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A NOTE ON BOUNDEDNESS PROPERTIES OF WRIGHT'S GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper we obtain some inequalities giving the boundedness properties for the Wright's generalized hypergeometric function which belong to the classes P(A,B) and R(A,B). The results besides yielding the inequalities obtained recently in [3] and [7], would also be applicable to special functions like, the Bessel- Maitland functions and Mittag-Leffler functions.

1. Introduction and Definitions

Let S denote the class of functions which are analytic in the unit disk $U = \{z : |z| < 1\}$.

The Wright's generalized hypergeometric function is defined by ([6,p. 50])

$$ {}_p\Psi_q \left[\begin{matrix} (\lambda_i, A_i)_{1,p} \; ; \\ (\mu_i, B_i)_{1,q} \; ; \end{matrix} \; z \right] = {}_p\Psi_q \left[\begin{matrix} (\lambda_1, A_1), \ldots, (\lambda_p, A_p) \; ; \\ (\mu_1, B_1), \ldots, (\mu_q, B_q) \; ; \end{matrix} \; z \right] $$

$$ = \sum_{n=0}^{\infty} \left(\prod_{i=1}^{p} \Gamma(\lambda_i + A_i n) \right) \left(\prod_{i=1}^{q} \Gamma(\mu_i + B_i n) \right)^{-1} \frac{z^n}{n!}, \qquad (1.1) $$

where $\lambda_i \in C$ ($i = 1,\ldots,p$) , $\mu_i \in C$ ($i=1,\ldots,q$), and the coefficients $A_i \in R_+$ ($i= 1,\ldots,p$) and $B_i \in R_+$ ($i=1,\ldots,q$) such that

$$1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \geq O \; (p,q \in N_0 = NU \; \{0\}). \tag{1.2}$$

For the subclass of functions of S normalized by $f(0) = f'(0) - 1 = 0$, a commonly studied integral operator considered by Bernardi ([1] and [2]) is

$$p_\gamma f(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z t^{\gamma-1} f(t)\, dt, \qquad (\gamma > 0\,). \tag{1.3}$$

A generalization of (1.3) was introduced in [4] and is defined by

$$\phi_\gamma^\alpha \; f(z) = \binom{\alpha + \gamma}{\gamma} \frac{\alpha}{z^\gamma} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha - 1} t^{\gamma - 1} f(t)\, dt, \quad (\alpha > 0, \gamma > 0) \tag{1.4}$$

and obviousely $\phi_\gamma^1 \; f(z) = \phi_\gamma \; f(z)$.

The symbol $\prec$ is the usual symbol of subordination. That is for

$f(z), g(z) \in S$, $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$

($w(0) = 0$, $| \, w(z) \, | < 1$ in U) such that $f(z) = g(w(z))$.

We denote by P(A,B) the set of functions

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n \, z^n \in S \;\; \text{if } h(z) \prec \frac{1 + Az}{1 + Bz},$$

where A and B are real numbers such that $-1 \leq B < A \leq 1$; $-1 \leq B \leq 0$.

Also, R(A,B) denotes the set of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n \, z^n \in S \text{ if } f'(z) \prec \frac{1 + Az}{1 + Bz}, \text{ for } z \in U.$$

If $A = 1 - 2\,\alpha$, $B = -1$, then the subclass of functions of S are denoted by $P^*(\alpha)$.

Our purpose in this paper is to obtain some inequalities for the function defined by (1.1) which belong to the classes P(A,B) and R(A,B) . The boundedness properties for the generalized hypergeometric function ${}_pF_q(z)$, as well as similar properties for the Bessel-Maitland and Mittag- Leffler functions follow as worthwhile consequences of our main inequalities.

2. Main Result

Let

$$\Delta = \left(\prod_{j=1}^{q} \Gamma(\mu_j)\right)\left(\prod_{j=1}^{p} \Gamma(\lambda_j)\right)^{-1} \tag{2.1}$$

The following result gives the inequalities for the function defined by (1.1):

Theorem 1. *Let* $\Delta \neq 0$, *and*

$$\Delta \; {}_p\Psi_q \left[\begin{matrix}(\lambda_i,\ A_i)_{1,p}\ ; \\ (\mu_i,\ B_i)_{1,q}\ ;\end{matrix}\ z\right] \in \mathrm{P}(\mathrm{A},\mathrm{B}), \text{ then}$$

(i)

$$\left|\frac{\prod_{i=1}^{p} \Gamma(\lambda_i + A_i n)}{\prod_{i=1}^{q} \Gamma(\mu_i + B_i n)}\right| \leq \frac{(A-B)\ n!}{|\Delta|}, \tag{2.2}$$

(ii)

$$\left|\ {}_p\Psi_q \left[\begin{matrix}(\lambda_i,\ A_i)_{1,p}\ ; \\ (\mu_i,\ B_i)_{1,q}\ ;\end{matrix}\ z\right]\right| \leq \frac{1}{|\Delta|}\left(\frac{1 + A\ |z|}{1 + B\ |z|}\right), \tag{2.3}$$

(iii)
$$Re\left\{ {}_p\Psi_q \left[\begin{matrix}(\lambda_i,\ A_i)_{1,p}\ ; \\ (\mu_i,\ B_i)_{1,q}\ ;\end{matrix}\ z\right]\right\} \geq \frac{1}{\Delta}\ \left(\frac{1 - A\ |z|}{1 - B\ |z|}\right), \tag{2.4}$$

(iv)
$$\left| {}_{p+1}\Psi_{q+1} \left[\begin{matrix}(\lambda_i,\ A_i)_{1,p}\ ,(1,1)\ ; \\ (\mu_i,\ B_i)_{1,q},\ (2,1)\ ;\end{matrix}\ z\right]\right|$$

$$\leq \begin{cases} \dfrac{A}{B|\Delta|}\left[1+\left(\dfrac{1}{A}-\dfrac{1}{B}\right)\dfrac{\log\ (1+B|z|)}{|z|}\right], & B\neq 0 \\ \dfrac{1}{|\Delta|}\left[1 + \dfrac{A\ |z|}{2}\right], & B = 0\ . \end{cases} \tag{2.5}$$

(v)
$$\left| {}_{p+1}\Psi_{q+1} \left[\begin{matrix}(\lambda_i,\ A_i)_{1,p}\ ,(1,1)\ ; \\ (\mu_i,\ B_i)_{1,q},\ (2,1)\ ;\end{matrix}\ z\right]\right|$$

$$\geq \begin{cases} \dfrac{A}{B|\Delta|}\left[1-\left(\dfrac{1}{A}-\dfrac{1}{B}\right)\dfrac{\log\ (1-B|z|)}{|z|}\right], & B\neq 0 \\ \dfrac{1}{|\Delta|}\left[1 - \dfrac{A\ |z|}{2}\right], & B = 0\ . \end{cases} \tag{2.6}$$

Proof. From [8], we know that if

$$G(z) = z + \sum_{n=2}^{\infty} a_n\ z^n \in R\ (A,\ B),\ \text{then}$$

$$|\ a_n\ |\ \leq\ \frac{A - B}{n}\ , \tag{2.7}$$

$$|\ G'(z)\ |\ \leq\ \frac{1 + A|z|}{1 + B|z|},\quad |\ z\ |\ <\ 1, \tag{2.8}$$

$$\text{Re}\ \{G'(z)\}\ \geq\ \frac{1 - A|z|}{1 - B|z|},\quad |\ z\ |\ <\ 1, \tag{2.9}$$

and $\int_0^{|z|} \frac{1 - At}{1 - Bt}\, dt \le |G(z)| \le \int_0^{|z|} \frac{1 + At}{1 + Bt}\, dt, \ |z| < 1 .$ (2.10)

Now for $\Delta_p\Psi_q \left[\begin{matrix} (\lambda_i, A_i)_{1,p} ; \\ (\mu_i, B_i)_{1,q} ; \end{matrix} z\right] \in$ P (A,B), let

$$G(z) = \Delta \int_0^z {}_p\Psi_q \left[\begin{matrix} (\lambda_i, A_i)_{1,p} ; \\ (\mu_i, B_i)_{1,q} ; \end{matrix} t\right] dt .$$

Then

$$G(z) = \Delta \sum_{n=0}^{z} \left(\prod_{i=1}^{p} \Gamma(\lambda_i + A_i n)\right)\left(\prod_{i=1}^{q} \Gamma(\mu_i + B_i n)\right)^{-1} \frac{z^{n+1}}{(n+1)!}$$

$$= z + \Delta \sum_{n=1}^{\infty} \left(\prod_{i=1}^{p} \Gamma(\lambda_i + A_i n)\right)\left(\prod_{i=1}^{q} \Gamma(\mu_i + B_i n)\right)^{-1} \frac{z^{n+1}}{(n+1)!} , \tag{2.11}$$

which belongs to the class R [A,B].

On comparing it with (2.7), we at once get the inequality (2.2).

Next from (2.11), we have

$$G'(z) = \Delta \sum_{n=0}^{\infty} \left(\prod_{i=1}^{p} \Gamma(\lambda_i + A_i n)\right)\left(\prod_{i=1}^{q} \Gamma(\mu_i + B_i n)\right)^{-1} \frac{z^{n}}{n!}$$

$$= \Delta\, {}_p\Psi_q \left[\begin{matrix} (\lambda_i, A_i)_{1,p} ; \\ (\mu_i, B_i)_{1,q} ; \end{matrix} z\right] . \tag{2.12}$$

Putting the value of G′(z) from (2.12) in (2.8) and (2.9), we get the inequalities

(2.3) and (2.4), respectively.

To establish (2.5) and (2.6), we express G(z) from (2.11) as

$$G(z) = z\,\Delta \sum_{n=0}^{\infty} \left(\prod_{i=1}^{p} \Gamma(\lambda_i + A_i n)\right)\left(\prod_{i=1}^{q} \Gamma(\mu_i + B_i n)\right)^{-1} \frac{\Gamma(1+n)}{\Gamma(2+n)} \frac{z^n}{n!}$$

$$= z\,\Delta\ {}_{p+1}\Psi_{q+1}\left[\begin{matrix}(\lambda_i, A_i)_{1,p}, (1,1)\ ; \\ (\mu_i, B_i)_{1,q}, (2,1)\ ;\end{matrix}\ z\right]. \tag{2.13}$$

Upon substituting the value of G(z) from (2.13) in (2.10) we get the desired inequalities (2.5) and (2.6), respectively.

3. Further Inequalties

Theorem 2. *Let* $\Delta \neq 0$, *and*

$$\Delta\ {}_p\Psi_q\left[\begin{matrix}(\lambda_i, A_i)_{1,p}\ ; \\ (\mu_i, B_i)_{1,q}\ ;\end{matrix}\ z\right] \in P(A,B) \text{ and } |z| \leq |B|, \text{ then}$$

$$\left| {}_p\Psi_q\left[\begin{matrix}(\lambda_i + A_i, A_i)_{1,p}\ ; \\ (\mu_i + B_i, B_i)_{1,q}\ ;\end{matrix}\ z\right] \right| \leq \frac{A - B}{|\Delta|\ (1-|B|\ |z|^2)}, \tag{3.1}$$

where Δ *is given by* (2.1).

Proof. Since

$$\Delta\ {}_p\Psi_q\left[\begin{matrix}(\lambda_i, A_i)_{1,p}\ ; \\ (\mu_i, B_i)_{1,q}\ ;\end{matrix}\ z\right]$$

$$= 1+\Delta \sum_{n=1}^{\infty} \left(\prod_{i=1}^{p} \Gamma(\lambda_i + A_i n)\right)\left(\prod_{i=1}^{q} \Gamma(\mu_i + B_i n)\right)^{-1} \frac{z^n}{n!},$$

it follows then that

$$\Delta \frac{d}{dz}\, {}_p\Psi_q \left[\begin{matrix}(\lambda_i, A_i)_{1,p} ; \\ (\mu_i, B_i)_{1,q} ;\end{matrix}\; z\right] = \Delta\, {}_p\Psi_q \left[\begin{matrix}(\lambda_i + A_i, A_i)_{1,p} ; \\ (\mu_i + B_i, B_i)_{1,q} ;\end{matrix}\; z\right]. \tag{3.2}$$

We recall the following result ([8]):

If $h(z) = 1 + \sum_{k=1}^{\infty} C_k z^k \in P(A,B)$, $|z| < 1$, then

$$|h'(z)| \leq \begin{cases} \dfrac{A - B}{(1-|B|\,|z|)^2} & |z| \leq |B|, \\[2ex] \dfrac{A-B}{(1 - |B|^2)(1 - |z|^2)} & |z| > |B|. \end{cases} \tag{3.3}$$

If we set

$$h(z) = \Delta\, {}_p\Psi_q \left[\begin{matrix}(\lambda_i, A_i)_{1,p} ; \\ (\mu_i, B_i)_{1,q} ;\end{matrix}\; z\right] \text{ in (3.3)},$$

and use (3.2) in the process, we are lead to the result (3.1).

Theorem 3. *Let* $\Delta \neq o$, and

$$\Delta\, z\, {}_p\Psi_q \left[\begin{matrix}(\lambda_i, A_i)_{1,p} ; \\ (\mu_i, B_i)_{1,q} ;\end{matrix}\; z\right] \in R\,(A,B),$$

then for $\operatorname{Re}(\gamma) > 0$ *and* $|z| < 1$:

$$\left| {}_{p+2}\Psi_{q+2} \left[\begin{matrix}(\lambda_i, A_i)_{1,p}, (2,1), (\gamma+1,1) ; \\ (\mu_i, B_i)_{1,q}, (1,1), (\gamma+\alpha+1,1) ;\end{matrix}\; z\right]\right|$$

$$\leq \left[\Delta\ \Gamma\ (\alpha + 1) \binom{\alpha + \gamma}{\gamma}\right]^{-1} \left(\frac{1 + A\ |z|}{1 + B\ |z|}\right), \tag{3.4}$$

and

$$Re \left\{ {}_{p+2}\Psi_{q+2} \left[\begin{matrix} (\lambda_i,\ A_i)_{1,p},\ (2,1),\ (\gamma+1,1)\ ; \\ (\mu_i,\ B_i)_{1,q},\ (1,1),\ (\gamma+\alpha+1,1)\ ; \end{matrix}\ z \right] \right\}$$

$$\geq \left[\Delta\ \Gamma\ (\alpha + 1) \binom{\alpha + \gamma}{\gamma}\right]^{-1} \left(\frac{1 - A\ |z|}{1 - B\ |z|}\right). \tag{3.5}$$

Proof. Using (1.1) and (1.4), we find that

$$\phi_\gamma^\alpha \left\{ \Delta\ z\ {}_p\Psi_q \left[\begin{matrix} (\lambda_i,\ A_i)_{1,p}\ ; \\ (\mu_i,\ B_i)_{1,q}\ ; \end{matrix}\ z \right] \right\}$$

$$= z + \Delta\Gamma(\alpha+1) \binom{\alpha+\gamma}{\gamma} \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{p} \Gamma\ (\lambda_i + A_i n)\ \Gamma\ (\gamma+1+n)}{\prod_{i=1}^{q} \Gamma\ (\mu_i + B_i n)\ \Gamma\ (\gamma+\alpha+1+n)} \frac{z^{n+1}}{n!}. \tag{3.6}$$

Then

$$\frac{d}{dz} \left\{ \phi_\gamma^\alpha \left(\Delta\ z\ {}_p\Psi_q \left[\begin{matrix} (\lambda_i,\ A_i)_{1,p}\ ; \\ (\mu_i,\ B_i)_{1,q}\ ; \end{matrix}\ z \right] \right) \right\}$$

$$= \Delta\Gamma(\alpha+1) \binom{\alpha+\gamma}{\gamma}\ {}_{p+2}\Psi_{q+2} \left[\begin{matrix} (\lambda_i,\ A_i)_{1,p},\ (2,1),\ (\gamma + 1,1)\ ; \\ (\mu_i,\ B_i)_{1,q},\ (1,1),\ (\gamma+\alpha+1,1)\ ; \end{matrix}\ z \right]. \tag{3.7}$$

Taking $G(z) = \phi_\gamma^\alpha \left(\Delta\ z\ {}_p\Psi_q \left[\begin{matrix} (\lambda_i,\ A_i)_{1,p}\ ; \\ (\mu_i,\ B_i)_{1,q}\ ; \end{matrix}\ z \right] \right)$

in (2.8) and (2.9), and using (3.7), we obtain the inequalities (3.4) and (3.5), respectively.

4. Some Consequences of Theorems 1-3

By specializing the parameters, we observe that for $A_j = 1$ (j=1,...,p) and $B_j = 1$ (j =1,...,q), the Wright's generalized hypergeometric function

$$ {}_p\Psi_q \left[\begin{matrix} (\lambda_j, 1)_{1,p} ; \\ (\mu_j, 1)_{1,q} ; \end{matrix} z \right] = {}_pF_q \left[\begin{matrix} \lambda_1, \ldots, \lambda_p ; \\ \mu_1, \ldots, \mu_q ; \end{matrix} z \right] . \tag{4.1} $$

Setting the parameters of the Wright's generalized hypergeometric function occurring in Theorems 1-2 in accordance with (4.1), we get the results obtained recently in [3] (Ths. 1-2, pp. 67-70). In addition to the choice of parameters indicated above for (4.1), if we also put $\alpha=1$ in Theorem 3, we are then lead to the other known result [3] (Th. 3, p. 71); see also [7]. Further, Theorems 1-2 can be applied to special functions like the Bessel-Maitland and Mittag-Leffler functions, and boundedness properties for these functions can be obtained. Indeed, by noting the relationships [7, p.51]:

$$ {}_0\Psi_1 \left[\begin{matrix} \overline{} ; \\ (1+\nu, \mu) ; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{z^n}{n!\, \Gamma(1+\nu+\mu n)} = J_\nu^\mu(-z), \tag{4.2} $$

and

$$ {}_1\Psi_1 \left[\begin{matrix} (1,1) ; \\ (\beta, \alpha) ; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = z^{(1-\beta)/\alpha} E_{\alpha;\beta}(z) , \tag{4.3} $$

where $J_\nu^\mu(z)$ and $E_{\alpha,\beta}(z)$ are the Bessel- Maitland and Mittag - Leffler functions, respectively, the corresponding relations can easily be deduced from the main results.

Thus for A=1-2α, B=-1, and setting the parameters of the function ${}_p\Psi_q\,[z]$ in Theorems 1-2 in accordance with the relations (4.2) and (4.3), we obtain the following results involving the functions $J_\nu^\mu(z)$ and $E_{\alpha,\beta}(z)$.

Corollary 1. *Let* $\Gamma(1+\nu)\ J_\nu^\mu\ (z) \in P^*(\alpha)$, *then*

(i) $$\left|\frac{1}{\Gamma(1+\nu+\mu n)}\right| \leq \frac{2\ (1-\alpha)\ n!}{|\Gamma\ (1+\nu)|}, \tag{4.4}$$

(ii) $$|J_\nu^\mu\ (z)| \leq \frac{1}{|\Gamma\ (1+\nu)|}\left(\frac{1+(1-2\alpha)\ |z|}{1-|z|}\right), \tag{4.5}$$

(iii) $$Re\{J_\nu^\mu\ (z)\} \geq \frac{1}{\Gamma\ (1+\nu)}\left(\frac{1-(1-2\alpha)\ |z|}{1+|z|}\right), \tag{4.6}$$

(iv) $$\left|{}_1\Psi_2\left[\begin{matrix}(1,\ 1) & ; \\ & & z \\ (1+\nu,\ \mu),(2,1); & \end{matrix}\right]\right|$$

$$\geq \frac{(2\alpha-1)}{|\Gamma(1+\nu)|}\left[1+\frac{2(1-\alpha)}{1-2\alpha}\cdot\frac{\log(1-|z|)}{|z|}\right], \tag{4.7}$$

(v) $$\left|{}_1\Psi_2\left[\begin{matrix}(1,\ 1) & ; \\ (1+\nu,\ \mu),(2,1); & \end{matrix}\ z\right]\right|$$

$$\geq \frac{(2\alpha-1)}{|\Gamma(1+\nu)|}\left[1-\frac{2(1-\alpha)}{1-2\alpha}\cdot\frac{\log(1+|z|)}{|z|}\right], \tag{4.8}$$

Corollary 2. *If* $\Gamma\ (1 + \nu)\ J_{\nu}^{\mu}\ (z) \in P^{*}\ (\alpha)$, *and* $|\, z\,| \leq 1$, *then*

$$\left|J_{1+\nu+\mu}^{\mu}\ (z)\right| \leq \frac{2(1-\alpha)}{|\Gamma(1+\nu)|(1+|z|^{2})} . \tag{4.9}$$

Corollary 3. *Let* $\Gamma(\beta)\, z^{(1-\beta)/\alpha}\, E_{\alpha,\beta}(z) \in P^{*}(\alpha)$, , *then*

(i) $$\left|\frac{1}{\Gamma\ (\beta + \alpha n)}\right| \leq \frac{2\ (1.-\alpha)\ n!}{|\Gamma(\beta)|} , \tag{4.10}$$

(ii) $$\left|z^{(1-\beta)/\alpha} E_{\alpha,\beta}\ (z)\right| \leq \frac{1}{|\Gamma(\beta)|}\left(\frac{1 + (1-2\alpha)\ |z|}{1 + |z|}\right) , \tag{4.11}$$

(iii) $$Re\left\{z^{(1-\beta)/\alpha} E_{\alpha,\beta}\ (z)\right\} \geq \frac{1}{|\Gamma(\beta)|}\left(\frac{1 - (1-2\alpha)|z|}{1 + |z|}\right) , \tag{4.12}$$

(iv) $$\left|{}_2\Psi_2\left[\begin{matrix}(1,1)\ , & (1,1)\ ; \\ (\beta,\alpha)\ , & (2,1)\ ;\end{matrix}\ z\right]\right|$$

$$\leq \frac{(2\ \alpha-1)}{|\ \Gamma\ (\beta)|}\left[1 + \frac{2(1-\alpha)}{1-2\ \alpha} \cdot \frac{\log(1-|z|)}{|z|}\right] \tag{4.13}$$

(v) $$\left|{}_2\Psi_2\left[\begin{matrix}(1,1)\ , & (1,1)\ ; \\ (\beta,\alpha)\ , & (2,1)\ ;\end{matrix}\ z\right]\right|$$

$$\geq \frac{(2\alpha - 1)}{|\Gamma\ (\beta)|}\left[1 - \frac{2\ (1-\alpha)}{1-2\alpha} \cdot \frac{\log\ (1+|z|)}{|z|}\right] , \tag{4.14}$$

Corollary 4. *If* $\Gamma(\beta)\ z^{(1-\beta)/\alpha} E_{\alpha,\beta}\ (z) \in P^{*}(\alpha)$, *and* $|\, z\,| \leq 1$, *then*

$$\left|{}_1\Psi_1\left[\begin{matrix}(2,1)\ ; \\ (\beta+\alpha,\alpha)\ ;\end{matrix}\ z\right]\right| \leq \frac{2\ (1-\alpha)}{|\Gamma(\beta)|\ (1 + |z|^{2})} .$$

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