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Unstable simple modes for a class of Kirchhoff equations (*)

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RÉSUMÉ. — Il est bien connu que les équations de Kirchhoff admettent infinies modes simples, c'est-à-dire solutions périodiques avec seulement un composant de Fourier dans les variables spatiales. Nous montrons que, pour quelque choix de la non-linéarité, ces modes simples sont instables à condition que leur énergie soit assez grande. Ce résultat, énoncé dans le cadre abstrait des espaces de Hilbert, et prouvé par réduction à un système de deux équations différentielles ordinaires du deuxième ordre, s'applique à équations différentielles ordinaires du deuxième ordre, s'applique à équations différentielles aux dérivées partielles de type de Kirchhoff sur domaines bornés de $\mathbb{R}^n$.

ABSTRACT. — It is well known that Kirchhoff equations admit infinitely many simple modes, *i.e.* time periodic solutions with only one Fourier component in the space variables. We prove that, for some choices of the nonlinearity, these simple modes are unstable provided that their energy is large enough. This result, stated in an abstract Hilbert space setting, and proved by reducing to a system of two second order ODEs, applies to PDEs of Kirchhoff type on bounded domains of $\mathbb{R}^n$.

1. Introduction

Let H be a real Hilbert space, with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$. Let A be a self-adjoint linear positive operator on H with dense domain $D(A)$

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(*i.e.* $\langle Au, u\rangle > 0$ for all $u \in D(A)$). We consider the evolution problem

$$u''(t) + m(|A^{1/2}u(t)|^2)Au(t) = 0, \tag{1.1}$$

where $m : [0, +\infty) \to (0, +\infty)$ is a C^1 function.

Equation (1.1) is an abstract setting of the hyperbolic PDE with a non-local non-linearity of Kirchhoff type

$$u_{tt} - m\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = 0, \qquad \text{in } \Omega \times \mathbb{R}, \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, ∇u is the gradient of u with respect to space variables, and Δ is the Laplace operator.

If Ω is an interval of the real line, this equation is a model for the small transversal vibrations of an elastic string.

In the case where H admits a complete orthogonal system made by eigenvectors of A (this is the case *e.g.* in the concrete situation of (1.2) if Ω is bounded), then (1.1) may be thought as a system of ODEs with infinitely many unknowns, namely the components of u.

A lot of papers have been written on equation (1.1) and (1.2) after Kirchhoff's monograph [6]: the interested reader can find appropriate references in the surveys [1] and [7]. We just recall that, at the present, the existence of global solutions for all initial data in C^∞ or in Sobolev spaces is still an open problem.

In this paper we consider a particular class of global solutions of (1.1). Let us assume that λ is an eigenvalue of A, and e_λ is a corresponding eigenvector, which we assume normalized so that $|e_\lambda| = 1$. If the initial data are multiples of e_λ, say

$$u(0) = w_0 e_\lambda, \qquad u'(0) = w_1 e_\lambda,$$

then the solution of (1.1) remains a multiple of e_λ for every $t \in \mathbb{R}$, *i.e.* we have that $u(t) = w(t)e_\lambda$, where $w(t)$ is the solution of the ODE

$$w''(t) + \lambda m(\lambda w^2(t))w(t) = 0, \qquad w(0) = w_0, \quad w'(0) = w_1.$$

Such solutions are called *simple modes* of equation (1.1), and are known to be time periodic under very general assumptions on m.

In this paper we prove instability of high energy simple modes for particular choices of m.

To this end, we can limit ourselves to consider the two-mode system

$$\begin{cases} w''(t) + \lambda\, m(\lambda w^2(t) + \mu z^2(t))w(t) = 0, \\ z''(t) + \mu\, m(\lambda w^2(t) + \mu z^2(t))z(t) = 0, \end{cases} \tag{1.3}$$

where μ is another eigenvalue of A, corresponding to an eigenvector e_μ such that $|e_\mu| = 1$, and $u(t) = w(t)e_\lambda + z(t)e_\mu$.

It is clear that simple modes are particular solutions of this system, corresponding to initial data with $z(0) = z'(0) = 0$. Moreover, if $w(t)$ is unstable as a solution of (1.3), then $w(t)e_\lambda$ is an unstable simple mode of (1.1).

In order to simplify the notation, let us set

$$\nu := \frac{\mu}{\lambda}, \qquad u(t) := \sqrt{\lambda}\, w\left(\frac{t}{\sqrt{\lambda}}\right), \qquad v(t) := \sqrt{\mu}\, z\left(\frac{t}{\sqrt{\lambda}}\right),$$

so that (1.3) is equivalent to

$$\begin{cases} u''(t) + m(u^2(t) + v^2(t))u(t) = 0, \\ v''(t) + \nu\, m(u^2(t) + v^2(t))v(t) = 0. \end{cases} \tag{1.4}$$

This system (as well as (1.3) and (1.1)) is Hamiltonian, with conserved energy

$$H(u, u', v, v') := \frac{1}{2}\left\{[u']^2 + \frac{[v']^2}{\nu} + M(u^2 + v^2)\right\},$$

where

$$M(r) := \int_0^r m(s)\, ds. \tag{1.5}$$

As far as we know, stability of simple modes was studied in at least three papers.

- Dickey [3] proved that simple modes are *linearly stable* provided that their energy is *small* enough. Roughly speaking, linearly stable means that $v(t) \equiv 0$ is a stable solution for the linearization of the second equation in (1.4).
- The authors proved in [4] that simple modes are *orbitally stable* (see section 2.1 for precise definitions) provided that their energy is *small* enough.

- CAZENAVE and WEISSLER [2] assumed that there exists $\alpha > 0$ such that
$$\lim_{\sigma \to +\infty} \frac{m(\sigma r)}{m(\sigma)} = r^{\alpha},$$
uniformly on bounded intervals (for example $m(r) = 1 + r^{\alpha}$). They showed that if
$$\nu \in \bigcup_{m \in \mathbb{N}} \left((m+1)\left((\alpha+1)m+1\right), (m+1)\left((\alpha+1)m+1+2\alpha\right)\right),$$
then every simple mode of (1.4) with *large* enough energy is *unstable.* If $\alpha = 1$, and Ω is an interval of the real line, this result implies the instability of every simple mode of (1.2) with large enough energy.

The case we consider in this paper is, in a certain sense, the limit of [2] as $\alpha \to +\infty$. Our main result is the following.

THEOREM 1.1.— *Let $\nu > 1$, and let $m : [0, +\infty) \to (0, +\infty)$ be a C^1 function such that*

(m1) *m is nondecreasing;*

(m2) *for every $r \in [0, 1)$ we have that*
$$\lim_{\sigma \to +\infty} \frac{m(\sigma r)}{m(\sigma)} = 0.$$

Then there exists E_0 such that, if $H(u_0, u_1, 0, 0) > E_0$, then the simple mode of (1.4) with $u(0) = u_0$, $u'(0) = u_1$ is unstable.

A simple example of function statisfying (m1) and (m2) is $m(r) = e^r$.

We point out in particular that we have instability for every $\nu > 1$, which in the original equation (1.1) corresponds to perturbing a simple mode with any higher frequency mode. Therefore this results can be applied to the n-dimensional PDE (1.2), obtaining the following.

COROLLARY 1.2.— *Let m be as in Theorem 1.1, and let $\Omega \subseteq \mathbb{R}^n$ be any bounded open set.*

Then every simple mode of (1.2) with large enough energy is unstable.

Let us make a few comments on our assumptions on m. What we actually use are properties (m3), (m4), and (m5), which we deduce from (m1) and (m2) in Lemma 3.1. It should be possible to deduce them also from weaker versions of (m1) and (m2), but this would only complicate proofs without introducing new ideas.

Assumption (m2) is also suggested by the following observation.

Remark 1.3. — Let m be a continuous positive function such that

$$\lim_{\sigma \to +\infty} \frac{m(\sigma r)}{m(\sigma)}$$

exists for every $r \in (0, 1)$. Since this limit is a multiplicative function, then there are only three possibilities:

- the limit is r^α for some $\alpha > 0$ (case considered in [2]);
- the limit is 0 for every $r \in (0, 1)$ (case considered in this paper);
- the limit is 1 for every $r \in (0, 1)$.

We conjecture that in this last case, simple modes with high energy are always stable.

As in previous literature, our proof considers the limiting form (for high energy) of the differential of the Poincaré map relative to a simple mode (see section 2.2). It is well known that this differential can be characterized (see section 2.3) using a Hill's equation, *i.e.* a second order linear differential equation $z'' + \nu a(t) z = 0$ with a time periodic coefficient $a(t)$ depending on the simple mode.

In previous works, this coefficient tends to a function $a_\infty(t)$ as the energy tends to $+\infty$. In our case, on the contrary, $a(t)$ tends to a measure, which concentrates in some points. Despite of this initial difficulty, the limit solutions can be explicitly computed (see section 3). For this reason, we think that this is a good situation where conjectures about high energy solutions of Kirchhoff equations (not only simple modes) can be tested.

2. Definitions and preliminaries

In this section we recall the notion of stability, and then we describe the Poincaré map P_k associated to a simple mode u_k of (1.4). We also characterize the differential L_k of P_k, and we recall how instability can be reduced to an algebraic condition on L_k.

The only assumptions on m which we need in this section are those required in order to have periodic simple modes: to this end, $m \in C^1$ and $m(r) > 0$ for $r > 0$ are enough.

We refer to [5] for general facts about dynamical and Hamiltonian systems, and to [2, 4] for specific results related to the particular system (1.4).

2.1. Stability

In this section we recall some definitions of stability from the classical theory of Hamiltonian systems. For the sake of simplicity, we adapt definitions to the case of simple modes for system (1.4).

Given a real number $k > 0$, let us consider the simple modes u_k of system (1.4) which solve the problem

$$u_k''(t) + m(u_k^2(t))u_k(t) = 0, \qquad u_k(0) = k, \quad u_k'(0) = 0. \tag{2.1}$$

We recall that u_k is a periodic function, and so we can assume $u_k(0) > 0$ and $u_k'(0) = 0$ without loss of generality. Moreover assuming that k is large is equivalent to assuming that the energy of u_k is large.

Now in the phase space $\mathbb{R}^4$ we consider the energy level

$$\mathcal{H}_k := \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ H(x_1, x_2, x_3, x_4) = H(k, 0, 0, 0)\right\},$$

and the orbit

$$\Gamma_k := \{(u_k(t), u_k'(t), 0, 0) : \ t \in \mathbb{R}\}.$$

DEFINITION 2.1.— The simple mode u_k is called *orbitally stable* if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every solution $(u(t), v(t))$ of system (1.4), the following property holds: if the initial datum $(u(0), u'(0), v(0), v'(0))$ belongs to a δ neighborhood of $(k, 0, 0, 0)$, then for every $t \in \mathbb{R}$ the point $(u(t), u'(t), v(t), v'(t))$ lies in an ε neighborhood of Γ_k.

DEFINITION 2.2.— The simple mode u_k is called *isoenergetically orbitally stable* if the condition of Definition 2.1 is satisfied with the restriction that $(u(0), u'(0), v(0), v'(0)) \in \mathcal{H}_k$.

It is obvious that orbital stability implies isoenergetical orbital stability. When in this paper we write that simple modes are unstable, we mean that they do not fulfil Definition 2.2.

We often use also that solutions of (1.4) are *reversible*: this means that if $(u(t), v(t))$ is any solution, then $(u(-t), v(-t))$ is another solution. Thanks to reversibility, istability of a simple mode can be proved by showing the existence of a non-periodic trajectory in $\mathcal{H}_k$ which is asymptotic to u_k as $t \to +\infty$.

2.2. The Poincaré map

Let u_k be the simple mode of (1.4) which solves (2.1). Let us consider the open set $\mathcal{U}_k \subseteq \mathbb{R}^2$ defined by

$$\mathcal{U}_k := \left\{(x,y) \in \mathbb{R}^2 : \; H(0,0,x,y) < H(k,0,0,0)\right\}. \tag{2.2}$$

For every $(x,y) \in \mathcal{U}_k$, let $\alpha(x,y) > 0$ be the unique positive number such that

$$H(\alpha(x,y),0,x,y) = H(k,0,0,0).$$

Let $(u(t),v(t))$ be the solution of system (1.4) with initial data

$$u(0) = \alpha(x,y), \quad u'(0) = 0, \quad v(0) = x, \quad v'(0) = y.$$

Finally, let $T := T(x,y)$ be the smallest $t > 0$ such that $u'(t) = 0$ and $u(t) > 0$. The existence of such a T is classical up to restricting $\mathcal{U}_k$.

The Poincaré map $P_k : \mathcal{U}_k \to \mathbb{R}^2$, relative to the simple mode u_k, is defined by

$$P_k(x,y) := (v(T),v'(T)).$$

We point out that both v and T depend on (x,y) and k.

When $(x,y) = (0,0)$, then $u(t) = u_k(t)$ and $v(t) = 0$ for every $t \in \mathbb{R}$. It follows that $P_k(0,0) = (0,0)$, *i.e.* $(0,0)$ is a fixed point of the Poincaré map.

The interested reader is referred to the quoted literature, and in particular to [4], for a heuristic description of the Poincaré map.

Now we recall the classical definition of stability of fixed points for planar maps.

DEFINITION 2.3.— Let $\mathcal{U} \subseteq \mathbb{R}^2$ be an open set containing $(0,0)$, and let $P : \mathcal{U} \to \mathbb{R}^2$ be a map such that $P(0,0) = (0,0)$. The fixed point $(0,0)$ is said to be *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(x,y) \in \mathcal{U}, \; \|(x,y)\| < \delta \implies \|P^n(x,y)\| < \varepsilon \qquad \forall n \in \mathbb{N},$$

where P^n denotes the n-th iteration of P.

The stability of u_k as a periodic solution is clearly related to the stability of $(0,0)$ as a fixed point of P_k. This relation is stated in (P5) of the following Proposition, where we recall the main properties of the Poincaré map associated to u_k.

PROPOSITION 2.4 (PROPERTIES OF P_k). — *For every $k > 0$, let u_k and P_k be as above.*

Then

(P1) $P_k \in C^1(\mathcal{U}_k, \mathbb{R}^2)$, *and* $P_k(0,0) = (0,0)$;

(P2) P_k *is area-preserving;*

(P3) *if* $P_k(x,y) = (a,b)$, *then* $P_k(a,-b) = (x,-y)$;

(P4) $P_k(-x,-y) = -P_k(x,y)$;

(P5) *the simple mode u_k is orbitally stable if and only if $(0,0)$ is a stable fixed point of P_k.*

We don't give here a proof of such properties, since they are well known in the literature on dynamical systems. We only remark that (P2) follows from the Hamiltonian character of the system, (P3) is a consequence of reversibility, while (P4) is a consequence of the following fact: if $(u(t), v(t))$ is a solution of (1.4), then $(-u(t), -v(t))$ is also a solution.

Thanks to (P5), instability of simple modes can be proved by verifying instability of a fixed point of a planar map. In next section, we see that this can be reduced to an algebraic condition on the differential of P_k.

2.3. Linearization of the Poincaré map

Let $L_k : \mathbb{R}^2 \to \mathbb{R}^2$ be the differential of P_k at $(0,0)$. In the case of system (1.4), the linear operator L_k can be characterized in the following way. Given $(x,y) \in \mathbb{R}^2$, let $v_k(t)$ be the solution of the linear problem

$$v_k''(t) + \nu\, m(u_k^2(t)) v_k(t) = 0, \qquad v_k(0) = x, \quad v_k'(0) = y, \tag{2.3}$$

which is the linearization of the second equation of system (1.4). Then we have that

$$L_k(x,y) := (v_k(\tau_k), v_k'(\tau_k)),$$

where τ_k is the period of u_k. We point out that v_k depends on x, y, and k, while τ_k depends only on k, and is given by the formula

$$\tau_k = 4 \int_0^k \frac{dx}{\sqrt{M(k^2) - M(x^2)}}. \tag{2.4}$$

We don't give the proof of this characterization, since it is completely analogous to the proof of [2, Proposition 2.1].

In the following Proposition, we state the main properties of L_k, and its relations to stability.

PROPOSITION 2.5 (PROPERTIES OF L_k). — *For every $k > 0$, let u_k, P_k and L_k be as above. Then*

(L1) $\det L_k = 1$;

(L2) *if L_k^{ij} are the entries of the matrix representing L_k in the canonical basis, then $L_k^{11} = L_k^{22}$;*

(L3) *if $|L_k^{11}| > 1$, then $(0,0)$ is an unstable fixed point of P_k.*

Property (L1) is typical of Hamiltonian systems, while (L2) follows from (P3) (for the details, see the proof of [2, Lemma 3.3]).

If $|L_k^{11}| > 1$, then by (L2) the absolute value of the trace of L_k is > 2. Together with (L1), this implies in particular that the eigenvalues of L_k are δ, δ^{-1} for some $\delta \in \mathbb{R}$ with $|\delta| > 1$. In this case, in the literature the fixed point $(0,0)$ of the Poincaré map is called *hyperbolic*, and it is known to be unstable. Indeed, one can prove that there exists a 2-dimensional submanifold $\mathcal{M}$ of the 3-dimensional energy level $\mathcal{H}_k$ such that every trajectory with initial data in $\mathcal{M}$ tends to a simple mode in $\mathcal{H}_k$, as $t \to +\infty$, with an exponential rate depending on $|\delta|$. Due to reversibility, this implies instability.

In conclusion, Theorem 1.1 is proved, provided we verify that $|L_k^{11}| > 1$ for k large enough. This descends from the following result, which will be proved in next section.

THEOREM 2.6. — *Let m be as in Theorem 1.1, and let L_k be as above.*

Then

$$\lim_{k \to +\infty} L_k^{11} = 1 + 8\nu(\nu - 1).$$

In particular, this limit is > 1 for every $\nu > 1$.

We point out that an estimate of L_k^{11} provides an estimate of the eigenvalues of L_k, hence an estimate on the "instability rate".

Remark 2.7. — As we have seen, linearization is a useful tool in order to prove instability of periodic trajectories. Just for completeness, we recall that, on the contrary, linearization is in general inconclusive in stability problems for Hamiltonian systems. In our case, for example, if for some k we find that $|L_k^{11}| < 1$, then we can conclude that u_k is linearly stable (see

[3]), but if we want to prove that u_k is orbitally stable, then further terms in the Taylor expansion of P_k near $(0,0)$ must be kept into account.

3. Proofs

This section is devoted to the proof of Theorem 2.6. The result is achieved in three steps: first we rescale problems (2.1) and (2.3), and then we pass to the limit as $k \to +\infty$ in the two rescaled problems. When passing to the limit, we use the following properties of m, which follow from (m1) and (m2).

LEMMA 3.1.— *Let $m : [0,+\infty) \to (0,+\infty)$ be a continuous function satisfying (m1) and (m2). Let M be as in (1.5).*

Then the following properties hold:

(m3) *if τ_k is defined as in (2.4), then*

$$\lim_{k\to+\infty} \frac{\tau_k}{k}\sqrt{M(k^2)} = 4;$$

(m4) *as $k \to +\infty$ we have that*

$$\frac{M(k^2 r)}{M(k^2)} \to 0 \qquad \textit{uniformly in } [0,\delta]$$

for every $\delta \in (0,1)$;

(m5) *if τ_k is defined as in (2.4), then as $k \to +\infty$ we have that*

$$\tau_k^2 m(k^2 r) \to 0 \qquad \textit{uniformly in } [0,\delta]$$

for every $\delta \in (0,1)$.

Proof. — With the substitution $x = ky$ in (2.4), we have that

$$\frac{\tau_k}{k}\sqrt{M(k^2)} = \frac{\sqrt{M(k^2)}}{k}\cdot 4\int_0^1 \frac{k\,dy}{\sqrt{M(k^2)-M(k^2y^2)}} = 4\int_0^1 \frac{dy}{\sqrt{1-\frac{M(k^2y^2)}{M(k^2)}}}. \tag{3.1}$$

Now we show that the limit of the last integral is 1. Indeed, by de l'Hopital's Theorem and (m2) we have that

$$\lim_{k\to+\infty}\frac{M(k^2y^2)}{M(k^2)} = \lim_{k\to+\infty}\frac{y^2 m(k^2y^2)}{m(k^2)} = 0, \qquad \forall y\in[0,1). \tag{3.2}$$

Moreover by Cauchy's Theorem

$$\frac{M(k^2y^2)}{M(k^2)} = \frac{M(k^2y^2) - M(0)}{M(k^2) - M(0)} = \frac{2\xi y^2 m(\xi^2 y^2)}{2\xi m(\xi^2)}$$

for some $\xi \in (0, k)$. By (m1) the last term is $\leqslant y^2$, so that the integrand in the last term of (3.1) is $\leqslant (1 - y^2)^{-1/2}$. Therefore we can apply Lebesgue's Theorem, and this proves (m3).

In order to prove (m4), we first remark that by (3.2) we have pointwise convergence in $[0, 1)$. Since the functions are monotone in r, and the limit is continuous, then convergence is uniform on compact subsets of $[0, 1)$.

By the same argument, (m5) is proved provided we show that $\tau_k^2 m(k^2 r) \to 0$ for every $r \in [0, 1)$. To this end, let us fix $\rho \in (r, 1)$. Then by (m1) we have that

$$M(k^2) = \int_0^{k^2} m(s)ds \geqslant \int_{\rho k^2}^{k^2} m(s)ds \geqslant k^2(1-\rho)m(k^2\rho),$$

so that

$$0 \leqslant \tau_k^2 m(k^2 r) = \frac{\tau_k^2 M(k^2)}{k^2} \cdot \frac{k^2 m(k^2 r)}{M(k^2)} \leqslant \frac{\tau_k^2 M(k^2)}{k^2} \cdot \frac{m(k^2 r)}{m(k^2\rho)} \cdot \frac{1}{1-\rho}.$$

Now the first factor in the last term tends to 16 by (m3). Moreover, setting $\sigma = k^2\rho$, by (m2) we have that

$$\lim_{k\to+\infty} \frac{m(k^2 r)}{m(k^2\rho)} = \lim_{\sigma\to+\infty} \frac{m(\sigma^2 r \rho^{-1})}{m(\sigma^2)} = 0.$$

This completes the proof of (m5). □

3.1. Rescaling

Let u_k be the simple mode which solves (2.1), and let v_k be the solution of (2.3) with $(x, y) = (1, 0)$. Setting

$$w_k(t) = \frac{u_k(\tau_k t)}{k}, \qquad z_k(t) = v_k(\tau_k t),$$

it turns out that w_k and z_k are solutions of

$$w_k''(t) + \tau_k^2 m(k^2 w_k^2(t)) w_k(t) = 0, \qquad w_k(0) = 1, \quad w_k'(0) = 0. \tag{3.3}$$

and

$$z_k''(t) + \nu\tau_k^2 m(k^2 w_k^2(t)) z_k(t) = 0, \qquad z_k(0) = 1, \quad z_k'(0) = 0. \tag{3.4}$$

Moreover

$$L_k^{11} = v_k(\tau_k) = z_k(1). \tag{3.5}$$

We also set

$$a_k(t) := \tau_k^2 m(k^2 w_k^2(t)). \tag{3.6}$$

In the sequel we need the following properties of w_k:

(w1) w_k is a 1-periodic function, and for every $t \in [0, 1/4]$,

$$w_k(t) = w_k(1-t) = -w_k(1/2-t) = -w_k(1/2+t);$$

(w2) $w_k(0) = w_k(1) = 1$, and $w_k(1/2) = -1$;

(w3) w_k is decreasing in $[0, 1/2]$ and increasing in $[1/2, 1]$;

(w4) for every $t \in [0, 1]$ we have that

$$\frac{k^2}{\tau_k^2} |w_k'(t)|^2 + M(k^2 w_k^2(t)) = M(k^2);$$

(w5) $|w_k(t)| \leqslant 1$ for every $t \in [0, 1]$;

(w6) $|w_k'(t)| \leqslant \frac{\tau_k}{k} \sqrt{M(k^2)}$ for every $t \in [0, 1]$.

Property (w1) and (w3) follow from the simmetries of u_k, while (w2) is a particular case of (w1). Moreover, (w4) follows from the conservation of the Hamiltonian for u_k, and (w5) and (w6) are consequences of (w4).

We also use the following properties of a_k, which are trivial consequences of (3.6) and (w1):

(a1) $a_k(t) \geqslant 0$ for every $t \in [0, 1]$;

(a2) for every $t \in [0, 1/4]$ we have that

$$a_k(t) = a_k(1-t) = a_k(1/2-t) = a_k(1/2+t).$$

Remark 3.2. — Up to now, no particular properties of m have been used. Indeed, (3.3) and (3.4) can be considered as the starting point for every asymptotic investigation of L_k, both for k large and for k small.

Assumptions on m are crucial when one considers the limit of w_k and z_k. For example, using on m the assumptions of [2] (cf. the introduction of this paper), it is possible to prove that w_k and z_k tends to functions w_∞ and z_∞ satisfying the system

$$\begin{cases} w''_\infty + \gamma \left|w_\infty\right|^{2\alpha} w_\infty = 0, & w_\infty(0) = 1, \quad w'_\infty(0) = 0, \\ z''_\infty + \nu\gamma \left|w_\infty\right|^{2\alpha} z_\infty = 0, & z_\infty(0) = 1, \quad z'_\infty(0) = 0, \end{cases}$$

where the constant γ is choosen so that the period of w_∞ is 1.

The same system was obtained in [2] through another rescaling, which doesn't work under our assumptions on m.

3.2. Asymptotic behaviour of simple modes

In this section we pass to the limit as $k \to +\infty$ in problem (3.3). The final goal of this section is to prove that, for every $\varepsilon \in (0, 1/4)$,

$$\lim_{k\to+\infty} \int_0^\varepsilon a_k(t)dt = 4; \tag{3.7}$$

$$\lim_{k\to+\infty} \int_\varepsilon^{\frac{1}{2}-\varepsilon} a_k(t)dt = 0. \tag{3.8}$$

Such estimates will be crucial in section 3.3, where we pass to the limit in problem (3.4). In order to prove (3.7) and (3.8), we first compute several limits involving w_k.

3.2.1. Asymptotic behaviour of w_k

We prove that

$$w_k \to w_\infty \qquad \text{uniformly in } [0,1], \tag{3.9}$$

where

$$w_\infty := \begin{cases} 1 - 4t & \text{if } t \in [0, 1/2], \\ -3 + 4t & \text{if } t \in [1/2, 1]. \end{cases} \tag{3.10}$$

Indeed, thanks to (w5), (w6), (m3), and Ascoli's Theorem, we have that w_k converges (up to subsequences) to a limit, which we denote by w_∞. By (w2) we know that

$$w_\infty(0) = w_\infty(1) = 1, \qquad w_\infty(1/2) = -1. \tag{3.11}$$

Moreover, passing to the limit in (w6) and using (m3), we find that w_∞ is Lipschitz continuous, with Lipschitz constant $\leqslant 4$. Together with (3.11), this implies that w_∞ has the form given in (3.10). Finally, since the limit doesn't depend on the subsequence, we have that the whole family w_k converges to w_∞.

3.2.2. Asymptotic behaviour of w'_k

We prove that

$$\lim_{k\to+\infty} w'_k(t) = -4 \qquad \forall t \in (0, 1/2). \tag{3.12}$$

Indeed, from (w4), we have that

$$|w'_k(t)|^2 = \frac{\tau_k^2}{k^2} M(k^2) \left\{1 - \frac{M(k^2 w_k^2(t))}{M(k^2)}\right\}.$$

Now from (m3) we know that the first factor in the right hand side tends to 16. Moreover, by (m4), (3.9), and (3.10) we have that

$$\lim_{k\to+\infty} \frac{M(k^2 w_k^2(t))}{M(k^2)} = 0 \qquad \forall t \in \left[\delta, \tfrac{1}{2} - \delta\right]$$

for every $\delta \in (0, 1/4)$. Letting $\delta \to 0^+$, we have proved that $|w'_k(t)| \to 4$ for every $t \in (0, 1/2)$. Since by (w3) the function $w'_k(t)$ is negative in $(0, 1/2)$, convergence (3.12) is proved.

Remark 3.3. — In the same way it can be proved that $w'_k(t) \to 4$ for every $t \in (1/2, 1)$.

An alternative (but a little technical) proof of (3.12) can be obtained also without mentioning properties of m, but using only (3.9), and the fact that w_k is concave in $[0, 1/4]$ and convex in $[1/4, 1/2]$.

3.2.3. Asymptotic behaviour of a_k

We first prove that for every $\varepsilon \in (0, 1/4)$ we have that

$$a_k \to 0 \qquad \text{uniformly in } \left[\varepsilon, \tfrac{1}{2} - \varepsilon\right], \tag{3.13}$$

which in particular proves (3.8). By definition of a_k, (3.13) follows from (3.9) and (m5).

Now it remains to prove (3.7). To this end, since $0 \leqslant w_k(t) \leqslant 1$ in $[0, 1/4]$, then we have that

$$\begin{aligned}\int_0^{\varepsilon} a_k(t)\,dt &\geqslant \int_0^{\varepsilon} a_k(t) w_k(t)\,dt \\ &= -\int_0^{\varepsilon} w_k''(t)\,dt = w_k'(0) - w_k'(\varepsilon) = -w_k'(\varepsilon).\end{aligned}$$

Letting $k \to +\infty$ and using (3.12), we find that

$$\liminf_{k\to+\infty} \int_0^{\varepsilon} a_k(t)\,dt \geqslant \liminf_{k\to+\infty} (-w_k'(\varepsilon)) = 4.$$

In order to obtain the opposite inequality, we fix $\sigma \in (0, \varepsilon)$. By (3.13) we have that a_k converges to 0 uniformly in $[\sigma, \varepsilon]$, hence

$$\begin{aligned}\limsup_{k\to+\infty} \int_0^{\varepsilon} a_k(t)\,dt \leqslant \limsup_{k\to+\infty} \int_0^{\sigma} a_k(t)\,dt \\ + \limsup_{k\to+\infty} \int_{\sigma}^{\varepsilon} a_k(t)\,dt = \limsup_{k\to+\infty} \int_0^{\sigma} a_k(t)\,dt.\end{aligned}$$

In order to estimate the last term, we recall that w_k is decreasing in $[0, \sigma]$, hence

$$\begin{aligned}-w_k'(\sigma) = -w_k'(\sigma) + w_k'(0) = -\int_0^{\sigma} w_k''(t)\,dt \\ = \int_0^{\sigma} a_k(t) w_k(t)\,dt \geqslant w_k(\sigma) \int_0^{\sigma} a_k(t)\,dt,\end{aligned}$$

so that by (3.9), (3.10), and (3.12),

$$\limsup_{k\to+\infty} \int_0^{\sigma} a_k(t)\,dt \leqslant \limsup_{k\to+\infty} \left(-\frac{w_k'(\sigma)}{w_k(\sigma)}\right) = -\frac{w_\infty'(\sigma)}{w_\infty(\sigma)} = \frac{4}{1-4\sigma}.$$

Letting $\sigma \to 0^+$, thesis is proved.

3.3. Asymptotic behaviour of linearized Poincaré map

In this section we consider the asymptotic behaviour (as $k \to +\infty$) of the solution z_k of the Cauchy problem (note that this problem is exactly (3.4))

$$z_k''(t) + \nu a_k(t) z_k(t) = 0, \qquad z_k(0) = 1, \quad z_k'(0) = 0, \tag{3.14}$$

where the coefficients $a_k(t)$ satisfy (a1), (a2), (3.7), and (3.8). We prove that

$$z_k \to z_\infty \qquad \text{uniformly in } [0,1], \tag{3.15}$$

where

$$z_\infty := \begin{cases} 1 - 4\nu t & \text{if } t \in [0, 1/2], \\ (1 + 4\nu - 8\nu^2) + 4\nu(4\nu - 3)t & \text{if } t \in [1/2, 1]. \end{cases} \tag{3.16}$$

In particular, setting $t = 1$, we have that

$$\lim_{k \to +\infty} z_k(1) = 1 + 8\nu(\nu - 1),$$

which by (3.5) is exactly the conclusion of Theorem 2.6.

3.3.1. Compactness of z_k

We show that there exists $k_0 > 0$, and constants C_1 and C_2 such that, for every $k > k_0$,

$$|z_k(t)| \leqslant C_1 \qquad \forall t \in [0,1], \tag{3.17}$$

$$|z_k'(t)| \leqslant C_2 \qquad \forall t \in [0,1]. \tag{3.18}$$

By Ascoli's Theorem, this proves in particular that z_k uniformly converges, up to subsequences, to a limit, which we denote by z_∞. If we prove that z_∞ can be characterized as in (3.16), then we have that the whole family z_k converges to z_∞.

In order to prove these estimates we first remark that from (3.7), (3.8), and (a2), it follows that, for every $\varepsilon \in (0, 1/4)$,

$$\lim_{k \to +\infty} \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} a_k(t)\, dt = 8, \tag{3.19}$$

$$\lim_{k \to +\infty} \int_{\frac{1}{2}+\varepsilon}^{1-\varepsilon} a_k(t)\, dt = 0. \tag{3.20}$$

Moreover, there exists a constant C_3 such that

$$\int_0^1 a_k(t)\, dt \leqslant C_3, \qquad \forall k > k_0. \tag{3.21}$$

Now let us set

$$E_k(t) = \frac{1}{2}\left\{|z_k'(t)|^2 + \nu\,|z_k(t)|^2\right\}.$$

With simple calculations, we find that

$$\begin{aligned} E_k'(t) &= \nu(1-a_k(t))z_k(t)z_k'(t) \\ &\leqslant \nu\,|1-a_k(t)|\cdot|z_k(t)|\cdot|z_k'(t)| \\ &\leqslant \sqrt{\nu}\,(1+a_k(t))\,E_k(t), \end{aligned}$$

hence, by Gronwall's Lemma and (3.21),

$$E_k(t) \leqslant E_k(0)\exp\left\{\sqrt{\nu}\int_0^t (1+a_k(t))dt\right\} \leqslant \frac{\nu}{2}\exp\left\{\sqrt{\nu}(1+C_3)\right\} \qquad \forall t \in [0,1].$$

By definition of E_k, inequalities (3.17) and (3.18) are proved.

3.3.2. Characterization of z_∞ for $t \in [0,1/2]$

We prove that

$$\lim_{k\to+\infty} z_k'(t) = -4\nu = z_\infty'(t) \qquad \forall t \in (0,1/2), \tag{3.22}$$

and

$$\lim_{k\to+\infty} z_k(t) = 1-4\nu t = z_\infty(t) \qquad \forall t \in [0,1/2]. \tag{3.23}$$

In order to compute the first limit, we fix ε such that $0<\varepsilon<\min\{t,1/2-t\}$. By (3.4) we have that

$$\begin{aligned} z_k'(t) &= z_k'(0) + \int_0^t z_k''(s)ds \\ &= -\nu\int_0^t a_k(s)z_k(s)ds \\ &= -\nu\int_0^\varepsilon a_k(s)z_k(s)ds - \nu\int_\varepsilon^t a_k(s)z_k(s)ds. \end{aligned} \tag{3.24}$$

Now let us compute the limit of the second summand. From (3.17) we have that

$$\left|\int_\varepsilon^t a_k(s)z_k(s)ds\right| \leqslant C_1\int_\varepsilon^t a_k(s)ds \leqslant C_1\int_\varepsilon^{\frac{1}{2}-\varepsilon} a_k(s)ds,$$

hence by (3.8)

$$\lim_{k\to+\infty} \nu \int_{\varepsilon}^{t} a_k(s)z_k(s)ds = 0 \tag{3.25}$$

for every fixed ε.

In order to estimate the first summand in (3.24), from $z_k(0) = 1$ and (3.18) we deduce that

$$1 - C_2\varepsilon \leqslant z_k(s) \leqslant 1 + C_2\varepsilon \qquad \forall s \in [0,\varepsilon],$$

hence

$$(1 - C_2\varepsilon) \int_0^{\varepsilon} a_k(s)ds \leqslant \int_0^{\varepsilon} a_k(s)z_k(s)ds \leqslant (1 + C_2\varepsilon) \int_0^{\varepsilon} a_k(s)ds.$$

By (3.7), (3.24), and (3.25), we therefore have that

$$-4\nu(1 - C_2\varepsilon) \leqslant \liminf_{k\to+\infty} z_k'(t) \leqslant \limsup_{k\to+\infty} z_k'(t) \leqslant -4\nu(1 + C_2\varepsilon).$$

Since ε is arbitrary, (3.22) follows letting $\varepsilon \to 0^+$.

In order to prove (3.23) we remark that

$$z_k(t) = z_k(0) + \int_0^t z_k'(s)ds \qquad \forall t \in [0, 1/2],$$

and then we pass to the limit in the integral (we can apply Lebesgue's Theorem by (3.18) and (3.22)).

3.3.3. Characterization of z_∞ for $t \in [1/2, 1]$

We prove that

$$\lim_{k\to+\infty} z_k'(t) = 4\nu(4\nu - 3) = z_\infty'(t) \qquad \forall t \in (1/2, 1), \tag{3.26}$$

and

$$\lim_{k\to+\infty} z_k(t) = (1 + 4\nu - 8\nu^2) + 4\nu(4\nu - 3)t = z_\infty(t) \qquad \forall t \in [1/2, 1]. \tag{3.27}$$

In order to compute the first limit, we fix ε such that $0 < \varepsilon < \min\{t - 1/2, 1 - t\}$. By (3.4) we have that

$$z_k'(t) \;=\; z_k'\left(\frac{1}{2} - \varepsilon\right) + \int_{\frac{1}{2}-\varepsilon}^{t} z_k''(s)ds$$

$$
\begin{aligned}
&= z_k'\left(\frac{1}{2}-\varepsilon\right)-\nu\int_{\frac{1}{2}-\varepsilon}^{t} a_k(s)z_k(s)ds \\
&= z_k'\left(\frac{1}{2}-\varepsilon\right)-\nu\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} a_k(s)z_k(s)ds-\nu\int_{\frac{1}{2}+\varepsilon}^{t} a_k(s)z_k(s)ds \quad (3.28)
\end{aligned}
$$

From (3.22) we know that the limit of the first summand is -4ν. Moreover, arguing as in the proof of (3.25), we can show that the limit of the third summand is 0.

In order to estimate the second summand in (3.28), from (3.18) we deduce that

$$
C_2\varepsilon \leqslant z_k(s)-z_k\left(\frac{1}{2}\right)\leqslant C_2\varepsilon \qquad \forall s\in[0,\varepsilon],
$$

hence

$$
\begin{aligned}
\left(z_k\left(\frac{1}{2}\right)-C_2\varepsilon\right)\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} a_k(s)ds &\leqslant \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} a_k(s)z_k(s)ds \\
&\leqslant \left(z_k\left(\frac{1}{2}\right)+C_2\varepsilon\right)\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} a_k(s)ds.
\end{aligned}
$$

By (3.19) and (3.28), we therefore have that

$$
\begin{aligned}
-4\nu-8\nu\left(z_\infty\left(\frac{1}{2}\right)-C_2\varepsilon\right) &\leqslant \liminf_{k\to+\infty} z_k'(t) \\
&\leqslant \limsup_{k\to+\infty} z_k'(t) \leqslant -4\nu-8\nu\left(z_\infty\left(\frac{1}{2}\right)+C_2\varepsilon\right).
\end{aligned}
$$

By (3.23), (3.26) follows letting $\varepsilon\to 0^+$.

In order to prove (3.27), we remark that

$$
z_k(t)=z_k\left(\frac{1}{2}\right)+\int_{\frac{1}{2}}^{t} z_k'(s)ds \qquad \forall t\in[0,1/2],
$$

and then we pass to the limit using the convergence for $t=1/2$ proved in (3.23), and Lebesgue's Theorem.

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