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Various kinds of sensitive singular perturbations

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Abstract

We consider variational problems of P. D. E. depending on a small parameter ε when the limit process $\varepsilon \downarrow 0$ implies vanishing of the higher order terms. The perturbation problem is said to be sensitive when the energy space of the limit problem is out of the distribution space, so that the limit problem is out of classical theory of P. D. E. We present here a review of the subject, including abstract convergence theorems and two very different model problems (the second one is presented for the first time). For each one we prove the sensitive character and we give a formal asymptotics for the behavior $\varepsilon \downarrow 0$.

1. Introduction

This paper is devoted to a review and some new results on a class of singular perturbations for variational problems of P. D. E. arising in thin shell theory (but the asymptotic behavior is highly pathological, so that examples only involve simplified model problems). We consider variational problems depending on $\varepsilon \downarrow 0$ of the form (2.4) hereafter, in an energy space V such that, for $\varepsilon > 0$ the bilinear form is continuous and coercive on V, whereas the loading is in the dual, $f \in V'$. At the limit $\varepsilon = 0$ the bilinear form changes drastically, and it is continuous and coercive on a larger space V_a. Sensitivity is concerned with the case when that space is so large that it is not contained in the distribution space (see [8] and [14]). Obviously, the limit problem is out of classical theory of P. D. E.; moreover, the variational problem only makes sense for loadings f in the dual V_a' which is "very small", not containing the space D of test functions of distributions.

Section 2 is devoted to abstract theory of singular perturbations. Classical results (for loading $f \in V_a'$ see [11] and [13]) are given in section 2.2. Moreover, Section 2.3 contains a result of D. Caillerie [3] which proves that there exist a space (denoted by V_A) where convergence of the solutions takes place for any $f \in V'$.

Section 3 contains comments, developments and examples to prepare the sequel. This includes consequences of sensitivity on finite element approximation (sect. 3.1), comments on the case when the limit form has a non - trivial kernel (and then the construction of V_a is not possible (sect. 3.2)) which is used later on to inspire the heuristics of the formal asymptotics in sect. 4.3. Moreover, sect. 3.3 recalls elements of Fourier transform of (non tempered) distributions, which sends to spaces of analytical functionals (out of the distribution space, see [10] and [12]) which are used later in sect. 4. Sequences of functions converging in the sense of analytical functionals (but not converging in the distribution sense) imply complexification (i.e. in some sense, "the graphs become infinitely complex"); examples of such situation, including cases which appear later in sensitive singular perturbations, are displayed in sect. 3.4.

Sect. 4 is based on [21], [20], [22] and [15]. It is devoted to an example of sensitive singular perturbation where the pathological character of the limit problem comes from the fact that there is a boundary condition not satisfying the Shapiro - Lopatinskii condition see [8], [14] and [17]. It constitutes a simplified model of thin shells in the case when a part of the boundary is free and the middle surface is elliptic. Convergence theorems and a formal asymptotics exhibiting complexification are given in sections 4.2 and 4.3, respectively.

Sect. 5 contains a new class of examples inspired by certain shell problems with edges. They are published by the first time. The sensitive character is proved in sect. 5.1. A formal asymptotics is given in sect. 5.2, showing that the solutions "explode to infinity" and are only holden by narrow boundary layers where the energy concentrates.

Notations are standard. We denote

$$\partial_k = \frac{\partial}{\partial x_k}, \quad k = 1, 2. \tag{1.1}$$

It should be noted that if u is a function that only depends on x_2, ∂_2 will also denote the derivative $u'(x_2)$. Moreover, $\|.\|_1$ will denote the H^1 norm and C will be a constant taking various irrelevant values.

2. Abstract theory of singular perturbations

Let us denote by V a Hilbert space, by a and b two bilinear continuous and symmetric forms satisfying:

$$\begin{cases} a(v,v) \geq 0, \\ a(v,v) = 0 \quad \Rightarrow \quad v = 0, \end{cases} \tag{2.1}$$

$$b(v,v) \geq 0. \tag{2.2}$$

The norm of V is defined by:

$$\|v\|_V^2 = a(v,v) + b(v,v). \tag{2.3}$$

We then consider, for a given $f \in V'$, the variational problem:

$$\begin{cases} \textit{Find } u^\varepsilon \in V \textit{ such that} \\ a(u^\varepsilon, v) + \varepsilon^2 b(u^\varepsilon, v) = \langle f, v \rangle_{V'V} \quad \forall v \in V. \end{cases} \tag{2.4}$$

From the obvious a priori estimates

$$c\varepsilon^2 \|v\|_V^2 \leq a(v,v) + \varepsilon^2 b(v,v) \leq C\|v\|_V^2$$

we see that, for each $\varepsilon > 0$, $a + \varepsilon^2 b$ is a continuous and coercive form on V and, consequently, there exists a unique solution $u^\varepsilon \in V$ of (2.4) .

For fixed $v \in V$, $a(v,w)$ is a functional continuous on V so that it may be expressed as the duality product of an element $Av \in V'$ by w:

$$a(v,w) = \langle Av, w \rangle_{V'V}. \tag{2.5}$$

Analogously

$$b(v,w) = \langle Bv, w \rangle_{V'V}. \tag{2.6}$$

Let us show that A is bounded (i.e. $A \in \mathcal{L}(V, V')$). Let us consider a sequence v_n such that $\|v_n\| = 1$. We have

$$|\langle Av_n, w \rangle| = |a(v_n, w)| \leq C\|w\|_V,$$

so that Av_n is bounded in V'. It then follows from the principle of uniform boundedness that Av_n is bounded in V' (see [16], corollary 1.12, p. 74 if necessary) so that A is bounded.

Lemma 2.1. *The operator $A \in \mathcal{L}(V, V')$ is injective.*

Proof. Let Av=0, then

$$0 = \langle Av, v \rangle_{V'V} = a(v,v) \quad \Rightarrow v = 0.$$

□

Now, let us define the following norms:

$$\|v\|_a = a(v,v)^{\frac{1}{2}}, \tag{2.7}$$

$$\|v\|_A = \|Av\|_{V'}. \tag{2.8}$$

From (2.1) and Lemma 2.1 we see that they are effectively norms. We shall denote by

$$\begin{cases} V_a \text{ the completion of } V \text{ with } \|.\|_a, \\ V_A \text{ the completion of } V \text{ with } \|.\|_A. \end{cases} \tag{2.9}$$

Lemma 2.2. *The operator A is a bijection from V on its range $\mathcal{R}(A)$ in V'.*

Proof. From the definition of $\mathcal{R}(A)$, A is surjective and, from Lemma 2.1 it is also injective. □

Lemma 2.3. *The range $\mathcal{R}(A)$ is dense in V'.*

Proof. If it is not dense, then there exists $f \in V'$ such that

$$(f, Av)_{V'} = 0 \quad \forall v \in V'. \tag{2.10}$$

The scalar product in V' is a continuous functional on V' which may be expressed as the duality product of an element $v^f \in V$ with $Av \in V'$, i.e. we have

$$(f, Av)_{V'} = \langle v^f, Av \rangle_{VV'}. \tag{2.11}$$

Then, by taking $v = v^f$ in (2.10) and recalling (2.5) and (2.11), we have

$$0 = \langle v^f, Av^f \rangle_{VV'} = a(v^f, v^f) \Rightarrow v^f = 0 \Rightarrow f = 0.$$

□

Lemma 2.4. *The operator $A \in \mathcal{L}(V, V')$ extends as a bijection $\tilde{A}$ from V_A on V'. Moreover, as V_A and V' are Hilbert spaces, $\tilde{A}$ is an isomorphism from V_A on V'.*

Proof. As $\mathcal{R}(A)$ is dense in V', V' is the completion of $\mathcal{R}(A)$ equipped with its norm. The Cauchy sequences in V' are of the form Av_n where v_n are Cauchy sequences in V_A hence the conclusion. □

Remark 2.5. It should be noticed that $A \in \mathcal{L}(V, V')$ or its extension $\tilde{A} \in \mathcal{L}(V_A, V')$ should not be confused with $\mathcal{A} \in \mathcal{L}(V_a, V_a')$ defined by

$$\langle \mathcal{A}v, w \rangle_{V_a' V_a} = a(v, w) \ \text{ for } v, w \in V_a \tag{2.12}$$

which is the operator in the limit problem.

Let us recall that

$$V \subset V_a \quad \Leftrightarrow \quad V_a' \subset V' \tag{2.13}$$

and

$$V \subset V_A. \tag{2.14}$$

2.1. **Limit problems**

Let $f \in V_a'$ be given. The variational formulation of the limit problem writes:

$$\begin{cases} \textit{Find } u^0 \in V_a \textit{ such that },\\ a(u^0, v) = \langle f, v\rangle_{V_a', V_a} \quad \forall v \in V_a. \end{cases} \tag{2.15}$$

This problem is automatically in the framework of the Lax-Milgram theorem, consequently, it has a unique solution in V_a and it is equivalent to

$$\begin{cases} \textit{Find } u^0 \in V_a \textit{ such that},\\ \mathcal{A}u^0 = f \quad \textit{in } V_a'. \end{cases} \tag{2.16}$$

Let be $f \in V'$. We consider the problem:

$$\widetilde{A}u^0 = f \ \ \textit{in} \ \ V'. \tag{2.17}$$

As $\tilde{A}$ is a bijection from V_A on V', this problem has a unique solution in V_A.

Remark 2.6. It is important to note the abstract character of the previous consideration. Indeed, it is possible that either (2.16) or (2.17) does not make sense in terms of equation (in the distributional sense or other one). Sensitivity exhibits such cases.

2.2. **Limit process. Variational theory**

Theorem 2.7. *Let be f in V_a'. Then,*

$$u^\varepsilon \ \to \ u^0 \ \textit{strongly in} \ \ V_a \tag{2.18}$$

where u^ε and u^0 are the solutions of (2.4) and (2.15), respectively.

Proof. Taking $v = u^\varepsilon$ in (2.4), we then have

$$a(u^\varepsilon, u^\varepsilon) + \varepsilon^2 b(u^\varepsilon, u^\varepsilon) = \langle f, u^\varepsilon\rangle_{V_a' V_a}. \tag{2.19}$$

From (2.1) and (2.2) we have

$$a(u^\varepsilon, u^\varepsilon) \le \langle f, u^\varepsilon\rangle_{V_a' V_a} \tag{2.20}$$

and

$$\varepsilon^2 b(u^\varepsilon, u^\varepsilon) \leq \langle f, u^\varepsilon \rangle_{V_a' V_a}. \tag{2.21}$$

From (2.20) we get

$$\|u^\varepsilon\|_{V_a} \leq \|f\|_{V_a'}$$

so that u^ε remains bounded in V_a. From the weak compactness of bounded sequences it follows that for a certain $u^* \in V_a$ and a subsequence (in fact we shall see that u^* is uniquely defined, so that the subsequence is the whole sequence)

$$u^\varepsilon \quad \to u^* \ \ \textit{weakly in } \ V_a. \tag{2.22}$$

Then, from (2.20), (2.21) and (2.22) we see that

$$\varepsilon^2 a(u^\varepsilon, u^\varepsilon) + \varepsilon^2 b(u^\varepsilon, u^\varepsilon) \leq \|f\|_{V_a'} \|u^\varepsilon\|_{V_a} \leq C\|f\|^2_{V_A'} \tag{2.23}$$

from which

$$\varepsilon^2 \|u^\varepsilon\|^2_V \leq C. \tag{2.24}$$

Now, let us fix v and take $\varepsilon \to 0$ in (2.4). We obtain, by using (2.22),

$$a(u^\varepsilon, v) \quad \to \quad a(u^*, v). \tag{2.25}$$

As we have

$$|\varepsilon^2 b(u^\varepsilon, v)| \leq \varepsilon^2 b(u^\varepsilon, u^\varepsilon)^{\frac{1}{2}} b(v, v)^{\frac{1}{2}}$$

and

$$b(u^\varepsilon, u^\varepsilon)^{\frac{1}{2}} \leq C\|u^\varepsilon\|_V \leq \frac{C}{\varepsilon}$$

we finally obtain

$$|\varepsilon^2 b(u^\varepsilon, v)| \leq \varepsilon^2 \frac{C'}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} 0, \tag{2.26}$$

so that, passing to the limit in (2.4), we have

$$a(u^*, v) = \langle f, v \rangle \ \ \forall v \in V \quad \Leftrightarrow \quad \forall v \in V_a.$$

Then, from the uniqueness of the solution of the limit problem, $u^* = u^0$, and we have

$$u^\varepsilon \to u^0 \qquad \textit{weakly in } \ V_a. \tag{2.27}$$

Let us show that $u^\varepsilon \to u^0$ *strongly in* V_a. We have:

$$\begin{aligned} & a(u^\varepsilon - u^0, u^\varepsilon - u^0) + \varepsilon^2 b(u^\varepsilon, u^\varepsilon) = \\ & \quad a(u^\varepsilon, u^\varepsilon) + \varepsilon^2 b(u^\varepsilon, u^\varepsilon) + a(u^0, u^0) - 2a(u^0, u^\varepsilon) = \\ & \quad \langle f, u^\varepsilon \rangle + \langle f, u^0 \rangle - 2a(u^0, u^\varepsilon) = -\langle f, u^\varepsilon \rangle + \langle f, u^0 \rangle = \\ & \quad \langle f, u^0 - u^\varepsilon \rangle \xrightarrow[\varepsilon \to 0]{} 0. \end{aligned} \tag{2.28}$$

consequently, $a(u^\varepsilon - u^0, u^\varepsilon - u^0) \underset{\varepsilon\to 0}{\longrightarrow} 0$, i.e. $u^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u^0$ strongly in V_a. □

Theorem 2.8. *When $f \in V'$ then there exists u^ε solution of (2.4). Let us now define the energy E^ε by*

$$2E^\varepsilon = a(u^\varepsilon, u^\varepsilon) + \varepsilon^2 b(u^\varepsilon, u^\varepsilon).$$

Then the necessary and sufficient condition for E^ε to be bounded is that $f \in V_a'$.

Proof. Let us show that the condition is sufficient. Let $f \in V_a'$ then, from (2.22)

$$2E^\varepsilon = \langle f, u^\varepsilon \rangle_{V_a'V_a} \le C\|u^\varepsilon\|_a \le C$$

which shows that E^ε is bounded. The condition is necessary: let us assume that E^ε is bounded i.e., that

$$a(u^\varepsilon, u^\varepsilon) + \varepsilon^2 b(u^\varepsilon, u^\varepsilon) \le C$$

then we have

$$a(u^\varepsilon, u^\varepsilon) \le C$$

and

$$\varepsilon^2 b(u^\varepsilon, u^\varepsilon) \le C\varepsilon^2 \|u^\varepsilon\|_V^2 \le C.$$

Then, for fixed v in (2.4) and $\varepsilon \to 0$, by reasoning as in the proof of theorem 2.7, we see that there exists $u^* \in V_a$ such that

$$a(u^*, v) = \langle f, v \rangle_{V'V} \qquad \forall v \in V.$$

We then see that the left hand side is a functional of v which is continuous in the V_a topology, so that the right-hand side is too, and this amounts to $f \in V_a'$. □

2.3. Limit process. Non variational theory

Theorem 2.9. *Let $f \in V'$, u^ε and u^0 the solutions of (2.4) and (2.17), respectively. Then, $u^\varepsilon \to u^0$ strongly in V_A.*

Proof. Taking $v = u^\varepsilon$ in (2.1), we obtain

$$a(u^\varepsilon, u^\varepsilon) + \varepsilon^2 b(u^\varepsilon, u^\varepsilon) = \langle f, u^\varepsilon \rangle_{V'V}$$

which may be written under the form

$$(1-\varepsilon^2) a(u^\varepsilon, u^\varepsilon) + \varepsilon^2 a(u^\varepsilon, u^\varepsilon) + \varepsilon^2 b(u^\varepsilon, u^\varepsilon) = \langle f, u^\varepsilon \rangle_{V'V}$$

and we see that

$$\varepsilon^2 a(u^\varepsilon, u^\varepsilon) + \varepsilon^2 b(u^\varepsilon, u^\varepsilon) \leq \langle f, u^\varepsilon \rangle_{V'V}$$

from which we get

$$\varepsilon^2 \|u^\varepsilon\|_V \leq C$$

and

$$\varepsilon^2 u^\varepsilon \to u^* \ \textit{weakly in } V, \tag{2.29}$$

in the sense of subsequences, for a certain $u^* \in V$. Let us show that $u^* = 0$ and that the convergence is strong. Let us write (2.4) under the form

$$Au^\varepsilon + \varepsilon^2 B u^\varepsilon = f \ \textit{ in } V'. \tag{2.30}$$

After multiplying by ε^2, we have

$$A\varepsilon^2 u^\varepsilon + \varepsilon^2 B \varepsilon^2 u^\varepsilon = \varepsilon^2 f \ \textit{ in } V',$$

where as $\varepsilon \to 0$:

$$A\varepsilon^2 u^\varepsilon \to Au^* \ \textit{ weakly in } V'$$

and

$$B\varepsilon^2 u^\varepsilon \to Bu^*.$$

We then have

$$Au^* = 0 \quad \textit{in } V'$$

from which

$$0 = \langle Au^*, u^* \rangle_{V'V} = a(u^*, u^*) \quad \Rightarrow \quad u^* = 0.$$

And (2.29) becomes

$$\varepsilon^2 u^\varepsilon \to 0 \ \textit{weakly in } V.$$

Let us show that the convergence is strong. Writing (2.4) with $v = \varepsilon^2 u^\varepsilon$, we have

$$\varepsilon^2 a(u^\varepsilon, u^\varepsilon) + b(\varepsilon^2 u^\varepsilon, \varepsilon^2 u^\varepsilon) = \langle f, \varepsilon^2 u^\varepsilon \rangle_{V'V}$$

and, as $\langle f, \varepsilon^2 u^\varepsilon \rangle_{V'V} \xrightarrow[\varepsilon \to 0]{} 0$,

$$\varepsilon^2 a(u^\varepsilon, u^\varepsilon) + b(\varepsilon^2 u^\varepsilon, \varepsilon^2 u^\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0,$$

or equivalently

$$(\varepsilon^2 - \varepsilon^4) a(u^\varepsilon, u^\varepsilon) + a(\varepsilon^2 u^\varepsilon, \varepsilon^2 u^\varepsilon) + b(\varepsilon^2 u^\varepsilon, \varepsilon^2 u^\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0$$

which implies

$$\varepsilon^2 u^\varepsilon \to 0 \ \textit{strongly in} \quad V.$$

Then,

$$B\varepsilon^2 u^\varepsilon \to 0 \ \textit{strongly in } V',$$

and passing to the limit in (2.30) we obtain

$$Au^\varepsilon \to f \ \textit{strongly in } V'$$

which may be written

$$\tilde{A}u^\varepsilon \to f \ \textit{strongly in } V'.$$

As $\tilde{A}$ is an isomorphism from V_A on V' (Lemma 2.4)

$$u^\varepsilon \to \tilde{A}^{-1}f \ \textit{strongly in } V_A.$$

But $\tilde{A}^{-1}f = u^0$ where u^0 is the solution of (2.17) and consequently $u^\varepsilon \to u^0$ strongly in V_A. □

3. General comments on sensitivity, analytic functionals and complexificating sequences.

3.1. Sensitivity

The above theory of singular perturbations is obviously abstract. Even in that framework, it appears that the limit process is "actually satisfying" only in the case $f \in V_a'$, where the limit problem (2.15) is variational i.e. of the same kind as the initial problem (2.4). When $f \notin V_a'$ the limit problem (2.16) is no longer variational so that something essential is lost in the limit process. This property is obviously associated with the property that the energy of the solution u^ε does not remain bounded as $\varepsilon \to 0$ (Theorem (2.8)), as the variational formulation is classically associated with minimization of energy. Moreover, in boundary value problems for partial differential equations, the variational theory is obviously associated with finite element approximation. It then appears that the limit process is only "well-behaved" in the case $f \in V_a$. In certain cases, this is an important drawback, as V_a' is "very small" and the pathologies associated with $f \notin V_a'$ are present for "almost any" f. In certain cases at the origin of the definition of sensitivity [14], V_a' is so small that it does not contain the space $\mathcal{D}(\Omega)$ of the test functions of distributions. Accordingly, $V_a \not\subset \mathcal{D}'$, i.e. it is so large that it has elements which are not distributions. In fact, it is worthwhile to define different levels of sensitivity:

Definition 3.1. Let E be a subspace of V'. The limit variational problem (2.15) and the singular perturbation (2.4) are said to be E - *sensitive* when

$$E \not\subset V_a'. \tag{3.1}$$

According to this definition, classical sensitivity defined in [14] is $\mathcal{D}$ - sensitivity.

Remark 3.2. In applications to boundary value problems, the loadings $f \in V'$ are not necessarily functions or distributions on a domain Ω. For instance, in Neumann problems loadings may be applied on the boundary and we may take as E a space of functions defined on the boundary; the corresponding sensitivity is then understood with respect to the boundary loadings.

In order to exhibit the practical consequences of E - sensitivity on numerical finite element approximation, let us consider V_h a discretization of V, i.e. a sequence $h \to 0$ of finite - dimensional subspaces of V satisfying

$$\begin{cases} v \in V \quad \Rightarrow \quad \exists v_h \in V_h \ \ \textit{such that} \\ v_h \to v \ \ \textit{strongly in} \ \ V \ \ \textit{as} \ \ h \to 0. \end{cases} \tag{3.2}$$

It is then classical that the problem (2.4) with fixed ε may be approximated by the discretization, i.e. there exists u_h^ε defined by

$$\begin{cases} u_h^\varepsilon \in V_h \ \ \textit{and} \\ a(u_h^\varepsilon, v) + \varepsilon^2 b(u_h^\varepsilon, v) = < f, v > \quad \forall v \in V_h \end{cases} \tag{3.3}$$

which enjoys the property

$$u_h^\varepsilon \to u^\varepsilon \ \ \textit{strongly in} \ \ V \ \ \textit{as} \ \ h \to 0. \tag{3.4}$$

We then have

Theorem 3.3. *Let the singular perturbation problem (2.4) be E - sensitive. Then, there exists $f \in E$ such that the convergence (3.4) in V_a is not uniform with respect to ε.*

This obviously means that, in order to have an error less than a given value in the finite element approximation (3.3), h must be taken smaller and smaller as $\varepsilon \to 0$. This (unpleasant!) property in finite element approximation is known as “numerical locking” (see [2]). It obviously amounts to saying that the discretization (3.2) is not adapted to the asymptotic process $\varepsilon \to 0$: accurate computation is more and more difficult as $\varepsilon \to 0$. In classical sensitive problems this locking occurs even for $f \in \mathcal{D}(\Omega)$. The proof of Theorem 3.3 is based on non - commutativity of the limit processes $h \to 0$ and $\varepsilon \to 0$ for $f \in E$ and may be seen in [8].

3.2. Recalling the case when $a(u, v)$ has a non-trivial kernel

Obviously, the hypothesis in the second line of (2.1) implies that $a(v, v)^{\frac{1}{2}}$ is a norm on V, which allows to construct, by completion, the various spaces involved in the abstract theory. As we shall see later, in order to make comparisons and heuristic reasonings, it will prove useful recalling the asymptotics in the case when that hypothesis is not fulfilled i.e. when $a(v, v)$ has a non-trivial kernel. In other words, when (2.1) is replaced by

$$a(v, v) \geq 0 \tag{3.5}$$

$$G = \{v; v \in V, a(v, v) = 0\} \neq \{0\} \tag{3.6}$$

we note that $a(v, v)^{\frac{1}{2}}$ is no longer a norm, but a semi-norm. Moreover, the Cauchy inequality for semi-norms allows to give the equivalent definition of G:

$$G = \{v; v \in V, a(v, w) = 0 \quad \forall w \in V\} \neq \{0\} \tag{3.7}$$

it then follows that G is a closed subspace of V, and so a Hilbert space with the topology induced by V. Moreover, on G the forms b and $a + b$ are the same. We note that making the change of unknowns

$$v^\varepsilon \mapsto \varepsilon^2 u^\varepsilon \tag{3.8}$$

the variational problem (2.4) becomes

$$\begin{cases} \textit{Find } v^\varepsilon \in V \ \textit{ such that} \\ \dfrac{1}{\varepsilon^2} a(v^\varepsilon, w) + b(v^\varepsilon, w) = \langle f, w \rangle \ \ \forall w \in V \end{cases} \tag{3.9}$$

which is a penalty problem. The solution u^ε converges to the solution of an analogous problem in the kernel G, of the form a. The asymptotics of the problem (2.4) is

Theorem 3.4. *For a fixed $f \in V'$, let u^ε be the solution of (2.4) under the hypotheses (2.3), (3.5) and (3.6). Then,*

$$\varepsilon^2 u^\varepsilon \to v^0 \quad \textit{strongly in } V$$

where v^0 is the unique solution of

$$\begin{cases} \textit{Find } v^0 \in G \ \textit{ such that} \\ b(v^0, w) = \langle f, w \rangle \ \ \forall w \in G. \end{cases}$$

The proof is somewhat classical, in particular it may be found in [19] along with examples in the context of shell theory.

3.3. Comments on distributions and analytical functionals

Let us consider, to fix ideas, functions (or distributions, or any other kind of generalized functions) on $\mathbb{R}$ of the variable x, as well as the corresponding Fourier transforms of the variable ξ. We shall mainly focus on singularities in the vicinity of $x = 0$.

There are many ways to define higher or lower degrees of singularity. The classical distributions

$$\delta(x), \delta'(x), \delta''(x), \ldots \tag{3.10}$$

belong to the Sobolev spaces of negative order $H^{-s}(\mathbb{R})$ for $s > 1/2, > 3/2, > 5/2, \ldots$, respectively, which enjoy the inclusion property

$$H^{-s} \subset H^{-r} \quad \text{for} \quad s < r \quad (s \text{ and } r \text{ in } \mathbb{R}). \tag{3.11}$$

We may define a function (or distribution) $f(x)$ to be more singular than another one $g(x)$ if g belongs to some H^{-s} which does not contain f. All that spaces are included in the space $\mathcal{D}'$ of distributions on $\mathbb{R}$.

Distributions enjoy the very important property (see [23]) that they are *locally of finite order*. This means that when a distribution is considered on a finite interval, it is the derivative of some order of a continuous function.

As a consequence, an expression of the form

$$u(x) = \sum_{k=0}^{+\infty} c_k \delta^{(k)}(x) \tag{3.12}$$

is not a distribution as obviously it is not of finite order in the vicinity of $x = 0$ (unless when the c_k vanish for sufficiently large values!). In other words, (3.10) are distributions *more and more singular*, but a linear combination of an infinity of them is *too much singular* and it is no longer a distribution. It should be noted that the expression

$$v(x) = \sum_{k=0}^{+\infty} \delta^{(k)}(x - k)$$

(where the successive $\delta^{(k)}$ are located at points tending to infinity) *is a distribution.* This is easily checked, and obviously $v(x)$ is of finite order on any finite interval.

Another way of defining higher and lower degrees of singularity is via the Fourier transforms. It is classical that the rapid decreasing of $\hat{f}(\xi)$, when $\xi \to +\infty$, measures the smoothness of $f(x)$. In the same way, the growing of $\hat{f}(\xi)$ at infinity measures the singularity of $f(x)$. As an example, the Fourier transforms of (3.10) are

$$1_\xi, i\xi, (i\xi)^2, \dots$$

It should be noticed that the formal Fourier transform of $u(x)$ in (3.12) is

$$\hat{u}(\xi) = \sum_{k=0}^{+\infty} c_k (i\xi)^k$$

which is a distribution (and even a function) provided that the sequence c_k decreases sufficiently fast as $k \nearrow \infty$. For instance, the function $\cosh(\xi)$ is entire, so that it may be represented by the series

$$\cosh(\xi) = \sum_{k=0}^{+\infty} \frac{\xi^{2k}}{(2k)!} \qquad \xi \in \mathbb{R}$$

Formally, it is the Fourier transform of the expression

$$u(x) = \sum_{k=0}^{+\infty} \frac{1}{(2k)!} (-i)^{2k} \delta^{(2k)}(x) \tag{3.13}$$

which, as we know, is not a distribution. In fact, this is a way to construct entities more general than the distributions. They are the (inverse) Fourier transforms of distributions, and constitute the spaces $\mathbb{Z}'(\mathbb{R}_x)$. The elements of $\mathbb{Z}'(\mathbb{R}_x)$ are analytic functionals. Let us explain this a little (the corresponding theory may be seen in [10]).

The space of test functions of distributions on $\mathbb{R}_\xi$ is $\mathcal{D}(\mathbb{R}_\xi)$, i.e. the space of indefinitely differentiable functions with compact support. It is a remarkable fact that if $\hat{\theta} \in \mathcal{D}(\mathbb{R}_\xi)$, then $\hat{\theta}(\xi)$ is not an analytic function (unless in the case $\hat{\theta} \equiv 0$). Indeed, if $\hat{\theta}(\xi)$ was analytic, as it vanishes for sufficiently large $|\xi|$, by analytic continuation, it should vanish everywhere. But, it is easily checked that the (inverse) Fourier transforms of $\hat{\theta} \in \mathcal{D}(\mathbb{R}_\xi)$ is a function $\theta(x)$ which is analytic (on $\mathbb{R}(x)$ and moreover on the complex plane of the variable x). Then, the (inverse) Fourier transform of $\mathcal{D}(\mathbb{R}_\xi)$ is a space of analytic functions named $\mathbb{Z}(\mathbb{R}_x)$. By duality, the (inverse) Fourier transform of the distributions of $\mathcal{D}'(\mathbb{R}_\xi)$ constitute the space $\mathbb{Z}'(\mathbb{R}_x)$, the elements of which are analytic functionals (i.e. functionals on the space $\mathbb{Z}(\mathbb{R}_x)$ of analytic functions).

Then, expressions as $u(x)$ in (3.13) are analytic functionals of $\mathbb{Z}'(\mathbb{R}_x)$. In fact, an expression as $u(x)$ in (3.13) is not a distribution because it cannot act upon any test function $\theta(x) \in \mathcal{D}(\mathbb{R}(x))$. Indeed, there are in $\mathcal{D}(\mathbb{R}(x))$ functions such that the derivatives of order k at $x = 0$ are such that the sum

$$\sum_{k=0}^{+\infty} \frac{1}{(2k)!}(-i)^{2k}\theta^{(2k)}(0)$$

diverge. As a consequence, $u(x)$ in (3.13) may only act on analytic test functions, which have derivatives at the origin $\theta^{(k)}(0)$ converging very quickly as $k \nearrow \infty$. In this way, entities as $u(x)$ in (3.13) may be defined in the context of analytic functionals, which generalize distributions.

But there is a drastic drawback when passing from distributions to analytic functionals of $\mathbb{Z}'(\mathbb{R}_x)$. The success of distributions relies on the fact that they inherit very many properties of functions. In particular, *distributions enjoy localization properties*: the value of a distribution at a point does not make sense, but its action on any neighbourhood of that point does (as the distribution may be tested with test function with support in that neighbourhood). Oppositely, in general, *analytic functionals do not enjoy localization properties*, as they may only act upon analytic functions (with support equal to the whole $\mathbb{R}_x$). We send to [5] for these properties and to [21] sect. 3 for an example showing that expressions analogous to (3.12) with singularities at different points may correspond to the same analytic functional.

3.4. Examples of sequences of functions converging to analytic functionals

In the context of singular perturbations, analytic functionals usually appear as limits of sequences of functions. In this subsection we shall display several examples of such sequences, exhibiting various interesting features.

Let us first consider the analytic functional $f(x)$ defined by its Fourier transform

$$\hat{f}(\xi) = \cosh\xi = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\xi^{2n}$$

which is in $\mathcal{D}'(\mathbb{R}_\xi)$ so that $f \in \mathbb{Z}'(\mathbb{R}_x)$. Formally, it is given by

$$f(x) = \delta(x) - \frac{\delta''(x)}{4!} + \frac{\delta^{(4)}(x)}{8!} + \cdots \tag{3.14}$$

Let us consider the approximation obtained by truncation of the Fourier transform. Specifically, we define

$$\hat{f}^\lambda(\xi) = \begin{cases} \hat{f}(\xi) & \textit{for } |\xi| < \lambda \\ 0 & \textit{for } |\xi| > \lambda \end{cases}$$

which, obviously, are tempered distributions such that

$$\hat{f}^\lambda(\xi) \to \hat{f}(\xi) \ \ \textit{in } \mathcal{D}'(\mathbb{R}_\xi) \ \ \textit{as } \lambda \to +\infty$$

so that the corresponding inverse Fourier transforms satisfy

$$f^\lambda(x) \to f(x) \ \ \textit{in } \mathbb{Z}'(\mathbb{R}_x) \ \ \textit{as } \lambda \to +\infty$$

and we easily obtain

$$\begin{aligned} f^\lambda(x) &= \frac{1}{2\pi}\int_{-\lambda}^{+\lambda} \cosh(\xi)\cos(\xi x)d\xi \qquad (3.15)\\ &= \frac{1}{2\pi(x^2+1)}\Big\{e^\lambda\left[\cos(\lambda x) + x\sin(\lambda x)\right] \\ &\qquad\qquad -e^{-\lambda}\left[\cos(\lambda x) - x\sin(\lambda x)\right]\Big\}. \end{aligned}$$

As we are interested in $\lambda \nearrow +\infty$, we shall discard the term in $e^{-\lambda}$, so that

$$f^\lambda(x) \approx \frac{e^\lambda}{2\pi}\psi^\lambda(x)$$

with

$$\psi^\lambda(x) = \frac{1}{x^2+1}\left(\cos(\lambda x) + x\sin(\lambda x)\right). \tag{3.16}$$

Obviously, the expression (3.16) may be considered as the sum of two terms

$$\frac{1}{x^2+1}\cos(\lambda x) \quad \text{and} \quad \frac{x}{x^2+1}\sin(\lambda x)$$

each one of them is the product of an "envelop" independent of λ by either $\cos(\lambda x)$ or $\sin(\lambda x)$, i.e. a sinusoidal function with wave length tending to zero as λ tends to infinity. This structure is clearly seen on figure 1. Clearly, as λ increases, the graph of that function becomes more and more dense and, at the limit, it occupies the whole region of the plane inside the

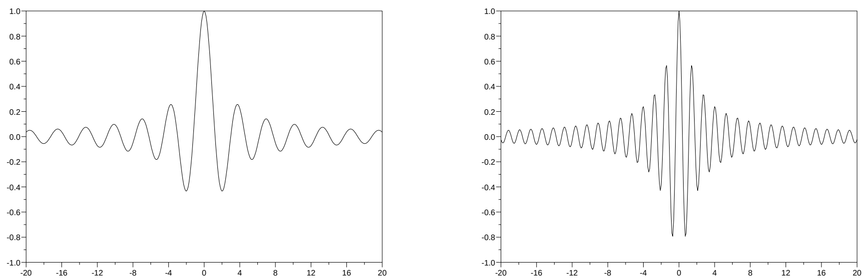

Figure 1. graphs of (3.16) with $\lambda = 2$ on the left and $\lambda = 5$ on the right

envelops. Moreover, on account of the factor $e^{\lambda}/2\pi$ in (3.16) the graph of $f^{\lambda}(x)$ occupies at the limit the whole plane.

Such is our first example of a sequence of functions f^{λ} tending to the non-zero analytic functional $f(x)$ defined in (3.14). Several features should be emphasized. First, the functional $f(x)$ defined by (3.14) is apparently a singularity at the origin; in fact, as we pointed out above, it has no support, and the sequence $f^{\lambda}(x)$ does not constitutes a layer in the vicinity of the origin, as the "envelop" is independent of λ. Another interesting property of the sequence $f^{\lambda}(x)$ is that it is clearly described by the two variables x (= macro-scale) and λx (= micro-scale). This situation is classical in homogenization theory, but there is a drastic difference between the two cases. In homogenization, after multiplying by a certain gauge function of λ, the sequence has a non-zero limit (in the weak topology of L^2 for instance), which is the (non-zero) limit of the homogenization. Oppositely, in the present situation, the sequence $f^{\lambda}(x)$ must be multiplied by the gauge function $e^{-\lambda}$ to remain bounded in $L^2(\mathbb{R}_x)$; and then it tends to zero either in the weak topology of $L^2(\mathbb{R}_x)$ or in $\mathcal{D}'(\mathbb{R}_\S)$. This is apparent in Figure 1, as after multiplying by test functions, $\lambda \to +\infty$ implies cancellation of the positive and negative parts of the integral. In other words, the sequence $f^{\lambda}(x)$ converges to the analytic functional $f(x)$ in the unusual topology of $\mathbb{Z}'(\mathbb{R}_x)$, but usual topologies such as $\mathcal{D}'(\mathbb{R}_x)$ are unable to describe the convergence to a non-zero limit. According to

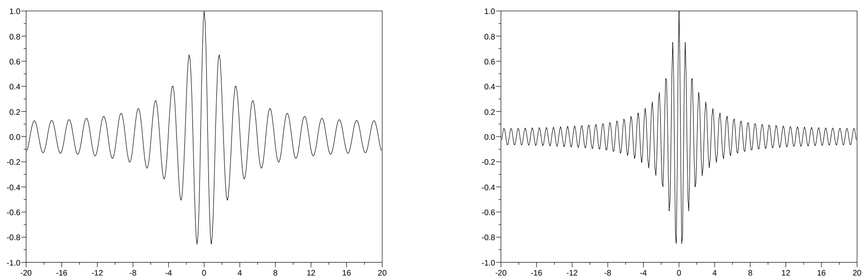

FIGURE 2. graphs of the normalized inverse Fourier transform of (3.17) in the exponential case, $x_2 + c = 1.5$, with $\lambda = 2$ on the left and $\lambda = 5$ on the right

the gauge function, the sequence remains unbounded or tends to zero. We shall say that <u>*a sequence of functions is complexifying*</u> when it has a non-zero limit in the $\mathbb{Z}'$ topology.

In order to display other examples of sequences more or less analogous to those appearing in the singular perturbation problem (see section 4), we first consider sequences $\hat{f}_\lambda(\xi)$ with support in the "outer region" $|\xi| > \lambda$ to be added to the previously truncated expressions (or analogous) in order to handle a "less drastic" truncation.

We carried out a few numerical experiments on the approximated $\hat{f}_\lambda(\xi)$ and $f_\lambda(x_1)$ using the free scientific computing package scilab (http://www.scilab.org).

We specifically consider

$$\hat{f}_\lambda(\xi) = \begin{cases} 0 & \text{for } |\xi| < \lambda \\ \frac{e^{(x_2+c-2)|\xi|}}{|\xi|^5} & \text{for } |\xi| \geq \lambda, \end{cases} \tag{3.17}$$

where x_2 and c are parameters (we shall see their meanings in section 4). It should be pointed out that as the supports are sent to infinity for $\lambda \to +\infty$, these sequences tend to zero in $\mathcal{D}'(\mathbb{R}_\xi)$ and the inverse Fourier transforms converge to zero in $\mathcal{Z}'(\mathbb{R}_{x_1})$ whatever gauge or normalization factors.

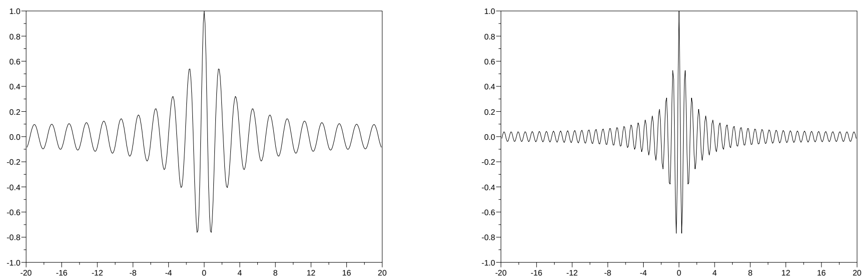

Figure 3. graphs of the normalized inverse Fourier transform of (3.17) in the algebraic case, $x_2 + c = 2$, with $\lambda = 2$ on the left and $\lambda = 5$ on the right

We shall consider the two values $x_2 + c = 1.5$ and $x_2 + c = 2$. We then note that the right hand side term of (3.17) are merely $\frac{e^{-0.5|\xi|}}{|\xi|^5}$ and $\frac{1}{|\xi|^5}$, so that they are exponentially or algebraically decreasing functions, respectively. We shall denote the two cases "exponential" and "algebraic", respectively.

The corresponding inverse Fourier transforms (normalized to take the value 1 at the origin) $f_\lambda(x_1)$ for $\lambda = 2$ and $\lambda = 5$ are displayed in Figure 2 in the exponential case and in Figure 3 in the algebraic case.

We observe that all of these inverse Fourier transforms exhibit "complexification" properties in the sense that as λ increases more and more "waves" are present between the "envelops". On the other hand, in the series of Figure 2 (i.e. in the exponential case), the envelops are almost the same for $\lambda = 2$, 5 whereas in the algebraic case (Figure 3) they are not. Defining the "significant support" as the region where the values of the envelops are more than a given fraction of its maximum value (1 here, because of the normalization), we see that in the algebraic case (Figure 3) the significant support shrinks as λ increases. The explanation of this property comes from the fact that the sequence (3.17) in the algebraic case is self-similar, while it is not in the exponential case. Specifically, for

$x_2 + c = 2$, the sequence (3.17) writes

$$\hat{f}_\lambda(\xi) = \lambda^{-5}\hat{\varphi}(\frac{|\xi|}{\lambda}), \tag{3.18}$$

where

$$\hat{\varphi}(y) = \begin{cases} 0 & \text{for } |y| < 1 \\ \frac{1}{|y|^5} & \text{for } |y| \geq 1. \end{cases} \tag{3.19}$$

Consequently, the inverse Fourier transforms are

$$f_\lambda(x_1) = \frac{\varphi(\lambda|x_1|)}{\lambda^4}, \tag{3.20}$$

where the shrinking character of the significant support is obvious.

As another example of a sequence converging in $\mathcal{Z}'$ to a non-zero limit, we now consider:

$$\hat{f}_\lambda(\xi) = \begin{cases} \frac{e^{(x_2+c)}}{8} & \text{for } |\xi| \leq 1 \\ \frac{e^{(x_2+c)|\xi|}}{8|\xi|^3} & \text{for } 1 \leq |\xi| \leq \lambda \\ \frac{e^{(x_2+c-2)|\xi|}}{\varepsilon^2|\xi|^5} & \text{for } \lambda \leq |\xi|, \end{cases} \tag{3.21}$$

where the small parameter ε is linked to λ by the relation:

$$\varepsilon^2\lambda^5 e^{2\lambda} = 8\lambda^3, \tag{3.22}$$

and x_2, c are, as before, parameters, which we shall consider in the "exponential case" $x_2 + c = 1.5$ and the "algebraic case" $x_2 + c = 2$. Let us explain a little the expression (3.21). The definition for $|\xi| < 1$ is merely a technical change to avoid the singularity in $|\xi|^{-3}$ at the origin. Roughly speaking, we may consider

$$\hat{f}_\lambda(\xi) = \frac{e^{(x_2+c)|\xi|}}{8|\xi|^3} \text{ for } |\xi| \leq \lambda, \tag{3.23}$$

so that the limit as λ tends to infinity is the function $\frac{e^{(x_2+c)|\xi|}}{8|\xi|^3}$, which increases exponentially for $|\xi| \to +\infty$. Then, this function is truncated in (3.21) replacing their values for $|\xi| > \lambda$ by those of an expression of the type (3.17) which takes the same values for $|\xi| = \lambda$ (as it is expressed by (3.22)).

The normalized inverse Fourier transforms for $\lambda = 2$ and $\lambda = 5$ in the exponential and algebraic cases are displayed in Figure 4 and Figure 5, respectively. The general trends of these series of graphs are analogous to those of the previous series (Figures 2 and 3).

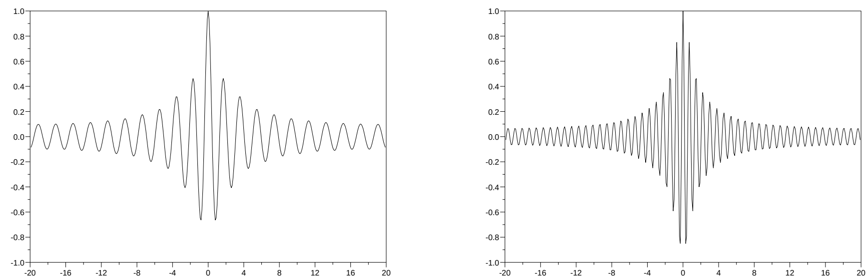

Figure 4. graphs of the normalized inverse Fourier transform of (3.21) in the exponential case, $x_2 + c = 1.5$, with $\lambda = 2$ on the left and $\lambda = 5$ on the right

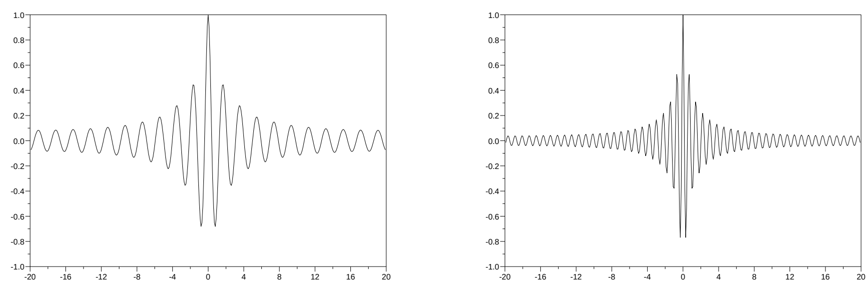

Figure 5. graphs of the normalized inverse Fourier transform of (3.21) in the algebraic case, $x_2 + c = 2$, with $\lambda = 2$ on the left and $\lambda = 5$ on the right

In all of the cases there is a complexification. In addition, the algebraic case (Figure 5) exhibits a phenomenon of contraction of the essential support. The phenomenon of complexification is not surprising as in our case the limits are elements of $\mathcal{Z}'(\mathbb{R}_{x_1})$ which are not in $\mathcal{D}'(\mathbb{R}_{x_1})$ (as the limit Fourier transform grows exponentially for $|\xi| \to +\infty$). Oppositely, the phenomenon of contraction of the significant support deserves of an explanation, as it is analogous to that of the self-similar layer (3.17) with $x_2 + c = 2$.

In order to get such an explanation, we shall quantify the influence of the various parts of a function. We shall use the L^2 norm, which is preserved by Fourier transform. Then, referring to (3.21) with $x_2 + c = 2$, it is easily seen that (on account of the relation (3.22) relating ε to λ):

$$\int_{|\xi|<\lambda} |\hat{f}_\lambda(\xi)|^2 d\xi = \mathcal{O}(e^{4\lambda}\lambda^{-6}) \tag{3.24}$$

$$\int_{|\xi|>\lambda} |\hat{f}_\lambda(\xi)|^2 d\xi = \mathcal{O}(\varepsilon^{-4}\lambda^{-9}) = \mathcal{O}(e^{4\lambda}\lambda^{-5}). \tag{3.25}$$

It then follows that in the algebraic case $x_2 + c = 2$, the L^2 norm of the region $|\xi| > \lambda$ of the function $\hat{f}_\lambda(\xi)$ in (3.21) is asymptotically large as λ tends to infinity with respect to the region $|\xi| < \lambda$. Accordingly, the inverse Fourier transforms are "almost the same" for (3.21) and (3.17).

Our last series of numerical experiments are related to those appearing in the singular perturbation (see section 4). They are concerned with the Fourier transforms

$$\hat{f}_\lambda(\xi) = \begin{cases} \frac{e^{(x_2+c)|\xi|}}{8+\varepsilon^2 e^2} & \text{if } |\xi| \le 1 \\ \frac{e^{(x_2+c)|\xi|}}{8|\xi|^3+\varepsilon^2|\xi|^5 e^{2|\xi|}} & \text{if } |\xi| > 1. \end{cases} \tag{3.26}$$

Once more, the special definition for $|\xi| < 1$ is merely a device to avoid the singularity at $\xi = 0$.

We observe that this function with small ε exhibits roughly two regions: for fixed ξ and small ε, it is approximately described by (3.26) with $\varepsilon = 0$. It is exponentially growing for $|\xi| \to +\infty$. Obviously the convergence to that limit as $\varepsilon \to 0$ is not uniform for $\xi \in \mathbb{R}$. For large $|\xi|$, the first term in the denominator is negligibly small with respect to the terms in ε^2, which may be considered to give a good approximation for large $|\xi|$. The transition between the two regions is clearly $|\xi| = \mathcal{O}(\lambda)$ with $\lambda = \lambda(\varepsilon)$ satisfying (3.22), which indicates that the two terms in the denominator of (3.26) are of the same order.

Then, we may expect that the numerical experiments with (3.26) should be similar to those of (3.21) with $\lambda = \lambda(\varepsilon)$ satisfying (3.22). There is, nevertheless an important difference. The transition between the two regions is sharper in (3.21), involving a jump of the first order derivative, whereas

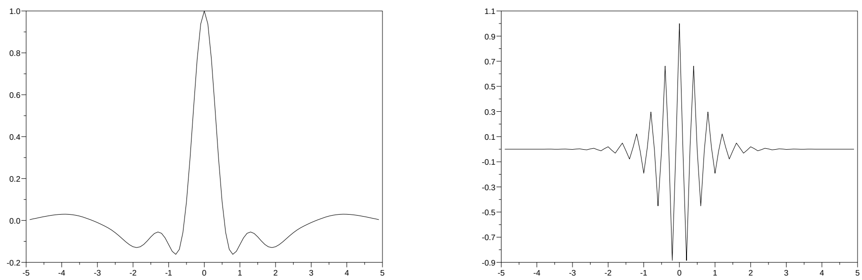

Figure 6. graphs of the normalized inverse Fourier transform of (3.26) in the exponential case, $x_2 + c = 1.5$, with $\lambda = 2$ on the left and $\lambda = 5$ on the right

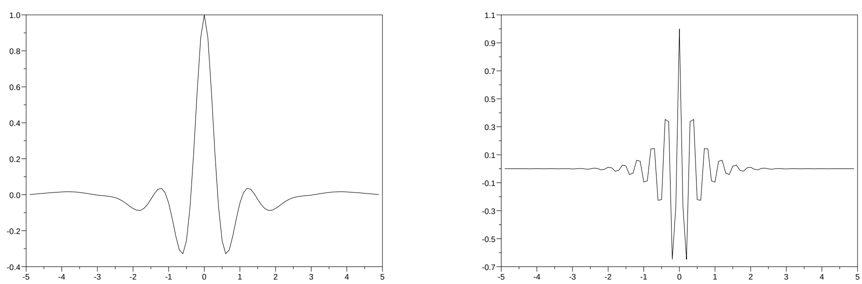

Figure 7. graphs of the normalized inverse Fourier transform of (3.26) in the algebraic case, $x_2 + c = 2$, with $\lambda = 2$ on the left and $\lambda = 5$ on the right

in (3.26) the transition is smooth. Accordingly, the asymptotic behaviour for $|x_1| \to +\infty$ of the inverse Fourier transforms is smoother in the case (3.26) than in the case (3.21). Specifically, it is classical (see [7] section 2.3 if necessary) that the jumps of the first order derivative of $\hat{f}_\lambda$ yields to a term in $|x_1|^{-2}$ at infinity which is present in the inverse Fourier transform of (3.21) but disapears in (3.26). Accordingly, we modified the abscisse's axe in Figure 6 and 7 with respect to the previous ones. We observe that the complexification still holds whereas the properties of the effective support are much less apparent than in the previous case.

4. First kind sensitivity

This section is concerned with an example where the sensitivity follows from "unadaptated boundary conditions" (that is to say not satisfying the Shapiro-Lopatinskii condition) in the limit problem.

4.1. An elliptic problem with non-classical boundary conditions

Let $\Omega = \mathbb{R} \times (0,1)$ be the infinite strip in the $\mathbb{R}^2$ plane of the variable $x = (x_1, x_2)$ and let a be the bilinear form given by:

$$a(u,v) = \int_\Omega \triangle u \, \triangle v dx. \tag{4.1}$$

We consider the following variational problem

$$\begin{cases} \text{Find } u \in V_a \text{ such that, } \forall v \in V_a \\ a(u,v) = \langle f, v \rangle, \end{cases} \tag{4.2}$$

where the space V_a is the "energy space" with the essential boundary conditions

$$v(x_1, 0) = \partial_2 v(x_1, 0) = 0, \tag{4.3}$$

which is defined as the completion with the norm $\|v\|_a = a(v,v)^{1/2}$ of the set of $H^2(\Omega)$ functions satisfying (4.3), while f is an element of the dual V_a'.

This problem exhibits several special features which we give as remarks.

Remark 4.1. We note that the energy space V_a is not a classical space. In fact, $\|v\|_a$ is a norm on $H^2(\Omega)$ (or any other space of sufficiently regular functions) with the essential boundary conditions (4.3). Indeed, when it vanishes, we have $\Delta v = 0$ with (4.3). This amounts to the Cauchy problem for the laplacian, which classically enjoys uniqueness (from the Holmgren local uniqueness theorem together with analytic continuation, see for instance [4]). Then, V_a is well defined in a somewhat abstract way. But obviously the Cauchy elliptic problems are ill - posed (so that "very large" v may correspond to "very small" Δv, see for instance [4] or [9]). In fact, V_a is a "very large space" not contained in the distribution space $D'(\Omega)$ (see [14] and [8]). This point will not be explicitly addressed here, but it will be (more or less) apparent from the forthcoming developments.

Moreover, after a formal integration by parts, we easily deduce that the classical formulation of problem (4.2) is:

$$\begin{cases} \triangle^2 u = f \text{ on } \Omega, \\ u(x_1,0) = \partial_2 u(x_1,0) = 0, \ \forall x_1 \in \mathbb{R}, \\ \triangle u(x_1,1) = \partial_2 \triangle u(x_1,1) = 0, \ \forall x_1 \in \mathbb{R}. \end{cases} \tag{4.4}$$

Remark 4.2. We note that under this "classical" (= non variational) form, the problem makes sense for more general loadings f, not necessarily contained in V_a'. We shall take it in the form

$$f(x_1,x_2) = \delta(x_1)F(x_2), \tag{4.5}$$

with (for instance) $F \in L^2(0,1)$, or

$$f(x_1,x_2) = \delta(x_1)\delta(x_2 - c), \tag{4.6}$$

with $0 < c < 1$ (or even $c = 1$ but in that case the "loading" is on the boundary and should be considered as a non-homogeneous boundary condition). Obviously, convolutions in x_1 or in x_2 allow considering very general loadings.

Remark 4.3. The boundary value problem (4.4) is not classical. It involves the new natural boundary conditions on $x_2 = 1$, but they do not satisfy the Shapiro - Lopatinskii condition (see for instance [1] or [6]). Indeed, considering the upper half plane (x_1,x_2), $x_2 \geq 0$, and the equation

$$\triangle^2 u(x_1,x_2) = (\partial_1^4 + 2\partial_1^2\partial_2^2 + \partial_2^4)u(x_1,x_2) = 0,$$

with the boundary conditions

$$\triangle u(x_1,0) = \partial_2 \triangle u(x_1,0) = 0,$$

and, taking the Fourier transform in the x_1 direction, we get:

$$(\xi^4 - 2\xi^2\partial_2^2 + \partial_2^4)\hat{u}(\xi,x_2) = 0,$$

with the boundary conditions

$$(\xi^2 - \partial_2^2)\hat{u}(\xi,1) = \partial_2(\xi^2 - \partial_2^2)\hat{u}(\xi,1) = 0.$$

It then appears that this problem has non zero-solutions in half plane, such as $\hat{u} = Ae^{-|\xi|x_2}$. This means that the Shapiro - Lopatinskii condition is not satisfied. Therefore, it is not a classical problem and it has no solution in general.

In order to avoid problems explained in Remark 4.3, we will first consider the x_1 Fourier transform of the previous boundary value problem (4.4) with a loading given by (4.6). The new problem is an ODE, which depends on a parameter ξ and which has solutions for any value of the parameter. Next, taking the inverse Fourier transform, we shall obtain solutions of (4.4) in the space of analytical functionals $\mathcal{Z}'$.

In that case, using the Fourier transform of (4.4) with respect to x_1 and denoting the Fourier transform of $u(x_1, x_2)$ by $\hat{u}(\xi, x_2)$, we obtain the following boundary value problem for $x_2 \in (0,1)$, which depends on the parameter $\xi \in \mathbb{R}$

$$\left\{ \begin{array}{l} (\partial_2^2 - \xi^2)(\partial_2^2 - \xi^2)\hat{u}(\xi, x_2) = \delta(x_2 - c), \ \forall x_2 \in (0,1) \\ \hat{u}(\xi, 0) = \partial_2 \hat{u}(\xi, 0) = 0, \\ (\partial_2^2 - \xi^2)\hat{u}(\xi, 1) = \partial_2(\partial_2^2 - \xi^2)\hat{u}(\xi, 1) = 0. \end{array} \right. \tag{4.7}$$

Since the solutions of $(\partial_2^2 - \xi^2)(\partial_2^2 - \xi^2)\hat{u}(\xi, x_2) = 0$, are linear combinations of

$$e^{|\xi| x_2},\ e^{-|\xi| x_2},\ x_2 e^{|\xi| x_2},\ x_2 e^{-|\xi| x_2}, \tag{4.8}$$

then, it is easily seen (see [15] for details) that the solutions of (4.7) are

$$\hat{u}(\xi, x_2) = \left\{ \begin{array}{lll} \hat{u}^-(\xi, x_2) & \text{if} & x_2 \in (0, c), \\ \hat{u}^+(\xi, x_2) & \text{if} & x_2 \in (c, 1), \end{array} \right. \tag{4.9}$$

where

$$\begin{aligned} \hat{u}^-(\xi, x_2) \ &= \ \frac{1}{2|\xi|^3} \cosh\,(|\xi| c)\, \sinh\,(|\xi| x_2) \\ &\quad - \frac{x_2}{2\xi^2} \cosh\,(|\xi|(x_2 - c)), \end{aligned} \tag{4.10}$$

$$\begin{aligned} \hat{u}^+(\xi, x_2) \ &= \ \hat{u}^-(\xi, x_2) + \frac{(x_2 - c)}{2\xi^2} \cosh(|\xi|(x_2 - c)) \\ &\quad - \frac{1}{2|\xi|^3} \sinh(|\xi|(x_2 - c)). \end{aligned} \tag{4.11}$$

We note that for all $x_2 \in]0,1[$, with $x_2 \neq c$, and for $|\xi| \to +\infty$, we have

$$\hat{u}(\xi, x_2) \approx \frac{1}{8|\xi|^3} e^{(c + x_2)|\xi|}. \tag{4.12}$$

Obviously, this expression was obtained for the loading (4.6), but in rather general cases, the asymptotic behavior for $|\xi| \to \infty$ is exponential, see [15].

4.2. Singular perturbation with non-classical boundary conditions in the limit problem

Let us now consider the variational problem depending on the parameter $\varepsilon > 0$ given by

$$\left\{ \begin{array}{l} \text{Find } u^\varepsilon \in V \text{ such that, } \forall v \in V \\ a(u^\varepsilon, v) + \varepsilon^2 b(u^\varepsilon, v) = \langle f, v \rangle, \end{array} \right. \tag{4.13}$$

where a is still given by (4.1) and b is such that

$$b(u,v) = \int_\Omega (\partial_1^3 u \partial_1^3 v + \partial_2^3 u \partial_2^3 v) dx. \tag{4.14}$$

In this case, the energy space V is the completion of the $H^3(\Omega)$ functions v satisfying the essential boundary conditions

$$v(x_1, 0) = \partial_2 v(x_1, 0) = \partial_2^2 v(x_1, 0) = 0, \tag{4.15}$$

with the norm $\|v\|_V^2 = a(v,v) + b(v,v)$.

Let $H^3_{\Gamma_0}(0,1)$ and $H^2_{\Gamma_0}(0,1)$ be defined as follows

$$H^3_{\Gamma_0}(0,1) = \{v \in H^3(0,1), \text{ s.t. } v(0) = \partial_2 v(0) = \partial_2^2 v(0) = 0\} \tag{4.16}$$

$$H^2_{\Gamma_0}(0,1) = \{v \in H^2(0,1), \text{ s.t. } v(0) = \partial_2 v(0) = 0\}. \tag{4.17}$$

We then are in the framework described in section 2.

In order to solve (4.13) by Fourier transform in x_1, we first obtain its classical formulation by an integration by parts in (x_1, x_2). Taking the x_1 Fourier transform, we obtain an ordinary differential equation for $x_2 \in (0,1)$ which depends on the parameter ξ. Then, the corresponding variational formulation, which follows from standard inegration by parts in x_2, is:

$$\left\{ \begin{array}{l} \text{Find } \hat{u}^\varepsilon(\xi) \in H^3_{\Gamma_0}(0,1) \text{ such that, } \forall v \in H^3_{\Gamma_0}(0,1) \\ \hat{a}(\hat{u}^\varepsilon, v) + \varepsilon^2 \hat{b}(\hat{u}^\varepsilon, v) = \langle F, v \rangle, \end{array} \right. \tag{4.18}$$

where for u, v in $H^2(0,1)$, $\hat{a}$ is defined by:

$$\hat{a}(u,v) = \langle (\partial_2^2 - \xi^2) u, (\partial_2^2 - \xi^2) v \rangle, \tag{4.19}$$

and similarly, for u, v in $H^3(0,1)$, $\hat{b}$ is the form defined by

$$\hat{b}(u,v) = |\xi|^6 \langle u, v\rangle + \langle \partial_2^3 u, \partial_2^3 v\rangle, \tag{4.20}$$

where $\langle , \rangle$ denotes the usual scalar product in $L^2(0,1)$.

The following result expresses the convergence of the Fourier transforms with fixed ξ (its proof can be found in [15]):

Lemma 4.4. *Let ξ be fixed and let $\hat{u}^\varepsilon(\xi)$, $\hat{u}(\xi)$ be the solutions of (4.18) and (4.7), respectively, then we have:*

$$\hat{u}^\varepsilon(\xi) \to \hat{u}(\xi) \text{ strongly in } H^2_{\Gamma_0}(0,1), \text{ as } \varepsilon \text{ goes to zero.} \tag{4.21}$$

The following theorem gives the convergence of $\hat{u}^\varepsilon$ in the distribution sense with respect to ξ (4.22). The inverse Fourier transform then gives the convergence of u^ε, (4.24).

Theorem 4.5. *Let $\hat{u}^\varepsilon$ and $\hat{u}$ be the solutions of (4.18) and (4.7), respectively. The following convergence holds, as ε goes to zero:*

$$\hat{u}^\varepsilon \to \hat{u} \text{ in } D'(\mathbb{R}_\xi; H^2_{\Gamma_0}(0,1)). \tag{4.22}$$

Moreover, for fixed x_2, we have:

$$\hat{u}^\varepsilon(.,x_2) \quad \to \quad \hat{u}(.,x_2) \text{ in } D'(\mathbb{R}_\xi), \tag{4.23}$$

$$u^\varepsilon(.,x_2) \quad \to \quad u(.,x_2) \text{ in } Z'(\mathbb{R}_{x_1}). \tag{4.24}$$

where u and u^ε are the solutions of (4.2) and (4.13), respectively.

4.3. Emergence of a new small parameter in the previous problem and formal asymptotics

This section has a formal character. Its goal is to get an easily understandable description of $\hat{u}^\varepsilon(\xi)$ and $u^\varepsilon(x_1)$ with small ε.

For obvious reasons, the limit properties of u^ε when $\varepsilon \to 0$ are more clear when considered in terms of the Fourier transform $\hat{u}^\varepsilon(\xi)$. Moreover, it follows from (4.18)-(4.20) with (4.5) (see [15] for details) that $\hat{u}^\varepsilon(\xi, x_2)$ with fixed small ε behaves when $|\xi| \to \infty$ in the form:

$$\hat{u}^\varepsilon(\xi) \approx \varepsilon^{-2}\xi^{-6}F, \tag{4.25}$$

whereas for moderately large values of $|\xi|$, we have, as in (4.12):

$$\hat{u}^\varepsilon(\xi, x_2) \approx K e^{(c+x_2)|\xi|}, \tag{4.26}$$

or other exponential functions, in general. In order to define a transition region between the two previous patterns, where we can neglect none of these two expressions with respect to the other, we must have

$$e^{(c+x_2)|\xi|} = \mathcal{O}(\varepsilon^{-2}). \tag{4.27}$$

Consequently, the characteristic frequency is of order:

$$|\xi| = \mathcal{O}(|log\varepsilon|). \tag{4.28}$$

We are now giving a heuristic approximate analysis of $\hat{u}^\varepsilon(\xi)$ for small ε and "moderately large" $|\xi|$. According to the previous considerations, it will be a good quantitative approximation of u^ε.

Let us consider again the problem (4.18) when ε and ξ are considered as parameters, with $\varepsilon \to 0$ and $|\xi| = \mathcal{O}(|log\varepsilon|)$. In order to allow explicit computations, we shall take a loading given by (4.6).

As it appears from section 3.2, the natural trend of the solution of the minimization problem is to avoid the term $\hat{a}(v, v)$ which is "expensive" in energy, with respect to the other, which bears the small factor ε^2. According to (4.19), the vanishing of the form $\hat{a}$ amounts to $(\partial_2^2 - |\xi|^2)v = 0$, i.e. v is in the two - dimensional space

$$v = \alpha e^{|\xi|x_2} + \beta e^{-|\xi|x_2} \tag{4.29}$$

on the whole interval $x_2 \in (0, 1)$. It then appears that, when imposing the three (essential) boundary conditions on $x_2 = 0$, see (4.16), the subspace reduces to the 0 vector, so that this first idea is too coarse for describing the asymptotics.

This (negative) result is natural, as the kernel of $a(v, v)$ reduces to $\{0\}$. Nevertheless, we shall see that it is possible to enlarge that subspace and to obtain special functions satisfying the essential boundary conditions and containing a very small amount of a-energy. The formal asymptotics consists in minimizing the energy in the space of that special functions.

We are then enlarging the previous subspace. To this end, we know that the (exact) solution satisfies the homogeneous equation

$$\Big((\partial_2^2 - \xi^2)(\partial_2^2 - \xi^2) - \varepsilon^2(\xi^6 + \partial_2^6)\Big)\hat{u}^\varepsilon(\xi, x_2) = 0 \tag{4.30}$$

on each of the intervals $(0, c)$ and $(c, 1)$. Then, on each one of these intervals, it is a linear combination of the six functions $e^{\lambda_i x_2}$ where λ_i are the roots of the equation:

$$(\lambda^2 - \xi^2)^2 - \varepsilon^2(\xi^6 + \lambda^6) = 0. \tag{4.31}$$

We are now solving approximatively this equation recalling that ε is small and $|\xi|$ moderately large. It immediately appears that there are two roots close to $|\xi|$, two roots close to $-|\xi|$, and two roots with very large modulus, approximatively equal to $1/\varepsilon$ and $-1/\varepsilon$. The two first assertions follow directly from (4.31) with $\varepsilon = 0$, whereas the last follows from the change of unknown $\lambda = \mu/\varepsilon$, which gives

$$(\mu^2 - \varepsilon^2\xi^2)^2 - \varepsilon^6\xi^6 + \mu^6 = 0, \tag{4.32}$$

and then taking $\varepsilon = 0$. Going on with our approximation, we may consider (see for instance [22] for details) that the two roots close to $|\xi|$ are in fact a double root, as well as the two close to $-|\xi|$. It means that, on each of the intervals $(0, c)$ and $(c, 1)$ we may consider, in addition to (4.29), functions of the form:

$$\gamma x_2 e^{|\xi| x_2} + \delta x_2 e^{-|\xi| x_2} + \zeta e^{\frac{-1}{\varepsilon} x_2} + \theta e^{\frac{1}{\varepsilon} x_2}. \tag{4.33}$$

Moreover, in the framework of our approximation, we observe that, as $|\xi|$ is large and ε small, the functions with coefficients γ and θ bear a large amount of energy associated with the form $\hat{a}$, and should be disregarded. As a result, at the present state, on each of the intervals $(0, c)$ and $(c, 1)$ we may consider, in addition to (4.29), functions of the form:

$$\delta x_2 e^{-|\xi| x_2} + \zeta e^{\frac{-1}{\varepsilon} x_2}. \tag{4.34}$$

But, as the functions must be in the space $H^3_{\Gamma_0}(0, 1)$, the traces of the functions and of the first and second order derivatives must be the same on both sides of $x_2 = c$ which are not concerned by the space of minimization). As these three conditions are automatically satisfied by (4.29), which is valid on the whole interval, we only must prescribe them on (4.34). This evidently shows that δ and θ should take the same value on both intervals. This gives, on the whole interval, $(0, 1)$, functions of the form

$$v = \alpha e^{|\xi| x_2} + \beta e^{-|\xi| x_2} + \delta x_2 e^{-|\xi| x_2} + \zeta e^{\frac{-1}{\varepsilon} x_2}. \tag{4.35}$$

We now have at our disposal a four - dimensional space (instead of the two-dimensional one (4.29)) and prescribing the three (essential) boundary conditions on $x_2 = 0$, see (4.16) we get:

$$\left\{ \begin{array}{lcl} \delta & = & \frac{1}{2|\xi|}(\frac{1}{\varepsilon^2} - \xi^2)\zeta, \\ \beta & = & \frac{1}{4\xi^2}(\frac{1}{\varepsilon} - 3|\xi|)(\frac{1}{\varepsilon} + |\xi|)\zeta, \\ \alpha & = & -\frac{1}{4\xi^2}(\frac{1}{\varepsilon} - |\xi|)^2\zeta. \end{array} \right. \tag{4.36}$$

and the space of minimization becomes the one - dimensional space

$$v(\xi, x_2) = A(\xi, \varepsilon) w^{\varepsilon}(\xi, x_2), \tag{4.37}$$

with

$$\begin{aligned} w^{\varepsilon}(\xi, x_2) = \Big((1-\varepsilon\xi)^2 \frac{\sinh\,(|\xi| x_2)}{|\xi|} &- (1-\varepsilon^2\xi^2) x_2 e^{-|\xi| x_2} \\ &- 2\varepsilon^2 |\xi| (e^{-\frac{1}{\varepsilon}x_2} - e^{-|\xi| x_2}) \Big). \end{aligned} \tag{4.38}$$

We note that, within our approximation, as ε is small, as well as $\varepsilon|\xi|$, we may also consider

$$w^{\varepsilon}_{app}(\xi, x_2) = \Big(\frac{\sinh\,(|\xi| x_2)}{|\xi|} - x_2 e^{-|\xi| x_2} - 2\varepsilon^2 |\xi| e^{-\frac{1}{\varepsilon}x_2} \Big) \tag{4.39}$$

where it should be noted that the last term is small with respect to the others, so that it should also be discarded; we only keep it in order to show that the boundary conditions at $x_2 = 0$ are (approximatively) satisfied; in fact, that term is a narrow boundary layer near $x_2 = 0$, but it will not play any role in the sequel.

The approximate solution of the minimization problem is now immediate as it is reduced to the one - dimensional space (4.37). Writing

$$\hat{u}^{\varepsilon}(\xi, x_2) = A(\xi, \varepsilon) w^{\varepsilon}(\xi, x_2), \tag{4.40}$$

and recalling that $\hat{u}^{\varepsilon}(\xi, x_2)$ is the solution of the variational problem given by (4.18), we deduce that

$$\hat{a}(\hat{u}^{\varepsilon}(\xi), \hat{u}^{\varepsilon}(\xi)) + \varepsilon^2 \hat{b}(\hat{u}^{\varepsilon}(\xi), \hat{u}^{\varepsilon}(\xi)) = \langle \hat{u}^{\varepsilon}(\xi), \delta(x_2 - c) \rangle,$$

so that

$$A(\xi, \varepsilon)^2 (\hat{a}(w^{\varepsilon}(\xi), w^{\varepsilon}(\xi)) + \varepsilon^2 \hat{b}(w^{\varepsilon}(\xi), w^{\varepsilon}(\xi))) = A(\xi, \varepsilon) \langle w^{\varepsilon}(\xi), \delta(x_2 - c) \rangle, \tag{4.41}$$

and then

$$A(\xi, \varepsilon) = \frac{w^{\varepsilon}(\xi, c)}{\hat{a}(w^{\varepsilon}(\xi), w^{\varepsilon}(\xi)) + \varepsilon^2 \hat{b}(w^{\varepsilon}(\xi), w^{\varepsilon}(\xi))}. \tag{4.42}$$

Furthermore, thanks to the fact that $\varepsilon|\xi| << 1$, the approximate expressions of $w^{\varepsilon}(\xi)$, $\hat{a}(w^{\varepsilon}(\xi), w^{\varepsilon}(\xi))$ and $\hat{b}(w^{\varepsilon}(\xi), w^{\varepsilon}(\xi))$ are (recall (4.39) and the comments after it):

$$w^{\varepsilon}(\xi, x_2) \approx \overline{w}_{\mathrm{app}}(\xi, x_2) = \frac{\sinh\,(|\xi| x_2)}{|\xi|} - x_2 e^{-|\xi| x_2}, \tag{4.43}$$

$$\hat{a}(w^\varepsilon(\xi), w^\varepsilon(\xi)) \approx 2|\xi|, \tag{4.44}$$

$$\hat{b}(w^\varepsilon(\xi), w^\varepsilon(\xi)) \approx \frac{|\xi|^3}{4} e^{2|\xi|}, \tag{4.45}$$

hence,

$$\hat{a}(w^\varepsilon(\xi), w^\varepsilon(\xi)) + \varepsilon^2 \hat{b}(w^\varepsilon(\xi), w^\varepsilon(\xi)) \approx 2|\xi| + \varepsilon^2 \frac{|\xi|^3}{4} e^{2|\xi|}. \tag{4.46}$$

Then (4.42) becomes

$$A(\xi, \varepsilon) = \frac{w^\varepsilon(\xi, c)}{2|\xi| + \varepsilon^2 \frac{|\xi|^3}{4} e^{2|\xi|}} \tag{4.47}$$

and the final expression of the approximate solution is

$$\hat{u}^\varepsilon(\xi, x_2) = \frac{w^\varepsilon(\xi, c) w^\varepsilon(\xi, x_2)}{2|\xi| + \varepsilon^2 \frac{|\xi|^3}{4} e^{2|\xi|}}, \tag{4.48}$$

where w^ε is given by (4.38) or even by its approximate expression (4.43).

Remark 4.6. It should be noticed that the previous result on the one-dimensional space (4.40) (which is the main one of our formal asymptotics) is independent of the point c of application of the point loading. Accordingly, it may be used for general loadings, which may be obtained by integration of elementary loadings with variable $c \in (0, 1)$.

Remark 4.7. When computing

$$\hat{a}(w^\varepsilon(\xi), w^\varepsilon(\xi)) = \int_0^1 \left|(\partial_2^2 - \xi^2) w^\varepsilon(\xi)(x_2)\right|^2 dx_2,$$

with (4.39), it is easily seen that the previous energy localizes in the vicinity of $x_2 = 0$ when $|\xi|$ tends to infinity. This result is evident form the fact that the leading part of the $\hat{a}(w^\varepsilon(\xi), w^\varepsilon(\xi))$ energy depends on the second term in the expression (4.39).

Oppositely, the $\hat{b}$ energy depends mainly on the first term of the expression (4.39) and the $\hat{b}$ energy localizes in the vicinity of $x_2 = 1$.

This is the phenomenon of "migration of energies" in the sense that as $|\xi|$ tends to infinity, the $\hat{a}$ and $\hat{b}$ energies migrate towards Γ_0 and Γ_1, respectively, whereas the open interval $(0, 1)$ tends to become free of energy.

Remark 4.8. Let us denote in this remark $\hat{u}_c^\varepsilon$ and $\hat{u}_{c'}^\varepsilon$ the solutions associated with δ loads located in c and c'. From our formal asymptotics, we deduce the reciprocity property

$$\hat{u}_c^\varepsilon(\xi, c') = \hat{u}_{c'}^\varepsilon(\xi, c).$$

Remark 4.9. According to Remark 4.6, in the case when the loading is $f(x_1, x_2) = \delta(x_1)F(x_2)$ with $F \in L^2(0,1)$, equations (4.41) and (4.42) become

$$A(\xi,\varepsilon)^2(\hat{a}(w^\varepsilon(\xi), w^\varepsilon(\xi)) + \varepsilon^2\hat{b}(w^\varepsilon(\xi), w^\varepsilon(\xi))) = A(\xi,\varepsilon)\langle F, \hat{w}^\varepsilon(\xi)\rangle \quad (4.49)$$

and

$$A(\xi,\varepsilon) = \frac{\int_0^1 F(x_2)w^\varepsilon(\xi, x_2)dx_2}{2|\xi| + \varepsilon^2\frac{|\xi|^3}{4}e^{2|\xi|}}. \quad (4.50)$$

respectively

We carried out a few numerical experiments on the approximated $\hat{u}^\varepsilon(\xi, 1)$ and $u^\varepsilon(x_1, 1)$. We observe that the principal term of (4.48), as $|\xi| \to \infty$, is precisely the last example of section 3, especifically (3.26). In fact, the example (3.26) was chosen for this reason. It appears that the graphs of the normalized inverse Fourier transforms of (3.26) and (4.48) are close when $\lambda = 2$ (see Figure 8 on the left) and practically identical when $\lambda = 5$ (see Figure 8 on the right). Therefore, the comments concerning complexification and behavior of the supports are analogous to those of (3.26).

On Figure 9 we represent $\hat{u}^\varepsilon(\xi, 1)$ (where $\hat{u}$ is given by (4.48) on the left and $u^\varepsilon(x_1, 1)$ on the right for three values of ε (we recall that ε and λ are related by (3.22)): $\varepsilon = 10^{-3}$, $\varepsilon = 10^{-4}$ and $\varepsilon = 10^{-5}$. In the left graphics, the abscisse-axis is $\xi \in [0, 32]$ while in the right one, it is $x_1 \in [0, 0.8]$. We observe that both $|\hat{u}^\varepsilon(\xi, 1)|$ and $|u^\varepsilon(x_1, 1)|$ (indeed we can not see $u^{10^{-3}}$ since it is too small in comparison with the two others) increase drastically as ε decreases. This is in good agreement with the fact that neither the limit of $\hat{u}^\varepsilon(\xi, 1)$ nor the limit of $u^\varepsilon(x_1, 1)$, as ε tends to zero, belong to the space S'.

In Figure 10 we displayed $u^\varepsilon(x_1, 0.75)$ and $u^\varepsilon(x_1, 0.95)$ for $\varepsilon = 10^{-3}$ and the loading $f(x_1, x_2) = \delta(x_1)\delta(x_2 - 0.75)$. The abscise-axis is $\xi \in [0, 8]$. The maximum of $|u^\varepsilon(x_1, 0.75)|$ and $|u^\varepsilon(x_1, 0.95)|$ are around 0.03 and 0.11, respectively. It is then apparent that the solution is much more singular in the vicinity of the boundary Γ_1 than on $x_1 = 0.75$ where the loading is applied. In fact, in the present situation, the singular behavior is somewhat

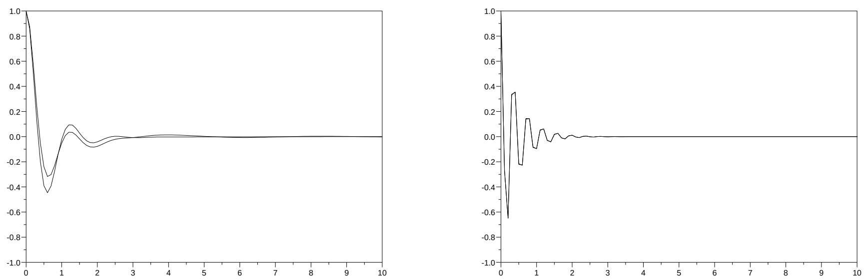

Figure 8. Normalized Inverse Fourier transforms of (3.26) and (4.48), with $x_2 + c = 1.5$ and $\lambda = 2$ on the left and $\lambda = 5$ on the right

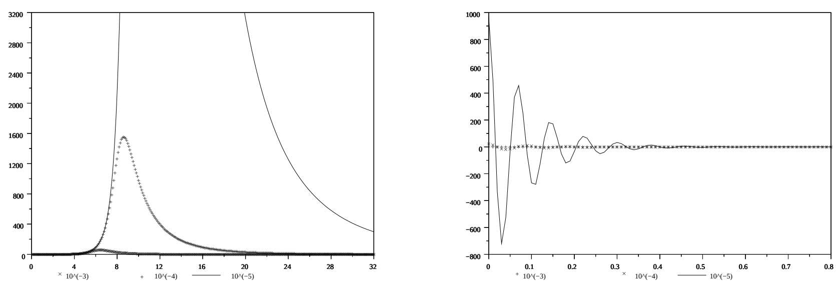

Figure 9. $\hat{u}^\varepsilon(\xi, 1)$ on the left and $u^\varepsilon(x_1, 1)$ on the right, for $\varepsilon = 10^{-3},\ 10^{-4}$ and 10^{-5}

"non local" as it is mainly localized in the vicinity of the boundary bearing the pathological boundary conditions rather than on the support of the loading.

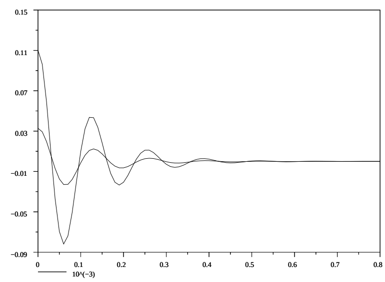

Figure 10. $u^\varepsilon(x_1, 0.95)$ and $u^\varepsilon(x_1, 0.75)$ for $\varepsilon = 10^{-3}$

5. Second kind sensitivity

This section is concerned with an example where the sensitivity follows from boundary conditions which lose their sense at the limit.

5.1. An example of model problem exhibiting second kind sensitivity

Let us consider the domain $\Omega = (0,1) \times (0,1)$. We shall denote by Γ_0, Γ_1 and Γ_2 the boundaries $\{x_1 = 0\}$, $\{x_1 = 1\}$ and $\{x_2 = 0\} \cup \{x_2 = 1\}$ respectively. Let now define the forms a and b as follows:

$$a(u,v) = \int_\Omega \partial_1 u_1 \partial_1 v_1 dx + \int_\Omega (\partial_2 u_1 - u_2)(\partial_2 v_1 - v_2) dx, \tag{5.1}$$

$$b(u,v) = \int_\Omega \partial_\alpha u_2 \partial_\alpha v_2 dx. \tag{5.2}$$

Let V be the space

$$V = \{v = (v_1, v_2) \in H^1(\Omega) \times H^1(\Omega);\ \ v_2 = 0 \ \ on\ \Gamma_1,\ \ v_1 + v_2 = 0 \ \ on\ \Gamma_0\} \tag{5.3}$$

Let us then consider the problem:

$$\left\{ \begin{array}{l} Find\ u \in V \ \ satisfying\ \forall v \in V \\ a(u^\varepsilon, v) + \varepsilon^2 b(u^\varepsilon, v) = \langle f, v\rangle = \int_\Omega (f_1 v_1 + f_2 v_2) dx. \end{array} \right. \tag{5.4}$$

We have:

Lemma 5.1. *The form $a+b$ is continuous and coercive on V.*

Proof. The continuity is obvious. As for the coerciveness, we have

$$(a+b)(v,v) = \|\partial_1 v_1\|_0^2 + \|\nabla v_2\|_0^2 + \|\partial_2 v_1 - v_2\|_0^2$$

where, from the Poincaré inequality with $v_2|_{\Gamma_1} = 0$,

$$\|\nabla v_2\|_0^2 \geq C\|v_2\|_1^2$$

Consequently, we have

$$(a+b)(v,v) \geq C_1[\|\partial_1 v_1\|_0^2 + \|v_2\|_1^2 + \|\partial_2 v_1 - v_2\|_0^2] \tag{5.5}$$

in particular,

$$(a+b)(v,v) \geq C_2[\|\partial_2 v_1 - v_2\|_0^2 + \|v_2\|_0^2] \tag{5.6}$$

Then, on account of the inequalities

$$\|\partial_2 v_1\|_0 \leq \|\partial_2 v_1 - v_2\|_0 + \|v_2\|_0 \Rightarrow \|\partial_2 v_1\|_0^2 \leq 2[\|\partial_2 v_1 - v_2\|_0^2 + \|v_2\|_0^2],$$

from (5.6) we see that

$$(a+b)(u,v) \geq C_3\|\partial_2 v_1\|_0^2. \tag{5.7}$$

At last, by combination of (5.5) and (5.7) we have

$$(a+b)(u,v) \geq C[\|v_2\|_1^2 + \|\nabla v_1\|_0^2]. \tag{5.8}$$

It is classical that the norm H^1 is equivalent to the semi norm defined by the norm of the gradient in L^2 plus any semi norm controlling additive constants which are disregarded by the first semi norm. In particular the equivalence of $\|\nabla v_1\|_0^2 + \|v_1|_{\Gamma_0}\|_0^2$ and $\|v_1\|_1^2$ is proved in [24], page 342. Consequently

$$\|v_1\|_1^2 \leq C_4[\|\nabla v_1\|_0^2 + \|v_1|_{\Gamma_0}\|_0^2], \tag{5.9}$$

where, as $v_1 = -v_2$ on Γ_0, we have

$$\|v_1|_{\Gamma_0}\|_0^2 = \|v_2|_{\Gamma_0}\|_0^2 \leq C_5\|v_2\|_1^2. \tag{5.10}$$

From (5.9) and (5.10) we then have

$$\|v_1\|_1^2 \leq C_6[\|\nabla v_1\|_0^2 + \|v_2\|_1^2] \tag{5.11}$$

and finally

$$(a+b)(u,v) \geq C'[\|v_2\|_1^2 + \|v_2\|_1^2], \tag{5.12}$$

hence the conclusion. □

Lemma 5.2. *The problem (5.4) is inhibited, in other words,* $a(v,v) = 0 \Rightarrow v = 0$.

Proof. Assume that $a(v,v)=0$, then

$$\begin{cases} \partial_1 v_1 = 0 \Rightarrow v_1 = \varphi(x_2) \in H^1(0,1) \\ v_2 = \partial_2 v_1 \Rightarrow v_2 = \varphi'(x_2) \end{cases}$$

With the boundary condition $v_2 = 0$ on Γ_1, we have $\varphi'(x_2) = 0 \Rightarrow \varphi(x_2) = Const.$ and, as $v_1 + v_2 = 0$ on Γ_0 with $v_1 = 0$, we have $\varphi = 0$, QED. □

We are then in the hypotheses of section 2.

Let us show that the problem (5.4) is a sensitive one. Specifically, $\mathcal{D}(\Omega)^2 \not\subset V_a'$.

Lemma 5.3. *The problem (5.4) is sensitive (and more precisely $\mathcal{D}(\Omega)^2$-sensitive).*

Proof. Let us take $f \in V_a' \cap \mathcal{D}(\Omega)^2$. We shall see that f must satisfy certain compatibility conditions, so that it is not any element of $\mathcal{D}(\Omega)^2$, proving that $\mathcal{D}(\Omega)^2 \not\subset V_a'$. Then, for that f, the limit problem (2.15) has a unique solution in V_a. Let us define

$$\begin{cases} \gamma_1(v) = \partial_1 v_1 \\ \gamma_2(v) = \partial_2 v_1 - v_2. \end{cases}$$

We see that for $(v_1, v_2) \in V_a$, even when it is not a distribution, according to the construction of V_a, $\gamma_1(u^0)$ and $\gamma_2(u^0)$ are well determined elements of $L^2(\Omega)$ and consequently, from (2.15), we have

$$\int_\Omega \gamma_1(u^0)\partial_1 v_1 dx + \int_\Omega \gamma_2(u^0)(\partial_2 v_1 - v_2)dx = \int_\Omega f_\alpha v_\alpha dx \quad \forall v \in V_a, \textit{in particular } \forall v \in V. \tag{5.13}$$

Taking, in particular, $v \in \mathcal{D}(\Omega)^2$ we have

$$\begin{cases} -\partial_1\gamma_1 - \partial_2\gamma_2 = f_1 \\ -\gamma_2 = f_2. \end{cases}$$

We then have

$$-\partial_1\gamma_1 = f_1 - \partial_2 f_2. \tag{5.14}$$

Moreover, provided that γ_1, $\gamma_2 \in L^2$, $f_1, f_2 \in L^2$, it is classical that the traces of $n_1\gamma_1 + n_2\gamma_2$ makes sense and vanishes as an element of the dual $H^{-\frac{1}{2}}(\partial\Omega)$ allowing integration by parts, see [16], p. 119-120 for analogous proof if necessary. Then, from (5.13), we have

$$\int_{\partial\Omega} (n_1\gamma_1 + n_2\gamma_2)v_1 d\sigma = 0 \tag{5.15}$$

which implies, as v_1 is arbitrary on $\partial\Omega$,

$$\gamma_1 = 0 \quad on\ \Gamma_0 \cup \Gamma_1 \tag{5.16}$$

$$\gamma_2 = 0 \quad on\ \Gamma_2. \tag{5.17}$$

From (5.14) and (5.16) we see that γ_1 satisfies the two boundary conditions only when the compatibility condition

$$\int_0^1 (f_1(x_1, x_2) - \partial_2 f_2(x_1, x_2))dx_1 = 0$$

is satisfied. This proves that $\mathcal{D}(\Omega)^2 \not\subset V_a'$ and that the problem is sensitive. □

Remark 5.4. We emphasize that we proved that the second condition of (2.1) is satisfied. This implies that in V_a, $a(v, v)^{1/2}$ is a norm so that

$$v \in V_a,\ a(v, v) = 0 \ \Rightarrow v = 0. \tag{5.18}$$

But V_a is not a space of distributions. We are now considering formally the limit problem in terms of equations and boundary conditions using the classical distribution theory (from which the expression "formall"). The problem is

$$\begin{cases} \text{Find } u \in V_a \text{ such that} \\ a(u, v) = \langle f, v\rangle \quad \forall v \in V_a \end{cases} \tag{5.19}$$

We observe that the equivalent to (5.18) does not hold true in this formal framework. Indeed,

$$a(v, v) = 0 \ \Rightarrow \begin{cases} \gamma_1(v) = 0 & \Rightarrow & \partial_1 v_1 = 0 \\ \gamma_2(v) = 0 & \Rightarrow & \partial_2 v_1 - v_2 = 0 \end{cases} \tag{5.20}$$

with the principal boundary conditions of this limit problem. But, in the present distributional framework, the principal boundary conditions do not make sense. Indeed, such boundary conditions are those inherited from those of V in the completion process which uses the norm $a(v, v)^{\frac{1}{2}}$. In that process, v_2 cannot have traces so that no principal boundary conditions appear in the limit problem (in the distributional framework). Consequently, in this context, there is a kernel formed by the solutions of (5.20) namely

$$v_1 = \varphi(x_2),\ \ v_2 = \varphi'(x_2) \tag{5.21}$$

We shall say that the functions (5.21) form the pseudo-kernel of the problem. This definition follows from the fact that the kernel is reduced to 0 in the non classical space V_a whereas in the distributional theory the kernel is defined by (5.21). This pseudo-kernel will be used in the formal asymptotics.

5.2. **Heuristic asymptotics of** u^ε

In this subsection, we shall modify the elements of the pseudo-kernel with boundary layers in order to obtain elements of V. Then, we shall minimize on the elements of the modified pseudo-kernel the energy $\frac{1}{2}(a(v,v)+\varepsilon^2 b(v,v)) - \langle f, v\rangle$.

The formal procedure of construction of boundary layers follows usual trends. Nevertheless, there are special features associated with different orders of differentiation of the unknowns u_1 and u_2 (note for instance the different powers of ε in u_1 and u_2 in (5.26) and (5.27) hereafter). More explanations on these procedures may be seen in [18].

In order to obtain the equations of the boundary layers, in the vicinity of $x_1 = 0$ and $x_1 = 1$ respectively, as usual, we define the dilated coordinate y^1 as follows:

$$y_1 = \frac{x_1}{\varepsilon}, \quad (resp.\ y_1 = \frac{1-x_1}{\varepsilon}). \tag{5.22}$$

We then have for the boundary layer at $x_1 = 0$

$$\partial_1 \equiv \frac{\partial}{\partial x_1} = \frac{\partial}{\varepsilon \partial y_1} \overset{definition}{\equiv} D_1 \tag{5.23}$$

and for the boundary layer at $x_1 = 1$

$$\partial_1 \equiv \frac{\partial}{\partial x_1} = -\frac{\partial}{\varepsilon \partial y_1} \overset{definition}{\equiv} D_1. \tag{5.24}$$

Boundary layer in the vicinity of $x_1 = 0$.

According to (5.21), the kernel is constituted by the elements

$$\mathbf{u}^\varphi = \begin{cases} u_1^\varphi = \varphi(x_2) \\ u_2^\varphi = \varphi'(x_2). \end{cases} \tag{5.25}$$

We define $\mathbf{u}^{\varphi\varepsilon}$ as follows:

$$\mathbf{u}^{\varphi\varepsilon} = \begin{cases} u_1^{\varphi\varepsilon} = u_1^\varphi(x_2) + \varepsilon U_1(y_1, x_2) + \cdots \\ u_2^{\varphi\varepsilon} = u_2^\varphi(x_2) + U_2(y_1, x_2) + \cdots \end{cases} \tag{5.26}$$

which must satisfy the boundary condition at $y_1 = 0$, i.e. on Γ_0. As it is classical in asymptotic expansions, we take test functions under the form

$$\mathbf{v}^\varepsilon = \begin{cases} v_1^\varepsilon = \varepsilon V_1(y_1, x_2) \\ v_2^\varepsilon = V_2(y_1, x_2) \end{cases} \tag{5.27}$$

i.e., with a structure analogous to that of the perturbations in (5.26). Here the V_α have bounded support in y_1 and x_2. In terms of y_1 and x_2 the expression $a(\mathbf{u}^{\varphi\varepsilon}, \mathbf{v}^\varepsilon) + \varepsilon^2 b(\mathbf{u}^{\varphi\varepsilon}, \mathbf{v}^\varepsilon) - \langle f, v\rangle$ writes

$$\begin{aligned}
&\int_{\mathcal{B}} [(D_1 U_1 + \cdots) D_1 V_1 + (U_2 + \cdots) V_2] \varepsilon dy_1 dx_2 + \\
&\qquad \varepsilon^2 \int_{\mathcal{B}} \left[[\frac{1}{\varepsilon} D_1 U_2 + \cdots] \frac{1}{\varepsilon} D_1 V_2 + (\varphi'' + \partial_2 U_2 + \cdots) \partial_2 V_2 \right] \varepsilon dy_1 dx_2 \\
&\qquad = \int_{\mathcal{B}} (f_2 V_2 + \cdots) \varepsilon dy_1 dx_2
\end{aligned} \tag{5.28}$$

where $\mathcal{B}$ is the infinite region $[0, +\infty] \times [0, 1]$ and the $\cdots$ denotes terms of smaller order. At the leading order we then have

$$\int_{\mathcal{B}} [D_1 U_1 D_1 V_1 + U_2 V_2 + D_1 U_2 D_1 V_2] dy_1 dx_2 = \int_{\mathcal{B}} f_2(0, x_2) V_2(0, x_2) dy_1 dx_2. \tag{5.29}$$

In the sequel we shall consider the case $f_2(0, x_2) \equiv 0$, i.e. the case when the boundary Γ_0 is free of loading (but it will be evident later that this hypothesis is unuseful as a consequence of the rescaling (5.41)).

From (5.29) we have

$$\left\{ \begin{array}{lll} -D_1^2 U_1 = 0 & \Rightarrow & U_1 = \alpha(x_2) y_1 + \beta(x_2) \\ U_2 - D_1^2 U_2 = 0 & \Rightarrow & U_2 = \gamma(x_2) e^{-y_1} + \delta(x_2) e^{y_1} \end{array} \right. \tag{5.30}$$

where $u_\alpha^{\varphi\varepsilon}$ must satisfy the boundary condition on Γ_0 and the matching condition as $y_1 \to +\infty$, i.e.

$$\begin{aligned}
&u_1^{\varphi\varepsilon}(0, x_2) + u_2^{\varphi\varepsilon}(0, x_2) = 0 \;\; on \;\; \Gamma_0 \Rightarrow \\
&\beta(x_2) + \gamma(x_2) + \delta(x_2) + \varphi(x_2) + \varphi'(x_2) = 0
\end{aligned} \tag{5.31}$$

and when $y_1 \to +\infty$

$$\mathbf{u}^{\varphi\varepsilon} \to \mathbf{u}^\varphi \Rightarrow \left\{ \begin{array}{l} U_1 \to 0 \Rightarrow \alpha(x_2) = 0, \beta(x_2) = 0 \\ U_2 \to 0 \Rightarrow \delta(x_2) = 0. \end{array} \right. \tag{5.32}$$

Then, from (5.31) we have

$$\gamma(x_2) = -(\varphi(x_2) + \varphi'(x_2)). \tag{5.33}$$

Finally, we get

$$\left\{ \begin{array}{l} U_1 = 0 \\ U_2 = -(\varphi(x_2) + \varphi'(x_2)) e^{-y_1}, \end{array} \right. \tag{5.34}$$

and then

$$\begin{cases} u_1^{\varphi\varepsilon} = \varphi(x_2) \\ u_2^{\varphi\varepsilon} = \varphi'(x_2) - [\varphi(x_2) + \varphi'(x_2)]e^{-y_1}. \end{cases} \tag{5.35}$$

Boundary layer in the vicinity of $x_1 = 1$

In the vicinity of Γ_1 with the boundary condition $u_2^\varepsilon = 0$ on Γ_1, analogous computations give:

$$\begin{cases} U_1 = 0 \\ U_2 = -\varphi'(x_2)e^{-y_1}, \end{cases} \tag{5.36}$$

and

$$\begin{cases} u_1^{\varphi\varepsilon} = \varphi(x_2) \\ u_2^{\varphi\varepsilon} = \varphi'(x_2)[1 - e^{-y_1}]. \end{cases} \tag{5.37}$$

Minimization

Let us search the solution u^ε in the space of the $v^{\varphi\varepsilon}$. Within our approximation, this amounts to (5.35) and (5.37) in the layers adjacent to Γ_0 and Γ_1 respectively, and merely (5.25) in Ω out of the layers. Searching for the solution amounts to minimizing the expression

$$E = \underbrace{a(u^{\varphi\varepsilon}, u^{\varphi\varepsilon})}_{nul\ out\ of\ the\ layers} + \varepsilon^2 \underbrace{b(u^{\varphi\varepsilon}, u^{\varphi\varepsilon})}_{of\ order\ \varepsilon^2\ out\ of\ the\ layers} - 2\langle f, u^{\varphi\varepsilon}\rangle \tag{5.38}$$

From (5.35) and (5.37), the leading orders give:

$$\int_0^1 dx_2 \int_0^{+\infty} \left[(\varphi + \varphi')e^{-y_1}\right]^2 \varepsilon dy^1 + \int_0^1 dx_2 \int_0^{+\infty} \left[\varphi' \exp(-y_1)\right]^2 \varepsilon dy_1$$
$$+\varepsilon^2 \int_0^1 dx_2 \int_0^{+\infty} \left[(\varphi + \varphi')\frac{e^{-y_1}}{\varepsilon}\right]^2 \varepsilon dy_1$$
$$+\varepsilon^2 \int_0^1 dx_2 \int_0^{+\infty} \left[\varphi' \frac{e^{-y_1}}{\varepsilon}\right]^2 \varepsilon dy_1 =$$
$$\int_0^1 \int_0^1 [f_1\varphi(x_2) + f_2\varphi'(x_2)]dx_1 dx_2 \tag{5.39}$$

so that

$$\varepsilon \int_0^1 \left[(\varphi + \varphi')^2 + \varphi'^2\right] dx_2 = \int_0^1 \int_0^1 (f_1\varphi + f_2\varphi')dx_1 dx_2 \tag{5.40}$$

Let us define

$$\psi = \varepsilon\varphi \tag{5.41}$$

then (5.40) becomes

$$\int_0^1 \left[(\psi+\psi')^2+\psi'^2\right] dx_2 = \int_0^1\int_0^1 (f_1\psi+f_2\psi')dx_1dx_2 \tag{5.42}$$

Let us show that the left hand side of (5.42) is coercive on $H^1(0,1)$. Indeed, from

$$2\psi\psi' \le \frac{\psi^2}{\lambda}+\lambda\psi'^2$$

we have

$$\psi^2+2\psi'^2+2\psi\psi' \ge \psi^2(1-\frac{1}{\lambda})+\psi'^2(2-\lambda)$$

and there exists λ such that

$$\int_0^1 \left[(\psi+\psi')^2+\psi'^2\right] dx_2 \ge C\int_0^1 \left[\psi^2+\psi'^2\right] dx_2.$$

hence the conclusion. In fact, (5.42) amounts to a continuous and coercive variational problem for the unknown $\psi \in H^1(0,1)$. Indeed, let us show that the right hand member is a linear continuous functional on $H^1(0,1)$. We have

$$\begin{aligned}\int_0^1 dx_2 \int_0^1 [f_1(x_1,x_2)\varphi(x_2)+f_1(x_1,x_2)\varphi'(x_2)]dx_1 \equiv\\ \int_0^1 \varphi(x_2)dx_2 \underbrace{\int_0^1 f_1(x_1,x_2)dx_1}_{F_1(x_2)}\\ +\int_0^1 \varphi'(x_2)dx_2 \underbrace{\int_0^1 f_2(x_1,x_2)dx_1}_{F_2(x_2)}\end{aligned} \tag{5.43}$$

and the continuity follows when F_1, F_2 are in $(H^1(0,1))'$ and consequently, for instance, with f_1 and f_2) are in $L^2((0,1)_{x_1};(H^1(0,1))')$.

The result of this formal asymptotics is that:

$$u^\varepsilon = \frac{1}{\varepsilon}u^{\psi\varepsilon}+... \tag{5.44}$$

(where ... denotes asymptotically smaller terms), where $u^{\psi\varepsilon}$ is defined by (5.35) and (5.37) with ψ instead of φ and $\psi \in H^1(0,1)$ is the solution of the variational problem:

$$\left\{ \begin{array}{l} \textit{Find } \psi \in H^1(0,1) \textit{ such that } \forall\theta \in H^1(0,1): \\ \int_0^1 [(\psi+\psi')(\theta+\theta') + \psi'\theta']dx_2 = \int_0^1\int_0^1 (f_1\theta_1 + f_2\theta_2)dx_1dx_2. \end{array} \right. \tag{5.45}$$

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