BLAISE PASCAL

RAFAEL DÍAZ, EDDY PARIGUAN

Symmetric quantum Weyl algebras

Symmetric quantum Weyl algebras

Rafael Díaz[1]

Eddy Pariguan[2]

Abstract

We study the symmetric powers of four algebras: q-oscillator algebra, q-Weyl algebra, h-Weyl algebra and $U(\mathfrak{sl}_2)$. We provide explicit formulae as well as combinatorial interpretation for the normal coordinates of products of arbitrary elements in the above algebras.

1 Introduction

This paper takes part in the time-honored tradition of studying an algebra by first choosing a "normal" or "standard" basis for it $\mathbb{B}$, and second, writing down explicit formulae and, if possible, combinatorial interpretation for the representation of the product of a finite number of elements in $\mathbb{B}$, as linear combination of elements in $\mathbb{B}$.

This method has been successfully applied to many algebras, most prominently in the theory of symmetric functions (see [7]). We shall deal with algebras given explicitly as the quotient of a free algebra, generated by a set of letters L, by a number of relations. We choose normal basis for our algebras by fixing an ordering of the set of the letters L, and defining $\mathbb{B}$ to be the set of normally ordered monomials, i.e., monomials in which the letters appearing in it respect the order of L.

We consider algebras of the form $\mathrm{Sym}^n(A)$, i.e, symmetric powers of certain algebras. Let us recall that for each $n \in \mathbb{N}$, there is a functor $\mathrm{Sym}^n \colon \mathbb{C}\text{-alg} \longrightarrow \mathbb{C}\text{-alg}$ from the category of associative $\mathbb{C}$-algebras into itself defined on objects as follows: if A is a $\mathbb{C}$-algebra, then $\mathrm{Sym}^n(A)$ denotes

---

the algebra whose underlying vector space is the n-th symmetric power of A: $\mathrm{Sym}^n(A) = (A^{\otimes n})/\langle a_1 \otimes \ldots \otimes a_n - a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n)} : a_i \in A, \sigma \in \mathbb{S}_n\rangle$, where $\mathbb{S}_n$ denotes the group of permutations on n letters. The product of m elements in $\mathrm{Sym}^n(A)$ is given by the rule

$$(n!)^{m-1} \prod_{i=1}^{m} \left(\overline{\bigotimes_{j=1}^{n}} a_{ij}\right) = \sum_{\sigma \in \{\mathrm{id}\} \times \mathbb{S}_n^{m-1}} \overline{\bigotimes_{j=1}^{n} \left(\prod_{i=1}^{m} a_{i\sigma_i^{-1}(j)}\right)} \tag{1.1}$$

for all $(a_{ij}) \in A^{[[1,m]] \times [[1,n]]}$. Notice that if A is an algebra, then $A^{\otimes n}$ is also an algebra. $\mathbb{S}_n$ acts on $A^{\otimes n}$ by algebra automorphisms, and thus we have a well defined invariant subalgebra $(A^{\otimes n})^{\mathbb{S}_n} \subseteq A^{\otimes n}$. The following result is proven in [8].

Theorem 1.1: *The map* $s : \mathrm{Sym}^n(A) \longrightarrow (A^{\otimes n})^{\mathbb{S}_n}$ *given by*

$$s\left(\overline{\bigotimes_{i=1}^{n}} a_i\right) = \frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} \overline{\bigotimes_{i=1}^{n}} a_{\sigma^{-1}(i)}, \quad \textit{for all} \ \ a_i \in A$$

defines an algebra isomorphism.

The main goal of this paper is the study of the symmetric powers of certain algebras that may be regarded as quantum analogues of the Weyl algebra. Let us recall the well-known

Definition 1.2: The algebra $\mathbb{C}\langle x, y\rangle[h]/\langle yx - xy - h\rangle$ is called the *Weyl algebra*.

This algebra admits a natural representation as indicated in the

Proposition 1.3: *The map* $\rho : \mathbb{C}\langle x, y\rangle[h]/\langle yx - xy - h\rangle \longrightarrow \mathrm{End}(\mathbb{C}[x, h])$ *given by* $\rho(x)(f) = xf$, $\rho(y)(f) = h\,\partial f/\partial x$, $\rho(h)(f) = hf$ *for any* $f \in \mathbb{C}[x, h]$ *defines a representation of the Weyl algebra.*

The symmetric powers of the Weyl algebra have been studied from several point of view in papers such as [1],[3],[6],[10]. Our interest in the subject arose from the construction of non-commutative solitonic states in string theory, based on the combinatorics of the annihilation $\frac{\partial}{\partial x}$ and creation $\hat{x}$ operators given in [3]. In [8] we gave explicit formula, as well as combinatorial interpretation for the normal coordinates of monomials $\partial^{a_1} x^{b_1} \ldots \partial^{a_n} x^{b_n}$. This formulae allow us to find explicit formulae for the product of a finite number of elements in the symmetric powers of the Weyl algebra. Looking at

Proposition 1.3 ones notices that the definition of Weyl algebra relies on the notion of the derivative operator $\frac{\partial}{\partial x}$ from classical infinitesimal calculus. The classical derivative $\frac{\partial}{\partial x}$ admits two well-known discrete deformations, the so called q-derivative ∂_q and the h-derivative ∂_h. The main topic of this paper is to introduce the corresponding q and h-analogues for the Weyl algebra, and generalize the results established in [8] to these new contexts.

For the q-calculus, we will actually introduce two q-analogues of the Weyl algebra: the q-oscillator algebra (section 2) and the q-Weyl algebra (section 3). Needless to say, the q-oscillator algebra also known as the q-boson algebra [11], and q-Weyl algebra are deeply related. The main difference between them is that while the q-oscillator is the algebra generated by ∂_q and $\hat{x}$, a third operator, the q-shift s_q is also present in the q-Weyl algebra. We believe that s_q is as fundamental as ∂_q and $\hat{x}$. The reason it has passed unnoticed in the classical case is that for $q = 1$, s_q is just the identity operator. For both q-analogues of the Weyl algebra we are able to find explicit formulae and combinatorial interpretation for the product rule in their symmetric powers algebras.

For the h-calculus, also known as the calculus of finite differences, we introduce the h-Weyl algebra in section 4. Besides the annihilator ∂_h and the creator $\hat{x}$ operators, also includes an h-shift operator s_h, which again reduces to the identity for $h = 0$. We give explicit formulae and combinatorial interpretation for the product rule in the symmetric powers of the h-Weyl algebra.

In section 5 we deal with an algebra of a different sort. Since our method has proven successful for dealing with the Weyl algebra (and it q and h-deformations); and it is known that the Weyl algebra is isomorphic to the universal enveloping algebra of the Heisenberg Lie algebra, it is an interest problem to apply our constructions for other Lie algebras. We consider here only the simplest case, that of $\mathfrak{sl}_2$. We give explicit formulae for the product rule in the symmetric powers of $U(\mathfrak{sl}_2)$.

Although some of the formulae in this paper are rather cumbersome, all of them are just the algebraic embodiment of fairly elementary combinatorial facts. The combinatorial statements will be further analyzed in [9].

Notations and conventions

- $\mathbb{N}$ denotes the set of natural numbers. For $x \in \mathbb{N}^n$ and $i \in \mathbb{N}$, we denote by $x_{<i}$ the vector $(x_1, \ldots, x_{i-1}) \in \mathbb{N}^{i-1}$, by $x_{\leq i}$ the vector $(x_1, \ldots, x_i) \in$

$\mathbb{N}^i$, by $x_{>i}$ the vector $(x_{i+1}, \ldots, x_n) \in \mathbb{N}^{n-i}$ and by $x_{\geq i}$ the vector $(x_i, \ldots, x_n) \in \mathbb{N}^{n-i+1}$.

- Given $(U, <)$ an ordered set and $s \in U$, we set $U_{>s} := \{u \in U : u > s\}$.

- $| \ | : \mathbb{N}^n \longrightarrow \mathbb{N}$ denotes the map such that $|x| := \sum_{i=1}^{n} x_i$, for all $x \in \mathbb{N}^n$.

- For a set X, $\sharp(X) :=$ cardinality of X, and $\mathbb{C}\langle X \rangle :=$ free associative algebra generated by X.

- Given natural numbers $a_1, \ldots, a_n \in \mathbb{N}$, $\min(a_1, \ldots, a_n)$ denotes the smallest number in the set $\{a_1, \ldots, a_n\}$.

- Given $n \in \mathbb{N}$, we set $[[1, n]] = \{1, \ldots, n\}$.

- Let S be a set and $A : [[1, m]] \times [[1, n]] \longrightarrow S$ an S-valued matrix. For $\sigma \in (\mathbb{S}_n)^m$ and $j \in [[1, n]]$, $A_j^\sigma : [[1, m]] \longrightarrow S$ denotes the map such that $A_j^\sigma(i) = A_{i\sigma_i^{-1}(j)}$, for all $i = 1, \ldots, m$.

- The q-analogue n integer is $[n] := \dfrac{1 - q^n}{1 - q}$. For $k \in \mathbb{N}$, we will use $[n]_k := [n][n-1] \ldots [n-k+1]$.

2 Symmetric q-oscillator algebra

In this section we define the q-oscillator algebra and study its symmetric powers. Let us introduce several fundamental operators in q-calculus (see [13] for a nice introduction to q-calculus).

Definition 2.1: The operators $\partial_q, s_q, \hat{x}, \hat{q}, \hat{h} : \mathbb{C}[x, q, h] \longrightarrow \mathbb{C}[x, q, h]$ are given as follows

$$\partial_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \qquad s_q(f)(x) = f(qx) \qquad \hat{q}(f) = qf$$
$$\hat{x}(f) = xf \qquad\qquad \hat{h}(f) = hf$$

for all $f \in \mathbb{C}[x, q, h]$. We call ∂_q the q-derivative and s_q the q-shift.

Definition 2.2: The algebra $\mathbb{C}\langle x, y \rangle[q, h]/I_{qo}$, where I_{qo} is the ideal generated by the relation $yx = qxy + h$ is called the *q-oscillator algebra*.

We have the following analogue of Proposition 1.3.

Proposition 2.3: *The map $\rho : \mathbb{C}\langle x, y \rangle[q, h]/I_{qo} \longrightarrow \mathrm{End}(\mathbb{C}[x, q, h])$ given by $\rho(x) = \hat{x}$, $\rho(y) = h\partial_q$, $\rho(q) = \hat{q}$ and $\rho(h) = \hat{h}$ defines a representation of the q-oscillator algebra.*

Notice that if we let $q \to 1$, y becomes central and we recover the Weyl algebra. We order the letters of the q-oscillator algebra as follows: $q < x < y < h$.

Assume we are given $A = (A_1, \ldots, A_n) \in (\mathbb{N}^2)^n$, and $A_i = (a_i, b_i) \in \mathbb{N}^2$, for $i \in [[1, n]]$. Set $X^{A_i} = x^{a_i} y^{b_i}$, for $i \in [[1, n]]$. We set $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, $b = (b_1, \ldots, b_n) \in \mathbb{N}^n$, $c = (c_1, \ldots, c_n) \in \mathbb{N}^n$ and $|A| = (|a|, |b|) \in \mathbb{N}^2$. Using this notation we have

Definition 2.4: The *normal coordinates* $N_{qo}(A, k)$ of

$$\prod_{i=1}^n X^{A_i} \in \mathbb{C}\langle x, y \rangle[h, q]/I_{qo}$$

are given by the identity

$$\prod_{i=1}^n X^{A_i} = \sum_{k=0}^{\min} N_{qo}(A, k) X^{|A|-(k,k)} h^k \tag{2.1}$$

where $\min = \min(|a|, |b|)$. For $k > \min$, we set $N_{qo}(A, k)$ to be equal to 0.

Let us introduce some notation needed to formulate Theorem 2.6 below which provides explicit formula for the normal coordinates $N_{qo}(A, k)$ of $\prod_{i=1}^n X^{A_i}$. Given $(A_1, \ldots, A_n) \in (\mathbb{N}^2)^n$ choose disjoint totally ordered sets $(U_i, <_i)$, $(V_i, <_i)$ such that $\sharp(U_i) = a_i$ and $\sharp(V_i) = b_i$, for $i \in [[1, n]]$. Define a total order set $(U \cup V \cup \{\infty\}, <)$, where $U = \cup_{i=1}^n U_i$, $V = \cup_{i=1}^n V_i$ and $\infty \notin U \cup V$, as follows: Given $u, v \in U \cup V \cup \{\infty\}$ we say that $u \leq v$ if and only if a least one of the following conditions hold

$$u \in V_i, v \in V_j \text{ and } i \leq j; \qquad u \in V_i, v \in U_j \text{ and } i \leq j;$$
$$u \in U_i, v \in V_j \text{ and } i \leq j; \qquad u, v \in U_i \text{ and } u \leq_i v;$$
$$u, v \in V_i \text{ and } u \leq_i v; \qquad v = \infty.$$

Given $k \in \mathbb{N}$, we let $P_k(U, V)$ be the set of all maps $p : V \longrightarrow U \cup \{\infty\}$ such that

- p restricted to $p^{-1}(U)$ is injective, and $\sharp(p^{-1}(U)) = k$.

- If $(v, p(v)) \in V_i \times U_j$ then $i < j$.

Figure 1 shows an example of such a map. We only show the finite part of p, all other points in V being mapped to ∞.
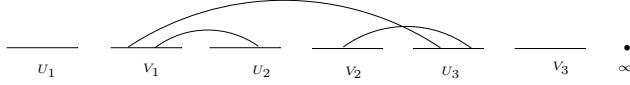

Figure 1: Combinatorial interpretation of N_{qo}.

Definition 2.5: The value of the *crossing number map* $c : P_k(U, V) \longrightarrow \mathbb{N}$ when evaluated on $p \in P_k(U, V)$ is given by

$$c(p) = \sharp(\{(s, t) \in V \times U | s < t < p(s), t \in p(U_{>s})^c\}).$$

Theorem 2.6: *For any* $A = (A_1, \ldots, A_n) \in (\mathbb{N}^2)^n$ *with* $A_i = (a_i, b_i) \in \mathbb{N}^2$ *for* $i \in [[1, n]]$ *and any* $k \in \mathbb{N}$, *we have that*

$$N_{qo}(A, k) = \sum_{p \in P_k(U, V)} q^{c(p)}.$$

PROOF: The proof is by induction. The only non-trivial case is the following

$$
\begin{aligned}
y \prod_{i=1}^{n} X^{A_i} \;=\; & \sum_{k=0}^{\min} q^{|a|-k} N_{qo}(A, k) X^{|A|-(k,k+1)} h^k \\
& + \sum_{k=0}^{\min} N_{qo}(A, k) \sum_{i=1}^{a-k} q^{i-1} X^{|A|-(k+1,k)} h^k \qquad (2.2)
\end{aligned}
$$

where $\min = \min(|a|, |b|)$. Normalizing the left-hand side of (2.2) we get a recursive relation

$$N_{qo}(((0, 1)A), k) = N_{qo}(A, k) + \sum_{i=1}^{a-k} q^{i-1} N_{qo}(A, k-1).$$

The other recursion needed being $N_{qo}(((1,0)A),k) = N_{qo}(A,k)$. It is straight-forward to check that $N_{qo}(A,k) = \displaystyle\sum_{p \in P_k(U,V)} q^{c(p)}$ satisfies both recursions. $\quad\square$

Consider the identity (2.1) in the representation of the q-oscillator algebra defined in Proposition 2.3. Apply both sides of the identity (2.1) to x^t for $t \in \mathbb{N}$, and use Theorem 2.6 to get the fundamental

Corollary 2.7: *Given* $(t,a,b) \in \mathbb{N} \times \mathbb{N}^n \times \mathbb{N}^n$ *the following identity holds*

$$\prod_{i=1}^{n} [t + |a_{>i}| - |b_{>i}|]_{b_i} = \sum_{p \in P_k(U,V)} q^{c(p)} [t]_{|b|-k}.$$

Our next theorem gives a fairly simple formula for the product of m elements in $\operatorname{Sym}^n(\mathbb{C}\langle x,y\rangle[q,h]/I_{qo})$. Fix a matrix $A : [[1,m]] \times [[1,n]] \longrightarrow \mathbb{N}^2$, $(A_{ij}) = ((a_{ij}),(b_{ij}))$. Recall that given $\sigma \in (\mathbb{S}_n)^m$ and $j \in [[1,n]]$, A_j^σ denotes the vector $(A_{1\sigma_1^{-1}(j)}, \ldots, A_{m\sigma_m^{-1}(j)}) \in (\mathbb{N}^2)^m$ and $X_j^{A_{ij}} = x_j^{a_{ij}} y_j^{b_{ij}}$. Set

$$|A_j^\sigma| = (|a_j^\sigma|, |b_j^\sigma|) \text{ where } |a_j^\sigma| = \sum_{i=1}^{m} a_{i\sigma_i^{-1}(j)} \text{ and } |b_j^\sigma| = \sum_{i=1}^{m} b_{i\sigma_i^{-1}(j)}.$$

Theorem 2.8: *For any* $A : [[1,m]] \times [[1,n]] \longrightarrow \mathbb{N}^2$, *the identity*

$$(n!)^{m-1} \prod_{i=1}^{m} \overline{\prod_{j=1}^{n} X_j^{A_{ij}}} = \sum_{\sigma,k} \left(\prod_{j=1}^{n} N_{qo}(A_j^\sigma, k_j) \right) \overline{\prod_{j=1}^{n} X_j^{|A_j^\sigma|-(k_j,k_j)}} h^{|k|}$$

where $\sigma \in \{\mathrm{id}\} \times \mathbb{S}_n^{m-1}$ *and* $k \in \mathbb{N}^n$, *holds in* $\operatorname{Sym}^n(\mathbb{C}\langle x,y\rangle[q,h]/I_{qo})$.

PROOF: Using the product rule given in (1.1), the identity (2.1) and the distributive property we obtain

$$(n!)^{m-1} \prod_{i=1}^{m} \overline{\prod_{j=1}^{n} X_j^{A_{ij}}} = \sum_{\sigma \in \{\mathrm{id}\} \times \mathbb{S}_n^{m-1}} \overline{\prod_{j=1}^{n} \prod_{i=1}^{m} X_j^{A_{i\sigma_i^{-1}(j)}}}$$

$$= \sum_{\sigma \in \{\mathrm{id}\} \times \mathbb{S}_n^{m-1}} \overline{\prod_{j=1}^{n} \left(\sum_{k=0}^{\min_j} N_{qo}(A_j^\sigma, k) x_j^{|a_j^\sigma|-k} y_j^{|b_j^\sigma|-k} h^k \right)}$$

$$= \sum_{\sigma,k} \left(\prod_{j=1}^{n} N_{qo}(A_j^\sigma, k_j) \right) \overline{\prod_{j=1}^{n} X_j^{|A_j^\sigma|-(k_j,k_j)}} h^{|k|}$$

where $\min_j = \min(|a_j^\sigma|, |b_j^\sigma|)$. □

3 Symmetric q-Weyl algebra

In this section we study the symmetric powers of the q-Weyl algebra.

Definition 3.1: The q-Weyl algebra is given by $\mathbb{C}\langle x, y, z\rangle[q]/I_q$, where I_q is the ideal generated by the following relations:

$$zx = xz + y \qquad yx = qxy \qquad zy = qyz$$

We have the following q-analogue of Proposition 1.3.

Proposition 3.2: *The map $\rho : \mathbb{C}\langle x, y, z\rangle[q]/I_q \longrightarrow \mathrm{End}(\mathbb{C}[x, q])$ given by $\rho(x) = \hat{x}$, $\rho(y) = s_q$, $\rho(z) = \partial_q$ and $\rho(q) = \hat{q}$ defines a representation of the q-Weyl algebra.*

Notice that if we let $q \to 1$, we recover the Weyl algebra. We order the letters of the q-Weyl algebra as follows: $q < x < y < z$. Given $a \in \mathbb{N}$ and $I \subset [[1, a]]$, we define the *crossing number of I* to be

$$\chi(I) := \sharp(\{(i, j) : i > j, i \in I, j \in I^c\}).$$

For $k \in \mathbb{N}$, we let $\chi_k : \mathbb{N} \longrightarrow \mathbb{N}$ the map given by $\chi_k(a) = \sum_{\substack{\sharp(I)=k \\ I \subset [[1,a]]}} q^{\chi(I)}$, for

all $a \in \mathbb{N}$. We have the following

Theorem 3.3: *Given $a, b \in \mathbb{N}$, the following identities hold in $\mathbb{C}\langle x, y, z\rangle[q]/I_q$*

1. $z^a x^b = \sum\limits_{k=0}^{\min} \chi_k(a)[b]_k x^{b-k} y^k z^{a-k}$, *where* $\min = \min(a, b)$.

2. $z^a y^b = q^{ab} y^b z^a$.

3. $y^a x^b = q^{ab} x^b y^a$.

194

PROOF: 2. and 3. are obvious, let us to prove 1. It should be clear that

$$z^a x^b = \sum_{I \subset [[1,a]]} [b]_{\sharp(I)} x^{b-\sharp(I)} \prod_{j=1}^{a} f_I(j),$$

where $f_I(j) = z$, if $j \notin I$ and $f_I(j) = y$, if $j \in I$. The normal form of $\prod_{j=1}^{a} f_I(j)$
is $q^{\chi(I)} y^{\sharp(I)} z^{a-\sharp(I)}$. Thus,

$$z^a x^b = \sum_{k=0}^{\min} \left(\sum_I q^{\chi(I)} \right) [b]_k x^{b-k} y^k z^{a-k} = \sum_{k=0}^{\min} \chi_k(a) [b]_k x^{b-k} y^k z^{a-k},$$

where $\min = \min(a, b)$, $I \subset [[1, a]]$ and $\sharp(I) = k$. $\qquad\square$

Assume we are given $A = (A_1, \dots, A_n) \in (\mathbb{N}^3)^n$, where $A_i = (a_i, b_i, c_i)$, for
$i \in [[1, n]]$, also $X^{A_i} = x^{a_i} y^{b_i} z^{c_i}$ for $i \in [[1, n]]$. We set $a = (a_1, \dots, a_n) \in \mathbb{N}^n$,
$b = (b_1, \dots, b_n) \in \mathbb{N}^n$, $c = (c_1, \dots, c_n) \in \mathbb{N}^n$ and $|A| = (|a|, |b|, |c|) \in \mathbb{N}^3$.
Using this notation, we have the

Definition 3.4: The *normal coordinates* $N_q(A, k)$ of

$$\prod_{i=1}^{n} X^{A_i} \in \mathbb{C}\langle x, y, z\rangle[q]/I_q$$

are given via the identity

$$\prod_{i=1}^{n} X^{A_i} = \sum_{k \in \mathbb{N}^{n-1}} N_q(A, k) X^{|A|+(-|k|,|k|,-|k|)} \tag{3.1}$$

where k runs over all vectors $k = (k_1, \dots, k_{n-1}) \in \mathbb{N}^{n-1}$ such that $0 \le k_i \le \min(|c_{\le i}| - |k_{<i}|, a_{i+1})$. We set $N_q(A, k) = 0$ for $k \in \mathbb{N}^{n-1}$ not satisfying the
previous conditions.

Our next theorem follows from Theorem 3.3 by induction. It gives an
explicit formula for the normal coordinates $N_q(A, k)$ in the q-Weyl algebra
of $\prod_{i=1}^{n} X^{A_i}$.

Theorem 3.5: *Let A, k be as in the previous definition, we have*

$$N_q(A, k) = q^{\sum_{i=1}^{n-1} \lambda(i)} \prod_{j=1}^{n-1} \chi_{k_j}(|c_{\leq j}| - |k_{<j}|)[a_{j+1}]_{k_j},$$

where $\lambda(i) = b_{i+1}(|c_{\leq i}| - |k_{\leq i}|) + (a_{i+1} - k_i)(|b_{\leq i}| + |k_{<i}|)$.

Applying both sides of the identity (3.1) in the representation of the q-Weyl algebra given in Proposition 3.2 to x^t and using Theorem 3.5, we obtain the remarkable

Corollary 3.6: *For any given $(t, a, b, c) \in \mathbb{N} \times \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^n$, the following identity holds*

$$\prod_{i=1}^{n} q^{\gamma(i)}[t + |a_{>i}| + |c_{>i}|]_{c_i} = \sum_k q^{\beta(k)} \left(\prod_{j=1}^{n-1} \chi_{k_j}(|c_{\leq j}| - |k_{<j}|)[a_{j+1}]_{k_j} \right) [t]_{|c|-|k|}$$

where $k \in \mathbb{N}^{n-1}$ such that $0 \leq k_i \leq \min(|c_{\leq i}| - |k_{<i}|, a_{i+1})$,

$\gamma(i) = b_i(t + |a_{>i}| - |c_{>i-1}|)$, *and* $\beta(k) = \left(\sum_{i=1}^{n-1} \lambda(i) \right) + (|b| + |k|)(t - |c| + |k|)$.

Our next theorem provides an explicit formula for the product of m elements in the algebra $\mathrm{Sym}^n(\mathbb{C}\langle x, y, z\rangle[q]/I_q)$. Fix $A : [[1, m]] \times [[1, n]] \longrightarrow \mathbb{N}^3$, with $(A_{ij}) = ((a_{ij}), (b_{ij}), (c_{ij}))$. Recall that given $\sigma \in \mathbb{S}_n^m$ and $j \in [[1, n]]$, A_j^σ denotes the vector $(A_{1\sigma_1^{-1}(j)}, \ldots, A_{m\sigma_m^{-1}(j)}) \in (\mathbb{N}^3)^m$ and set $X_j^{A_{ij}} = x_j^{a_{ij}} y_j^{b_{ij}} z_j^{c_{ij}}$, for $j \in [[1, n]]$. Set $A_j^\sigma = (|a_j^\sigma|, |b_j^\sigma|, |c_j^\sigma|)$, where $|a_j^\sigma| = \sum_{i=1}^{m} a_{i\sigma_i^{-1}(j)}$ and similarly for $|b_j^\sigma|$ and $|c_j^\sigma|$. We have the following:

Theorem 3.7: *For any $A : [[1, m]] \times [[1, n]] \longrightarrow \mathbb{N}^3$, the identity*

$$(n!)^{m-1} \prod_{i=1}^{m} \overline{\prod_{j=1}^{n} X_j^{A_{ij}}} = \sum_{\sigma, k} \left(\prod_{j=1}^{n} N_q(A_j^\sigma, k^j) \right) \overline{\prod_{j=1}^{n} X_j^{|A_j^\sigma| + (-|k^j|, |k^j|, -|k^j|)}}$$

where $\sigma \in \{\mathrm{id}\} \times \mathbb{S}_n^{m-1}$ and $k = (k^1, \ldots, k^n) \in (\mathbb{N}^{m-1})^n$, holds in $\mathrm{Sym}^n(\mathbb{C}\langle x, y, z\rangle[q]/I_q)$.

PROOF:

$$(n!)^{m-1} \prod_{i=1}^{m} \overline{\prod_{j=1}^{n} X_j^{A_{ij}}} = \sum_{\sigma \in \{\mathrm{id}\} \times \mathbb{S}_n^{m-1}} \prod_{j=1}^{n} \overline{\prod_{i=1}^{m} X_j^{A_{i\sigma_i^{-1}(j)}}}$$

$$
\begin{aligned}
&= \overline{\sum_\sigma \prod_{j=1}^n \left(\sum_{k=0}^{\min_j} N_q(A_j^\sigma, k) x_j^{|a_j^\sigma|-|k|} y_j^{|b_j^\sigma|+|k|} z_j^{|c_j^\sigma|-|k|} \right)} \\
&= \sum_{\sigma,k} \left(\prod_{j=1}^n N_q(A_j^\sigma, k^j) \right) \overline{\prod_{j=1}^n X_j^{|A_j^\sigma|+(-|k^j|,|k^j|,-|k^j|)}},
\end{aligned}
$$

where $\min_j = \min(|a_j^\sigma|, |b_j^\sigma|, |c_j^\sigma|)$. $\qquad\qquad\square$

4 Symmetric h-Weyl algebra

In this section we introduce the h-analogue of the Weyl algebra in the h-calculus, and study its symmetric powers. A basic introduction to h-calculus may be found in [13].

Definition 4.1: The operators $\partial_h, s_h, \hat{x}, \hat{h} : \mathbb{C}[x,h] \longrightarrow \mathbb{C}[x,h]$ are given by

$$
\partial_h f(x) = \frac{f(x+h)-f(x)}{h} \qquad s_h(f)(x) = f(x+h) \qquad \hat{x}(f) = xf \qquad \hat{h}(f) = hf
$$

for $f \in \mathbb{C}[x,h]$. We call ∂_h the h-derivative and s_h the h-shift.

Definition 4.2: The h-*Weyl algebra* is the algebra $\mathbb{C}\langle x,y,z\rangle[h]/I_h$, where I_h is the ideal generated by the following relations:

$$
yx = xy + z \qquad zx = xz + zh \qquad yz = zy
$$

Proposition 4.3: *The map* $\rho : \mathbb{C}\langle x,y,z\rangle[h]/I_h \longrightarrow \mathrm{End}(\mathbb{C}[x,h])$ *given by* $\rho(x) = \hat{x}$, $\rho(y) = \partial_h$, $\rho(z) = s_h$ *and* $\rho(h) = \hat{h}$ *defines a representation of the* h-*Weyl algebra.*

Notice that if we let $h \to 0$, z becomes a central element and we recover the Weyl algebra. We order the letters on the h-Weyl algebra as follows $x < y < z < h$. Also, for $a \in \mathbb{N}$ and $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, we set $\binom{a}{k} := \frac{a!}{\prod k_i!(a-|k|)!}$. With this notation, we have the

Theorem 4.4: *Given* $a, b \in \mathbb{N}$, *the following identities hold in* $\mathbb{C}\langle x,y,z\rangle[h]/I_h$

1. $z^a x^b = \displaystyle\sum_{k=0}^{b} \binom{b}{k} a^k x^{b-k} z^a h^k$.

2. $z^a y^b = y^b z^a$.

3. $y^a x^b = \displaystyle\sum_{k \in \mathbb{N}^a} \binom{b}{k} x^{b-|k|} y^{a-s(k)} z^{s(k)} h^{|k|-s(k)}$, where $k \in \mathbb{N}^a$ is such that $0 \le |k| \le b$ and $s(k) = \sharp(\{i : k_i \ne 0\})$.

PROOF: 2. is obvious and 3. is similar to 1. We prove 1. by induction. It is easy to check that $z^a x = x z^a + a z^a h$. Furthermore,

$$
\begin{aligned}
z^a x^{b+1} &= \sum_{k=0}^{b} \binom{b}{k} a^k x^{b-k} (z^a x) h^k \\
&= \sum_{k=0}^{b} \binom{b}{k} a^k x^{b+1-k} z^a h^k + \sum_{k=1}^{b+1} \binom{b}{k-1} a^k x^{b+1-k} z^a h^k \\
&= \sum_{k=0}^{b+1} \binom{b+1}{k} a^k x^{b+1-k} z^a h^k.
\end{aligned}
$$

$\square$

Assume we are given $A = (A_1, \ldots, A_n) \in (\mathbb{N}^3)^n$ and $A_i = (a_i, b_i, c_i)$, for $i \in [[1, n]]$. Set $X^{A_i} = x^{a_i} y^{b_i} z^{c_i}$ for $i \in [[1, n]]$. We set $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, $b = (b_1, \ldots, b_n) \in \mathbb{N}^n$, $c = (c_1, \ldots, c_n) \in \mathbb{N}^n$ and $|A| = (|a|, |b|, |c|) \in \mathbb{N}^3$. Using this notation, we have the

Definition 4.5: The *normal coordinates* $N_h(A, p, q)$ of

$$
\prod_{i=1}^{n} X^{A_i} \in \mathbb{C}\langle x, y, z\rangle[h]/I_h
$$

are given via the identity

$$
\prod_{i=1}^{n} X^{A_i} = \sum_{p,q} N_h(A, p, q) X^{r(A,p,q)} h^{|q|+|p|-s(p)} \tag{4.1}
$$

where the sum runs over all vectors $q = (q_1, \ldots, q_{n-1}) \in \mathbb{N}^{n-1}$ such that $0 \le q_j \le a_{j+1}$ and $p = (p_1, \ldots, p_{n-1})$ with $p_j \in \mathbb{N}^{|b_{\le j}| - |s(p_{<j})|}$.

Also, $r(A, p, q) = (|a| - |p| - |q|, |b| - s(p), |c| + s(p))$, where $s(p) = \sum_{j=1}^{n-1} s(p_j)$. We set $N_h(A, p, q) = 0$ for p, q not satisfying the previous conditions.

The condition for p and q in the definition above might seem unmotivated. They appear naturally in the course of the proof of Theorem 4.6 below, which is proved using induction and Theorem 4.4.

Theorem 4.6: *Let A, p and q be as in the previous definition, we have*

$$N_h(A, p, q) = \prod_{i=1}^{n-1} \binom{a_{i+1}}{q_i} \binom{a_{i+1} - q_i}{p_i} (|c_{\leq i}| + |s(p_{<i})|)^{q_i}.$$

Figure 2 illustrates the combinatorial interpretation of Theorem 4.6. As we try to moves the z's or the y's above the 'x', a subset of the x may get killed. The z's do not die in this process but the y's do turning themselves into z' s.
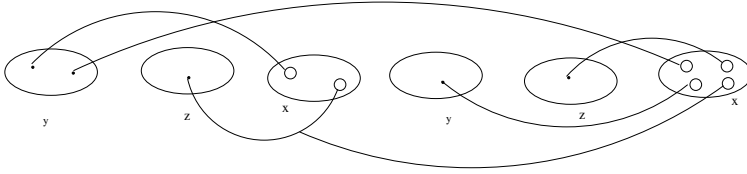

Figure 2: Combinatorial interpretation of N_h.

Our next result provides an explicit formula for the product of m elements in the algebra $\mathrm{Sym}^n(\mathbb{C}\langle x, y, z\rangle[h]/I_h)$. Fix $A : [[1, m]] \times [[1, n]] \longrightarrow \mathbb{N}^3$, with $(A_{ij}) = ((a_{ij}), (b_{ij}), (c_{ij}))$. Recall that given $\sigma \in \mathbb{S}_n^m$ and $j \in [[1, n]]$, A_j^σ denotes the vector $(A_{1\sigma_1^{-1}(j)}, \ldots, A_{m\sigma_m^{-1}(j)}) \in (\mathbb{N}^3)^m$ and set $X_j^{A_{ij}} = x_j^{a_{ij}} y_j^{b_{ij}} z_j^{c_{ij}}$, for $j \in [[1, n]]$. Set $A_j^\sigma = (|a_j^\sigma|, |b_j^\sigma|, |c_j^\sigma|)$, where $|a_j^\sigma| = \sum_{i=1}^{m} a_{i\sigma_i^{-1}(j)}$ and similarly for $|b_j^\sigma|$ and $|c_j^\sigma|$.

Theorem 4.7: *For any $A : [[1, m]] \times [[1, n]] \longrightarrow \mathbb{N}^3$, the identity*

$$(n!)^{m-1} \prod_{i=1}^{m} \overline{\prod_{j=1}^{n} X_j^{A_{ij}}} = \sum_{\sigma, p, q} \left(\prod_{j=1}^{n} N_h(A_j^\sigma, p^j, q^j) \right) \overline{\prod_{j=1}^{n} X_j^{r(A_j^\sigma, p^j, q^j)}} h^{k_j(p,q)}$$

holds in $\mathrm{Sym}^n(\mathbb{C}\langle x, y, z\rangle/I_h)$, *where* $\sigma \in \{\mathrm{id}\} \times \mathbb{S}_n^{m-1}$, p^j, q^j *are such that* (A_j^σ, p^j, q^j) *satisfy the condition of the definition above.* $r(A_j^\sigma, p^j, q^j) = (|a_j^\sigma| - |p^j| - |q^j|, |b_j^\sigma| - s(p^j), |c_j^\sigma| + s(p^j))$ *and* $k_j(p, q) = |q^j| - |p^j| - s(p^j)$.

Theorem 4.7 is proven similarly to Theorem 3.7.

5   Deformation quantization of $(\mathfrak{sl}_2^*)^n/\mathbb{S}_n$

We denote by $\mathfrak{sl}_2$ the Lie algebra of all 2×2 complex matrices of trace zero. $\mathfrak{sl}_2^*$ is the dual vector space. It carries a natural structure of Poisson manifold. We consider a deformation quantization of the Poisson orbifold $(\mathfrak{sl}_2^*)^n/\mathbb{S}_n$. It is proven in [4] that the quantized algebra of the Poisson manifold $\mathfrak{sl}_2^*$ is isomorphic to $U(\mathfrak{sl}_2)$ the universal enveloping algebra of $\mathfrak{sl}_2$, after setting the formal parameter $\hbar$ appearing in [4] to be 1. Thus we regard $(U(\mathfrak{sl}_2)^{\otimes n})^{\mathbb{S}_n} \cong \mathrm{Sym}^n(U(\mathfrak{sl}_2))$ as the quantized algebra associated to the Poisson orbifold $(\mathfrak{sl}_2^*)^n/\mathbb{S}_n$. It is well-known that $U(\mathfrak{sl}_2)$ can be identified with the algebra $\mathbb{C}\langle x, y, z\rangle/I_{\mathfrak{sl}_2}$ where $I_{\mathfrak{sl}_2}$ is the ideal generated by the following relations:

$$zx = xz + y \qquad yx = xy - 2x \qquad zy = yz - 2z$$

The next result can be found in [2],[5].

Proposition 5.1: *The map* $\rho : \mathbb{C}\langle x, y, z\rangle/I_{\mathfrak{sl}_2} \longrightarrow \mathrm{End}(\mathbb{C}[x_1, x_2])$ *given by* $\rho(x) = x_2 \dfrac{\partial}{\partial x_1}$, $\rho(y) = x_1 \dfrac{\partial}{\partial x_1} - x_2 \dfrac{\partial}{\partial x_2}$ *and* $\rho(z) = x_1 \dfrac{\partial}{\partial x_2}$ *defines a representation of the algebra* $\mathbb{C}\langle x, y, z\rangle/I_{\mathfrak{sl}_2}$.

Given $s, n \in \mathbb{N}$ with $0 \leq s \leq n$, the s-th elementary symmetric function $\displaystyle\sum_{1 \leq i_1 < \ldots < i_s \leq n} x_{i_1} \ldots x_{i_s}$ on variables $x_1, \ldots, x_n$ is denoted by $e_s^n(x_1, \ldots, x_n)$. For $b \in \mathbb{N}$, the notation $e_s^n(b) := e_s^n(b, b-1, \ldots, b-n+1)$ we will used. Given $a, n \in \mathbb{N}$ such that $a \leq n$, we set $(a)_n = a(a-1)\ldots(a-n+1)$.

Theorem 5.2: *Given* $a, b \in \mathbb{N}$, *the following identities hold in* $\mathbb{C}\langle x, y, z\rangle/I_{\mathfrak{sl}_2}$

1. $z^a x^b = \displaystyle\sum_{s,k} \frac{(a)_k (b)_k}{k!} e_{k-s}^k(-a - b + 2k) x^{b-k} y^s z^{a-k}$,

    where the sum runs over all $k, s \in \mathbb{N}$ *such that* $0 \leq s \leq k \leq \min(a, b)$.

2. $z^a y^b = \displaystyle\sum_{k=0}^{b} \binom{b}{k} (-2a)^k y^{b-k} z^a$.

3. $y^a x^b = \sum_{k=0}^{a} \binom{a}{k} (-2b)^k x^b y^{a-k}.$

Proof: Formula 1. is proved by induction. It is equivalent to another formula for the normalization of $z^a x^b$ given in [12]. 2. and 3. are similar to Theorem 4.4, part 1. Formula 2. express the fact that as we try to move the z's above the y's some of the y's may get killed. This argument justify the $\binom{b}{k}$ factor. The a^k factor arises from the fact that each 'y' may be killed for any of the z's. The $(-2)^k$ factor follows from the fact that each killing of a 'y' is weighted by a -2. $\qquad\square$

Assume we are given $A = (A_1, \ldots, A_n) \in (\mathbb{N}^3)^n$ and $A_i = (a_i, b_i, c_i)$, for $i \in [[1, n]]$. Set $X^{A_i} = x^{a_i} y^{b_i} z^{c_i}$ for $i \in [[1, n]]$, furthermore set $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, $b = (b_1, \ldots, b_n) \in \mathbb{N}^n$, $c = (c_1, \ldots, c_n) \in \mathbb{N}^n$ and $|A| = (|a|, |b|, |c|) \in \mathbb{N}^3$. Using this notation, we have the

Definition 5.3: The *normal coordinates* $N_{\mathfrak{sl}_2}(A, k, s, p, q)$ of $\prod_{i=1}^{n} X^{A_i}$ in the algebra $\mathbb{C}\langle x, y, z \rangle / I_{\mathfrak{sl}_2}$ are given via the identity

$$\prod_{i=1}^{n} X^{A_i} = \sum_{k,s,p,q} N_{\mathfrak{sl}_2}(A, k, s, p, q) X^{r(A,k,s,p,q)} \tag{5.1}$$

where $k, s, p, q \in \mathbb{N}^{n-1}$ are such that $0 \le s_i \le k_i \le \min(|c_{\le i}| - |k_{<i}|, a_{i+1})$, $0 \le p_i \le b_{i+1}$, $0 \le q_i \le |b_{\le i}| + |s_{<i}| - |p_{<i}| - |q_{<i}|$, for $i \in [[1, n-1]]$. Moreover, $r(A, k, s, p, q) = (|a| - |k|, |b| + |s| - |p| - |q|, |c| - |k|)$. We set $N_{\mathfrak{sl}_2}(A, k, s, p, q) = 0$ for k, s, p, q not satisfying the previous conditions.

Theorem 5.4 provides an explicit formula for the normal coordinates $N_{\mathfrak{sl}_2}(A, k, s, t)$ of $\prod_{i=1}^{n} X^{A_i}$. Its proof goes by induction using Theorem 5.2.

Theorem 5.4: *With the notation of the definition above, we have*

$$N_{\mathfrak{sl}_2}(A, k, s, p, q) = (-2)^{|p|+|q|} \prod_{i=1}^{n-1} \alpha_i \beta_i \gamma_i \binom{b_{i+1}}{p_i} (|c_{\le i}| - |k_{\le i}|)^{p_i} (a_{i+1} - k_i)^{q_i},$$

where $\alpha_i = \dfrac{(|c_{\le i}| - |k_{<i}|)_{k_i} (a_{i+1})_{k_i}}{k_i!}$, $\beta_i = e_{k_i - s_i}^{k_i}(-a_{i+1} - |c_{\le i}| + |k_{<i}| + 2k_i)$, *and* $\gamma_i = \dbinom{|b_{\le i}| + |s_{<i}| - |p_{<i}| - |q_{<i}|}{q_i}$, *for all* $i \in [[1, n-1]]$.

Our final result provides an explicit formula for the product of m elements in the algebra $\mathrm{Sym}^n(\mathbb{C}\langle x, y, z\rangle / I_{\mathfrak{sl}_2})$. Fix $A : [[1, m]] \times [[1, n]] \longrightarrow \mathbb{N}^3$, with $(A_{ij}) = ((a_{ij}), (b_{ij}), (c_{ij}))$. Recall that given $\sigma \in \mathbb{S}_n^m$ and $j \in [[1, n]]$, A_j^σ denotes the vector $(A_{1\sigma_1^{-1}(j)}, \dots, A_{m\sigma_m^{-1}(j)}) \in (\mathbb{N}^3)^m$. Set $X_j^{A_{ij}} = x_j^{a_{ij}} y_j^{b_{ij}} z_j^{c_{ij}}$, for $j \in [[1, n]]$ and $|A_j^\sigma| = (|a_j^\sigma|, |b_j^\sigma|, |c_j^\sigma|)$, where $|a_j^\sigma| = \sum_{i=1}^{m} a_{i\sigma_i^{-1}(j)}$ and similarly for $|b_j^\sigma|$, $|c_j^\sigma|$, and $k, s, p, q \in (\mathbb{N}^{m-1})^n$.

Theorem 5.5: *For any $A : [[1, m]] \times [[1, n]] \longrightarrow \mathbb{N}^3$, the identity*

$$(n!)^{m-1} \prod_{i=1}^{m} \overline{\prod_{j=1}^{n} X_j^{A_{ij}}} = \sum_{\sigma, k, s, p, q} \left(\prod_{j=1}^{n} N_{\mathfrak{sl}_2}(A_j^\sigma, k^j, s^j, p^j, q^j) \right) \overline{\prod_{j=1}^{n} X_j^{r_j(A_j^\sigma, k^j, s^j, p^j, q^j)}}$$

holds in $\mathrm{Sym}^n(\mathbb{C}\langle x, y, z\rangle / I_{\mathfrak{sl}_2})$, where $k = (k^1, \dots, k^n) \in (\mathbb{N}^{m-1})^n$, and similar for s, p, q, $r_j(A_j^\sigma, k^j, s^j, p^j, q^j) = (|a_j^\sigma| - |k^j|, |b_j^\sigma| + |s^j| - |p^j| - |q^j|, |c_j^\sigma| - |k^j|$, and $\sigma \in \{id\} \times \mathbb{S}_n^{m-1}$.

Theorem 5.5 is proven similarly to Theorem 3.7.

Acknowledgment

References

[1] J. Alev, T.J. Hodges, and J.D. Velez. Fixed rings of the Weyl algebra $A_1(\mathbb{C})$. *Journal of algebra*, 130:83–96, 1990.

[2] Christian Kassel. *Quantum groups*. Springer-Velarg, New York, 1995.

[3] Emil Martinec and Gregory Moore. Noncommutative Solitons on Orbifolds. hep-th/0101199, 2001.

[4] M. Kontsevich. Deformation Quantization of Poisson Manifolds I. math. q-alg/97090401, 1997.

[5] Leonid Korogodski and Yan Soibelman. Algebras of functions on Quantum groups. part I. *Mathematical surveys and monographs*, 56, 1996.

[6] Pavel Etingof and Victor Ginzburg. Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. *Invent. Math*, 147(2):243–348, 2002.

[7] Peter Doubilet. On the foundations of combinatorial theory. VII: Symmetric functions through the theory of distribution and occupancy. *Gian-Carlo Rota on Combinatorics. Introductory papers and commentaries*, pages 402–422, 1995.

[8] Rafael Díaz and Eddy Pariguan. Quantum symmetric functions. math.QA/0312494, 2003.

[9] Rafael Díaz and Eddy Pariguan. Super, quantum and non-commutative species. Work in progress, 2004.

[10] Rajesh Gopakumar, Shiraz Minwalla, and Andrew Strominger. Non-commutative Solitons. *J. High Energy Phys. JHEP*, 05-020, 2000.

[11] A.I. Solomon. *Phys.Lett. A 196*, 126(29), 1994.

[12] Victor Kac. *Infinite dimensional Lie algebras.* Cambridge University Press, New York, 1990.

[13] Victor Kac and Pokman Cheung. *Quantum Calculus.* Springer-Velarg, New York, 2002.

Rafael Díaz
Instituto Venezolano de Investigaciones Científicas
Departamento de Matemáticas
Altos de Pipe.
Caracas 21827
Venezuela
radiaz@ivic.ve

Eddy Pariguan
Universidad Central de Venezuela
Departamento de Matemáticas
Los Chaguaramos
Caracas 1020
Venezuela
eddyp@euler.ciens.ucv.ve