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On the asymptotic behaviour of stationary gaussian processes

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by
Rita Giuliano Antonini (§)

0. Introduction

Let $(X_t)_{t>0}$ be a real stationary gaussian process such that, for each t, X_t is an N(0,1) distributed random variable; suppose moreover that $(X_t)_{t>0}$ has continuous paths and let r(t) be its covariance function. Under these assumptions, MARCUS [1] has shown that

$$(0.1) \qquad \limsup_{t \to \infty} \frac{|X_t|}{\sqrt{2 \log t}} \le 1 \qquad \text{a.s.}$$

Moreover, by a result of PICKANDS ([2], Th. 5.2), assuming in addition that

$$(0.2) \qquad \lim_{t \to \infty} r(t) \log t = 0,$$

and putting

$$Z_t = \sup_{0 \le s \le t} X_s \,,$$

one has

$$\liminf_{t \to \infty} \left[Z_t - \sqrt{2 \log t} \right] \ge 0 \quad \text{a.s.}$$

Now this relation yields

$$(0.3) \qquad \liminf_{t \to \infty} \frac{Z_t}{\sqrt{2 \log t}} \ge 1 \qquad \text{a.s.}$$

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Using (0.1) and (0.3) one can easily show the following

(0.4) THEOREM. *Under Pickands' assumptions, one has*

$$\limsup_{t \to \infty} \frac{X_t}{\sqrt{2 \log t}} = \limsup_{t \to \infty} \frac{|X_t|}{\sqrt{2 \log t}} = 1 \qquad \text{a.s.,}$$

so that a. s. the limit set (as $t \to \infty$ *) of the process*

$$Y_t = \frac{X_t}{\sqrt{\log t}}$$

is the interval

$$S = \left\{ y \in \mathbb{R} : \frac{1}{2} y^2 \le 1 \right\}.$$

The preceding remarks are due to G. LETTA.

In this paper we consider the process $\left(Y_t \right)_{t > 0}$ defined by

$$Y_t = \frac{X_t}{\varphi(t)},$$

where φ is a function verifying suitable assumptions; under a condition on r(t) weaker than (0.2), we show that a. s. the limit set of $\left(Y_t \right)$ as $t \to \infty$ is the interval

$$S_M = \left\{ y \in \mathbb{R} : \frac{1}{2} y^2 \le M \right\},$$

and we characterize the number M in terms of φ.
We point out that our result can be easily extended to the multidimensional case, as is sketched in section 4.
We are indebted to the referee for having simplified and improved the proofs: in particular we owe to him the use of theorem (2.3).

1. Assumptions and main theorem

Let $\left(X_t\right)_{t>0}$ be a real stationary gaussian process such that, for every t, X_t is N(0,1) distributed. Let r be the covariance function; we assume that

(1.1) $\left(X_t\right)_{t>0}$ has continuous sample paths with probability one;

(1.2) $r(n) = O\left(\dfrac{1}{\log n}\right)$ as $n \to \infty$ $(n \in \mathbb{N})$ (mixing condition).

Let now $\varphi:(0,+\infty) \to \mathbb{R}^+$ be a non decreasing function, such that $\lim\limits_{t\to\infty} \varphi(t) = +\infty$.

Consider the process $\left(Y_t\right)_{t>0}$ defined by

$$Y_t = \frac{X_t}{\varphi(t)}.$$

The main result of this paper is the following

(1.3) THEOREM. *Let*

$$M = \limsup_{n\to\infty} \frac{\log\left(\displaystyle\sum_{k=1}^{n} \frac{1}{\varphi(k)}\right)}{\varphi^2(n)}.$$

Obviously we have $0 \le M \le \infty$.
Let now

$$S_M = \left\{ y \in \mathbb{R} : \frac{1}{2}y^2 \le M \right\},$$

where we interpret S_M as $\{0\}$ (resp. $\mathbb{R}$) if $M = 0$ (resp. ∞).
Then a. s. the limit set of $\left(Y_t\right)$ as $t \to \infty$ is S_M.

The two cases M=0 and $M = \infty$ will be discussed in section 3.

In the case $0<M<\infty$, the proof will be carried out in the following two steps:

(1.4) PROPOSITION. *The limit set of* (Y_t) *is a. s. contained in* S_M.

(1.5) PROPOSITION. *Each point of* S_M *is a limit point of* (Y_t).

As to proposition (1.4), it is enough to notice that

$$M \geq \limsup_{n\to\infty} \frac{\log \frac{n}{\varphi(n)}}{\varphi^2(n)} = \limsup_{n\to\infty} \frac{\log n}{\varphi^2(n)} = \limsup_{n\to\infty} \frac{\log(n+1)}{\varphi^2(n)} \geq$$

$$\geq \limsup_{t\to\infty} \frac{\log t}{\varphi^2(t)} \quad,$$

because

$$\frac{\log t}{\varphi^2(t)} \leq \frac{\log([t]+1)}{\varphi^2([t])} \quad.$$

Hence, it follows from result (0.1) of Marcus that

$$\limsup_{t\to\infty} \frac{|X_t|}{\varphi(t)} \leq \sqrt{2M} \quad \text{a.s.}$$

Proposition (1.5) is proved in section 2.

2. The proof of (1.5).

It is easy to prove the following result, which implies (1.5):

(2.1) PROPOSITION. *Under the mixing condition* (1.2), *almost surely each point of* $[-\sqrt{2M},\sqrt{2M}]$ *is a limit point of* $(Y_n)_n$ *(and thus of* $(Y_t)_{t>0}$).

PROOF.

As a first step, we show that $\displaystyle\limsup_{n\to\infty} \frac{X_n}{\varphi(n)} = \sqrt{2M}$, a. s. This amounts to saying that

$$P\Big(X_n > \sqrt{2\lambda}\, \varphi(n) \text{ i. o.}\Big) = 0 \text{ or } 1$$

according as λ is greater or smaller than M. Now, recalling the properties of Dirichlet series (see appendix) this is immediate from the following

(2.2) THEOREM. *Let* $\big(X_n\big)_{n\geq 1}$ *be a gaussian stationary sequence, with zero mean and unit variance. Assume that the covariance function* r(n) *satisfies the mixing condition (1.2). Let* $(\psi(n))_{n\geq 1}$ *be a non decreasing sequence of positive numbers, with* $\lim_{n\to\infty} \psi(n) = +\infty$. *Then*

$$P\Big(X_n > \psi(n) \text{ i.o.}\Big) = 0 \text{ or } 1$$

according as the sum

$$\sum_{n=n_0}^{\infty} \frac{1}{\psi(n)} \exp\left(-\frac{\psi^2(n)}{2}\right)$$

is finite or infinite.

This theorem has been proved by P. K. PATHAK and C. QUALLS ([3], Th. B, pag. 190).
Now the statement of Proposition (2.1) follows from a general result on real centered gaussian sequences:

(2.3) THEOREM. *Let* $(Z_n)_n$ *be a real centered gaussian sequence satisfying the condition* $\lim_{n\to\infty} E\big(|Z_n|^2\big) = 0$. *Then there exists* c *in* $[0,+\infty]$ *such that* $\limsup_{n\to\infty} Z_n = c$ *a. s., and almost surely the limit set of* $(Z_n)_n$ *is equal to* [- c, c].

Indeed, in our setting, we have $c = \sqrt{2M}$.

This result can be found for example in M. TALAGRAND [4], where the fact that the limit set is not random is proved at the beginning of section III, and the rest in Lemma 7.

3. The cases M=0 and M=∞.

(3.1) M=0.

In this case we have

$$\limsup_{t \to \infty} \frac{\log t}{\varphi^2(t)} = 0,$$

and the desired result follows from (0.1).

(3.2) M=∞.

That every $y \in \mathbb{R}$ is a limit point is a straightforward consequence of (2.3).

4. Extension to the multidimensional case.

Let now $\left(X_t\right)_{t \rhd 0}$ be a d-dimensional stationary gaussian process such that, for each t, X_t is an N(0,I)-distributed random vector.

Let $r(t) = \left(r^{(h,k)}(t)\right)_{h,k=1,\dots,d}$ be its covariance function , where

$$r^{(h,k)}(t) = cov\left(X_s^{(h)}, X_{s+t}^{(k)}\right)$$

(obviously $r(0) = I$).

Suppose that the following assumptions are verified

(4.1) X_t has continuous sample paths with probability one;

(4.2) for every h, k = 1,...,d, $r^{(h,k)}(n) = O\left(\frac{1}{\log n}\right)$ as $n \to \infty$.

Put as above

$$Y_t = \frac{X_t}{\varphi(t)}$$

and

$$S_M = \left\{ x \in \mathbb{R}^d : \frac{1}{2} \|x\|^2 \leq M \right\}.$$

Then , as in the real case, a. s. S_M is the limit set of (Y_t) as $t \to +\infty$.

Indeed, proposition (1.4) may be immediately extended by standard arguments.

As to Proposition (1.5), we already know that the limit set is not random and. a. s. contained in S_M. Moreover it is closed and balanced a. s. by Lemma 7 of [4] (which holds in $\mathbb{R}^d$); so it is enough to prove that, for every unit vector $z = (z_1,...,z_d)$, we have

$$\limsup_{n \to +\infty} \frac{1}{\varphi(n)} \sum_{k=1}^{d} z_k X_n^{(k)} = \sqrt{2M} \quad \text{a. s.}$$

By condition (4.2), this follows from (2.2).

Appendix: a particular kind of Dirichlet series.

Let φ be a non decreasing function defined on $(1,+\infty)$, with

$$\lim_{x \to +\infty} \varphi(x) = +\infty$$

and consider the Dirichlet series

$$S(\lambda) = \sum_{n=1}^{+\infty} a_n \exp(-\lambda \varphi^2(n)),$$

with $(a_n) \subset \mathbb{R}^+$.

Put

$$\Lambda = \limsup_{n \to \infty} \frac{\log\left(\sum_{k=1}^{n} a_k\right)}{\varphi^2(n)} \geq 0;$$

$$\Lambda' = \sup \{\lambda \in \mathbb{R} \; ; \; S(\lambda) = +\infty \}.$$

Λ is not always equal to Λ'. However, it is well known (see [5] for example) that $\Lambda = \Lambda'$ if $\sum_{n=1}^{\infty} a_n = \infty$.

Consider now the Dirichlet series

$$S(\lambda) = \sum_{n=1}^{+\infty} \frac{1}{\varphi(n)} \exp(-\lambda \varphi^2(n)).$$

In this case we do have $\Lambda = \Lambda'$ (indeed, if $\sum_{n=1}^{+\infty} \frac{1}{\varphi(n)} < \infty$, then $\Lambda = 0$ and $S(\lambda)$ diverges for every $\lambda < 0$).

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