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CLAUDE MITSCHI

MICHAEL F. SINGER

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Solvable-by-finite groups as differential Galois groups (*)

CLAUDE MITSCHI (†), MICHAEL F. SINGER (‡)

RÉSUMÉ. — Nous résolvons le problème inverse de la théorie de Galois différentielle au-dessus d'un corps différentiel $k = C(x)$, où C désigne un corps algébriquement clos de caractéristique zéro, pour tout groupe algébrique linéaire G dont la composante connexe de l'identité est résoluble. Nous démontrons que pour tout espace principal homogène V sur G, irréductible sur k, on peut étendre la dérivation d/dx de k à $k(V)$ de sorte que $k(V)$ soit une extension de Picard-Vessiot de k admettant G comme groupe de Galois. Notre démonstration est constructive à l'étape près d'un problème de plongement de théorie de Galois classique sur $C(x)$.

ABSTRACT. — We solve the inverse problem of differential Galois theory over the differential field $k = C(x)$, where C is an algebraic closed field of characteristic zero, for linear algebraic groups G over C with a solvable connected component of the identity. We show that for any k-irreducible principal homogeneous space V for G, the derivation d/dx of k can be extended on $k(V)$ in such a way that $k(V)$ is a Picard-Vessiot extension of k with Galois group G. Our proof is constructive up to an embedding problem of classical Galois theory over $C(x)$.

1. Introduction

In this paper we prove a special case of the following inverse problem in differential Galois theory.

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(†) Institut de Recherche Mathématique Avancée, Université Louis Pasteur et C.N.R.S., 7 rue René Descartes, 67084 Strasbourg Cedex, France.
mitschi@math.u-strasbg.fr.
(‡) North Carolina State University, Department of Mathematics, Box 8205, Raleigh, NC 27695-8205, USA, singer@msri.org. The second author would like to thank the Institut de Recherche Mathématique Avancée, Université Louis Pasteur et C.N.R.S. for its hospitality and support during the preparation of this paper. The preparation was also partially supported by NSF Grant CCR-9731507 and NSF Grant CCR-0096842.

THEOREM 1.1.— *Let $k = C(x)$, C an algebraically closed field*[1]*, G a linear algebraic group defined over C and V a k-irreducible principal homogeneous space for G. Then there exists a derivation on $k(V)$ extending $\frac{d}{dx}$ such that $k(V)$ is a Picard-Vessiot extension of k with Galois group G, where the action of the Galois group on $k(V)$ corresponds to the action of G on V.*

For certain classes of linear algebraic groups (including connected groups and those with a semisimple identity component) it was proved by one of the authors [22] that there exists at least one homogeneous space V satisfying the conclusions of Theorem 1.1. We note (*cf.* [21]) that when G is a *connected* linear algebraic group then all k-irreducible principal homogeneous spaces for G are isomorphic to G. Thus, in [14], the authors give a constructive proof of the Theorem in this special case . For arbitrary linear groups and $C = \mathbf{C}$, the complex numbers, C. and M. Tretkoff [23] and J.-P. Ramis ([19], [20]) showed the existence of such a space (we refer the reader to [14] for a further history of the problem prior to 1996). The general case has recently been established by J. Hartmann [6][2].

The results in the present paper were achieved in 1998 and presented at the Conference on Differential Galois Theory at C.I.R.M., Luminy, in February 1999, then appeared as a preprint available at http://www.math.ncsu.edu/~singer). Since these results are referenced in [6] we think it is appropriate to publish this. Most of the basic facts that we use are now well-known and presented in slightly different form in [15], [16] and [17]. We have nonetheless not significantly changed our original presentation except for the order of some of the sections and the addition of examples.

In this paper we first show how to reduce the proof of Theorem 1.1 to solving the inverse problem for simpler cases, in particular when the given group G is a semidirect product of its identity component by a finite group. We shall use these reduction steps to show that the theorem is true for any linear algebraic group with G^0 solvable, that is, for solvable-by-finite linear algebraic groups.

The rest of the paper is organized as follows. In Section 2 we give basic results concerning principal homogeneous spaces. In Section 3 we consider the inverse problem for finite groups and in Section 4 we reduce the problem to semidirect products. In Section 5 we review basic facts from differential

(1) All fields in this paper will be assumed to be of characteristic zero.

(2) Although the results of [6] and [23] state the existence of one homogeneous space satisfying the conclusions of Theorem 1.1 , the proofs can be easily modified to prove the full theorem (when $C = \mathbf{C}$ in the cases of [23]). We thank D. Bertrand for pointing this out to us.

Galois theory and in Section 6 we give a criterion for realizing a linear algebraic group as a Galois group. In Section 7 we give useful reductions of the problem and in Section 8 we present a proof of Theorem 1.1 for solvable-by-finite groups, constructive up to the Riemann Existence Theorem in classical Galois theory.

2. General facts about groups and homogeneous spaces

Let G be a linear algebraic group defined over a field k. A *k-homogeneous space* for G is a k-affine variety together with a morphism $G \times V \mapsto V$ of k-varieties inducing a transitive action of $G(\bar{k})$ on $V(\bar{k})$, where $\overline{k}$ denotes the algebraic closure of k. If moreover this action is faithful, V is called a *principal* k-homogeneous space for G, or *G-torsor*.

Note that the group G is itself a G-torsor, called the *trivial* torsor. One can also define, in an obvious way, the notion of G-morphisms of homogeneous spaces for G.

It is a well-known fact that the set of G-torsors (up to G-isomorphism) maps bijectively to the first Galois cohomology set $H^1(k, G)$ (*cf.*, [21] Chapter III.1).

We will use the following result, known to be true for any perfect field k of dimension $\leqslant 1$ (*cf.* [21], Ch. II.3.1).

PROPOSITION 2.1.— *Let k be a function field of one variable over an algebraically closed field C of characteristic zero and G a linear algebraic group defined over k with identity component G^o.*

1. *If G is k-connected, then any G-torsor over k is trivial.*
2. *For any k-homogeneous space W for G, there exist a G-torsor V over k and a G-morphism $\pi : V \to W$.*
3. *If W is a G/G^o-torsor over k then there exists a unique G-torsor V and a G-morphism $\pi : V \to W$, where the action of G on W is induced by the projection $\pi : G \to G/G^o$. Furthermore, W is the geometric quotient of V under the action of G^o and V is k-irreducible whenever W is.*

Proof. — Statement 1 follows from the discussion after Théorème 1' of Chapter III.2.3 of [21]. Statement 2. is Théorème 3 of Chapter III.2.4 of [21]. The first part of Statement 3. follows from Corollaire 3 of Chapter III.2.4 of [21], which states that the canonical map $\rho : H^1(k, G) \to H^1(k, G/G^o)$ is a bijection. The other part follows from the fact that $\mid W(\bar{k}) \mid = \mid G/G^o \mid$.

3. Torsors for a finite group

Before we characterize k-irreducible torsors for a finite group, we need the following result, a key lemma in our construction of a solution to the inverse problem.

LEMMA 3.1.— *Let k be a field, $\varphi : H \to \mathrm{GL}_n(k)$ a representation of the finite group H and K a finite Galois extension of k with Galois group H. Then there exists $w \in \mathrm{GL}_n(K)$ such that ${}^h w = w\varphi(h)$ for all $h \in H$, where ${}^h w$ denotes the Galois action of $h \in \mathrm{Gal}(K/k)$ on the entries of w. Furthermore, if $k = C(x)$, then there exist $g_1, \ldots, g_n \in K$, linearly independent over C, such that $({}^h g_1, \ldots, {}^h g_n) = (g_1, \ldots, g_n)\varphi(h)$ for all $h \in H$.*

Proof.— The inclusion $\varphi(H) \subset \mathrm{GL}_n(k)$ defines a cocycle in $H^1(\mathrm{Gal}(K/k), \mathrm{GL}_n(K))$, and this latter set is trivial (*cf.*[12], p. 549). Therefore, there exists a matrix $w \in \mathrm{GL}_n(K)$ such that ${}^h w = w\varphi(h)$.

To prove the second statement, let $w \in GL_n(K)$ be such that ${}^h w = w\varphi(h)$ for all $h \in H$. We claim that there exist polynomials $p_1, \ldots, p_m \in k$ such that the entries of $(p_1, \ldots, p_m)w = (g_1, \ldots, g_m)$ are linearly independent over C. To see this, let $Y_1, \ldots, Y_m$ be differential indeterminates. If $(Y_1, \ldots, Y_m)w(c_1, \ldots, c_m)^T = 0$ for $(c_1, \ldots, c_m) \in C^m$ then $w(c_1, \ldots, c_m)^T = 0$, so $(c_1, \ldots, c_m) = 0$. Therefore, the differential polynomial $\mathrm{Wr}((Y_1, \ldots, Y_m)Z)$ in the Y_i's, where Wr denotes the wronskian determinant, is non identically zero. This implies ([18], p. 35) that there exist polynomials $p_1, \ldots, p_m \in k$ such that $\mathrm{Wr}((p_1, \ldots, p_m)w) \neq 0$. For this choice of p_i, one then sees that $(g_1, \ldots, g_m)$ are C-linearly independent and satisfy the conclusion of the lemma. □

An immediate consequence of this result is the following

COROLLARY 3.2.— *With the same notation and $k = C(x)$, any C-finite dimensional H-module is isomorphic to an H-submodule V of the H-module K.*

PROPOSITION 3.3.— *Let k be a field and H a finite group.*

1. *A field K is a Galois extension of k with group H if and only if $K = k(W)$ for some k-irreducible H-torsor W.*
2. *Assume that $k = C(x)$ where C is an algebraically closed field and let $\phi : H \to H'$ be a surjective homomorphism. If W' is a k-irreducible H'-torsor then there exist a k-irreducible H-torsor W and a map $\Phi : W \to W'$ such that $\Phi(wh) = \Phi(w)\phi(h)$ for all $\bar{k}$-points $w \in W$ and all $h \in H$.*

Proof. — 1. For each $h \in H$ let $\rho_h : W \to W$ be the morphism of k-varieties corresponding to the map $\rho_h(w) = wh$ on geometrical points $w \in W$. Let $\rho_h^* : k(W) \to k(W)$ be the induced map on the function field. In this way H acts faithfully as a group of automorphisms of $k(W)$ over k. Since the fixed field of H is k, we conclude that H is the Galois group of K over k.

Conversely, let K be a Galois extension of k with group H, let $|H| = n$ and let H act on itself via the regular representation. This allows us to represent H as a subgroup of permutation matrices of $\mathrm{GL}_n(k)$. Lemma 3.1 states that there exists $w \in \mathrm{GL}_n(K)$ such that ${}^h w = wh$ for all $h \in H$. The set $W = \{wh\}_{h \in H}$ is an algebraic set that is clearly invariant under the action of the Galois group. Therefore W is an H-torsor, defined over k. Since the action of the Galois group is transitive, W is irreducible and each point is a specialization of any other point. Therefore $k(W) = k(w) \subset K$. Since the only element of H leaving $k(w)$ fixed is the identity, we have that $k(w) = K$.

2. If $K' = k(W')$ then H' is the Galois group of K' over k. Theorem 7.13 of [24] implies that there exists a Galois extension K of k with Galois group H such that $K' \subset K$ and restricting automorphisms of K to K' corresponds to the homomorphism $\phi : H \to H'$. The field K is the function field of a k-irreducible H-torsor W and the inclusion $K' \subset K$ yields a morphism $\Phi : W \to W'$ as desired. $\square$

Note. — All of the above results are purely algebraic except for Proposition 3.3.2. Theorem 7.13 of [24] uses the Riemann Existence Theorem. Note that if we let $H' = (1)$, Proposition 3.3.2 implies that any finite group is a Galois group over $C(x)$. We know of no proof of this latter result that avoids analytic considerations.

4. Reduction to semidirect products

We will show that it is enough to prove Theorem 1.1 for groups that are semidirect products of their identity component G^o by the finite group G/G^o of their connected components and for homogeneous spaces of the form $W \times G^o$ where W is a k-irreducible G/G^o-torsor. The reduction to this case is made in several steps.

1. In ([25], p.142) or ([3], lemme 5.11 p.152) it is shown that any linear algebraic group G over k is (on $\bar{k}$-points) of the form $H \cdot G^o$ where H is a finite group and G^o is the identity component of G (this result is sometimes quoted as Platonov's theorem). Therefore, there

exists a surjective homomorphism $\psi : \bar{G} := H \ltimes G^o \to G$ and in particular $H' = G/G^o$ is the homomorphic image of H. Let V be a k-irreducible G-torsor and let $W' = V//G^o$ be the associated k-irreducible homogeneous space for $H' = G/G^o$. We wish to construct a k-irreducible $\bar{G}$-torsor $\bar{V}$ and a surjective morphism $\Psi : \bar{V} \to V$ such that $\Psi(vg) = \Psi(v)\psi(g)$ for all $\bar{k}$-points $v \in \bar{V}$ and $g \in \bar{G}$.

Proposition 3.3.2 implies that there exists a k-irreducible H-torsor W and a surjective morphism $\Phi : W \to W'$ such that $\Phi(wh) = \Phi(w)\psi(h)$ for all $\bar{k}$-points $w \in W$ and all $h \in H$. Proposition 2.1 states that there is a unique k-irreducible $\bar{G}$-torsor $\bar{V}$ and surjective $\bar{G}$-morphism $\xi : \bar{V} \to W$. The kernel J of ψ is a finite group so the quotient $\bar{V}//J$ exists and is a $\bar{G}/J = G$-torsor. By uniqueness, $\bar{V}//J$ is G-isomorphic to V so the composit $\bar{V} \to \bar{V}//J \simeq V$ gives the desired map Ψ.

2. Suppose there exists a derivation D on $k(\bar{V})$ satisfying the conclusion of Theorem 1.1 for the group $\bar{G}$ and a k-irreducible $\bar{V}$-torsor. Note that in the above, V was identified with $\bar{V}//J$ where J is the kernel J of the map $\psi : \bar{G} \to G$. Therefore, $k(V)$ is the fixed field of the normal subgroup J and so is again a Picard-Vessiot extension with Galois group G.

3. Let $G = H \ltimes G^o$. Let V be a k-irreducible G-torsor and let W be the associated k-irreducible G/G^o-torsor. The k-irreducible variety $W \times G^o$ has the structure of a homogeneous space for $G = H \ltimes G^o$ where the action of $H \ltimes G^o$ on $W \times G^o$ is given by $(w, g)(h', g') = (wh', h'^{-1}gh'g')$. Since this latter space is also inducing the G/G^o-torsor W, Proposition 2.1 implies that $W \times G^o$ is G-isomorphic to V.

We have thus reduced the problem to the case where G is a semidirect product of its identity component G^o by a finite group H and V is of the form $W \times G^o$ where W is a k-irreducible H-torsor.

5. The logarithmic derivative

In this section we review some of the results of [10], [11] which we used in [14]. Let k be a differential field of characteristic zero with algebraically closed field of constants C. As in [10], [11], we shall let $\mathbf{U}$ denote a fixed universal differential extension of k, that is, a field such that if K is a finitely generated differential extension of k, then there is a k-differential embedding of K into $\mathbf{U}$. Using elementary facts from the theory of differential ideals and Zorn's Lemma, one can show that such a field exists ([8], p. 134). Note that if K is a finitely generated differential extension of k and I a differential ideal in $K\{y_1, \dots, y_n\}$, $1 \notin I$, then I has a zero in $\mathbf{U}^n$. Without further mention, we shall assume that all fields under consideration are subfields of $\mathbf{U}$.

Let $G \subset \mathrm{GL}_n$ be a connected linear algebraic group defined over a field C and F any field containing C. If $\mathcal{G}$ denotes the Lie algebra of G (over C) one can identify $\mathcal{G}(F) = F \otimes_F \mathcal{G}$ with the space $\{A \in \mathrm{gl}_n(F) \mid 1+\epsilon A \in G(F[\epsilon])\}$, where the F-algebra $F[\epsilon]$ is defined by $\epsilon^2 = 0$ (*cf.* [4], see also [8] ex.1 p. 329, or [15], [17]).

Kolchin [8] and Kovacic [10], [11] introduced the *logarithmic derivative* defined by $l\delta(g) = g'g^{-1} \in \mathrm{gl}_n(\mathbf{U})$ for any $\mathbf{U}$-point g of $G \subset \mathrm{GL}_n$. The next propositions (*cf.* [8], [10], [11], [15], [17]) describe useful properties of the logarithmic derivative. We shall use the following notation. Let $f : \mathbf{U}^n \to \mathbf{U}^m$ be a polynomial map and $u, v \in \mathbf{U}^n$. We will use $df_u[v]$ to denote the derivative of f at u applied to v. For example, if $m = 1$, $df_u[v] = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(u)v_i$. Note that Taylor's Formula implies $f(u + \epsilon v) = f(u) + \epsilon df_u[v]$.

PROPOSITION 5.1.— *With notation as before, let G be a connected linear algebraic group defined over C. Then*

1. *$l\delta(g) \in \mathcal{G}(\mathbf{U})$ for all $g \in G(\mathbf{U})$*

2. *For any $A \in \mathcal{G}(\mathbf{U})$, there exists $g \in G(\mathbf{U})$ such that $l\delta(g) = A$. If $A \in \mathcal{G}(k)$ one can furthermore select $g \in G(\mathbf{U})$ such that $k(g)$ is a Picard-Vessiot extension of k (that is, such that the field of constants of $k(g)$ is C).*

3. *For any $g \in G(\mathbf{U})$ such that $k(g)$ is a Picard-Vessiot extension of k, the Galois group G' of $k(g)$ over k is a closed subgroup of G and the action of G' on g is given by right multiplication.*

Proof.— To show 1 we must show that $1 + \epsilon g'g^{-1} \in G(\mathbf{U}[\epsilon])$. To do this it is enough to show that $(1 + \epsilon g'g^{-1})g = g + \epsilon g' \in G(\mathbf{U}[\epsilon])$. Let P be a polynomial in the radical ideal defining G. Using Taylor's formula and the chain rule we clearly have $P(g+\epsilon g') = P(g)+\epsilon dP_g[g'] = P(g)+\epsilon(P(g))' = 0$ since $P(g) = 0$.

Statement 2 is just a restatement of Corollary 4.3 (and some facts from its proof) of [15]. Let $I \subset R = C[X_{1,1}, \dots, X_{n,n}, \frac{1}{\det}]$ be the radical ideal defining G. We extend the derivation on k to a derivation on $R \otimes k$ by setting $(X_{i,j})' = A(X_{i,j})$.

We first claim that $I \otimes k$ is a differential ideal in $R \otimes k$. To see this let $f \in I$. Note that $(f(X))' = df_X[AX]$ where $X = (X_{i,j})$. Therefore, to show that $f' \in I$, we will show that $df_g[Ag] = 0$ for all $g \in G(C)$. Since $A \in \mathcal{G}(k)$, we have that $1+\epsilon A \in G(k[\epsilon])$ and so $g+\epsilon Ag \in G(k[\epsilon])$. Therefore $0 = f(g+\epsilon Ag) = f(g) + \epsilon df_g[Ag]$ and $f(g) = 0$ implies that $df_g[Ag] = 0$.

Since $I \otimes k$ is a differential ideal, it is contained in a maximal differential ideal P, which is furthermore prime. The quotient field $\tilde{K}$ of $(R \otimes k)/P$ is a Picard-Vessiot extension of k. The image g of X in this field is a $\tilde{K}$-point of G by construction.

To prove assertion 3 note that if $\sigma \in \mathrm{Gal}(k(g)/k)$ then $\sigma(g) = g \cdot [\sigma]$ for some $[\sigma] \in \mathrm{GL}_n(C)$. Since $g \in G$, g is a zero of the ideal I and so $\sigma(g)$ is also a zero of I, hence $\sigma(g) \in G$. We can therefore conclude that $[\sigma] = g^{-1}\sigma(g) \in G$. $\square$

The following properties of the logarithmic derivative will help reduce the inverse problem to simpler cases.

PROPOSITION 5.2.— *Let $G, \tilde{G}$ be connected C-groups and let $f : G \to \tilde{G}$ be a C-morphism. If $g \in G(\mathbf{U})$ then*

$$l\delta(f(g)) = df_e[l\delta(g)]$$

Proof.— Let $l\delta(g) = A$, and let e denote the identity element of G. We see that $f(g+\epsilon Ag) = f(e+\epsilon A)f(g) = (f(e)+\epsilon df_e[A])f(g) = f(g) + \epsilon(df_e[A])f(g)$. We also have that $f(g+\epsilon Ag) = f(g)+\epsilon df_g[Ag] = f(g) + \epsilon df_g[g'] = f(g)+\epsilon(f(g))'$. Therefore $(f(g))' = (df_e[A])f(g)$, so $l\delta(f(g)) = df_e[A]$. $\square$

PROPOSITION 5.3.— *Let $f : G \to \tilde{G}$ be a surjective morphism of connected C-groups. Let $k(g)$ be a Picard-Vessiot extension of k with $g \in G(\mathbf{U})$ and with $g' = Ag$, $A \in \mathcal{G}(k)$. Then $k(f(g))$ is a Picard-Vessiot extension of k with $(f(g))' = df_e[A]f(g)$ and with Galois group $f(\mathrm{Gal}(k(f(g))/k)) \subset \tilde{G}$.*

Furthermore, assume that if H is any C-subgroup of G satisfying $f(H) = \tilde{G}$ then $H = G$. If $\mathrm{Gal}(k(f(g))/k) = \tilde{G}$, then $\mathrm{Gal}(k(g)/k) = G$.

Proof.— Proposition 5.2 implies that $f(g)$ satisfies the differential equation $Y' = df_e[A]Y$. Therefore $k(f(g)) \subset k(g)$ is generated by a fundamental set of solutions of a linear differential equation and so is a Picard-Vessiot extension of k. We end the proof of the first assertion by showing that the subgroup of $\mathrm{Gal}(k(g)/k)$ leaving $k(f(g))$ fixed is precisely the kernel of f in $\mathrm{Gal}(k(g)/k) \subset G(C)$. Proposition 5.1.3 implies that the Galois group of $k(g)$ over k acts on g by right multiplication. For $\sigma \in \mathrm{Gal}(k(g)/k)$, $f(\sigma(g)) = f(g\sigma) = f(g)f(\sigma)$. Therefore, σ leaves $f(g)$ invariant if and only if $f(\sigma)$ is the identity.

Now assume that if H is any C-subgroup of G satisfying $f(H) = \tilde{G}$ then $H = G$, and assume that the Galois group of $k(f(g))$ over k is $\tilde{G}$. Since this Galois group is $f(\mathrm{Gal}(k(g)/k))$, we must have $\mathrm{Gal}(k(g)/k) = G$. □

6. A criterion for realizing a group as a Galois group

Let k be a differential field with field of constants C, and E a Picard-Vessiot extension of k with Galois group G. It is a known fact (*cf.* [1], [5], [8], [13], [15], [17]) that E is the function field $k(V)$ of some k-irreducible G-torsor where the action of the Galois group on E is the same as the action resulting from $G(C)$ acting on V. We may furthermore write $E = k(v)$ for some E-point $v \in V$. For any element σ of the Galois group of E and any E-point $y \in V$ we shall denote by ${}^{\sigma}y$ the (differential Galois) action of σ on y, and by $y.\sigma$ the (translation) action of σ on y via the G-torsor V. We then have ${}^{\sigma}v = v \cdot \sigma$ for all $\sigma \in G(C)$.

Let us now restrict our attention to the field $k = C(x)$ and groups of the form $G = H \ltimes G^o$ as above (see Section 2). In this case, $V = W \times G^o$ for some k-irreducible H-torsor W. The field $K = k(W) \subset E$ is the fixed field of G^o and we may write $K = k(w)$ for some K-point $w \in W$ and $E = K(g) = k(w, g)$ for some E-point $g \in G^o$. For notational convenience, we think of G as being a subgroup of some GL_n and its Lie algebra $\mathcal{G}$ as being a subalgebra of the Lie algebra gl_n of all $n \times n$ matrices. In this case $A = g'g^{-1} \in \mathrm{gl}_n(K)$ since the entries of A are invariant under the action of the (constant) group G^o. For $(\sigma, \tau) \in G(C) = H(C) \ltimes G^o(C)$, we have that

$$^{(\sigma,\tau)}(w, g) = (w \cdot \sigma, \sigma^{-1} g \sigma \tau) ,$$

and in particular for $\sigma \in H(C)$,

$$^{\sigma}g = \sigma^{-1} g \sigma .$$

This last equation shows that for $\sigma \in H(C)$,

$$^{\sigma}A \;=\; ^{\sigma}(g'g^{-1}) \;=\; (\sigma^{-1}g\sigma)'\sigma^{-1}g^{-1}\sigma \;=\; \sigma^{-1}A\sigma.$$

DEFINITION 6.1.— *Let K be a Galois extension of k with Galois group H. Let V be a right H-module over k. We consider $K \otimes_k V$ as a left H-module via the action $\sigma \cdot a \otimes v = \sigma(a) \otimes v$ and as a right H-module via the action $a \otimes v \cdot \sigma = a \otimes (v \cdot \sigma)$ for any $\sigma \in H$. We say an element $u \in K \otimes_k V$ is H-*equivariant *if $\sigma \cdot u = u \cdot \sigma$ for all $\sigma \in H$.*

With notation as above, consider $V = \mathcal{G}$ as a right H-module via $v \mapsto h^{-1}vh$ for all $h \in H$ and $v \in \mathcal{G}$. We have then shown

Proposition 6.2. — *Let E be a Picard-Vessiot extension of k with Galois group $G = H \ltimes G^o$. Then*

1. *$K = E^{G^o}$ is the function field of a k-irreducible H-torsor (and so K is a Galois extension of k with Galois group H),*
2. *E is a Picard-Vessiot extension of K for an equation of the form $y' = Ay$ where A is an H-equivariant element of $\mathcal{G}(K)$. Furthermore the Galois group of E over K is G^o.*

The converse of this result gives the following criterion, that we will use to construct equations with a given Galois group.

Proposition 6.3. — *Let k be a differential field of characteristic zero with algebraically closed field of constants C. Let $G = H \ltimes G^o \subset \mathrm{GL}_n$ be an algebraic group defined over C, with H finite and G^o connected with Lie algebra $\mathcal{G}$. Let W be a k-irreducible H-torsor and let $K = k(W)$.*

Let $A \in \mathcal{G}(K)$ and assume that

1. *A is H-equivariant*
2. *The Picard-Vessiot extension E of K corresponding to the equation $y' = Ay$ has Galois group G^o.*

Then E is the function field of the k-irreducible G-torsor $W \times G^o$ and a Picard-Vessiot extension of k with Galois group G. Furthermore the action of the Galois group corresponds to the action of G on E induced by the action of G on $W \times G^o$.

Note that by Proposition 5.1 the condition $A \in \mathcal{G}$ implies that $E = K(g)$ for some $g \in G^o$ with the action of G^o on g given by right mutliplication. The condition that the Galois group is G^o implies that g is a generic point of G^o.

Proof. — Lemma 3.1 states that there exists a matrix $w \in \mathrm{GL}_n(K)$ such that for any $\sigma \in H \subset \mathrm{GL}_n(C)$ we have that ${}^{\sigma}w = w\sigma$. Note that $K = k(w)$ since $k(w) \subset K$ and the Galois group of K over $k(w)$ is trivial.

We may write $E = K(g)$ where $g \in G^o$ satisfies $g' = Ag$. This implies that $E = k(w, wg)$. By assumption the constant subfield of E is C.

Furthermore, $Y = \mathrm{diag}(w, wg) \in \mathrm{GL}_{2n}(E)$ satisfies $Y' = \bar{A}Y$ where

$$\bar{A} = \begin{pmatrix} w'w^{-1} & 0 \\ 0 & w'w^{-1} + wAw^{-1} \end{pmatrix}$$

Clearly $w'w^{-1}$ and $w'w^{-1} + wAw^{-1}$ lie in K. A calculation shows that both of these are invariant under the action of H and so both must lie in k. Therefore E is a Picard-Vessiot extension of k. Since $\mathrm{Gal}(E/K) = G^o$ and $\mathrm{Gal}(K/k) = H$ we have an exact sequence of groups:

$$(1) \to G^o \to \mathrm{Gal}(E/k) \to H \to (1)\ .$$

Since G^o is connected, the field K is algebraically closed in E and so E is the quotient field of $K \otimes_C C(G^o)$. Any automorphism σ of K over k gives an automorphism $\tilde{\sigma} = \sigma \otimes \mathrm{Ad}(\sigma)$ of $K \otimes_C C(G^o)$ (where $\mathrm{Ad}(\sigma)(g) = \sigma^{-1} g \sigma$) and therefore of E. The map $\Phi(\sigma) = \tilde{\sigma}$ gives an injective homomorphism of H to $\mathrm{Gal}(E/k)$. Since $\tilde{\sigma}(w) = w \cdot \sigma$ and $\tilde{\sigma}(wg) = wg \cdot \sigma$ we have that $\Phi(\sigma) = \mathrm{diag}(\sigma, \sigma)$. The image of G^o in $\mathrm{Gal}(E/k)$ is $\mathrm{diag}(I, G^o)$. Therefore $\mathrm{Gal}(E/k)$ is isomorphic to $H \ltimes G^o$. $\square$

EXAMPLE 6.4.— *Consider the semidirect product* $G = \mathbf{Z}/2\mathbf{Z} \ltimes C^*$ *represented in* $\mathrm{GL}_2(C)$ *as*

$$G = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in C^* \right\} \cup \left\{ \begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix}, a \in C^* \right\}.$$

Let $k = C(x)$ *and* $K = k(\sqrt{x})$. *The Galois group* H *of* K *over* k *is* $\mathbf{Z}/2\mathbf{Z}$. *If we identify* H *with*

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

then the matrix

$$w = \begin{pmatrix} 1 + \sqrt{x} & 1 - \sqrt{x} \\ 1 - \sqrt{x} & 1 + \sqrt{x} \end{pmatrix}$$

satisfies ${}^{\sigma}w = w\sigma$ *for all* $\sigma \in H$. *The matrix*

$$A = \begin{pmatrix} \sqrt{x} & 0 \\ 0 & -\sqrt{x} \end{pmatrix}$$

is H*-equivariant and the Galois group of* $Y' = AY$ *over* K *is*

$$G^o = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in C^* \right\} \simeq C^*.$$

A calculation shows that

$$w'w^{-1} = \frac{1}{4}\begin{pmatrix} \frac{1}{x} & \frac{-1}{x} \\ \frac{-1}{x} & \frac{1}{x} \end{pmatrix}$$

and

$$w'w^{-1} + wAw^{-1} = \frac{1}{4}\begin{pmatrix} \frac{1+2x+2x^2}{x} & \frac{-1-2x+2x^2}{x} \\ \frac{-1+2x-2x^2}{x} & \frac{1-2x-2x^2}{x} \end{pmatrix}.$$

Thus the differential equation $Y' = \bar{A}Y$ where

$$\bar{A} = \begin{pmatrix} w'w^{-1} & 0 \\ 0 & w'w^{-1} + wAw^{-1} \end{pmatrix}$$

realizes $\mathbf{Z}/2\mathbf{Z} \ltimes C^*$ as its differential Galois group over $C(x)$.

In the next sections we shall need the following

LEMMA 6.5.— *Let k, K, H and V be as in Definition 6.1. Let W be a right H-submodule of V and $\pi : V \to V/W$ be the associated projection map. For any H-equivariant $u \in K \otimes (V/W)$ there exists an H-equivariant $w \in K \otimes V$ such that $1 \otimes \pi(w) = u$.*

Proof.— Let $\tilde{u} \in K \otimes V$ be any element such that $1 \otimes \pi(\tilde{u}) = u$. The element $w = \frac{1}{|H|}\sum_{\sigma \in H} \sigma \cdot \tilde{u} \cdot \sigma^{-1}$ satisfies the conclusion of the lemma. □

7. Reduction steps

Our purpose, reached so far only in the case of a solvable identity component G^o, is to give a proof as constructive as possible of Theorem 1.1. The idea is to mimick the proof we gave in [14] for connected groups, now over a finite extension K of $C(x)$, taking into account the equivariance condition under the action of the finite Galois group of this extension.

In this section we recall the reduction steps of [14], inspired by Kovacic's strategy in ([10], [11]).

In the following, we will use the purely group theoretic fact that *if G is a connected linear algebraic group defined over C with unipotent radical R_u and if $\rho : G \to G/[R_u, R_u]$ is the quotient homomorphism, then the only C-subgroup H of G with $\rho(H) = \rho(G)$ is G* (*cf.*, Lemma 7 of [11]). The following corollary of Proposition 5.3 then reduces the inverse problem for arbitrary connected linear groups to the same problem for connected linear groups whose unipotent radical is commutative.

COROLLARY 7.1. — *Let G be a C-connected linear algebraic group with unipotent radical R_u. Let $\rho : G \to \tilde{G} = G/[R_u, R_u]$ and let $B \in \tilde{\mathcal{G}}(K)$, where $\tilde{\mathcal{G}}$ is the Lie algebra of $\tilde{G}$. Assume that the Galois group of $Y' = BY$ is $\tilde{G}$. Then for any $A \in \mathcal{G}(K)$, where $\mathcal{G}$ is the Lie algebra of G, such that $d\rho_e[A] = B$, the Galois group of $Y' = AY$ is G.*

Proof. — Let A be as in the hypotheses and let $K(g)$ be the Picard-Vessiot extension of K for the equation $Y' = AY$. The element $\rho(g)$ satisfies the equation $Y' = BY$ so the field $K(\rho(g))$ is a Picard-Vessiot extension for this equation. Furthermore, the Galois group of $K(\rho(g))$ over K is $\tilde{G}$. Since G is the only algebraic subgroup of G mapping surjectively onto $\tilde{G}$, the conclusion follows from Proposition 5.3. □

Let G be a connected group with commutative unipotent radical U. We shall recall the conditions of Kovacic that allow us to lift a solution of the inverse problem for G/U to a solution of the inverse problem for G. These conditions are based on the *Levi decomposition of G* as a semidirect product $G = U \rtimes P$ where P is a reductive group ([7], p. 117).

To motivate Kovacic's conditions, let $L = K(g)$ be a Picard-Vessiot extension of K with Galois group G and g a L-point of G. We may write $g = up$ with $u \in U$, $p \in P$. Note that since $\text{tr.deg.}_K K(p) \leqslant \dim P$, $\text{tr.deg.}_{K(p)} K(up) \leqslant \dim U$ and $\dim G = \text{tr.deg.}_K K(g) = \text{tr.deg.}_K K(p) + \text{tr.deg.}_{K(p)} K(up)$, we have that $\text{tr.deg.}_K K(p) = \dim P$ and $\text{tr.deg.}_{K(p)} K(up) = \dim U$. In particular this implies that $K(p)$ is a Picard-Vessiot extension of K with Galois group P. Let $\mathcal{G}, \mathcal{P}$, and $\mathcal{U}$ be the Lie algebras of G, P, and U respectively and let $l\delta(g) = A_G \in \mathcal{G}$ and $l\delta(p) = A_P \in \mathcal{P}$. Calculating, we find

$$\begin{aligned} l\delta(g) &= (up)'(up)^{-1} \\ &= (p \cdot p^{-1}up)'(p \cdot p^{-1}up)^{-1} \\ &= p'p^{-1} + p(l\delta(p^{-1}up))p^{-1}. \end{aligned}$$

If $\tilde{p}$ is any element such that $K(\tilde{p})$ is a Picard-Vessiot extension of K and $l\delta(p) = l\delta(\tilde{p})$, then a simple calculation shows that $\tilde{p} = pc$ for some $c \in P(C)$ and also that $\tilde{p}(l\delta(\tilde{p}^{-1}u\tilde{p}))\tilde{p}^{-1} = p(l\delta(p^{-1}up))p^{-1}$. We therefore define

$$l_{A_P}\delta(u) = p(l\delta(p^{-1}up))p^{-1}$$

where p is any element of P such that $l\delta(p) = A_P$ and the constants of $K(p)$ are C. Note that $\mathcal{U}$ is left fixed by any automorphism of $\mathcal{G}$, so $l_{A_P}\delta(u) \in \mathcal{U}$. The following result of Kovacic shows that one can reverse the process (*cf.*, [10], Proposition 13; [11], Proposition 19; [14], Proposition 2.6).

PROPOSITION 7.2. — *Let G be a connected linear algebraic C-group and let $G = U \rtimes P$ be a Levi decomposition. Assume that U is commutative and that $A_P \in \mathcal{P}(K)$, $A_U \in \mathcal{U}(K)$ satisfy*

1. *There exists an element $p \in P$ such that $l\delta(p) = A_P$ and $K(p)$ is a Picard-Vessiot extension of K with Galois group P.*
2. *There exists an element $u \in U$ such that $l_{A_P}\delta(u) = A_U$, and such that the field of constants of $K(u,p)$ is C.*
3. *The map $\sigma \mapsto p^{-1}u^{-1}\sigma(u)p$ is a C-isomorphism from the differential Galois group $\mathrm{Gal}(K(up)/K(p))$ onto $U(C)$.*

Then $l\delta(up) = A_U + A_P$ and the Galois group of $K(up)$ is G.

The condition $l_{A_P}\delta(u) = A_U$ can be described in a simple way. To do this note that U, being a commutative unipotent group, is the isomorphic image, via the exponential map, of the vector group of its Lie algebra $\mathcal{U}$. Furthermore, it is easy to show that $(\exp\gamma)' = \gamma' \exp(\gamma)$ for any $\gamma \in \mathcal{U}$. This implies that $l\delta(\exp\gamma) = \gamma'$ and if we identify U with the vector group of $\mathcal{U}$ via the exponential map, we have that $l\delta(g) = g'$ for all $g \in U$.

Via the identification of U with $\mathcal{U}$ the action of P on U by conjugation induces the (adjoint) representation $\rho : P \to \mathrm{GL}(U)$ and the corresponding representation $d\rho : \mathcal{P} \to \mathrm{gl}(U)$ on Lie algebras. For A_U, A_P, u, p as in Proposition 7.2, we have

$$\begin{aligned} l_{A_P}\delta(u) &= p(l\delta(p^{-1}up))p^{-1} \\ &= \rho(p)\cdot(\rho(p^{-1})\cdot u)' \\ &= u' - l\delta(\rho(p))\cdot u \\ &= u' - d\rho(A_P)\cdot u \ . \end{aligned}$$

Kovacic is able to refine Proposition 7.2 in the following way. Since P is reductive we may write U as a sum of irreducible P-modules. Grouping isomorphic copies, we write $U = U_1^{r_1} \oplus \cdots \oplus U_s^{r_s}$, where the U_i are non-isomorphic P-modules. For all $i = 1, \ldots, s$ let $\rho_i : P \to \mathrm{GL}(U_i)$ be the representation of P on the simple module U_i and $d\rho_i : \mathcal{P} \to \mathrm{gl}(U_i)$ the corresponding representation on Lie algebras. We denote by $\rho_i^{r_i}$ and $d\rho_i^{r_i}$ the corresponding representations on the powers. As before, we shall identify each U_i and its Lie algebra $\mathcal{U}_i$ with some C^{ν_i}. Kovacic shows ([11], Proposition 19):

PROPOSITION 7.3. — *Let G be a connected linear algebraic group defined over an algebraically closed field C and let K be a differential field* *with field of constants C. Assume that $G = (U_1^{r_1} \oplus \cdots \oplus U_s^{r_s}) \rtimes P$ as above. Let $A_P \in \mathcal{P}(K)$ and $A_i \in (\mathcal{U}_i^{r_i})(K)$, $i = 1, \ldots, s$, be such that*

1. *There exists an element $p \in P$ with $l\delta(p) = A_P$ such that $K(p)$ is a Picard-Vessiot extension of K with Galois group P.*
2. *There exist elements $u_i \in U_i^{r_i}$ with $u_i' - d\rho_i^{r_i}(A_P)u_i = A_i$, such that the field of constants of $K(u_i, p)$ is C.*
3. *The map $\sigma \mapsto \rho_i^{r_i}(p^{-1}) \cdot (\sigma(u_i) - u_i)$ is a C-isomorphism from* Gal$(K(u_ip)/K(p))$ *onto $U_i^{r_i}(C)$).*

Then for $u = u_1 + \ldots + u_s$, $l\delta(up) = A_1 + \ldots + A_s + A_P$ and $K(up)$ is a Picard-Vessiot extension of K with Galois group G.

We shall also need criteria to find A_U and A_P. We do not know a general criterion for realizing constructively an arbitrary reductive group as a Galois group over a general differential field. The following is a criterion for realizing tori over any differential field (*cf.*[10] Proposition 15). It is a consequence of the Kolchin-Ostrowski Theorem (*cf.* [9]). If T is a torus defined over C, we identify $T(C)$ with the l-fold product $C^* \times \cdots \times C^*$ and the Lie algebra $\mathcal{T}_C$ of $T(C)$ with the l-fold sum $C \oplus \cdots \oplus C$. With this identification, the logarithmic derivative of an element in T becomes $l\delta(\alpha_1, \ldots, \alpha_l) = (\alpha_1'/\alpha_1, \ldots, \alpha_l'/\alpha_l)$.

PROPOSITION 7.4.— *Let T be as above and F a differential field containing C. Let $(a_1, \ldots, a_l) \in \mathcal{T}_F = F^l$. A necessary and sufficient condition that $(\tau_1, \ldots, \tau_l)$ realize T over F is that there exists no relation of the form $n_1\tau_1 + \ldots + n_l\tau_l = f'/f$ with $n_i \in \mathbf{Z}$ not all zero and $f \in F$.*

Finally, Kovacic gives a criterion for finding elements satisfying Proposition 7.3. Let $L_{A_P,\rho} : K^m \to K^m$, where $m = \dim_C(\mathcal{U})$, be the map defined by $L_{A_P,\rho}(v) = v' - d\rho(A_P) \cdot v$ and let $\pi : K^m \to K^m / L_{A_P,\rho}(K^m)$ be the quotient homomorphism of C-vector spaces. Kovacic shows (*cf.* [11], Proposition 20; [14], Proposition 2.11)

PROPOSITION 7.5.— *With notation as in Proposition 7.3, assume that $s = 1$ and $r_1 = r$. If $A_P \in \mathcal{P}(K)$ satisfies condition 1 above, then $A_1 = (a_1, \ldots, a_r) \in \mathcal{U}_1^r(K)$ satisfies conditions 2 and 3 if and only if $\pi a_1, \ldots, \pi a_r$ are linearly independent over C.*

8. Solvable-by-finite groups

We are now able to prove Theorem 1.1 for linear algebraic groups G, defined over C, with a solvable identity component G^o. We shall keep the notation of the previous sections. We may assume that G is a semidirect product $G = H \ltimes G^o$, where H is a finite group, and that G^o has a Levi-decomposition $G^o = U \ltimes P$, where the unipotent radical U is commutative and P is a torus. We have the decomposition $U = U_1^{r_1} \oplus \cdots \oplus U_s^{r_s}$ of U into nonisomorphic irreducible P-modules U_i and since P is a torus we know that all the U_i are one-dimensional. We shall moreover assume that U_1 is the trivial P-module (and so allow the possibility that $r_1 = 0$). Let K denote a finite extension of $k = C(x)$ with Galois group H. To satisfy the criterion of Proposition 6.3 we need to find $A_P \in \mathcal{P}(K), A_i \in \mathcal{U}_i^{r_i}(K)$ satisfying conditions 1, 2 and 3 of Proposition 7.3 and such that A_P, A_1, and $A_2+\cdots+A_s$ are H-equivariant. Before we show how to select these elements, we must select H-invariant C-subspaces of K which are H-isomorphic to $\mathcal{P}(C)$, to $\tilde{W}_1 = \mathcal{U}_1^{r_1}$ and to $\tilde{W}_2 = \mathcal{U}_2^{r_2} \oplus \cdots \oplus \mathcal{U}_s^{r_s}$ respectively.

Corollary 3.2 implies that there exist H-invariant C-subspaces W_0, W_1 and W_2 of K, isomorphic as H-modules to $\mathcal{P}(C)$, $\tilde{W}_1$ and $\tilde{W}_2$ respectively. We need to adjust these spaces so that the poles of their elements have certain properties. Let $\mathbf{Y}$ be the curve corresponding to the function field K and assume that $\mathbf{Y}$ is unramified over ∞.

1. After multiplying by a suitable rational function $r \in C(x)$, we may assume that all the elements of W_2 are regular on $\mathbf{Y}$ except possibly at points above ∞. Multiplication by such an element does not change the property of being an H-module isomorphic to $\tilde{W}_2$. Let $e \geqslant 2$ be a positive integer such that any element of W_2 has a pole of order at most $e-1$ at any point above ∞.
2. After multiplying by a suitable rational function, we may assume that the nonzero elements of W_0 are regular on $\mathbf{Y}$ except at points above ∞ and at such points they have poles of order at least e. Again, this does not change the property of being H-isomorphic to $\tilde{W}_0$.
3. Since H is reductive, we may (if $r_1 > 0$) write $W_1 = \oplus_{i=1}^t W_{1,i}$ where each $W_{1,i}$ is H-irreducible. Let $c_1, \ldots, c_t$ be finite points of $\mathbf{P}^1$ such that $\mathbf{Y}$ is not ramified above the c_i and such that the elements of W_1 are regular above all the c_i. Let k_i be an integer such that all the elements of $V_i = (x-c_i)^{k_i} W_{1,i}$ have poles of order at most 1 above c_i and at least one element has a pole of order precisely 1 at some point above c_i. The space $\bar{W}_1 = \oplus_{i=1}^t V_i$ is again H-isomorphic to $\mathcal{U}_1^{r_1}$. Abusing notation, we may therefore assume that $W_1 = \oplus_{i=1}^t W_{1,i}$ where, for each i, the elements of $W_{1,i}$ have poles of order at most 1 at c_i, are regular at $c_j, j \neq i$, and some element of $W_{1,i}$ has a simple pole at c_i.

We now show how to select A_P, A_1, and $A_2 + \cdots + A_s$.

8.1. Choice of A_P

Let $g_1, \ldots, g_l$ be an H-equivariant C-basis of W_0. We claim that if $d_i \in C$ are such that $d_1g_1 + \ldots + d_lg_l = f'/f$ for some $f \in K$, then all $d_i = 0$. Note that any element f'/f has a zero of order at least one at any point above ∞. Since $d_1g_1 + \ldots + d_lg_l$ is either zero or has a pole of order at least e, we must have $d_1g_1 + \ldots + d_lg_l = 0$, hence $d_i = 0$ for all $i = 1, \ldots, l$. Proposition 7.4 implies that for $A_P = (g_1, \ldots, g_l) \in K^l = \mathcal{P}(K)$, $Y' = A_PY$ has Galois group P over K. Since ${}^{\sigma}(g_1, \ldots, g_l) = (g_1, \ldots, g_l) \cdot \sigma$ for all $\sigma \in \mathrm{Gal}(K/k) = H$, A_P is H-equivariant and satisfies condition 1 of Proposition 7.3.

8.2. Choice of A_1

As before, we write $W_1 = \oplus_{i=1}^{t} W_{1,i}$ if $r_1 > 0$. For each i, let $\{f_{ij}\}_{1 \leqslant j \leqslant n_i}$ be an H-equivariant C-basis of $W_{1,i}$ and let $A_1 = (f_{11}, \ldots f_{tn_t}) \in \mathcal{U}_1^{r_1}(K) = K^{r_1}$. We claim that A_1 satisfies conditions 2 and 3 of Proposition 7.3.

Recall that ρ_1 is the trivial representation of P. Therefore $L_{A_{P,\rho_1}}(v) = v'$ for all $v \in \mathcal{U}_1(K) = K$. Assume that there exist constants c_{ij} such that $\sum c_{ij}f_{ij} = L_{A_{P,\rho_1}}(\phi) = \phi'$ for some $\phi \in K$. Letting $\gamma_i = \sum_{j=1}^{n_i} c_{ij}f_{ij}$, we have $\sum_{i=1}^{t} \gamma_i = \phi'$. For each i, let $S_i = \{f \in W_{1,i} \mid f + u = v' \text{ for some } u \in W_{1,1} + \ldots + \widehat{W_{1,i}} + \ldots + W_{1,t}, v \in K\}$. Note that each S_i is an H-invariant subspace of $W_{1,i}$ and $\gamma_i \in S_i$. The elements of $W_{1,j}$, $j \neq i$ are all regular above c_i and an element of the form ϕ' cannot have a simple pole above c_i. Therefore, the element of $W_{1,i}$ having a simple pole above c_i is not in S_i. Since $W_{1,i}$ is irreducible, we must have that $S_i = \{0\}$. Therefore $\gamma_i = 0$. Since, for each i, the f_{ij} form a basis of W_{1i} we have that all $c_{ij} = 0$.

We therefore have that $A_1 = (f_{11}, \ldots, f_{tn_t}) \in \mathcal{U}_1^{r_1}(K)$ is H-equivariant and satisfies conditions 2 and 3 of Proposition 7.3.

8.3. Choice of $\oplus_{i=2}^{s} A_i$

For each i, $2 \leqslant i \leqslant s$ let $\{\varphi_{ij}\}_{1 \leqslant j \leqslant r_i}$ be an H-equivariant basis of the C-subspace of W_2 isomorphic to $\mathcal{U}_i^{r_i}$. The Lie algebra $\mathcal{P}$ acts on $\mathcal{U}_i$ via a nonzero character of the form $d\rho_i((x_1, \ldots x_l)) = \sum_j n_{ij}x_j$ for some integers n_{ij}. We shall show that, for each i, the element $A_i = (\varphi_{i1}, \ldots, \varphi_{ir_i}) \in \mathcal{U}_i^{r_i}(K) = K^{r_i}$ satisfies conditions 2 and 3 of Proposition 7.3 . To do this we must show that if $c_1, \ldots, c_{r_i}$ are constants such that for some $y \in K$, $y' - (\sum n_{ij}g_j)y = \sum c_j\varphi_{ij}$ where $A_P = (g_1, \ldots, g_l)$, then each $c_j = 0$. Note that the g_j and the φ_{ij} are regular except above ∞. Therefore any such y must also be regular at all points except those above ∞. Therefore there is a point above ∞ such that the order of y is at most 0. At such a point $y' - (\sum n_{ij} g_j)y$ will have a pole of order at least e, if it is not zero, whereas $\sum c_j \varphi_{ij}$ will have a pole of order at most $e-1$. We get $y' - (\sum n_j g_j)y = \sum c_j \varphi_{ij} = 0$. Since the φ_{ij} form a basis, we conclude that all $c_j = 0$.

The H-equivariant elements A_P and A_i, $i = 1 \ldots, s$ fulfill the conditions of Proposition 7.3. This ends the proof of Theorem 1.1 for groups with a solvable identity component.

EXAMPLE 8.1.— *We shall apply the method of this section to the subgroup $G = \mathbf{Z}/2\mathbf{Z} \ltimes (C^* \ltimes (C \oplus C))$ of $\mathrm{GL}_4(\mathrm{C})$ defined as*

$$G = \left\{ \begin{pmatrix} a & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}, a \in C^*,\ b, c \in C \right\} \cup \left\{ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Using the above notation, we have

$$P = \left\{ p_a = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, a \in C^* \right\}$$

$$W_1 = \left\{ u_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}, c \in C \right\}$$

$$W_2 = \left\{ v_b = \begin{pmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, b \in C \right\}$$

$$H = \left\{ \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \varepsilon = \pm 1 \right\}.$$

Note that the action of H on P is trivial whereas the action of H on W_1 and W_2 sends an element to its inverse. The action of P on W_1 is trivial while $p_a v_b p_a^{-1} = v_{ab}$ for all $p_a \in P$ and $v_b \in W_2$.

Let $k = C(x)$ and $K = k(t)$ where $t^2 = \frac{x+1}{x}$. Note that this has no ramification above ∞. A calculation shows that the following choices satisfy conditions 1, 2 and 3.

1. *W_2 is the C-span of xt. This element has a pole of order 1 at points above ∞.*

2. *W_0 is the C-span of x^2. This element has a pole of order 2 at points above ∞.*

3. *W_1 is the C-span of $\frac{1}{x-2}t$.*

We therefore have that the matrix

$$A = \begin{pmatrix} x^2 & xt & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{t}{x-2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

is H-equivariant. As in Example 6.4 one can find $\bar{A} \in \mathrm{gl}_8(k)$ such that the differential equation $Y' = \bar{A}Y$ realizes G as its Galois group over $k = C(x)$.

9. Final remark

In the special case of a solvable identity component G^o, note that our realization of G as a differential Galois group is constructive up to a finite embedding problem of classical Galois theory over $C(x)$. Let us briefly recall this procedure.

Given G and an irreducible G-torsor V over $k = C(x)$, and W the induced (irreducible) G/G^o-torsor, let $K \supset k(W)$ be a solution of the finite embedding problem for $H \to G/G^o$, where H is a finite subgroup of G mapping surjectively to G/G^o (with respect to $\bar{k}$-points). We construct an irreducible H-torsor W' such that $K = k(W')$ and such that W' induces W for G/G^o. There is a unique derivation ∂ extending $\frac{d}{dx}$ on $k(W)$ and K, and we extend ∂ constructively to a derivation $\bar{\partial}$ on $k(W' \times G^o)$ realizing $H \ltimes G^o$ as the Galois group of the Picard-Vessiot extension $k(W' \times G^o)$ of k. The G-stable subfield $k(V)$ of $k(W' \times G^o)$ is then a Picard-Vessiot extension of k with Galois group G.

In particular, if we are given a field K, algebraic over k with finite Galois group H, we can effectively realize any group of the form $H \ltimes G^o$, G^o connected and solvable, as a differential Galois group over k.

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