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ARENS ALGEBRAS, ASSOCIATED WITH COMMUTATIVE VON NEUMANN ALGEBRAS

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1. Introduction. Let (Ω, Σ, μ) be a measurable space with a finite measure, $L^p(\mu) = L^p(\Omega, \Sigma, \mu)$ the Banach space of all μ-measurable complex functions on Ω, integrable with the degree, $p \in [1, +\infty)$. R. Arens [1] introduced and studied the set $L^\omega(\mu) = \bigcap\limits_{1 \leq p < \infty} L^p(\mu)$. He showed, in particular, that $L^\omega(\mu)$ is a complete locally-convex metrizable algebra with respect to "t" topology generated by the system of norms $\|f\|_p = \left(\int\limits_\Omega |f|^p \, d\mu\right)^{1/p}$, $p \geq 1$. Later G.R. Allan [2] observed that $(L^\omega(\mu), t)$ is a GB^*-algebra with the unit ball $B_0 = \{f \in L^\infty : \|f\|_p \leq 1\}$. Further investigation of properties of the Arens algebra $L^\omega(\mu)$ was made by S.J. Bhaft [3,4]. He described the ideals of the algebrà $L^\omega(\mu)$ and considered some classes of homomorphisim of this algebra. B.S. Zakirov [5] showed that $L^\omega(\mu)$ is an EW^*-algebra and gave an example of two measures, μ and ν, on an atomic Boolean algebra, for which the algebras $L^\omega(\mu)$ and $L^\omega(\nu)$ are not isomorphic. It is clear that the problem of complete classification of the Arens algebras arises. Speaking more preciesly, what conditions should be imposed on measures μ and ν for the corresponding Arens algebras to be isomorphic? It is natural to solve this problem in the class of equivalent measures. Therefore instead of a measurable space with a measure, one should consider a commutative von Neumann algebra M with faithful normal finite traces μ and ν on M and study the problem of $*$-isomorphism of EW^*- algebras $L^\omega(M;\mu) = \bigcap\limits_{1 \leq p < \infty} L^p(M;\mu)$ and $L^\omega(M, \nu)$

The present article gives the complete solution of the mentioned problem, a classification of the normalized Boolean algebras from the book by D.A. Vladimirov [6] being considerably used. All neccessary notations and results from the theory of von Neumann algebras are taken from [7] and the theory of integration on von Neumann algebras is from [8].

2. Preliminaries. Let M be an arbitrary von Neumann algebra, μ a faithful normal finite trace on M, $P(M)$ the lattice of all projections of M. Let $K(M,\mu)$ be the $*$-algebra of all μ-measurable operators affiliated with M [8].

In the commutative case, when $M = L^\infty(\Omega,\Sigma,\mu)$ and $\mu(x) = \int\limits_\Omega x\,d\mu$, where (Ω,Σ,μ) is a measurable space, the algebra $K(M,\mu)$ coincides with the algebra of all measurable complex functions on (Ω,Σ,μ).

For every set $A \subset K(M,\mu)$ we shall denote by A_h (respectively, by A_+) the set of all self-adjoint (respectively, positive self-adjoint) operators from A. The partial order in $K_h(M,\mu)$ generated by the positive cone $K_+(M,\mu)$ will be denoted by $x \leq y$.

Put $M(x) = \sup\{\mu(y) | 0 \leq y \leq x, y \in M\}$ for every $x \in K_+(M,\mu)$. Let $p \in [1,\infty)$ and $L^p(M,\mu) = \{x \in K(M,\mu) | \mu(|x|^p) < \infty\}$, where $|x| = (x^*x)^{1/2}$. The set $L^p(M,\mu)$ is a subspace in $K(M,\mu)$ and the function $\|x\|_p = \mu(|x|^p)^{1/p}$ is a Banach norm on $L^p(M,\mu)$ [9]. Moreover,

1. $\|x\|_p = \|x^*\|_p = \|xu\|_p$ for all $x \in L^p(M,\mu)$ and a unitary element $u \in M$;

2. If $|x| \leq |y|$, $x \in K(M,\mu)$, $y \in L^p(M,\mu)$, then $x \in L^p(M,\mu)$ and $\|x\|_p \leq \|y\|_p$;

3. If $x \in L^p(M,\mu), y \in L^q(M,\mu)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $1 < p,q,r < \infty$, then $xy \in L^r(M,\mu)$ and $\|xy\|_r \leq \|x\|_p \|y\|_q$.

From these properties of the norm $\|\cdot\|_p$ it follows that the set $L^\omega(M,\mu) = \bigcap\limits_{1 \leq p < \infty} L_p(M,\mu)$ is a $*$-subalgebra in $K(M,\mu)$, and $M \subset L^\omega(M,\mu)$. It was shown in [5] that $M = L^\omega(M,\mu)$ if and only if $\dim M < \infty$. Furthermore, since $L^\omega(M,\mu)$ is a solid $*$-subalgebra in $K(M,\mu)$ (e.g. the inequality $|x| \leq |y|, x \in K(M,\mu), y \in L^\omega(M,\mu)$ implies $x \in L^\omega(M,\mu)$), $L^\omega(M,\mu)$ is an EW^*-algebra, the bounded part of which coincides with M [10].

Now we cite from [6] some information which will be used in the seqnel.

Let X be an arbitrary complete Boolean algebra, $e \in X$, $X_e = [0,e] = \{g \in X | \ g \leq e\}$. The minimal cardinality of the set which is dense in X_e in the (o)-topology will be denoted $\tau(X_e)$. An infinite complete Boolean algebra X is called homogeneous, if $\tau(X_e) = \tau(X_g)$ for any non-zero $e, g \in X$. The cardinality of $\tau(X) = \tau(X_{\mathbf{1}})$ where $\mathbf{1}$ – is the unit of the Boolean algebra X is called a weight of a homogeneous Boolean algebra X.

Let μ be a strictly positive countably additive measure on X. If $\mu(\mathbf{1}) = 1$, then the pair (X, μ) is called a normalized Boolean algebra. It was shown in [6] that for any cardinal number τ there existed a complete homogeneous normalized Boolean algebra X with the weight $\tau(X) = \tau$. The next theorem gives a criterion of isomorphism of two homogeneous normalized Boolean algebras.

Theorem ([6]). *Let (X, μ) and (Y, ν) be homogeneous normalized Boolean algebras. The following conditions are equivalent:*

1) $\tau(X) = \tau(Y)$;
2) *There exists an isomorphism $\varphi : X \to Y$ for which $\nu(\varphi(x)) = \mu(x)$ for all $x \in X$.*

This theorem enables us to describe the class of von Neumann algebras for which the existence of $*$-isomorphism between the Arens algebras $L^{\omega}(M, \mu)$ and $L^{\omega}(N, \nu)$ is equivalent to isomorphism between M and N.

Proposition 1. *Let M and N be commutative von Neumann algebras, the Boolean algebras $P(M)$ and $P(N)$ of which are homogeneous, and let μ and ν be faithful normal finite traces on M and N, respectively. The following conditions are equivalent:*

1) *The Arens algebras $L^{\omega}(M, \mu)$ and $L^{\omega}(N, \nu)$ are $*$-isomorphic;*
2) *The von Neumann algebras M and N – are $*$-isomorphic;*
3) $\tau(P(M)) = \tau(P(N))$.

Proof. Sience $L^{\omega}(M, \mu)$ and $L^{\omega}(N, \nu)$ are EW^*-algebras the bounded parts of which coincide with M and N respectively, restriction on M of any $*$-isomorphism from $L^{\omega}(M, \mu)$ on $L^{\omega}(N, \nu)$ is a $*$-isomorphism from M on N. On the other hand if the von Neumann algebras M and N are $*$-isomorphic, then their Boolean algebras of projectors are also isomorphic and therefore, in this case, $\tau(P(M)) = \tau(P(N))$.

Now suppose that $\tau(P(M)) = \tau(P(N))$ and assume $\mu'(x) = \mu(x)/\mu(\mathbf{1})$, $\nu'(y) = \nu(y)/\nu(\mathbf{1})$, $x \in M$, $y \in N$. According to the theorem 1, there exists an isomorphism of Boolean algebras $\varphi : X \to Y$ for which $\nu(\varphi(x)) = \mu'(x)$ for all $x \in X$. This isomorphism extends to a $*$-isomorphism $\Phi : K(M, \mu) \to K(N, \nu)$ (See [11]): At the same time $\mu'(x) = \nu'(\Phi(x))$ for all $x \in L^1(M, \mu')$. Since $\mu'(|x|^p) = \nu'(\Phi(|x|^p)) = \nu'(|\Phi(x)|^p)$ we have $\Phi(L^p(M, \mu)) = \Phi(L^p(M, \mu')) = L^p(N, \nu') = L^p(N, \nu)$ for all $p \geq 1$. Hence $\Phi(L^{\omega}(M, \mu)) = L^{\omega}(N, \nu)$.

Corolary. *Let M and N be non-atomic commutative von Neumann algebras on separable Hilbert spaces, μ and ν faithful normal finite traces on M and N, respectively. Then the Arens algebras $L^\omega(M,\mu)$ and $L^\omega(N,\nu)$ are $*$-isomorphic.*

Proof. At first, show that if M acts on a separable Hilbert space H, then the Banach space $(L^r(M,\mu), \|\cdot\|_r)$ is also separable. To start one should note that in this case the strong topology is metrizable on the unit ball M_1 of the algebra M ([12] p.24). In addition, the convergence $x_\alpha \xrightarrow{so} 0$ in the strong topology in M_1 is equivalent to the convergence $\mu(x_\alpha^* x_\alpha) \to 0$ ([12] p.130).

Thus, for any sequence of $\{x_n\} \subset M$ and $x \in M$ the convergence $x_n \xrightarrow{so} x$ implies $\sup \|x_n\|_M < \infty$ and $\|x_n - x\|_2 \to 0$, where $\|\cdot\|_M$ is a C^*-norm in M. Hence, on any ball $M_n = \{x \in M | \|x\|_M \leq n\}$ the strong topology coincides with the topology induced from $L_2(M,\mu)$. Since H is separable, there exists a countable set $X_n \subset M$ which is dense in M_n in the strong topology ([13], p.568). Hence the countable set $X = \bigcup_{n=1}^{\infty} X_n$ is dense in M in the topology induced from $L_2(M,\mu)$. Since M is dense in $(L_2(M,\mu), \|\cdot\|_2)$, $(L_2(M,\mu), \|\cdot\|_2)$ is separable.

There is one thing left to say: the (o)-topology in $(P(M),\mu)$ coincides with the topology induced from $(L^2(M,\mu), \|\cdot\|_2)$. Therefore, the $P(M)$ is a non-atomic Boolean algebra which is separable in the (o)-topology. Hence it is homogeneous [6]. Similarly, $P(N)$ is a non-atomic Boolean algebra and $\tau(P(M)) = \tau(P(N))$. According to the proposition 1, the Arens algebras $L^\omega(M,\mu)$ and $L^\omega(N,\nu)$ – are $*$-isomorphic.

Let (X,μ) be an arbitrary complete non-atomic normalized Boolean algebra. It was shown in [6] that there is a sequence $\{e_n\}$ of non-zero pairwise disjoint elements for which the Boolean algebras $[0,e_n]$ are homogeneous and $\tau_n = \tau([0,e_n]) < \tau_{n+1}$, $n = 1,2,\ldots$ This collection is determined uniquely and the matrix

$$\begin{pmatrix} \tau_1 & \tau_2 & \cdots \\ \mu(e_1) & \mu(e_2) & \cdots \end{pmatrix}$$

is called the passport of the Boolean algebra (X,μ)

The following theorem will be used for investigation of isomorphisms of Arens algebra.

Theorem 2 [6]. *Let (X, μ) and (Y, ν) be complete non-atomic normalized Boolean algebras. The following conditions are equivalent.*

1. *There exists an isomorphism $\varphi : X \to Y$ for which $\nu(\varphi(x)) = \mu(x)$ for all $x \in X$.*
2. *The passports of the Boolean algebras (X, μ) and (Y, ν) coincide.*

3. Main results. A von Neuman algebra M is called σ-finite if it admits at most countable family of orthogonal projections. On any σ-finite von Neumann algebra M, there exists a normal state, in particular, if M is commutative, then its Boolean algebra of projections $P(M)$ is a normed one. The next theorem discribes the class of commutative σ-finite von Neumann algebras M for which the Arens algebras $L^\omega(M, \mu)$ and $L^\omega(M, \nu)$ are $*$-isomorphic for any faithful normal finite traces of μ and ν on M.

Theorem 3. *For a commutative σ-finite von Neumann algebra M the following conditions are equivalent:*

1. *The Arens algebras $L^\omega(M, \mu)$ and $L^\omega(M, \nu)$ are $*$-isomorphic for any faithful normal finite traces μ and ν on M.*
2. *$M = M_0 + \sum_{i=1}^{n} M_i$, where M_0 is a finite-dimentional commutative von Neumann algebra, M_i is an infinite-dimensional commutative von Neumann algebra in which the lattice of projections $P(M_i)$ is a homogeneous Boolean algebra and $\tau_i = \tau(P(M_i) < \tau_{i+1}$, $i = 1, \dots, n-1$ (the summand M_0 are $\sum_{i=1}^{n} M_i$ may be absent).*

Proof. 1) $\to$ 2). Let Δ be the set of all atoms in $P(M)$ and $e = \sup \Delta$. Suppose that Δ is a countable set. Then $M_0 = eM$ coincides with the algebra ℓ_∞ of all bounded sequences of complex numbers. Denote the atoms in $P(\ell_\infty)$ by $q_u = (0, \dots, 0, 1, 0, \dots)$. Consider two faithful normal finite traces μ and ν on M, for which $\mu(q_n) = n^{-2}$, $\nu(q_n) = e^{-2n}$ and $\mu(x) = \nu(x)$ for all $x \in (\mathbf{1} - e)M$. Suppose, that a $*$-isomorphism Φ from $L^\omega(M, \nu)$ on $L^\omega(M, \mu)$ exists. Since $\Phi(M_0) = M_0$, we have $\Phi(L^\omega(M_0, \nu) = L^\omega(M_0, \mu)$. Choose $x \in K(M_0, \nu)$ such that $xq_n = 2^n$. The series

$$\sum_{n=1}^{\infty} \frac{2^{pn}}{e^{2^n}} = \nu(|x|^p)$$

converges for all $p \geq 1$. Therefore $x \in L^\omega(M_0, \nu)$ and, so $\Phi(x) \in L^\omega(M_0, \nu)$. Since $M_0 = l_\infty$, the $*$-isomorphism Φ is generated by some bijection π of the set of natural numbers. It means that $\Phi(x) = \Phi(\{2^n\}) = \{2^{\pi(n)}\} = y \in L^\omega(M_0, \mu)$. In particular,

$$\nu(|y|) = \sum_{n=1}^{\infty} 2^{\pi(n)} n^{-2} < \infty$$

which is wrong. Hence, a set Δ is either finite or empty.

Now suppose that in the Boolean algebra $P((\mathbf{1}-e)M)$ there is a countable set $\{e_n\}$ of disjoint elements, for which the algebras $X_n = P(e_n M)$ are homogeneous and $\tau_n = \tau(X_n) < \tau_{n+1}$. Choose two faithful normal finite traces μ and ν on M such that $\mu(e_n) = n^{-2}$, $\nu(e_n) = e^{-2^n}$ and $\mu(x) = \nu(x)$ for all $x \in M_0$. Let Φ be a $*$-isomorphisms from $L^\omega(M, \nu)$ on $L^\omega(M, \mu)$. Then $\Phi((\mathbf{1} - e)M) = (\mathbf{1} - e)M$ and, since weights τ_n are different, $\Phi(e_n M) = e_n(M)$ (See [6]). Choose $x \in K((\mathbf{1}-e)M, \nu)$ such that $xe_n = 2^n e_n$. Then $x \in L^\omega((\mathbf{1} - e)M, \nu)$, $\Phi(x) = x$ and

$$\mu(|\Phi(x)|) = \sum_{n=1}^{\infty} 2^n n^{-2} = \infty,$$

i.e. $\Phi(x)$ does not belong to $L^\omega(M, \nu)$.

The obtained contradiction implies that the set $\{e_n\}$ is at most countable.

2) $\to$ 1). Let $M = M_0 + \sum_{i=1}^{n} M_i$, where M_0 is finite-dimensional and M_i is infinite dimensional commutative von Neumann algebra, the Boolean algebra $P(M_i)$ being homogeneous, $\tau_i < \tau_{i+1}$, $i = 1, \dots, n-1$.

Take arbitrary faitful normal traces μ and ν on M. As $\dim M_0 < \infty$, $L^\omega(M_0, \mu) = M_0 = L^\omega(M_0, \nu)$. According to the proposition 1 a $*$-isomorphism Φ_i from $L^\omega(M, \mu)$ on $L^\omega(M_i, \nu)$ exists. Each element x from $L^\omega(M, \mu)$ is represented as $x = x_0 + \sum_{i=1}^{n} x_i$, where $x_0 \in M_0 = L^\omega(M_0, \mu)$, $x_i \in L^\omega(M_i, \mu)$, $i = 1, \dots, n$. It is obvious that $\Phi(x) = x_0 + \sum_{i=1}^{n} \Phi_i(x_i)$ is a $*$-isomorphism from $L^\omega(M, \mu)$ on $L^\omega(M, \nu)$. The theorem is proved.

Using theorem 3, it is easy to construct an example of a non-atomic commutative von Neumann algebra M with traces μ and ν, such that the Arens algebras $L^\omega(M, \mu)$ and $L^\omega(M, \nu)$ are isomorphic, while there is no $*$-isomorhism φ from M on M, for which $\nu \circ \varphi = \mu$. Indeed, assume that

$M = M_1 + M_2$, where M_1, M_2 are non-atomic commutative σ-finite von Neumann algebras in which the lattice of projections form homogeneous Boolean algebras and $\tau(P(M_1)) < \tau(P(M_2))$. Identify M_1 with the subalgebra $e_1 M_1$ and M_2 with $(\mathbf{I} - e_1)M_1$, $e_1 \in P(M)$. Let μ be an arbitrary faithful normal finite trace on M, $\mu(\mathbf{I}) = 1$. Assume that

$$\nu(x) = p(\mu(e_1)^{-1}\mu(xe_1) + q(\mu(\mathbf{I} - e_1))^{-1}\mu(x(\mathbf{I} - e_1)),$$

$x \in M$, $p, q > 0$, $p + q = 1$. It is evident that ν is a faithful normal finite trace on M. Choose p and q such that $\mu(e_1) \neq \nu(e_1) = p$, $\mu(\mathbf{I} - e_1) \neq \nu(\mathbf{I} - e_1) = q$. According to the theorem 2, there is no $*$-isomorphism $\varphi : M \to M$ for which $\nu \circ \varphi = \mu$. At the same time, according to the theorem 3, the Arens algebras $L^\omega(M, \mu)$ and $L^\omega(M, \nu)$ are $*$-isomorphic.

Now, let us find out when the Arens algebras coincide for different traces. Let μ and ν be two faithful normal finite traces on a commutative von Neumann algebra M. Denote by $h = \frac{d\mu}{d\nu}$ the Radon-Nikodim derivate of the trace μ relative ν, i.e. h is the element from $L^1_+(M, \nu)$ for which $\mu(x) = \nu(hx)$ for all $x \in M$.

It is clear that the element x from $K(M, \mu)$ belongs to $L^1(M, \mu)$ if and only if $hx \in L^1(M, \nu)$. In this case the equality $\mu(x) = \nu(hx)$ holds.

Proposition 2. *$L^\omega(M, \nu) \subset L^\omega(M, \mu)$ if only if*

$$h \in \bigcup_{1<p\leq\infty} L^p(M, \nu),$$

where $L^\infty(M, \nu)$ is identified with M.

Proof. Let $L^\omega(M, \nu) \subset L^\omega(M, \mu) \subset L^1(M, \mu)$. Then $\mu(x) = \nu(hx)$ for all $x \in L^\omega(M, \nu)$, and μ is a positive linear functional on $L^\omega(M, \nu)$. Since $L^\omega(M, \nu)$ is a complete metrizable locally-convex algebra with respect to the t-topology generated by the system of norms $\left\{\|x\|_p = \left(\nu(|x|^p)\right)^{1/p}\right\}_{p\geq 1}$ (see[3]) and involution in $L^\omega(M, \nu)$ is continuous in this topology, μ is continuous [14]. It was shown in [3] that the dual space of $(L^\omega(M, \nu)t)$ may be identified with $\bigcup_{1<p\leq\infty} L^p(M, \nu)$. Hence one can find such $y \in L^p(M, \nu)$ for some $p \in (1, \infty]$ that $\nu(hx) = \mu(x) = \nu(yx)$ for all $x \in L^\omega(M, \nu)$. It means that $h = y$ and $h \in \bigcup_{1<p\leq\infty} L^p(M, \nu)$.

Conversely, if $h \in L^p(M,\nu)$ for some $p \in (1,\infty]$, then $\nu(hx)$ is a t-continuous linear functional on $L^\omega(M,\nu)$(See[3]) and therefore $\mu(|x|^q) = \nu(h|x|^q) < \infty$ for any $x \in L^p(M,\nu)$ and $q \geq 1$; we recall that $|x|^q \in L^\omega(M,\nu)$ for all $x \in L^\omega(M,\nu)$ and $q \geq 1$. Thus,

$$L^\omega(M,\nu) \subset \bigcap_{q\geq 1} L^q(M,\mu) = L^\omega(M,\mu)$$

The following criterion of coincidence of the algebras $L^\omega(M,\mu)$ and $L^\omega(M,\nu)$ arises from the proposition 2.

Theorem 4. *Let μ,ν be faithful normal finite traces on a commutative von Neumann algebra M. Then $L^\omega(M,\mu) = L^\omega(M,\nu)$ if only if*

$$\frac{d\mu}{d\nu} \in \bigcup_{1<p\leq\infty} L^p(M,\nu) \text{ and } \frac{d\nu}{d\mu} \in \bigcup_{1<p\leq\infty} L^p(M,\mu).$$

Remarks.

1. In the example constructed after theorem 3 $L^\omega(M,\mu) = L^\omega(M,\nu)$ since

$$\frac{d\mu}{d\nu} = \mu(e_1)p^{-1}e_1 + \mu(1-e_1)q^{-1}(1-e_1)).$$

Now everything is ready to obtain the criterion of $*$-isomorphism of the Arens algebras $L^\omega(M,\mu)$ and $L^\omega(M,\nu)$. Let M be an arbitrary non-atomic commutative σ-finite von Neumann algebra. According to [6] the Boolean algebra $P(M)$ of projections M possesses uniquely determined collection $\{e_r\}$ of non-zero pairwise disjoint elements for which the Boolean algebras $X_n = \{e \in P(M) : e \leq e_n\}$ are homogeneous and $\tau(X_n) < \tau(X_{n+1})$. Assume that the collection $\{e_n\}$ is infinite otherwise all Arens algebras $L^\omega(M,\mu)$ are $*$-isomorphic (see theorem 3).

Theorem 5. *Let μ and ν be faithful normal finite traces on a non-atomic commutative σ-finite von Neumann algebra M. The following conditions are equivalent:*

1) The Arens algebras $L^\omega(M,\mu)$ and $L^\omega(M,\nu)$ are $*$-isomorphic;

2) There are such $p,q \in (1,\infty]$ that

$$\sum_{n=1}^{\infty} \mu_n^p \nu_n^{1-p} < \infty, \quad \sum_{n=1}^{\infty} \nu_n^q \mu_n^{1-q} < \infty$$

in the case $p \neq \infty$, $q \neq \infty$, *and* $\sup_{n\geq 1} |\mu_n \nu_n^{-1}| < \infty$ *if* $p = \infty$, $\sup_{n\geq 1} |\nu_n \mu_n^{-1}|$ $< \infty$ *if* $q = \infty$.

Proof. 1) $\to$ 2). Let Φ be a $*$-isomorhism from $L^{\omega}(M,\mu)$ on $L^{\omega}(M,\nu)$. Since all $\tau(x_n)$ are different, $\Phi(e_n\mu) = e_n\mu$.

Denote by N the atomic von Neumann subalgebra of all elements x from M, for which $xe_n = \lambda_n$ for some complex numbers λ_n, $n = 1, \ldots$. It is evident that N is identified with the algebra l_∞ of all bounded sequences of complex numbers. Since $\Phi(e_n) = e_n$, $n = 1, 2, \ldots$,it follows that $\Phi(z) = z$ for all $z \in N$. If $z \in L^{\omega}(N,\mu) \bigcap K(N,\mu) = L^{\omega}(N,\mu)$, $z \geq 0$, then $z = \sup_{m\geq 1} z \sum_{n=1}^{m} e_n$, and $\left(z \sum_{n=1}^{m} e_n\right) \in N_+$. Therefore,

$$\Phi(z) = \sup_{m\geq 1} \Phi(z \sum_{n=1}^{m} e_n) = \sup_{m\geq 1} z \sum_{n=1}^{m} e_n = z.$$

Thus the restriction of Φ on $L^{\omega}(N,\mu)$ coincides with the identity mapping. It means that $L^{\omega}(N,\nu) = \Phi(L^{\omega}(N,\mu)) = L^{\omega}(N,\mu)$.

Therefore, according to the theorem 4 $h \in \bigcup_{1<p\leq\infty} L^p(N,\nu)$, and $h^{-1} \in \bigcup_{1<p\leq\infty} L^p(N,\mu)$, where h is the Radon-Nikodym's derivative of the trace μ relative the trace ν, being considered in N. So using the equality $he_n = \mu_n \nu_n^{-1} e_n$, $n = 1, 2, \ldots$, the required inequalities follow from the condition 2).

2) $\to$ 1). Let the inequalities from the condition 2) hold. Consider the faithful normal finite trace on M given by the equality

$$\lambda(x) = \sum_{n=1}^{\infty} \nu_n \mu_n^{-1} \mu(e_n x), \ x \in M.$$

Since x_n is a homogeneous Boolean algebra and $\lambda(e_n) = \nu_n = \nu(e_n)$, using the proof of proposition 1, construct a $*$-isomorphism $\Phi_n : K(e_nM,\nu) \to K(e_nM,\lambda)$ for which $\nu(y) = \lambda(\Phi_n(y))$ for all $y \in L^1(e_nM,\nu)$. For each $x \in K(M,\nu)$ denote by $\psi(\lambda)$ such an element from $K(M,\lambda)$ for which $e_n\psi(x) = \Phi_n(e_nx)$. It is evident that ψ is a $*$-isomorphism from $K(M,\nu)$ on $K(M,\lambda)$. At the same time, if $x \in L^1_+(M,\nu)$, then

$$\nu(x) = \sum_{n=1}^{\infty} \nu(e_n x) = \sum_{n=1}^{\infty} \lambda(\Phi_n(e_n x)) =$$

$$\sum_{n=1}^{\infty} \lambda(e_n \psi(x)) = \lambda(\psi(x)),$$

therefore $\psi(L^\omega(M,\nu)) = L^\omega(M,\lambda)$.

Let is show that $L^\omega(M,\lambda) = L^\omega(M,\mu)$. Let h be such an element from $K(M,\mu)$ that $he_n = \mu_n \nu_n^{-1} e_n$. For every $x \in M$ we have

$$\lambda(hx) = \sum_{n=1}^{\infty} \lambda(he_n x) = \sum_{n=1}^{\infty} \mu_n \nu_n^{-1} \lambda(e_n x) =$$
$$= \sum_{n=1}^{\infty} \mu(e_n x) = \mu(x),$$

therefore $h = \frac{d\mu}{d\lambda}$. According to the inequalities from the condition 2, we obtain that

$$h^{-1} \in \bigcup_{1<p\leq\infty} L^p(M,\mu).$$

If $\sup\limits_{n\geq 1}(\mu_n \nu_n^{-1}) < \infty$, then $h \in M$.

Suppose that $\sum_{n=1}^{\infty} \mu_n^p \nu_n^{1-p} < \infty$ for some $p \in (1,\infty)$. Then

$$\lambda(h^p) = \sum_{n=1}^{\infty} \nu_n \mu_n^{-1} \mu(e_n h^p) = \sum_{n=1}^{\infty} \mu_n^p \nu_n^{1-p} < \infty.$$

Thus,

$$h \in \bigcup_{1<p\leq\infty} L^p(M,\lambda)$$

and, using the theorem 4, we get $L^\omega(M,\lambda) = L^\omega(M,\mu)$.

Therefore $\psi(L^\omega(M,\nu)) = L^\omega(M,\mu)$.

Remarks 2. Repeating the argument from the proof of the theorem 5, it is easy to obtain the following criterion of $*$-isomorphism of the Arens algebras $L^\omega(l_\infty,\mu)$ and $L^\omega(l_\infty,\nu)$:

Let μ and ν be faithful normal finite traces on a infinite dimensional atomic commutative von Neumann algebra N, $\{q_n\}_{n=1}^{\infty}$ – the set of all atoms in $P(N)$, $\mu_n = \mu(q_n)$, $\nu_n = \nu(q_n)$, $n = 1, 2, \ldots$. Then, the Arens algebras

$L^\omega(N,\mu)$ and $L^\omega(N,\nu)$ are $*$-isomorphic only in the case when there are such $p,\ q \in (1,\infty)$ and permutation π of a set of natural numbers, that

$$\sum_{n=1}^{\infty} \mu_n^p \nu_{\pi(n)}^{1-p} < \infty, \quad \sum_{n=1}^{\infty} \nu_{\pi(n)}^q \mu_n^{1-q} < \infty, \quad \text{in the case } \ p,q \in (1,\infty)$$

and $\sup_{n\geq 1} |\mu_n \nu_n^{-1}| < \infty$ if $p=\infty$, $\sup_{n\geq 1} |\nu_n \mu_n^{-1}| < \infty$ if $q=\infty$.

3. Any von Neumann algebra M is represented as $M = M_1 + M_2$, where M is an atomic von Neumann algebra and M_2 is a non-atomic von Neumann algebra. Moreover, if Φ is a $*$-automorphism of M, then $\Phi(M_1) = M_1$ and $\Phi(M_2) = M_2$. Therefore theorem 5 and Remark 2 give criterion of isomorphism of Arens algebras for arbitrary commutative σ-finite von Neumann algebras

References

1. Arens R. The space L^ω and convex topological rings. Bull. Amer. Math. Soc. 1946. v.52 p.931-935.
2. Allan G.R. On a class of locally convex algebras. Proc. London Math. Soc. 1967. v.17, n.3, p.91-114.
3. Bhatt S.J. On Arens algebras L^ω Glas. Mat. 1980, v.15, n.35, p.305-312.
4. Bhatt S.J. On Arens algebras L^ω II. Glas. Mat. 1981. v.16, n.36, p.297-306.
5. Zakirov B.S. Non-commutative Arens algebras (Russian). Uzbekski Matematicheski Jurnal. 1997. N 1.
6. Vladimirov D.A. The Boolean algebras (Russian). "Nauka". Moskow. 1969.
7. Takesaki M. Theory of operator algebras I. New York. Springer. 1979. 415 p.
8. Fack T., Kosaki H. Generalized s-numbers of τ-measureables operator Pacific J. Math. 1986, v.123, p.269-300.
9. Inone A. On a class of unbounded operator algebras II. Pacific J.Math. 1976, v.66, n.2, p.411-431.
10. Chilin V.I., Zakirov B.S. The abstract characterization of EW^*-algebras (Russian). Funct. analys i ego prilozh. 1991, V.25, p.76-78.

11. Sarymsakov T.A., Ajupov Sh.A., Hadzhiev Dzh., Chilin V.I. The ordered algebras (Russian). Tashkent, "Fan". 1984
12. Stratila S., Zsido L. Lectures on von Neumann algebras. Abacus Press 1979, 480 p.
13. Naimark M.A. The normed rings (Russian). Moskow, "Nauka". 1968.
14. Schaeffer H.H. Topological vector spaces. Mc. Millan. London. 1966

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