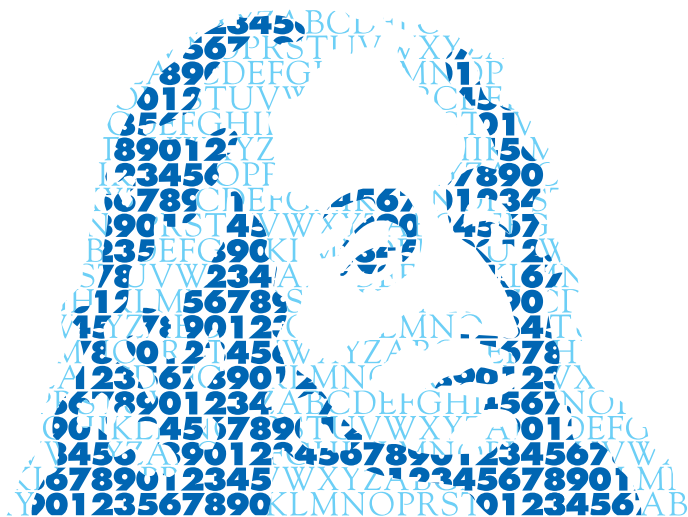


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# The spin-statistics relation in nonrelativistic quantum mechanics and projective modules

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## Abstract

In this work we consider non-relativistic quantum mechanics, obtained from a classical configuration space  $\mathcal{Q}$  of indistinguishable particles. Following an approach proposed in [8], wave functions are regarded as elements of suitable projective modules over  $C(\mathcal{Q})$ . We take furthermore into account the  $G$ -Theory point of view (cf. [4, 10]) where the role of group action is particularly emphasized. As an example illustrating the method, the case of two particles is worked out in detail. Previous works (cf. [2, 3]) aiming at a proof of a spin-statistics theorem for non-relativistic quantum mechanics are re-considered from the point of view of our approach, enabling us to clarify several points.

## 1 Introduction

Being a consequence of the general principles of relativistic QFT, the spin-statistics (SS) theorem has found rigorous proofs in the context of axiomatic QFT, as well as in that of algebraic QFT.

In spite of various efforts, this has not been the case in non relativistic quantum mechanics (NRQM). However, it would be interesting to find such a proof, which does not rely as heavily on concepts of relativistic QFT as the established ones, for several reasons. There are many examples of phenomena taking place outside the relativistic realm which depend essentially on the SS relation for its description, making such a proof desirable. On the other hand, this new sought-after approach to SS could also be of benefit for the understanding of QFT itself. For instance, a proof which does not make use

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of the complexification of the full Lorentz group could provide hints towards the understanding of SS in more general situations, such as theories where a background gravitational field is present. It could also provide guidance for the development of theories on non commutative spaces. There is also a motivation coming from the idea that *quantum indistinguishability*, if correctly incorporated into quantum theory, might lead to a better understanding of SS.

Usually, arguments along these lines are based on what can be called the “*configuration space approach*”, since in one form or another they make use of configuration space techniques. For example, in one of the first works of this kind, Laidlaw and DeWitt found out in [5] that when applying the path integral formalism to a system consisting of a finite number of non-relativistic, identical spin zero particles in three spatial dimensions, the topology of the corresponding configuration space imposed certain restrictions on the propagator. From this, they were able to deduce that only particles obeying Fermi or Bose statistics were allowed (note that this Fermi-Bose alternative is an input in the standard proofs of axiomatic QFT). Leinaas and Myrheim considered a similar situation in [6], in an analysis that was motivated by the relevance of indistinguishability to Gibb’s paradox. They argued that if a quantum theory is obtained after a process of quantization on a classical configuration space  $\mathcal{Q}$ , then indistinguishability should be incorporated in  $\mathcal{Q}$  right from the beginning. Mathematically, they considered wave functions to be sections of vector bundles on a space where permuted configurations were identified. By physical reasons, these vector bundles should be equipped with a flat connection, whose holonomy was shown to describe the effect of particle exchange. They reproduced the results of [5] by obtaining the Fermi-Bose alternative in three dimensional space for spinless particles. But, in addition, they also found that in one and two dimensions the statistics parameter could, in principle, take infinitely many values. In that same work, they remarked that their results could provide a geometrical basis for the derivation of the SS theorem. A lot of work based on this kind of “topological arguments” has been done since then, but most of the results that have been obtained are based on assumptions which go beyond NRQM. It should also be said that in several cases the argumentations remain at a classical level, having no clear interpretation in terms of quantum mechanics.

More recently, Berry and Robbins provided in [2] an explicit construction in which the quantum mechanics of two identical particles is formulated along the lines described in [6], leading to the physically correct SS relation. Since

their result is based on a particular construction of what they call a “transported spin basis”, one cannot consider it as a proof from first principles -as the authors themselves have recognized, since there are various alternatives for the construction of such a spin basis leading to different statistics signs (cf.[3])- . Nevertheless, the construction is interesting for its own sake, and raises several questions that deserve to be considered.

In this paper we consider the SS relation (for two particles) from an algebraic point of view, by studying projective modules over  $C(\mathcal{Q})$  (by the Serre-Swan theorem these modules can be interpreted as modules of sections on vector bundles over  $\mathcal{Q}$ ). Additionally, we assume the  $G$ -Theory point of view for the consideration of the symmetries of the problem. This allows us to arrive at a precise and explicit formulation of the SS problem, in which various known results can be reproduced in a clear and efficient way. As an example illustrating the relevance of the proposed approach, we make a comparison with the Berry-Robbins construction and show how various points can be clarified.

The paper is organized as follows. In section **2** we present the method through a detailed discussion of the two particle case. In section **3** the work of Berry and Robbins is briefly reviewed and a comparison with the approach presented in section **2** is made. Finally, we present in section **4** a brief discussion of the results.

## 2 Projective modules and the configuration space approach to SS: an example.

The classical configuration space of a system of  $N$  identical particles moving in  $\mathbb{R}^3$  is defined as a quotient space,  $\mathcal{Q}_N = \tilde{\mathcal{Q}}_N/S_N$ , obtained from the natural action of the permutation group  $S_N$  on the space

$$\tilde{\mathcal{Q}}_N = \{(r_1, \dots, r_N) \in \mathbb{R}^{3N} | r_i \neq r_j \text{ for all pairs}(i, j)\}. \quad (2.1)$$

The non-coincidence condition  $r_i \neq r_j$  is included in the definition of  $\tilde{\mathcal{Q}}_N$  in order to make  $\mathcal{Q}_N$  a manifold. Following [6], we consider wave functions to be given by square integrable sections of some vector bundle on  $\mathcal{Q}_N$ .

In this work we will restrict ourselves to the case  $N = 2$ . Here, after performing a transformation to center of mass and relative coordinates, one sees that  $\mathcal{Q}_2$  is of the same homotopy type as a two-sphere  $S^2$ , this latter

representing the space of normalized relative coordinates of the two particles. Under exchange, the relative coordinate  $r$  goes to  $-r$ , so that after quotienting out by the action of  $S_2 \cong \mathbb{Z}_2$ , we obtain a projective plane. For our purpose it is therefore enough to consider  $\tilde{\mathcal{Q}}_2 = S^2$  and  $\mathcal{Q}_2 = \mathbb{RP}^2$  for the configuration space. The sphere will be considered as embedded in  $\mathbb{R}^3$ . Points on it will be denoted by  $x = (x_1, x_2, x_3)$  and, accordingly, points in the projective plane will be denoted by  $[x] = \{x, -x\}$ .

Define now  $\mathcal{A} := C(S^2)$ . The  $\mathbb{Z}_2$ -action on  $S^2$  induces one on  $\mathcal{A}$ , leading to a decomposition into subspaces of even and odd functions:

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-. \tag{2.2}$$

$\mathcal{A}_+$ , the subalgebra of even functions, is easily seen to be isomorphic to  $C(\mathbb{RP}^2)$ . For the description of the spin degrees of freedom, we will certainly need a representation of  $SU(2)$  taking into account the transformation properties of the wave functions under rotations. Since  $S^2$  is a homogeneous space for  $SU(2)$  and consequently  $\mathcal{A}$  carries an  $SU(2)$  representation, the isomorphism  $\mathcal{A}_+ \cong C(\mathbb{RP}^2)$  offers the possibility of constructing projective modules corresponding to  $SU(2)$ -equivariant bundles over  $\mathbb{RP}^2$ , just by exploiting the rotational symmetry of the sphere. The construction of a projective module having these properties has already been carried out in [8], but for the sake of completeness we reproduce it here. Let us denote with  $V^j$ , as usual, the  $(2j+1)$ -dimensional irreducible  $SU(2)$  representation, so that  $\mathcal{A} \cong \bigoplus_{j \in \mathbb{N}_0} V^j$ . Consider the tensor product representation  $\mathcal{A} \otimes V^1$ . Decomposing it into irreducibles, we obtain:

$$\mathcal{A} \otimes V^1 \cong \left( \bigoplus_{j \in \mathbb{N}_0} V^j \right) \otimes V^1 \cong V^1 \oplus (V^0 \oplus V^1 \oplus V^2) \oplus (V^1 \oplus V^2 \oplus V^3) \oplus \dots$$

Note that the trivial representation  $V^0$  appears only once in the decomposition. Thus, there is a unique scalar element, up to normalization, with respect to this representation. Using  $\mathcal{A} \otimes V^1 \cong \mathcal{A}^3$ , this scalar element is given, in terms of spherical harmonics, by the (normalized) vector

$$|\psi\rangle := \sqrt{\frac{4\pi}{3}} \begin{pmatrix} Y_{1,1} \\ -Y_{1,0} \\ Y_{1,-1} \end{pmatrix} \tag{2.3}$$

This  $\mathcal{A}$ -valued vector has remarkable properties. For instance, we have the following result.

**Proposition 2.1:** (cf.[8]) Define a projector on  $\mathcal{A}^3$  by  $p := |\psi\rangle\langle\psi|$ . Then, the following isomorphism of  $\mathcal{A}_+$ -modules holds:  $p(\mathcal{A}_+^3) \cong \mathcal{A}_-$ .

By the Serre-Swan theorem, there is a line bundle  $L_-$  on  $\mathbb{R}\mathbb{P}^2$  whose module of sections is isomorphic to  $p(\mathcal{A}_+^3)$ . Since  $p(\mathcal{A}_+^3)$  is, by construction,  $SU(2)$ -equivariant (cf.[8]), it follows that  $L_-$  is also  $SU(2)$ -equivariant. In the following proposition we give a proof of this fact in a form suitable for the applications in the next section.

**Proposition 2.2:** The line bundle  $L_-$  is  $SU(2)$ -equivariant.

PROOF: First, note that the total space  $E(L_-)$  of the bundle is defined, as a set, as  $E(L_-) := \{([x], \lambda|\psi(x)) \in \mathbb{R}P^2 \times \mathbb{C}^3 \mid \lambda \in \mathbb{C}, x \in [x]\}$ . Therefore,  $L_-$  can be regarded as a subbundle of the trivial bundle  $\mathbb{R}\mathbb{P}^2 \times \mathbb{C}^3$ , with projection map  $\pi([x], \lambda|\psi(x)) := [x]$ . Since  $|\psi\rangle$  is invariant under  $SU(2)$  we know that for every  $g \in SU(2)$  and  $x \in S^2$  the relation  $\mathcal{D}^{(1)}(g)|\psi(x)\rangle = |\psi(g \cdot x)\rangle$  holds. This can be used to define an  $SU(2)$  action on  $E(L_-)$ . Indeed, since  $|\psi(x)\rangle (= -|\psi(-x)\rangle)$  spans the fiber over  $[x]$ , we see that the action of an element  $g \in SU(2)$  on  $y = ([x], \lambda|\psi(x)) \in \pi^{-1}([x])$  can be correctly defined by setting  $\tau_g(y) := ([gx], \lambda|\psi(g \cdot x))$ .  $\square$

**Remarks 2.3:**

1. It is well known that -up to equivalence- there are only two complex line bundles on  $\mathbb{R}\mathbb{P}^2$ . One of them is the trivial one,  $L_+$ , and the other is  $L_-$ . The corresponding modules of sections  $\Gamma(L_\pm)$  are isomorphic, respectively, to  $\mathcal{A}_\pm$ . Higher rank bundles can always be written as sums of these bundles.
2. The Grassmann connection naturally associated to  $p(\mathcal{A}_+^3)$  is defined as  $\nabla = p dp$ . It can be shown, by direct computation, that  $\nabla$  has vanishing curvature. The corresponding holonomy group is  $\mathbb{Z}_2$ .
3. The choice of  $\mathcal{A}_-$  as Hilbert space corresponds, according to the discussion in the introduction, to the description of a system of two *identical* spin zero particles obeying Fermi statistics. Those obeying Bose statistics are of course described by  $\mathcal{A}_+$  and, by the first remark above, there are no other possibilities. Note that this result is obtained from an intrinsic treatment of indistinguishability, where no use of a symmetrization postulate is made.

We thus see how the Fermi-Bose alternative for scalar particles is obtained as a direct consequence of the topology of the configuration space. This is a well known result, but we have discussed it in order to illustrate the method. The usefulness of our approach will be clear when we consider higher values of the spin, since in that case the Fermi-Bose alternative has no direct relation to the topology of the configuration space: the  $SU(2)$ -equivariance of bundles on it must be used. In fact, in the general case the Fermi-Bose alternative follows from the requirement of a well defined transformation law for the wave functions under rotations, compatible with the exchange of particles (cf.[7, 9]).

Now we consider the relation between the  $SU(2)$  action on  $L_-$  and parallel transport with respect to  $\nabla = p dp$ . For the proof of the following proposition, it is convenient to consider the equivalent projector  $\tilde{p} := U^\dagger p U$ , where

$$U = \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ -1/\sqrt{2} & -i/\sqrt{2} & 0 \end{pmatrix}.$$

All the previous formulae for  $p$  (or  $L_-$ ) in terms of  $|\psi\rangle$  remain valid for  $\tilde{p}$  upon replacing  $|\psi\rangle$  by  $|\phi\rangle := U^\dagger|\psi\rangle$  in them.

**Proposition 2.4:** *Let  $\gamma : [0, 1] \rightarrow SU(2)$  be a smooth path such that  $\gamma(0) = e$ . Given  $[x^{(0)}] \equiv \{x^{(0)}, -x^{(0)}\} \in \mathbb{RP}^2$  and  $y \in E(L_-)$  with  $\pi(y) = [x^{(0)}]$ , define  $\alpha : [0, 1] \rightarrow \mathbb{RP}^2$  through  $\alpha(t) = [\gamma(t) \cdot x^{(0)}]$ . Then it follows that  $t \mapsto \tau_{\gamma(t)}(y)$  is a section along  $\alpha$ , parallel with respect to the connection  $\tilde{\nabla} = \tilde{p} d\tilde{p}$ .*

PROOF: Pick one representative of  $[x^{(0)}]$ , say  $x^{(0)}$ . From the description of  $E(L_-)$  given in the proof of proposition 2.2, we know that there exists  $\lambda \in \mathbb{C}$  such that  $y = ([x^{(0)}], \lambda|\phi(x^{(0)})\rangle)$  (this  $\lambda$  is unique, since a choice of representative has been made). By letting  $SU(2)$  act on the sphere, we obtain three real functions  $t \mapsto x_i(t)$ ,  $i \in \{1, 2, 3\}$ , namely, the components of  $x(t) := \gamma(t) \cdot x^{(0)}$ . Now, note that the columns of the projector  $\tilde{p}$  give place to sections  $e_i \in \Gamma(L_-)$ ,  $i \in \{1, 2, 3\}$ , that are generators for the module. Writing  $e_i(t)$  for  $e_i \circ \alpha(t)$ , we can define  $s(t) := \sum_{i=1}^3 x_i(t)e_i(t)$ , a section along  $\alpha$ . It is easily verified that  $s(t) \equiv \tau_{\gamma(t)}(y)$ . That  $s$  is parallel along  $\alpha$  follows directly from the explicit form of the connection:  $\tilde{\nabla} e_i = \sum_{j=1}^3 da_{ij} \otimes e_j$ , where  $a_{ij}([x]) := x_i x_j$ .  $\square$

We finish this section with some comments about the  $N > 2$  case. Recall

that because of proposition 2.1 we may regard functions on  $\tilde{\mathcal{Q}}_2$  as sections on (flat) vector bundles over  $\mathcal{Q}_2$ . Looking back at equation (2.2), we realize that the decomposition of  $C(\tilde{\mathcal{Q}}_2)$  into  $C(\mathcal{Q}_2)$ -submodules induced by the  $\mathbb{Z}_2$  action on it provides a complete description of all flat bundles over  $\mathcal{Q}_2$ . This assertion remains valid for  $N > 2$ :

**Proposition 2.5:** *The  $C(\mathcal{Q}_N)$ -submodules of the algebra  $C(\tilde{\mathcal{Q}}_N)$  obtained from the  $S_N$  action on  $\tilde{\mathcal{Q}}_N$  are finitely generated and projective and the natural connections associated to them are flat.*

Since only the case  $N=2$  will be needed for the discussion of the next sections, the reader is referred elsewhere [9] for the proof of the proposition. A remark on how the construction of proposition 2.1 may be extended to arbitrary  $N$  is however in order, because the way we have decomposed  $\mathcal{A}$  into  $\mathcal{A}_+$ -projective modules (the case  $N=2$ ), depends strongly on the natural  $SU(2)$ -action available in this case. For general  $N$ , we have a free action of  $S_N$  on  $\tilde{\mathcal{Q}}_N$  (see Eq.2.1), which gives place to a representation of  $S_N$  on  $C(\tilde{\mathcal{Q}}_N)$  (this representation turns out to be closely related to the regular representation). Now, since  $\tilde{\mathcal{Q}}_N$  is the universal covering space of  $\mathcal{Q}_N$ , it is not difficult (by means of a suitably chosen partition of unity) to find explicit expressions for the transformation properties of functions in each irreducible subspace of  $C(\tilde{\mathcal{Q}}_N)$ . This in turn allows one to show that each such subspace is, in fact, a finitely generated and projective module over  $C(\mathcal{Q}_N)$ .

With this result at hand, the study of the general case and of its relevance to the SS problem can be carried out systematically. Nevertheless, in order to obtain a *proof* of the SS theorem within the present approach, further assumptions, motivated from physics, are needed. A proposal in this direction is being carried out and will be published shortly [7].

### 3 Comparison with the Berry-Robbins approach.

Now we proceed to make a comparison between the approach explained in the last section and that of Berry and Robbins (BR). We will be mainly concerned with the relation between the projector  $p$  of proposition 2.1 and the transported spin basis of BR, on one hand, and with the singlevaluedness assumption of BR, on the other. Some remarks will be made about the spin operators defined in BR in relation to proposition 2.4.



### 3.1 Relation between $p$ and the BR construction.

We begin by briefly recalling the essential points of the BR construction. We refer the reader to their work (cf.[2, 3]) for details on the construction and also for the notation, which we follow here. Consider two identical particles of spin  $S$ . Let the label  $M$  stand for the -ordered- pair of eigenvalues  $\{m_1, m_2\}$  of the spin of the two particles in a given direction. If  $M$  corresponds to a given pair  $\{m_1, m_2\}$ , then (following BR) we denote with  $\bar{M}$  the label corresponding to the permuted pair  $\{m_2, m_1\}$ . In the BR approach, the usual spin basis  $\{|M\rangle\}_M$  is replaced by a new, position dependent one,  $\{|M(r)\rangle\}_M$ . The transported spin vectors are obtained from the usual “fixed” ones by means of a position dependent unitary transformation  $U$ , constructed with the help of Schwinger’s representation of spin, and acting on  $\mathbb{C}^{N_S}$ , where  $N_S = \frac{1}{6}(4S + 1)(4S + 2)(4S + 3)$ . The main properties of the resulting basis are the following:

(i) The map

$$\begin{aligned} S^2 &\longrightarrow \mathbb{C}^{N_S} \\ r &\longmapsto |M(r)\rangle := U(r)|M\rangle \end{aligned}$$

is well defined and smooth for all  $M$ . (Note that  $U$  is really an  $SU(2)$  representation, so its matrix components are not functions on  $S^2$ . Only when acting on the “physical” vectors of the form  $|M\rangle$  on  $\mathbb{C}^{N_S}$ , does one obtain a vector (at each  $r$ ) whose components can be regarded as functions on  $S^2$ ).

(ii) The following “exchange” property holds:

$$|\bar{M}(-r)\rangle = (-1)^{2S}|M(r)\rangle. \tag{3.1}$$

(iii) The “parallel transport” condition  $\langle M'(r(t)) | \frac{d}{dt} M(r(t)) \rangle = 0$  is satisfied for all  $M, M'$  and any smooth curve  $t \mapsto r(t)$ .

The wave function is then expressed in terms of the transported basis,

$$|\Psi(r)\rangle = \sum_M \Psi_M(r) |M(r)\rangle, \tag{3.2}$$

and the following condition is *imposed* on it:

$$|\Psi(r)\rangle \stackrel{!}{=} |\Psi(-r)\rangle. \tag{3.3}$$

An immediate consequence of this is that the coefficient functions must satisfy the relation  $\Psi_{\overline{M}}(-r) = (-1)^{2S}\Psi_M(r)$ , which is the usual form of the SS relation. The task of (3.3) is to incorporate the indistinguishability of the particles in the formalism, but we shall postpone the discussion of this point to section 3.2. Our concern for the moment is to find out “where does the wave function live” because, in the words of BR, what we are doing with this construction is “*setting up quantum mechanics on a ‘two-spin bundle’, whose six-dimensional base is the configuration space  $r_1, r_2$  with exchanged configurations identified and coincidences  $r_1 = r_2$  excluded (...). The fibres are the two-spin Hilbert spaces spanned by the transported basis  $|M(r)\rangle$ . The full Hilbert space consists of global sections of the bundle, i.e. singlevalued wave functions*” [2]. Therefore, the wave function  $|\Psi(r)\rangle$  should be regarded as a section of some vector bundle over  $\mathbb{RP}^2$ . What we want to do first is to find out which bundle this is.

In order to accomplish this task, we perform a change from the basis  $\{|M\rangle\}_M$  to a basis of total angular momentum  $\{|J, m_J\rangle\}_{J, m_J}$ , according to the Clebsch-Gordan decomposition, and write the transported spin basis in terms of this new basis. The bundle these new transported vectors generate can then be easily identified. Let us consider, in order to be concrete, the  $S = 1/2$  case, for which  $N_S = 10$ . A basis for the space acted on by  $U(r)$  can be written down in terms of the four oscillator operators (cf. [2, 3])  $a_i^\dagger, b_j^\dagger$  ( $i, j = 1, 2$ ). We shall use the following one:

$$\begin{aligned} |1, 1\rangle^{(-1)} &= \frac{(a_1^\dagger)^2}{\sqrt{2}}|0\rangle, & |1, 1\rangle^{(0)} &= a_1^\dagger a_2^\dagger |0\rangle, & |1, 1\rangle^{(1)} &= \frac{(a_2^\dagger)^2}{\sqrt{2}}|0\rangle, \\ |1, 0\rangle^{(-1)} &= \frac{(b_1^\dagger)^2}{\sqrt{2}}|0\rangle, & |1, 0\rangle^{(0)} &= b_1^\dagger b_2^\dagger |0\rangle, & |1, 0\rangle^{(1)} &= \frac{(b_2^\dagger)^2}{\sqrt{2}}|0\rangle, \\ |1, -1\rangle^{(-1)} &= a_1^\dagger b_1^\dagger |0\rangle, & |1, -1\rangle^{(0)} &= \frac{a_1^\dagger b_2^\dagger + b_1^\dagger a_2^\dagger}{\sqrt{2}}|0\rangle, & |1, -1\rangle^{(1)} &= a_2^\dagger b_2^\dagger |0\rangle, \\ & & |0, 0\rangle &= \frac{a_1^\dagger b_2^\dagger - b_1^\dagger a_2^\dagger}{\sqrt{2}}|0\rangle. \end{aligned}$$

For the transported spin basis one then obtains

$$\begin{aligned} |J = 1, m_J(r)\rangle &:= U(r)|J = 1, m_J\rangle^{(0)} \\ &= \sum_{\mu=-1}^1 W(r)_{0, \mu} |J = 1, m_J\rangle^{(\mu)}, \quad (m_J = 1, 0, -1), \end{aligned}$$

and

$$|J = 0, 0(r)\rangle := U(r)|J = 0, 0\rangle = |J = 0, 0\rangle,$$

where

$$W(\vec{r}) := \begin{pmatrix} \cos^2 \frac{\theta}{2} & e^{i\varphi} \frac{\sin \theta}{\sqrt{2}} & e^{2i\varphi} \frac{\sin^2 \theta}{2} \\ -e^{-i\varphi} \frac{\sin \theta}{\sqrt{2}} & \cos \theta & e^{i\varphi} \frac{\sin \theta}{\sqrt{2}} \\ e^{-2i\varphi} \frac{\sin^2 \theta}{2} & -e^{-i\varphi} \frac{\sin \theta}{\sqrt{2}} & \cos^2 \frac{\theta}{2} \end{pmatrix}.$$

To identify the corresponding bundle, we follow the remark -quoted above- that the transported vectors, evaluated at the point  $\pm r$ , span the fiber over  $[r]$ . From the last equations it is clear that the singlet component of the wave function will lie in a trivial line bundle and that the (line) bundles corresponding to the triplet components are all equivalent. The projection operator onto the vector space spanned by  $|J = 1, m_J(r)\rangle$  will have the same form for all  $m_J$ , so we may just define  $P^{(J=1)}(r) := W^t(r)P_0W^*(r)$ , with  $(P_0)_{ij} = \delta_{2,i}\delta_{2,j}$  ( $i, j = 1, 2, 3$ ). This leads to:

$$P^{(J=1)}(r) = \begin{pmatrix} \frac{1}{2} \sin^2 \theta & -\frac{1}{\sqrt{2}} \sin \theta \cos \theta e^{-i\varphi} & -\frac{1}{2} \sin^2 \theta e^{-2i\varphi} \\ -\frac{1}{\sqrt{2}} \sin \theta \cos \theta e^{i\varphi} & \cos^2 \theta & \frac{1}{\sqrt{2}} \sin \theta \cos \theta e^{-i\varphi} \\ -\frac{1}{2} \sin^2 \theta e^{2i\varphi} & \frac{1}{\sqrt{2}} \sin \theta \cos \theta e^{i\varphi} & \frac{1}{2} \sin^2 \theta \end{pmatrix}.$$

From equation (2.3) we see that this projector is exactly the same as the one defined in proposition 2.1:  $P^{(J=1)} \equiv p$ . This means that the vector bundle in question is  $L_- \oplus L_- \oplus L_- \oplus L_+$ . The wave function must therefore be a section of this bundle or, equivalently, an element of the  $\mathcal{A}_+$ -module  $\mathcal{A}_-^3 \oplus \mathcal{A}_+$ . The case of general spin can be handled in a similar way, so we will not consider it here.

### 3.2 The singlevaluedness condition.

At first glance, as implied by (3.2), the wave function is given by a map  $S^2 \rightarrow \mathbb{C}^{N_S}$ . It is only because of (3.3) that we may consider its domain to be  $\mathbb{RP}^2$ . But, in which sense and to what extent does the imposition of this condition really define  $|\Psi(r)\rangle$  as a section of a bundle over  $\mathbb{RP}^2$ ?

Denote with  $L_{m_J}^J$  the bundle over  $S^2$  whose fiber over  $r$  is the complex line spanned by  $|J, m_J(r)\rangle$  (note that, since  $|J, m_J(r)\rangle \neq 0$  for all  $r$ ,  $L_{m_J}^J$  is trivial). Put  $\Psi(r) := (r, |\Psi(r)\rangle)$  and define  $\eta^S := \bigoplus_{J=0}^S (\bigoplus_{m_J=-J}^J L_{m_J}^J)$ . Because of (3.2), we have  $\Psi \in \Gamma(\eta^S)$ . But  $\eta^S$  is a bundle over  $S^2$ , not over  $\mathbb{RP}^2$ , as we need. A possible way out of this problem is to specify a  $\mathbb{Z}_2$ -action on  $\eta^S$  and then to construct the quotient  $\eta^S/\mathbb{Z}_2$ . This is justified by the following well-known fact ( $M$  is a manifold with a free  $G$ -action and  $G$ , for our purposes, a finite group):

**Proposition 3.1:** (cf.[1]) *If  $M$  is  $G$ -free  $G$ -vector bundles over  $M$  correspond bijectively to vector bundles over  $M/G$  by  $\eta \rightarrow \eta/G$ .*

In the case of the *line* bundle  $L_{m_J}^J$ , there are exactly two such possible  $\mathbb{Z}_2$ -structures given, say, by actions  $\tau_{\pm}$ . Quotienting out by  $\tau_{\pm}$  one obtains  $L_{m_J}^J/\tau_{\pm} \cong L_{\pm}$ , but there is no *a priori* way of choosing between  $\tau_-$  and  $\tau_+$ . Nevertheless, there is something particular in the way  $L_{m_J}^J$  has been constructed: because of the “exchange” property (3.1), we have  $|J, m_J(-r)\rangle = (-1)^J |J, m_J(r)\rangle$ . This relation suggests somehow a choice of action, according to whether  $J$  is even or odd <sup>1</sup>. This is in fact true, in a certain sense (to be explained), that involves the “singlevaluedness” condition (3.3). But before that we have to answer the following question: if we can pass from  $\eta^S$  to a bundle over  $\mathbb{R}P^2$  by specifying a  $\mathbb{Z}_2$ -action on  $\eta^S$  and taking the quotient, what is the procedure to follow with  $\Psi$ ? The answer is easily obtained through of a reformulation of proposition 3.1, as explained below.

Consider again the situation of proposition 3.1. Let  $q : M \rightarrow M/G$  be the quotient map. The proposition says that if  $\eta$  is a  $G$ -vector bundle over  $M$  (with action  $\tau$ ), then  $\eta/\tau$  is a bundle over  $M/G$  and  $q^*(\eta/\tau) \cong_G \eta$ . The equivalence is an isomorphism of  $G$ -bundles:  $\eta$  as  $G$ -bundle with respect to  $\tau$ , and  $q^*(\eta/\tau)$  with respect to the  $G$ -action naturally inherited from the pull-back operation. On the other hand, if  $\xi$  is a bundle over  $M/G$ , the induced  $G$ -action on  $q^*(\xi)$  makes it a  $G$ -bundle, and then  $q^*(\xi)/G \cong \xi$ . These isomorphisms allow one to work on  $M$ , considering  $G$ -bundles on it, in order to describe bundles on  $M/G$ . But of course we must always take the additional structure carried by  $\eta$  into account, if we want to “regard” it as a bundle over  $M/G$ . A convenient way of doing this, which at the same time answers the question posed above, consists in considering, instead of bundles, the respective modules of sections. In this setting, the pull-back operation leads to the following isomorphism of  $C(M)$ -modules:  $\Gamma(q^*\xi) \cong C(M) \otimes_{C(M/G)} \Gamma(\xi)$ . Recalling now that  $C(M)$  has a decomposition into  $C(M/G)$ -submodules we expect, when we regard  $\Gamma(q^*\xi)$  as a  $C(M/G)$ -module, to find a submodule inside it which is isomorphic to  $\Gamma(\xi)$ . This is true, and the submodule we are looking for can be characterized in the following way. First notice that the natural  $G$ -action on  $q^*(\xi)$  induces one on  $\Gamma(q^*\xi)$ . Then we have:

**Proposition 3.2:** (cf.[9]) *The space of  $G$ -invariant sections of  $\Gamma(q^*\xi)$  is iso-*

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<sup>1</sup>This relation is also the reason why the projector  $P^{(J=1)}$  defines a module over  $\mathcal{A}_+$ , since from it we get  $P^{(J=1)}(-r) = P^{(J=1)}(r)$ .

morphic, as a  $C(M/G)$ -module, to  $\Gamma(\xi)$ :

$$\Gamma(\xi) \cong \Gamma^{inv}(q^*\xi) = \{s \in \Gamma(q^*\xi) \mid g \cdot s = s \text{ for all } g \in G\}.$$

Similarly, if  $\eta$  is a  $G$ -bundle on  $M$  (with action  $\tau$ ), then the space of  $\tau$ -invariant sections of  $\Gamma(\eta)$  is isomorphic, as a  $C(M/G)$ -module, to  $\Gamma(\eta/\tau)$ .

We thus arrive at the conclusion that, in order to regard  $\Psi(\in \Gamma(\eta^S))$  as a section on a bundle over  $\mathbb{RP}^2$ , we must: (i) Specify a  $\mathbb{Z}_2$ -action  $\tau$  on  $\eta^S$  and (ii) Require that  $\Psi$  be a  $\tau$ -invariant section. Regarding (i), there are two choices<sup>2</sup>, that can be described as follows. Let  $t$  denote the non trivial element of  $\mathbb{Z}_2$  and choose an integer  $\tilde{K}$ . Recalling that the total space of  $\eta^S$  is given by  $E(\eta^S) = \{(r, \sum_{J,m_J} \lambda_{J,m_J} |J, m_J(r)\rangle) \mid r \in S^2, \lambda_{J,m_J} \in \mathbb{C}\}$ , we may define an action  $\tau : \mathbb{Z}_2 \times E(\eta^S) \rightarrow E(\eta^S)$  by setting

$$\tau_t(r, \lambda |J, m_J(r)\rangle) := \left(-r, (-1)^{2S-J+\tilde{K}} \lambda |J, m_J(-r)\rangle\right). \quad (3.4)$$

The induced action on the section  $\Psi$  gives:

$$\begin{aligned} (t \cdot \Psi)(r) &:= \tau_t \Psi(-r) = \tau_t \left(-r, \sum_{J,m_J} \Psi_{j,m}(-r) |j, m(-r)\rangle\right) \\ &= \left(r, \sum_{J,m_J} \Psi_{J,m_J}(-r) (-1)^{2S-J+\tilde{K}} |J, m_J(r)\rangle\right). \end{aligned}$$

Taking now (ii) into account, we must require  $t \cdot \Psi = \Psi$ . This implies  $\Psi_{J,m_J}(-r) = (-1)^{2S-J+\tilde{K}} \Psi_{J,m_J}(r)$ . We are now in a position to discuss the singlevaluedness condition (3.3). Let us consider a spin basis with the exchange property  $|\overline{M}(-r)\rangle = (-1)^K |M(r)\rangle$ . In the  $\{J, m_J\}$  basis this corresponds to

$$|J, m_J(-r)\rangle = (-1)^{2S-J+K} |J, m_J(r)\rangle. \quad (3.5)$$

We then see that, with a basis satisfying (3.5), imposing (3.3) amounts to require  $\Psi$  to be an invariant section with respect to the action (3.4), *provided* we choose  $K = \tilde{K}$ .

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<sup>2</sup>On  $\mathbb{RP}^2$  there are only two equivalence classes of bundles of a given rank  $k+1$ , represented respectively by  $L_+^{k+1}$  and  $L_- \oplus L_+^k$  (notice that  $L_-^2$  is trivial). For the SS problem, the choice of connection is also relevant. This is closely related to a choice of representative for the class of the bundle. But for each class only one choice is compatible with the Fermi-Bose alternative. These are the ones we are considering.

But note that for the definition of  $\tau$  in (3.4) a previously specified relation between  $|J, m_J(r)\rangle$  and  $|J, m_J(-r)\rangle$  is not needed. In fact, not even a dependence of the basis on  $r$  is required, given that  $\eta^S$  is anyway a trivial bundle. Hence, (3.5) seems not to have a further meaning. Its only role is to ensure that the fibers of  $\eta^S$  at opposite points on the sphere *coincide*, given that  $\eta^S$  was constructed as a twisted (yet trivial) subbundle of a trivial bundle. It then makes sense to “compare” the values of the section  $\Psi$  at different points, as is tacitly assumed in (3.3).

Due to proposition 3.2, there is an isomorphism  $\Phi : \Gamma(\eta^S/\tau) \rightarrow \Gamma^{inv(\tau)}(\eta^S)$  of  $C(\mathbb{R}\mathbb{P}^2)$ -modules. The condition  $t \cdot \Psi = \Psi$  (w.r.t  $\tau$ ) guarantees that  $\Psi = \Phi(\sigma)$  for a unique  $\sigma \in \Gamma(\eta^S/\tau)$ . In particular, note that if in (3.5) we choose  $K = \tilde{K} + 1$ , then that *same* section  $\sigma$  will be represented on the sphere by a function  $|\Psi'\rangle$  satisfying  $|\Psi'(-r)\rangle = -|\Psi'(r)\rangle$  and hence in contradiction with the singlevaluedness assumption.

### 3.3 Spin operators

In the BR approach, spin operators do also depend on  $r$ . As with the spin basis, they are defined making use of the map  $U$ :

$$S_i(r) := U(r)S_iU^\dagger(r). \quad (3.6)$$

The spin operators are defined in such a way that they act linearly -as the physically correct representation- on each fiber (recall that in general the fibers are isomorphic to  $V^s \otimes V^s$ ). In order to relate this definition to our approach, let us recall that, for a given value of  $S$ , the two bundles corresponding to Fermi and Bose statistics are  $SU(2)$ -equivariant. Now, if under finite rotations the wave function transforms according to such an  $SU(2)$ -action, then the spin operators should correspond to an infinitesimal version of the corresponding  $SU(2)$ -action. In the definition of such infinitesimal operators one must be careful that only the *spin* degrees of freedom are being described. One would perhaps expect that such a requirement imposes a restriction on the admissible bundles where the wave function is supposed to be defined. But this is not the case: a consistent description of identical particles having the physically wrong statistics is in fact possible within the present approach. This may be regarded as a further indication that additional physical requirements are really needed for a proof of SS in NRQM.

Let us, as an example, consider the case of two spinless particles obeying fermionic statistics. In that case, as we have seen, the wave function is defined

on  $L_-$ . Let  $\nabla$  be the corresponding flat connection and  $\tau$  the  $SU(2)$ -action. Consider the integral curve  $\gamma_i(t)$  of the projection to  $\mathbb{RP}^2$  of the vector field  $L_i$  on the sphere, with  $\gamma_i(0) = [x]$ . Given  $[x] \in \mathbb{RP}^2$ , there is for  $t$  small enough, an element  $g_t \in SU(2)$  (unique up to elements in the stability group of  $[x]$ ) such that  $g_t \cdot [x] = \gamma_i(t)$ . Given  $y$  a vector in the fiber over  $[x]$ , consider  $\tau_{g_t}(y)$ . Parallel-transport this vector from  $g_t \cdot [x]$  back to  $[x]$  and call  $y_t$  the result. The local spin operator  $S_i([x])$  corresponding to a rotation about the  $i^{\text{th}}$  axis can then be defined through

$$S_i([x])(y) := \lim_{t \rightarrow 0} \frac{i}{t} (y_t - y). \quad (3.7)$$

But because of proposition 2.4 the operators  $S_i([x])$  are all equal to zero, as required for scalar particles.

In the case of general  $S$ , the bundles corresponding to Fermi and Bose statistics carry respective flat connections and  $SU(2)$  actions. We can therefore take (3.7) as a definition of spin operators. Again because of proposition 2.4, one sees that no inconsistency arises in the non-physical case. On the other hand, for the bundle corresponding to the physically correct SS relation, one gets the same operators defined in BR by means of (3.6).

## 4 Discussion

In this work we have tried to approach the SS problem from a new point of view which, although formally equivalent to the more familiar ones, enables a clear formulation and understanding of the problem. This has been illustrated through a comparison with the BR construction, where our formalism proves to be a much more natural one (cf. section 3.2). Particular attention has been devoted to the meaning of their singlevaluedness condition, which we have shown to be misleading. The reason for this is that their construction is actually performed in the universal cover  $\tilde{\mathcal{Q}}$  of the configuration space  $\mathcal{Q}$ . We have shown in a precise way what are the requirements that allow us to regard sections defined on bundles over  $\tilde{\mathcal{Q}}$  as sections defined on bundles over  $\mathcal{Q}$ . Our approach also settles the question of how many different constructions of the BR kind exist. Indeed, the decomposition of  $C(\tilde{\mathcal{Q}})$  into  $C(\mathcal{Q})$ -submodules already contains all the necessary information. Moreover, it allows one to work directly with *functions* on  $\tilde{\mathcal{Q}}$ , thus making the construction of a transported spin basis unnecessary.

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