m,t)-H(z,t)}{m-z} = x_g(m,t)\ .$$

These equalities follow from the definition of H and x_g. As $x_g(\cdot,t)$ is non-decreasing and by definition of $M_*(x,t)$, one gets $m \geq M_*(x,t)$. In the same way, $m \leq M^*(x,t)$. As $M_*(x,t), M^*(x,t)$ are consecutive in $\mathcal{E}_t$, *i.e.* $(M_*(x,t), M^*(x,t)) \cap \mathcal{E}_t = \emptyset$, one gets $m \in \{M_*(x,t), M^*(x,t)\}$.
2) $\forall\, t$, one has $m_*(t), m^*(t) \in \mathcal{E}_t$ and $\big(m_*(t), m^*(t)\big) \cap \mathcal{E}_t = \emptyset$. For $s > t, \mathcal{E}_s \subset \mathcal{E}_t$ (see proposition 3.2); so $\big(m_*(t), m^*(t)\big) \cap \mathcal{E}_s = \emptyset$. This implies that $m_*(s), m^*(s) \notin \big(m_*(t), m^*(t)\big)$. As $\big[m_*(t), m^*(t)\big] \cap \big[m_*(s), m^*(s)\big] \supset \{m\}$, one gets $\big[m_*(t), m^*(t)\big] \subset \big[m_*(s), m^*(s)\big]$.
3) This is a consequence of the monotonicity of the functions, with assertion (b) of next proposition. □

Proposition 4.3. (Regularity of M_* and M^*)

(a) $x < x' \implies M^*(x,t) \leq M_*(x',t) \quad \forall\, t.$

(b) *If m is a value of adherence of $M_*(x',s)$ or $M^*(x',s)$ as (x',s) tends to (x,t), then* $\quad m \in [M_*(x,t), M^*(x,t)]$

(c) $\forall\,(x,t), \quad M_*(x+0,t) = M^*(x+0,t) = M^*(x,t),$
$\qquad\qquad M_*(x-0,t) = M^*(x-0,t) = M_*(x,t).$

(d) *M_* (resp. M^*) is continuous in (x,t) if and only if $M_*(x,t) = M^*(x,t)$.*

Furthermore, $\displaystyle \lim_{x\to-\infty} M^*(x,t) = 0\,, \qquad \lim_{x\to+\infty} M^*(x,t) = 1.$

(e) *If $M_*(x,t) < M^*(x,t)$, then $\forall\, m \in \big(M_*(x,t),\, M^*(x,t)\big)$,*

$$\frac{A(m) - A(M_*(x,t))}{m - M_*(x,t)} \geq \frac{A(m) - A(M^*(x,t))}{m - M^*(x,t)}\ .$$

One can see that $M^*(\cdot,t)$ is a c.d.f.

Proof. (a) If $x < x'$, then for all $m < M^*(x,t)$, one has $x_\varepsilon(m,t) < x'$; so by definition, $M_*(x',t) \geq M^*(x,t)$.

For assertion (b), one uses the fact that for all (x,t), the function $m \mapsto \varphi(m,t) - x(m-a)$ reaches its lower bound in $[M_*(x,t),\ M^*(x,t)]$:

$$\begin{aligned}\varphi(M^*(x,t),t) - x\big(M^*(x,t)-a\big) &= \varphi(M_*(x,t),t) - x\big(M_*(x,t)-a\big)\,;\\ \varphi(m,t) - x(m-a) &\geq \varphi(M_*(x,t),t) - x\big(M_*(x,t)-a\big)\ \ \forall\, m\,;\\ \varphi(m,t) - x(m-a) &> \varphi(M_*(x,t),t) - x\big(M_*(x,t)-a\big)\\ &\qquad\qquad \forall\, m \notin [M_*(x,t),\ M^*(x,t)]\,.\end{aligned}$$

Let (m,x,t) be a value of adherence of $\big(M_*(x',s),x',s\big)$ or $\big(M^*(x',s),x',s\big)$. One has $\varphi(m',s) - x'(m'-a) \geq \varphi(M_*(x',s),s) - x'\big(M_*(x',s)-a\big) \quad \forall\, m'$. Then, by continuity, $\varphi(m',t) - x(m'-a) \geq \varphi(m,t) - x(m-a) \quad \forall\, m'$, which means that $m \in [M_*(x,t),\ M^*(x,t)]$.
Assertions (c) and (d) are immediate consequences of (b).
For assertion (e) : let us define, for all s,

$$d_1(s) = \frac{\varphi(m,s) - \varphi(M_*(x,t),s)}{m - M_*(x,t)}\ ,\quad d_2(s) = \frac{\varphi(M^*(x,t),s) - \varphi(m,s)}{M^*(x,t) - m}\ .$$

One has $d_1(t) \geq d_2(t)$ by definition of $M_*(x,t), M^*(x,t)$. As $d_1(0) \leq F_g^{-1}(m) \leq F_d^{-1}(m) \leq d_2(0)$, one gets the desired result : $t^{-1}(d_1(t) - d_1(0)) \geq t^{-1}(d_2(t) - d_2(0))$. □

Forward flow indexed by initial positions

Now, we define precisely a cluster at time t as the set of all initial particles $F_d^{-1}(m), F_g^{-1}(m')$ which have the same position $p = x_d(m,t) = x_g(m',t)$. Let ξ_t be the set of these clusters.

Proposition 4.4. (Construction of clusters) *The set ξ_t is a partition of the initial support $\mathcal{S}$. Furthermore, the slope of a segment (or simply a derivative) on the graph of $H(\cdot,t)$, given by abscissas $[m_1,m_2]$, is the position at time t of a cluster $\Big[F_\epsilon^{-1}(m_1), F_{\epsilon'}^{-1}(m_2)\Big] \cap \mathcal{S}$ which has mass $m_2 - m_1$, where $\epsilon,\epsilon' \in \{g,d\}$ are given as follows :*

1) *$m_1 < m_2$ consecutive in $\mathcal{E}_t$ (case of massive clusters) :*

$m_1 \backslash m_2$	$x_g < x_d$	$x_g = x_d$
$x_g < x_d$	$\left[F_d^{-1}(m_1), F_g^{-1}(m_2)\right] \cap \mathcal{S}$	$\left[F_d^{-1}(m_1), F_d^{-1}(m_2)\right] \cap \mathcal{S}$
$x_g = x_d$	$\left[F_g^{-1}(m_1), F_g^{-1}(m_2)\right] \cap \mathcal{S}$	$\left[F_g^{-1}(m_1), F_d^{-1}(m_2)\right] \cap \mathcal{S}$

2) $m_1 = m_2 = m \in \mathcal{E}_t^- \cap \mathcal{E}_t^+,\ x_g(m,t) = x_d(m,t)$ *(first case of massless clusters) :* $\{F_g^{-1}(m), F_d^{-1}(m)\}$ *is a cluster not isolated on the left nor on the right in* ξ_t*; the two particles (if not confused) stick together at time* t *and were not shocked before.*

3) $m_1 = m_2 = m,\ x_g(m,t) < x_d(m,t)$ *(second case of massless clusters) :*

If $m \in \mathcal{E}_t^- \cap \mathcal{E}_t^+$*, then* $\{F_g^{-1}(m)\}, \{F_d^{-1}(m)\} \in \xi_t$ *and these particles, separated by a vacuum, were not shocked until time* t*.*

If $m \in \mathcal{E}_t^- \backslash \mathcal{E}_t^+$*, then* $\{F_g^{-1}(m)\} \in \xi_t$ *and this particle was not shocked until time* t *(and* $F_d^{-1}(m)$ *is in a massive cluster given in 1)).*

If $m \in \mathcal{E}_t^+ \backslash \mathcal{E}_t^-$*, then* $\{F_d^{-1}(m)\} \in \xi_t$ *and this particle was not shocked until time* t *(and* $F_g^{-1}(m)$ *is in a massive cluster given in 1)).*

These results arise immediately from the above expressions of x_g, x_d. As a consequence, we get the generalized variational principle (GVP) introduced in [5]. If $P_0(G) > 0$, let us define

$$\mathrm{C}(G,t) := (P_0(G))^{-1} \int_G [\eta + t u_0(\eta)]\, P_0(\mathrm{d}\eta)\ .$$

Proposition 4.5. (GVP) *For all cluster* $[\alpha, \beta] \cap \mathcal{S}$ *at time* t*, we have* $\forall\ y_1 < \alpha < y_2$ *s.t.* $P_0([y_1, \alpha))P_0([\alpha, y_2)) > 0$ *:*

$$\mathrm{C}([y_1, \alpha), t) < \mathrm{C}([\alpha, y_2), t)\ ; \qquad (\mathrm{GVP})_{\mathrm{g}}$$

$\forall\ y_1 < \beta < y_2$ *s.t.* $P_0((y_1, \beta])P_0((\beta, y_2]) > 0$ *:*

$$\mathrm{C}((y_1, \beta], t) < \mathrm{C}((\beta, y_2], t)\ ; \qquad (\mathrm{GVP})_{\mathrm{d}}$$

$\forall\, y \in [\alpha,\beta]$ s.t. $P_0([\alpha,y))P_0((y,\beta]) > 0$:

$$\mathrm{C}([\alpha,y),t) \geq \mathrm{C}([\alpha,\beta],t) \geq \mathrm{C}((y,\beta],t) \tag{4.5}$$

and one or all the intervals can be closed in y; if $P_0([\alpha,\beta]) > 0$, then

$$\alpha + tu_0(\alpha) \geq \mathrm{C}([\alpha,\beta],t) \geq \beta + tu_0(\beta)\ . \tag{4.6}$$

Proof. In proposition 4.4, $[\alpha,\beta]$ is given by $m_1 = F_0(\alpha-0) =: M_*(x,t)$, $m_2 = F_0(\beta) =: M^*(x,t)$, with $x = x_d(m_1,t) = x_g(m_2,t)$. As in the proof of proposition 4.3, one has

$$\varphi(m,t) - \varphi(M_*(x,t),t) \geq x(m - M_*(x,t)) \quad \forall\, m\ ,$$
$$\varphi(m,t) - \varphi(M_*(x,t),t) > x(m - M_*(x,t)) \quad \forall\, m \notin [M_*(x,t),\ M^*(x,t)]\ .$$

Then one gets $(\mathrm{GVP})_g$, $(\mathrm{GVP})_d$ and (4.5), using the change of variable, for all locally integrable f and $m < m'$: if $m, m' \in F_0(\mathbb{R})$,

$$\int_m^{m'} f(F_0^{-1}(z))dz = \int_{\left(F_g^{-1}(m),F_g^{-1}(m')\right]} f(\eta)P_0(d\eta)\ ;$$

if $m \in F_0(\mathbb{R}-0),\ m' \in F_0(\mathbb{R})$,

$$\int_m^{m'} f(F_0^{-1}(z))dz = \int_{\left[F_d^{-1}(m),F_g^{-1}(m')\right]} f(\eta)P_0(d\eta)\ .$$

For (4.6), if $\alpha \in \mathcal{S}_0 \cup \mathcal{S}_+$ and $\beta \in \mathcal{S}_0 \cup \mathcal{S}_-$, one gets the result from (4.5) when $y \downarrow \alpha$ and $y' \uparrow \beta$. If $\alpha \notin \mathcal{S}_0 \cup \mathcal{S}_+$, then $\alpha = F_g^{-1}(F_0(\alpha))$ and $x_g(F_0(\alpha),t) = \alpha + tu_0(\alpha) = x_d(F_0(\alpha),t) = \mathrm{C}([\alpha,\beta],t)$ by construction of the cluster. In the same way, $\mathrm{C}([\alpha,\beta],t) = \beta + tu_0(\beta)$ if $\beta \notin \mathcal{S}_0 \cup \mathcal{S}_-$. □

Let us define

$$\phi\,(x,t,P_0,u_0) = \begin{cases} x_g(F_0(x),t) & \text{if} \quad x \in \mathcal{S}_- \cup \mathcal{S}_0\ , \\ x_d(F_0(x),t) & \text{if} \quad x \in \mathcal{S}_+ \backslash (\mathcal{S}_- \cup \mathcal{S}_0)\ . \end{cases}$$

This function defines the trajectories of particles since for $x \in \mathcal{S}_- \cup \mathcal{S}_0$, one has $x = F_g^{-1}(F_0(x))$; if not, $x = F_d^{-1}(F_0(x))$.

Corollary 4.6. *1) For all t, the function $x \mapsto \phi(x,t,P_0,u_0)$ is continuous and non-decreasing on $\mathcal{S}$.*
2) $\forall\,(x,t) \in \mathcal{S} \times \mathbb{R}_+$,

$$\phi(x,t,P_0,u_0) = \begin{cases} \mathrm{C}([\alpha,\beta],t) & \textit{if} \quad x \in [\alpha,\beta] \cap \mathcal{S} \ \ \textit{massive cluster} \\ x + tu_0(x) & \textit{otherwise}\ . \end{cases}$$

3) Take $C \geq ||u_0||_\infty$. $\forall x \in \mathcal{S}, \forall s,t$:

$$|\phi(x,t,P_0,u_0) - \phi(x,s,P_0,u_0)| \leq C|t-s| \ ,$$

$$\lim_{h\to 0+} \frac{\phi(x,t+h,P_0,u_0) - \phi(x,t,P_0,u_0)}{h} = v(x,t,P_0,u_0) \ ,$$

$$v(x,t,P_0,u_0) = \begin{cases} \dfrac{\int_{[\alpha,\beta]} u_0(\eta) P_0(\mathrm{d}\eta)}{P_0([\alpha,\beta])} & \text{if } x \in [\alpha,\beta]\cap\mathcal{S} \text{ massive cluster} \\ u_0(x) & \text{for massless } \{x\} \in \xi_t. \end{cases}$$

Moreover, for all $x \in \mathcal{S}$, the function $t \mapsto v(x,t,P_0,u_0)$ is càdlàg.
4) For any cluster $[\alpha,\beta]\cap\mathcal{S} \in \xi_t$, the trajectory $s \in [0,t] \mapsto \phi(\alpha,s,P_0,u_0)$ is concave, and the trajectory $s \in [0,t] \mapsto \phi(\beta,s,P_0,u_0)$ is convex.

Proof. Any cluster $[\alpha,\beta]\cap\mathcal{S}$ is given, in proposition 4.4, by $m_1 = F_0(\alpha - 0)$, $m_2 = F_0(\beta)$.
1) If $\alpha \in \mathcal{S}_-$, then $\alpha = F_g^{-1}(m_1) = F_g^{-1}(F_0(\alpha))$, $m_1 \in \mathcal{E}_t^-$ and

$$\begin{aligned} \lim_{x\uparrow\alpha} x_g(F_0(x),t) = \lim_{x\uparrow\alpha} x_d(F_0(x),t) = \lim_{x\uparrow\alpha} \phi(x,t,P_0,u_0) = x_g(m_1,t) \\ = \alpha + tu_0(\alpha) = x_g(F_0(\alpha),t) = \phi(\alpha,t,P_0,u_0) \ . \end{aligned}$$

Idem for $\beta \in \mathcal{S}_+ : \phi(\beta,t,P_0,u_0) = \beta + tu_0(\beta) = \lim_{y\downarrow\beta} \phi(y,t,P_0,u_0)$.

Moreover, for $x < y$ in $\mathcal{S}$ s.t. $F_0(x) < F_0(y)$, one has $\phi(x,t,P_0,u_0) \leq x_d(F_0(x),t) \leq x_g(F_0(y),t) \leq \phi(y,t,P_0,u_0)$. If $F_0(x) = F_0(y)$, then $x \in \mathcal{S}_-\cup\mathcal{S}_0$, $y \in \mathcal{S}_+\setminus(\mathcal{S}_-\cup\mathcal{S}_0)$ so $\phi(x,t,P_0,u_0) = x_g(F_0(x),t) \leq x_d(F_0(y),t) = \phi(y,t,P_0,u_0)$.
2) For $x \in [\alpha,\beta]\cap\mathcal{S}$, one has thus $\phi(x,t,P_0,u_0) = x + tu_0(x)$ if $P_0([\alpha,\beta]) = 0$. If not, one has $m_1 \leq F_0(x-0) \leq F_0(x) \leq m_2$, and then $x_d(m_1,t) = \phi(x,t,P_0,u_0) = x_g(m_2,t) = \mathrm{C}([F_d^{-1}(m_1),F_g^{-1}(m_2)],t) = \mathrm{C}([\alpha,\beta],t)$.
Part 3) is an application of theorem 4.1.
4) Let us show that $s \in [0,t] \mapsto v(\alpha,s,P_0,u_0)$ is non-increasing. Define $[\alpha(s),\beta(s)]\cap\mathcal{S} = \{y \in \mathcal{S} \ : \ \phi(y,s,P_0,u_0) = \phi(\alpha,s,P_0,u_0)\}$. As clusters grow up, one has $\alpha = \alpha(s') = \alpha(s) \leq \beta(s) \leq \beta(s') \leq \beta$, $\forall\, s < s' \leq t$.
If $P_0([\alpha,\beta(s)]) > 0$, then

$$v(\alpha,r,P_0,u_0) = \frac{\int_{[\alpha,\beta(r)]} u_0(\eta)P_0(d\eta)}{P_0([\alpha,\beta(r)])} \qquad \forall\, r \in \{s,s'\} \ .$$

One has (from (4.5)) $\mathrm{C}([\alpha,\beta(s)],s') \geq \mathrm{C}([\alpha,\beta(s')],s')$, with

$$\mathrm{C}([\alpha,\beta(r)],s') = \mathrm{C}([\alpha,\beta(r)],0) + s'v(\alpha,r,P_0,u_0) \qquad \forall\, r \in \{s,s'\} \ ,$$

$C([\alpha, \beta(s)], 0) \leq C([\alpha, \beta(s')], 0)$. Hence $v(\alpha, s, P_0, u_0) \geq v(\alpha, s', P_0, u_0)$.
If $P_0([\alpha, \beta(s)]) = 0 < P_0([\alpha, \beta(s')])$ with $\alpha = \beta(s)$, the result comes from $v(\alpha, s, P_0, u_0) = u_0(\alpha)$, $\alpha + s' u_0(\alpha) \geq C([\alpha, \beta(s')], s')$ given by (4.6).
If $P_0([\alpha, \beta(s')]) = 0$ with $\alpha = \beta(s)$, then $v(\alpha, s, P_0, u_0) = v(\alpha, s', P_0, u_0) = u_0(\alpha)$.
If $P_0([\alpha, \beta(s)]) = 0$ with $\alpha < \beta(s)$, then $v(\alpha, s, P_0, u_0)$ is not (well) defined; but there can exist atmost one such s.

The proof of the convexity of the trajectory of β is analogous. □

These trajectories then give a butterfly with folded wings (**Fig. 1**). In contrast, the endpoints of a vacuum give a butterfly with spread wings (**Fig. 2**); the whole interior of the butterfly is then a vacuum.

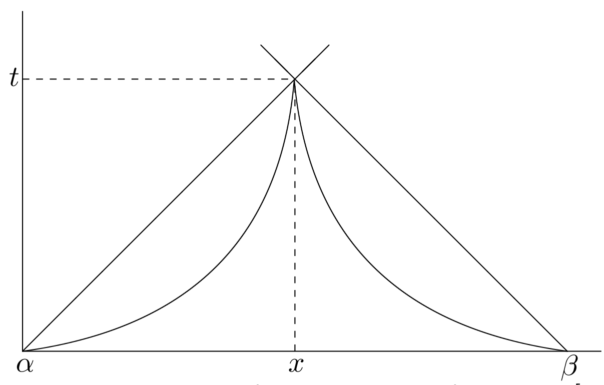

Fig. 1. Trajectories of the endpoints of a cluster $[\alpha, \beta]$ before their collision at position x and time t.

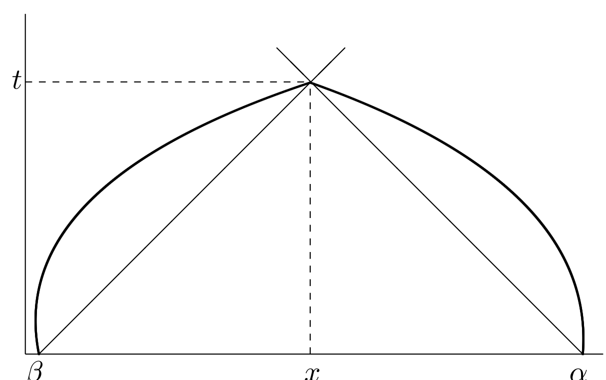

Fig. 2. Trajectories of the endpoints of a vacuum (β, α) before their collision at position x and time t.

Discontinuity lines

Here, we exhibit and describe some discontinuity lines of the entropy solution M of (1.2), *i.e.* curves of discontinuity points $(x(t),t)$ of M; they are usually used in the resolution of this equation ([2]).

As a consequence of proposition 2.3, the function $(x,t) \mapsto M(x,t) := M^*(x,t) = P(X_t \le x)$ given in the proof of theorem 2.1, is the entropy solution of (1.2), with $X_t(y) = \phi(y,t,P_0,u_0)$. Then, the trajectory of a massive particle y, illustrated in the above figures, is a discontinuity line of M, since $M(X_t(y)+0,t)-M(X_t(y)-0,t) = M^*(X_t(y),t)-M_*(X_t(y),t) \ge F_0(y)-F_0(y-0) = P_0(\{y\})$. All the discontinuity lines of the entropy solution are not known *a priori*. We make the conjecture that they coincide with the trajectories $t \mapsto X_t(y)$, motivated by the following property of the entropy solution (see [2]) : for all discontinuity line $t \mapsto x(t)$,

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{A(M^*(x(t),t)) - A(M_*(x(t),t))}{M^*(x(t),t) - M_*(x(t),t)} = \frac{\int_{[\alpha(t),\beta(t)]} u_0(\eta) P_0(\mathrm{d}\eta)}{P_0([\alpha(t),\beta(t)])} \ ,$$

with $[\alpha(t),\beta(t)] \cap \mathcal{S} = \{X_t = x(t)\}$. Hence

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}X_t}{\mathrm{d}t}(y) \ , \quad \forall\, y \in [\alpha(t),\beta(t)] \cap \mathcal{S} \ .$$

5. **Proof of the main results**

We recall that the probability P_0 and the velocity u_0 satisfy the negative jump condition (2.1), and F_0 is the c.d.f. of P_0; for $G \subset \mathbb{R}$ such that $P_0(G) > 0$, $\mathrm{C}(G,t) = (P_0(G))^{-1} \int_G [\eta + t u_0(\eta)]\, P_0(\mathrm{d}\eta)$.

5.1. **Flow characterization and Lipschitz continuity**

Proof of Theorem 2.4. We give the proof with $P = P_0,\ u = u_0$. The sticky particles flow $\phi(\cdot,\cdot,P_0,u_0)$ already satisfies these four properties. Let Φ be a function which satisfies these four properties. Property 2) shows that for all (x,t), there exist $\alpha,\beta \in \mathcal{S}$ such that

$$E(x,t) := \{y \in \mathcal{S} \,:\, \Phi(y,t) = \Phi(x,t)\} = [\alpha,\beta] \cap \mathcal{S} \ .$$

We will show that $E(x,t) \in \xi_t$. First, let us show that each right endpoint β_1 (of an $E(z,t)$) and each left endpoint α_2 (of an $E(z',t)$) such that $P_0((\beta_1,\alpha_2)) > 0$, satisfy $\Phi(\beta_1,t) < \mathrm{C}((\beta_1,\alpha_2),t) < \Phi(\alpha_2,t)$. Let the stochastic process $(X_t(x) := \Phi(x,t)\,,\ t \geq 0)$ be defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_0)$ and $P_t = P_0 X_t^{-1}$, $D = S_0(P_t)$. It easy to see that for all t,

$$\begin{aligned} X_t &= \mathrm{E}[X_0 + tu_0(X_0)/X_t]1_D(X_t) + (X_0 + tu_0(X_0))1_{D^c}(X_t) \\ &= \mathrm{E}[X_0 + tu_0(X_0)/X_t] \end{aligned}$$

since for $X_t(y) \notin D$, $X_t(y) = y + tu_0(y)$ if $y \in (\mathcal{S}_-)\Delta(\mathcal{S}_+)$, and $X_t(y) = y + tu_0^-(y) = y + tu_0^+(y)$ if $y \in \mathcal{S}_- \cap \mathcal{S}_+$; in the last case, (2.1) implies $u_0^-(y) = u_0^+(y) = u_0(y)$.

So, using the fact that $\mathrm{C}(G,t) = \mathrm{E}[X_0 + tu_0(X_0)/X_0 \in G]$ for massive G, we get the desired result from

$$\begin{aligned} \mathrm{C}((\beta_1, \alpha_2), t) &= \mathrm{E}[X_0 + tu_0(X_0)/\beta_1 < X_0 < \alpha_2] \\ &= \mathrm{E}[X_0 + tu_0(X_0)/\Phi(\beta_1, t) < X_t < \Phi(\alpha_2, t)] \\ &= \mathrm{E}[X_t/\Phi(\beta_1, t) < X_t < \Phi(\alpha_2, t)]\ . \end{aligned}$$

For all $y_1 < \alpha < y_2$ s.t. $P_0([y_1, \alpha))P_0([\alpha, y_2)) > 0$, let β_1 be the right endpoint of $E(y_1, t)$, and α_2 be the left endpoint of $E(y_2, t)$. Suppose that $P_0([y_1, \beta_1])P_0((\beta_1, \alpha)) > 0$. One has $\beta_1 < \alpha \leq \alpha_2$ and $\Phi(\beta_1, t) < \mathrm{C}((\beta_1, \alpha), t) < \Phi(\alpha, t)$. Because of the fourth property $\mathrm{C}([y_1, \beta_1], t) \leq \Phi(\beta_1, t)$, this gives $\mathrm{C}([y_1, \beta_1], t) < \mathrm{C}((\beta_1, \alpha), t) < \Phi(\alpha, t)$, so one gets from barycenter calculus $\mathrm{C}([y_1, \alpha), t) < \Phi(\alpha, t)$. When $P_0([y_1, \beta_1])P_0((\beta_1, \alpha)) = 0$, the result is the same. If $y_2 \leq \beta$, the fourth property says $\Phi(\alpha, t) \leq \mathrm{C}([\alpha, y_2), t)$. If $\beta < y_2$ with $P_0((\beta, y_2)) > 0$, then $\beta < \alpha_2$ and this leads (as for y_1, α) to $\Phi(\beta, t) < \mathrm{C}((\beta, y_2), t)$. Using $\Phi(\beta, t) = \mathrm{C}([\alpha, \beta], t)$ if $P_0([\alpha, \beta]) > 0$, one gets from barycenter calculus

$$\mathrm{C}([y_1, \alpha), t) < \mathrm{C}([\alpha, y_2), t),$$

which means that α satisfies $(\mathrm{GVP})_g$ (defined in proposition 4.5).

In the same way, β satisfies $(\mathrm{GVP})_d$ at time t, and these two endpoints satisfy the whole proposition 4.5 (the fourth property gives (4.5) which implies (4.6)). There exist $[a, b] \cap \mathcal{S}$, $[a', b'] \cap \mathcal{S} \in \xi_t$ such that $\alpha \in [a, b]$, $\beta \in [a', b']$. Suppose that $[a, b] = [a', b']$.

If $P_0([a, b]) = 0$, then $\Phi(x, t) = x + tu_0(x) = \phi(x, t)$. If $P_0([a, b]) > 0$, only one of the probabilities $P_0([a, \alpha))$, $P_0([\alpha, \beta])$ or $P_0((\beta, b])$ is positive; in the contrary case, $\mathrm{C}([a, \alpha), t) < \mathrm{C}([\alpha, b], t)$ or $\mathrm{C}([a, \beta], t) < \mathrm{C}((\beta, b], t)$ would contradict property 4) for $[a, b]$. Now, suppose that $P_0([a, \alpha)) > 0 = P_0([\alpha, b])$. One has $a < \alpha$, and then $\alpha \notin \mathcal{S}_-$; indeed, if $\alpha \in \mathcal{S}_-$, there exit α_n, $\beta_n \uparrow \alpha$ s.t. $E(\alpha_n, t) = [\alpha_n, \beta_n] \cap \mathcal{S}$, $P_0([a, \alpha_n))P_0([\alpha_n, \alpha)) > 0$, and then $\mathrm{C}([a,\alpha_n),t) < \mathrm{C}([\alpha_n,b],t)$, since α_n satisfies $(\mathrm{GVP})_g$. Necessarily, $\alpha \in \mathcal{S}_+$, so $\alpha = \beta = b = x$. Thus $\Phi(x,t) = x + tu_0(x)$, and as in the proof of (4.6), $\phi(x,t) = \mathrm{C}([a,x],t) = x + tu_0(x)$. One gets the same result, either when $P_0([a,\beta]) = 0 < P_0((\beta,b])$ (with $\Phi(x,t) = \phi(x,t) = a + tu_0(a)$), or when $P_0([\alpha,\beta]) > 0 = P_0([a,\alpha) \cup (\beta,b])$ (with $\Phi(x,t) = \phi(x,t) = \mathrm{C}([\alpha,\beta],t)$).

In the same way (as for x), one has $\Phi(a,t) = \phi(a,t) = \phi(b,t) = \Phi(b,t)$, which means that $\alpha = a$, $\beta = b$.

With the same arguments, the case $[a,b] \neq [a',b']$, *i.e.* $b < a'$, is in contradiction with the fourth property. □

In order to prove theorem 2.1, we define the process $(x,t) \mapsto Y_t(x) = \phi(x,t,P_0,u_0)$ on the probability space $(\mathbb{R}\,,\mathcal{B}(\mathbb{R})\,,P_0)$, and the processes $(m,t) \mapsto X_t^-(m) = x_g(m,t)$, $X_t^+(m) = x_d(m,t)$ on the probability space $((0,1)\,,\mathcal{B}\,,\lambda)$, where λ is the Lebesgue measure. It is clear that the processes (X_t^-), (X_t^+) are not distinguishable : a.s., $X_t^- = X_t^+ =: X_t \ \forall\, t \geq 0$, and $X_t = \phi(X_0,t,P_0,u_0) \ \forall\, t \geq 0$.

Both processes (X_t) and (Y_t) are illustrations of theorem 2.1.

Proof of theorem 2.1. 1) In the case of $X_0 = F_0^{-1}$, we consider the previous process (X_t). By the definition (4.3) of M^*, one has $X_t = F_t^{-1}$ where $F_t := M^*(\cdot,t)$ is the c.d.f. of X_t. Theorem 4.1 shows that (X_t) is almost surely Lipschitz continuous. Moreover, if $D := \{x : F_t(x) > F_t(x-0)\}$, then using theorem 4.1 and the same argument as in the previous proof, one gets

$$\begin{aligned}\frac{\mathrm{d}X_t}{\mathrm{d}t} = u(X_t,t) &= u_0(X_0)1_{D^c}(X_t) + \mathrm{E}[u_0(X_0)/X_t]1_D(X_t)\\ &= \mathrm{E}[u_0(X_0)/X_t].\end{aligned}$$

In the general case of $PX_0^{-1} = P_0$, the process $X_t := \phi(X_0,t,P_0,u_0)$ is also Lipschitz continuous and we get again the same result from corollary 4.6.

2) Now, we show the negative jump condition (2.1) for $P_t := PX_t^{-1}$ and $u_t := u(\cdot,t)$. Let $\mathcal{S},\mathcal{S}_0,\mathcal{S}_-,\mathcal{S}_+$ be the support of P_0 and its subsets defined in section 2. We recall that the functions u_t^-, u_t^+ are respectively defined on $\mathcal{S}_-(P_t)$, $\mathcal{S}_+(P_t)$ by

$$u_t^-(x) = \limsup_{x'\to x-0} \frac{\int_{[x',x)} u_t(\eta)P_t(\mathrm{d}\eta)}{P_t([x',x))}\ , \quad u_t^+(x) = \liminf_{x'\to x+0} \frac{\int_{(x,x']} u_t(\eta)P_t(\mathrm{d}\eta)}{P_t((x,x'])}\ .$$

One has $\mathcal{S}_-(P_t) \subset \phi(\mathcal{S}_-, t, P_0, u_0)$, $\mathcal{S}_+(P_t) \subset \phi(\mathcal{S}_+, t, P_0, u_0)$. If $x \in \mathcal{S}_-(P_t)$ and $G(x,t) := \{y \in \mathcal{S} \,:\, \phi(y,t,P_0,u_0) = x\} = [\alpha,\beta] \cap \mathcal{S}$ then $\alpha \in \mathcal{S}_-$. Thus, writing $G(x',t) = [\alpha',\beta'] \cap \mathcal{S}$ in the term defining u_t^-, one gets

$$\frac{\int_{\{x' \leq X_t < x\}} u_t(X_t)\mathrm{d}P}{P(\{x' \leq X_t < x\})} = \frac{\int_{\{x' \leq X_t < x\}} u_0(X_0)\mathrm{d}P}{P(\{x' \leq X_t < x\})} = \frac{\int_{\{\alpha' \leq X_0 < \alpha\}} u_0(X_0)\mathrm{d}P}{P(\{\alpha' \leq X_0 < \alpha\})} ,$$

$$u_t^-(x) = \limsup_{\alpha' \to \alpha - 0} \frac{\int_{[\alpha',\alpha)} u_0(\eta)P_0(\mathrm{d}\eta)}{P_0([\alpha',\alpha))} = u_0^-(\alpha) = u_0(\alpha) \, . \qquad (5.1)$$

If $x \in \mathcal{S}_-(P_t) \cap \mathcal{S}_0(P_t)$, then (4.6) gives $\alpha + tu_t^-(x) \geq x = \mathrm{C}([\alpha,\beta],0) + tu_t(x)$, so $u_t^-(x) \geq u_t(x)$. In the same way, $u_t(x) \geq u_t^+(x)$ for $x \in \mathcal{S}_+(P_t) \cap \mathcal{S}_0(P_t)$, since

$$u_t^+(x) = \liminf_{\beta' \to \beta + 0} \frac{\int_{(\beta,\beta']} u_0(\eta)P_0(\mathrm{d}\eta)}{P_0((\beta,\beta'])} = u_0^+(\beta) = u_0(\beta) \, . \qquad (5.2)$$

If $x \in \mathcal{S}_-(P_t) \cap \mathcal{S}_+(P_t) \backslash \mathcal{S}_0(P_t)$, one has $u_t^-(x) = u_0(\alpha) \geq u_0(\beta) = u_t^+(x)$. If $\alpha = \beta$, then $u_t(x) = u_0(\alpha) = u_t^-(x) = u_t^+(x) =$. If $\alpha < \beta$, then $u_0(\alpha) > u_0(\beta)$ and the velocity is not defined, but one can define $u_t(x) \in [u_0(\beta), u_0(\alpha)]$; the set of such x's is at most countable.

For $x \in \mathcal{S}_-(P_t) \backslash (\mathcal{S}_+(P_t) \cup \mathcal{S}_0(P_t))$, $\alpha = \beta \in \mathcal{S}_- \backslash (\mathcal{S}_+ \cup \mathcal{S}_0)$, so $u_t(x) = u_0(\alpha) = u_t^-(x)$. In the same way, for $x \in \mathcal{S}_+(P_t) \backslash (\mathcal{S}_-(P_t) \cup \mathcal{S}_0(P_t))$, $u_t(x) = u_t^+(x) = u_0(\beta)$.

3) The flow $(x,s) \mapsto \phi(x,s,P_t,u_t)$ is then well defined. Let us define

$$(x,s) \mapsto \psi(x,s) := \phi(y,t+s,P_0,u_0) \quad \forall\, y \;\text{ s.t. }\; \phi(y,t,P_0,u_0) = x \, .$$

This function is also well defined because clusters grow up. The following lemma 5.1 then implies that $\psi = \phi(\cdot,\cdot,P_t,u_t)$. □

Lemma 5.1. *Let $(P_t, u_t, t \geq 0)$ be defined as in theorem 2.1. If a function ψ is such that*

$$\psi(\phi(y,t,P_0,u_0),s) = \phi(y,t+s,P_0,u_0) \quad \forall\,(y,s) \in \mathcal{S}(P_0) \times \mathbb{R}_+ \, ,$$

then $\psi(x,s) = \phi(x,s,P_t,u_t) \quad \forall\,(x,s) \in \mathcal{S}(P_t) \times \mathbb{R}_+$.

Proof. Let us show that $\psi = \phi(\cdot,\cdot,P_t,u_t)$, by proving that ψ satisfies the characteristic properties of theorem 2.4 with $P = P_t, u = u_t$. Properties 1) and 2) are immediate. For property 3) while $P_t(\{a \,:\, \psi(a,s) = \psi(x,s)\}) > 0$, we remark that

$$\psi(\cdot,s) = \mathrm{E}_{P_t}[\psi_0 + su_t(\psi_0)/\psi(\cdot,s)] \quad \forall\, s$$

on $(\mathbb{R}, P_t)$, where ψ_0 is the identity function. This comes from the process $(X_t(x) = \phi(x, t, P_0, u_0)\,, t \geq 0)$ defined on $(\mathbb{R}, P_0)$, which satisfies for all s, t, $u_t(X_t) = \mathrm{E}_{P_0}[u_0(X_0)/X_t]$, $\sigma(X_{t+s}) \subset \sigma(X_t)$, so $\mathrm{E}_{P_0}[u_0(X_0)/X_{t+s}] = \mathrm{E}_{P_0}[u_t(X_t)/X_{t+s}]$. The fact that $X_t = \mathrm{E}_{P_0}[X_0+tu_0(X_0)/X_t]$ and $P_0X_t^{-1} = P_t$ then leads to

$$\begin{aligned}\psi(X_t,\, s) = X_{t+s} &= \mathrm{E}_{P_0}[X_0 + tu_0(X_0)/X_{t+s}] + s\mathrm{E}_{P_0}[u_0(X_0)/X_{t+s}]\\ &= \mathrm{E}_{P_0}[X_t/X_{t+s}] + s\mathrm{E}_{P_0}[u_t(X_t)/X_{t+s}]\\ &= \mathrm{E}_{P_0}[X_t + su_t(X_t)/\psi(X_t,\, s)] = \mathrm{E}_{P_t}[\psi_0 + su_t(\psi_0)/\psi(\cdot, s)](X_t)\;.\end{aligned}$$

What about elements $B(x, s) := \{a \in \mathcal{S}(P_t) \,:\, \psi(a, s) = \psi(x, s)\}$ such that $P_t(B(x, s)) = 0$? As $\psi(\cdot, s)$ is continuous and non-decreasing, there exist $a, b \in \mathcal{S}(P_t)$ such that $B(x, s) = [a, b] \cap \mathcal{S}(P_t)$. One has $\{y \in \mathcal{S} \,:\, \phi(y, t+s, P_0, u_0) = \psi(x, s)\} =: [\alpha, \beta] \cap \mathcal{S} \in \xi_{t+s}$ with $a = \phi(\alpha, t, P_0, u_0)$, $b = \phi(\beta, t, P_0, u_0)$, $P_t([a, b]) = P_0([\alpha, \beta])$. If $P_t([a, b]) = P_0([\alpha, \beta]) = 0$, there are two cases.

i) $x = a \in \mathcal{S}_-(P_t)$: in this case, a is an accumulation on the left of elements of $\mathcal{S}(P_t) = \phi(\mathcal{S}, t, P_0, u_0)$ and $\alpha \in \mathcal{S}_-$, so

$$a = \phi(\alpha, t, P_0, u_0) = \alpha + tu_0(\alpha)\,, \quad \phi(\alpha, t+s, P_0, u_0) = \alpha + (t+s)u_0(\alpha)\;.$$

We have already seen in (5.1) that in this case $u_t^-(a) = u_0(\alpha)$. So $\psi(a, s) = a + su_t^-(a)$.

ii) $x = b \in \mathcal{S}_+(P_t)$: using (5.2), we get in the same way as in the previous case $\psi(b, s) = b + su_t^+(b)$.

Now, let us show that $B(x, t) = [a, b] \cap \mathcal{S}(P_t)$ satisfies the fourth property. Suppose that $P_t([a, x])P_t([x, b]) > 0$. Using previous α, β, we have

$$\begin{aligned}[\alpha_1, \beta_1] \cap \mathcal{S} &:= \{y \in \mathcal{S} \,:\, \phi(y, t, P_0, u_0) = x\}\\ &\subset \{y \in \mathcal{S} \,:\, a \leq \phi(y, t, P_0, u_0) \leq b\}\\ &= [\alpha, \beta] \cap \mathcal{S}\end{aligned}$$

with $P_t([a, x]) = P_0([\alpha, \beta_1])$. The fourth characteristic property of $\phi(\cdot, t+s, P_0, u_0)$ for $\beta_1 \in [\alpha, \beta]\cap\mathcal{S} \in \xi_{t+s}$ means that $\phi(\alpha, t+s, P_0, u_0) \leq \mathrm{E}_{P_0}[X_0 + (t+s)u_0(X_0)/\alpha \leq X_0 \leq \beta_1]$. As

$$\begin{aligned}&\mathrm{E}_{P_0}[X_0 + (t+s)u_0(X_0)/\alpha \leq X_0 \leq \beta_1]\\ &= E_{P_0}[X_0 + (t+s)u_0(X_0)/a \leq X_t \leq x] = E_{P_0}[X_t + su_t(X_t)/a \leq X_t \leq x]\\ &= E_{P_t}[\psi_0 + su_t(\psi_0)/a \leq \psi_0 \leq x]\;,\end{aligned}$$

we get $\psi(a,s) = \phi(\alpha, t+s, P_0, u_0) \le \mathrm{E}_{P_t}[\psi_0 + su_t(\psi_0)/a \le \psi_0 \le x]$. In the same way, $\psi(a,s) \ge \mathrm{E}_{P_t}[\psi_0 + su_t(\psi_0)/x \le \psi_0 \le b]$ which is the fourth characteristic property of $\psi(\cdot, s)$ for $[a,b] \cap \mathcal{S}(P_t) = B(x,t)$. We conclude from theorem 2.4 that $\psi = \phi(\cdot,\cdot,P_t,u_t)$. □

Remark 5.2. The stochastic process $X_t := \phi(X_0, t, P_0, u_0)$ is such that $X_t = \mathrm{E}[X_0 + tu_0(X_0)/X_t]$, simply because of the third characteristic property in theorem 2.4.

Proof of Proposition 2.5. From the definition of u_t, it is clear that

$$||u_t||_\infty \le ||u_0||_\infty \le C$$

for all $t \ge 0$. One has

$$\begin{aligned}
\big|\phi(x,s',P_t,u_t) - \phi(y,s,P_t,u_t)\big| \\
&\le \big|\phi(x,s',P_t,u_t) - \phi(x,s,P_t,u_t)\big| \\
&\quad + \big|\phi(x,s,P_t,u_t) - \phi(y,s,P_t,u_t)\big| \\
&\le C|s'-s| + \big|\phi(x,s,P_t,u_t) - \phi(y,s,P_t,u_t)\big|.
\end{aligned}$$

If $x < y$ in $\mathcal{S}(P_t)$, define

$$\begin{aligned}
[\alpha_1, \beta_1] \cap \mathcal{S}(P_t) &:= \{a \in \mathcal{S}(P_t) : \phi(a,s,P_t,u_t) = \phi(x,s,P_t,u_t)\}\,, \\
[\alpha_2, \beta_2] \cap \mathcal{S}(P_t) &:= \{a \in \mathcal{S}(P_t) : \phi(a,s,P_t,u_t) = \phi(y,s,P_t,u_t)\}\,.
\end{aligned}$$

If $\phi(x,s,P_t,u_t) \ne \phi(y,s,P_t,u_t)$, then $x \le \beta_1 < \alpha_2 \le y$ and one gets by definition of massless clusters or from proposition 4.5 (replacing (P_0,u_0) by (P_t,u_t)) : $\beta_1 + su_t(\beta_1) \le \phi(x,s,P_t,u_t) < \phi(y,s,P_t,u_t) \le \alpha_2 + su_t(\alpha_2)$. So

$$\begin{aligned}
\big|\phi(x,s,P_t,u_t) - \phi(y,s,P_t,u_t)\big| &\le \alpha_2 - \beta_1 + s[u_t(\alpha_2) - u_t(\beta_1)]\,, \\
\big|\phi(x,s',P_t,u_t) - \phi(y,s,P_t,u_t)\big| &\le C|s'-s| + |x-y| \\
&\quad + s \sup_{x \le \beta \le \alpha \le y} \big(u_t(\alpha) - u_t(\beta)\big)\,.
\end{aligned}$$

Now, we show (2.3). For $t > 0$ define

$$\begin{aligned}
[\alpha_1, \beta_1] \cap \mathcal{S}(P_0) :&= \{a \in \mathcal{S}(P_0) : \phi(a,t,P_0,u_0) = x_1\} \\
[\alpha_2, \beta_2] \cap \mathcal{S}(P_0) :&= \{a \in \mathcal{S}(P_0) : \phi(a,t,P_0,u_0) = x_2\}\,.
\end{aligned}$$

If $x_1 < x_2$ in $\mathcal{S}_0(P_t)$, one has

$$\begin{aligned} x_2 - x_1 &= \phi(\beta_2, t, P_0, u_0) - \phi(\beta_1, t, P_0, u_0) \\ &= \mathrm{C}([\alpha_2, \beta_2], 0) - \mathrm{C}([\alpha_1, \beta_1], 0) + t[u_t(x_2) - u_t(x_1)] \,. \end{aligned}$$

As $\mathrm{C}([\alpha_2, \beta_2], 0) \geq \alpha_2 > \beta_1 \geq \mathrm{C}([\alpha_1, \beta_1], 0)$, one gets (2.3). If $x_i \notin \mathcal{S}_0(P_t)$, then $x_i = \alpha_i + tu_0(\alpha_i) = \beta_i + tu_0(\beta_i)$, with $u_t(x_i) \in [u_0(\beta_i), u_0(\alpha_i)]$. □

These results can easily be generalized to the case of infinite mass (modeled by a Radon measure P_0) with unbounded velocity function. In [6], an extension of the flow for Radon measures was already done.

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Octave Moutsinga
Université des Sciences et Techniques de
Masuku
Faculté des Sciences - Dpt
Mathématiques et Informatique
BP 943 Franceville, Gabon.
octavemoutsing-pro@yahoo.fr