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On a Galoisian Approach to the Splitting of Separatrices(*)

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RÉSUMÉ. — Pour les systèmes hamiltoniens analytiques de deux degrés de liberté avec une orbite homoclinique associée à un point d'équilibre selle-centre, nous faisons le rapport entre deux différents critères de non-intégrabilité : le critère (algébrique) donné par la théorie de Galois différentielle (une version sophistiquée du théorème de non-intégrabilité de Ziglin sur les équations aux variations complexes et analytiques attachées à une courbe intégrale particulière) et un théorème de Lerman sur l'existence des orbites homocliniques transversales dans la partie réelle de l'espace des phases. Pour obtenir ce résultat, on utilise une interprétation du théorème de Lerman donné par Grotta-Ragazzo.

ABSTRACT. — For two degrees of freedom analytic Hamiltonian systems with a homoclinic orbit to a saddle-center equilibrium point, we make the connection between two different approaches to non-integrability: the (algebraic) Galois differential approach (a sophisticated version of Ziglin's non-integrability theorem on the complex analytical variational equations associated to a particular integral curve) and a theorem of Lerman about the existence of transversal homoclinic orbits in the real part of the phase space. In order to accomplish this we use an interpretation given by Grotta-Ragazzo of Lerman's theorem.

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1. Introduction

The motivation of this work is to clarify the relations between the real chaotic dynamics of non integrable Hamiltonian systems and the purely algebraic Galois differential criteria of non integrability based on the analysis (in the complex phase space) of the variational equations along a particular integral curve. This problem was posed in [13] (Sect. 6).

Concretely, and as a first step to understand the above problem, we consider the relatively simple situation of a two degrees of freedom Hamiltonian system with a (real) homoclinic orbit contained in an invariant plane and asymptotic to a center-saddle equilibrium point. In this situation Lerman [9] gives a necessary criteria (in terms of some kind of asymptotic monodromy matrix of the normal variational equations along the homoclinic orbit) for the non existence of transversal homoclinic orbits associated to the invariant manifolds of the Lyapounov orbits around the equilibrium point (i.e., real "dynamical integrability" in a neigbourhood of the homoclinic orbit). This condition was interpreted by Grotta-Ragazzo [4] in terms of a global monodromy matrix of the algebraic normal variational equation in the complex phase space and he conjectured the existence of some kind of relation between the Lerman's theorem and Ziglin's non integrability criteria about the monodromy of the normal variational equations ([17], [18]). The present paper is devoted to clarify this relation. Instead of Ziglin's original theorem we prefer to work with a more general theory in terms of the differential Galois group of the variational equations ([2], [12], [1], [11]). This theory (as stated in [11]) roughly says that a necessary condition for (complex analytical) complete integrability is the solvability (in the Galois differential sense) of the variational equations along any integral curve. In fact the identity component of the Galois group must be abelian (see Theorem 2).

The main result of our paper says that (under suitable assumptions of complex analitycity) the two above necessary conditions for integrability are indeed the same, when we restrict the analysis to a complex neighbourhood of the real homoclinic orbit (Prop. 3). The Section 6 is devoted to a detailed analysis of an example with a normal variational equation of Lamé type.

2. Differential Galois Theory

The Galois differential theory for linear differential equations is the Picard–Vessiot Theory. We shall expose the minimum of definitions and results of this theory ([6], [10], [15]).

A differential field K is a field with a derivative $\mathrm{d}/\mathrm{d}t := {}'$, i.e., an additive mapping that satisfies the Leibnitz rule. Examples are $\mathcal{M}(\mathcal{R})$ (meromorphic functions over a Riemann surface $\mathcal{R}$) with an holomorphic tangent vector field X as derivative, $\mathbb{C}(t) = \mathcal{M}(\mathbf{P}^1)$, $\mathbb{C}\{t\}[t^{-1}]$ (convergent Laurent series), $\mathbb{C}[[t]][t^{-1}]$ (formal Laurent series). We observe that between the above fields there are some inclusions.

We can define (differential) subfields, (differential) extensions in a direct way by impossing that the inclusions must commutes with the derivation. Analogously, an (differential) automorphism in K is an automorphism that commutes with the derivative. The field of constants of K is the kernel of the derivative. In the above examples it is $\mathbb{C}$. From now on we will suppose that this is the case.

Let

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = A\xi\,, \quad A \in \mathrm{Mat}(m, K) \tag{1}$$

be a linear differential equation with coefficients in the differential field K.

We shall proceed to associate to (1) the so called Picard–Vessiot extension of K. The Picard–Vessiot extension L of (1) is an extension of K, such that if $u_1, \ldots, u_m$ is a "fundamental" system of solutions of the equation (1) (i.e., linearly independent over $\mathbb{C}$), then $L = K(u_{ij})$ (rational functions in K in the coefficients of the "fundamental" matrix $(u_1 \ldots u_m)$ and its derivatives). This is the extension of K generated by K together with u_{ij}. We observe that L is a differential field (by (1)). The existence and uniqueness (except by isomorphism) of the Picard–Vessiot extensions is proved by Kolchin (in the analytical case, $K = \mathcal{M}(\mathcal{R})$, this result is essentially the existence and uniqueness theorem for linear differential equations).

As in the classical Galois theory we define the Galois group of (1) $G := \mathrm{Gal}_K(L)$ as the group of all the (differential) automorphims of L which leave fixed the elements of K. This group is isomorphic to an algebraic linear group over $\mathbb{C}$, i.e., a subgroup of $\mathrm{GL}(m, \mathbb{C})$ whose matrix coefficients satisfy polynomial equations over $\mathbb{C}$.

By the normality of the Picard–Vessiot extensions it is proved that the Galois correspondence (group-extension) works well in this theory.

THEOREM 1.— *Let L/K be the Picard-Vessiot extension associated to a linear differential equation. Then there is a $1-1$ correspondence between the intermediary differential fields $K \subset M \subset L$ and the algebraic subgroups $H \subset G := \mathrm{Gal}_K(L)$, such that $H = \mathrm{Gal}_M(L)$. Furthermore, the normal extensions M/K correspond to the normal subgroups $H \subset \mathrm{Gal}_K(L)$ and $G/H = \mathrm{Gal}_K(M)$.*

As a corollary when we consider the algebraic closure $\overline{K}$ (of K in L), $\mathrm{Gal}_K(\overline{K}) = G/G_0$, where $G_0 = \mathrm{Gal}_{\overline{K}}(L)$ is the connected component of the Galois group, G, containing the identity. Note that this identity component corresponds to the transcendental part of the Picard–Vessiot extension. Here we have used the Zariski topology: a closed set is, by definition, the set of common zeros of polynomials.

Another consequence of the theorem is that if $\Lambda \subset \mathcal{R}$ is a Riemann surface contained in $\mathcal{R}$ and L is a Picard–Vessiot extension of $\mathcal{M}(\mathcal{R})$, then $\mathrm{Gal}_{\mathcal{M}(\Lambda)}(L) \subset \mathrm{Gal}_{\mathcal{M}(\mathcal{R})}(L)$.

We will say that a linear differential equation is (Picard–Vessiot) solvable if we can obtain its Picard–Vessiot extension and, hence, the general solution, by adjunction to K of integrals, exponentials of integrals or algebraic functions of elements of K (the usual terminology is that in this case the Picard–Vessiot extension is of Liouville type). Then, it can be proved, that the equation is solvable if, and only if, the identity component G_0 is a solvable group. In particular, if the identity component is abelian, the equation is solvable.

Furthermore, the relation between the monodromy and the Galois group (in the analytic case) is as follows. The monodromy group is contained in the Galois group and if the equation is of fuchsian class (i.e., has only regular singularities), then the monodromy group is dense in the Galois group (in the Zariski topology).

We recall here the classification of the algebraic subgroups of $\mathrm{SL}(2,\mathbb{C})$. From [8, p. 7] (or [5, p. 32]), it is possible to prove the following result.

PROPOSITION 1.— *Any algebraic subgroup G of $\mathrm{SL}(2,\mathbb{C})$ is conjugated to one of the following types:*

1) finite, $G_0 = \{\mathbf{1}\}$, where $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

2) $G = G_0 = \left\{ \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}, \mu \in \mathbb{C} \right\}$;

3) $G = \left\{ \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix}, \lambda \text{ is a root of unity}, \mu \in \mathbb{C} \right\}$,

$G_0 = \left\{ \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}, \mu \in \mathbb{C} \right\}$;

4) $G = G_0 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbb{C}^* \right\}$;

5) $G = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix} \lambda, \beta \in \mathbb{C}^* \right\}$,

$G_0 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbb{C}^* \right\}$;

6) $G = G_0 = \left\{ \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbb{C}^*, \mu \in \mathbb{C} \right\}$;

7) $G = G_0 = \mathrm{SL}(2, \mathbb{C})$.

We remark that the identity component G_0 is abelian in the cases 1)-5).

3. Non integrability and Galois Differential Theory

Let us now consider a complex analytic symplectic manifold of dimension $2n$ and X_H a holomorphic Hamiltonian system defined on it. Let Γ the Riemann surface corresponding to an integral curve $z = z(t)$ (which is not an equilibrium point) of X_H. Then we can obtain the variational equations (VE) along Γ,

$$\dot{\eta} = X'_H\big(z(t)\big)\,\eta\,.$$

By using the linear first integral $\mathrm{d}H\big(z(t)\big)$ of the VE it is possible to reduce it by one degree of freedom, and obtain the so called normal variational equation (NVE)

$$\dot{\xi} = JS(t)\xi\,,$$

where, as usual,

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

is the matrix of the symplectic form (of dimension $2(n-1)$). Furthermore the coefficients of the matrix $S(t)$ are holomorphic on Γ.

Now, we shall complete the Riemann surface Γ with some equilibrium points and (possibly) the point at infinity, in such a way, that the coefficients of the matrix $S(t)$ are meromorphic on this extended Riemann surface $\overline{\Gamma} \supset \Gamma$. So, the field of coefficients K of the NVE is the field of meromorphic functions on $\overline{\Gamma}$. To be more precise, $\overline{\Gamma}$ is contained in the Riemann surface defined by the desingularization of the analytical (in general singular) curve C in the phase space given by the integral curve $z = z(t)$ with their adherent equilibrium points, the singularities of the Hamiltonian system and the points at infinity. For more information see [11]. Anyway, in Sections 5 and 6 we shall make explicit all the neccesary details in our particular case.

Then, in the above situation, it is proved in [11] the following result.

THEOREM 2.— *Assume there are n meromorphic first integrals of X_H in involution and independent in a neigbourhood of the analytical curve C, not necessarily on C itself, and the points at the infinity of the NVE are regular singularities. Then the identity component of the Galois group of the NVE is an abelian symplectic group.*

We note that, for two degrees of freedom and if the NVE is of fuchsian type, the above result is contained in [2] and [12] (in fact, this particular case is the only result that we will need in this paper).

Furthermore, the conclusion of the theorem is the same if we restrict the NVE to some domain Λ of $\overline{\Gamma}$ and the Galois group of this restricted equation is, by the galoisian correspondence, an algebraic subgroup of the Galois total group of the NVE (Sect. 2).

To end this section, we shall give an intrinsic Galois differential criterium for a second order linear differential equation to be symplectic. We say that the equation (with coefficients P and Q in a differential field K)

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + P(t)\frac{\mathrm{d}\xi}{\mathrm{d}t} + Q(t)\xi = 0 \tag{2}$$

is symplectic (or Hamiltonian) if it has a Galois group contained in $\mathrm{SL}(2,\mathbb{C})$. Now, for all σ in the Galois group, the Wronskian $W \in K$ if, and only if, $W = \sigma(W) = \det(\sigma)W$, which is equivalent to $\det(\sigma) = 1$. From this it is easy to prove that the above equation is symplectic if, and only if, $P = d'/d$, $d \in K$ (it suffices to consider the Abel equation $W' + PW = 0$ and then $W = 1/d$). For more information on the galoisian approach to linear Hamiltonian systems ([1], [11]).

4. Grotta-Ragazzo interpretation of Lerman's theorem

Let X_H be a two degrees of freedom real analytic Hamiltonian system with an homoclinic orbit $\Gamma_{\mathbb{R}}$ to a saddle-center equilibrium point.

Let Φ be the flow map along the homoclinic orbit,between two points in the homoclinic orbit and contained in a small enough neigborhood U of the equilibrium point.

THEOREM 3 (cf. [9]).— *There are suitable coordinates in U such that in these coordinates, the linearized flow is given by*

$$\mathrm{d}\Phi = \begin{pmatrix} P & Q \\ S & R \end{pmatrix},$$

being R the 2×2 matrix corresponding to the normal variational equation along $\Gamma_{\mathbb{R}}$. Now assume that the stable and unstable invariant manifolds of every (small enough) Lyapounov orbit are the same. Then R must be a rotation.

From now on in this paper we will restrict to classical Hamiltonian systems $H = (1/2)(y_1^2 + y_2^2) + V(x_1, x_2)$. Then if the homoclinic orbit $\Gamma_{\mathbb{R}}$ is contained in an invariant plane (x_1, y_1), we can write the Hamiltonian as

$$H = \frac{1}{2}\,(y_1^2 + y_2^2) + \varphi(x_1) + \frac{1}{2}\,\alpha(x_1)x_2^2 + \text{h.o.t.}(x_2), \tag{3}$$

being y_i the canonically conjugated momentum to the position x_i, and

$$\varphi(x_1) = -\frac{1}{2}\,\nu^2 x_1^2 + \text{h.o.t.}(x_1), \quad \alpha(x_1) = \omega^2 + \text{h.o.t.}(x_1)\,, \tag{4}$$

with ν and ω non-vanishing real parameters.

In his interpretation of Lerman's Theorem, Grotta-Ragazzo considered the (complexified) NVE along the (complex) homoclinic orbit Γ (Γ: $x_1 = x_1(t)$, $y_1 = y_1(t)$, $x_2 = y_2 = 0$)

$$\ddot{\xi} + \alpha\big(x_1(t)\big)\xi = 0\,.$$

By the change of independent variables $x := x_1(t)$, he obtain the "algebraic" form of the NVE

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}x^2} + \frac{\varphi'(x)}{2\varphi(x)}\,\frac{\mathrm{d}\xi}{\mathrm{d}x} - \frac{\alpha(x)}{2\varphi(x)}\xi = 0\,. \tag{5}$$

Among the real singularities of this variational equation there are the equilibrium point $x_1 = x = 0$ and the branching point (of the covering $t \to x$)

$$\left(x_1 = x = e\,,\; y_1 = \big(2\varphi(e)\big)^{1/2} = 0\right)$$

corresponding to the zero velocity point of the homoclinic orbit. Let σ be the closed simple arc (element of the fundamental group) in the x-plane surrounding only the singularities $x = 0$, $x = e$ and m_σ the monodromy matrix of the above equation along σ. Then Grotta-Ragazzo obtained the following result (cf. [4, Theorem 8]).

THEOREM 4.— *The matrix R in Theorem 3 is a rotation if, and only if, $m_\sigma^2 = \mathbf{1}$ (identity).*

In his proof Grotta-Ragazzo used the relation between the monodromy m_σ and the reflexion coefficient of the NVE.

5. Differential Galois approach

In this section we shall give the relation between the Grotta-Ragazzo result in the last section (Theorem 4) and the Galois differential obstruction to integrability (Theorem 2). In order to get this, we start with a reformulation of Theorem 4.

So, let X_H be a two degrees of freedom hamiltonian system with a saddle-center equilibrium point and an homoclinic orbit $\Gamma_{\mathbb{R}}$ to this point contained in an invariant plane $x_2 = 0$, $y_2 = 0$.

Now we consider this real Hamiltonian system as the restriction to the real domain of a complex holomorphic Hamiltonian system (with complex time), as in Section 3. If we add the origin to the homoclinic orbit, then we get a complex analytic singular curve. The origin is the singular point and by desingularization one obtains a nonsingular (in a neighbourhood of origin) analytic curve $\overline{\Gamma}$. On $\overline{\Gamma}$ there are two points, 0^+ and 0^-, corresponding to the origin. We note that the homoclinic orbit is, up to first order, defined by $x_1 y_1 = 0$, while the desingularized curve is defined by the pair of disconnected lines $x_1 = 0$, $y_1 = 0$ with two points at the origin. These points are, in the temporal parametrization, $t = +\infty$ and $t = -\infty$ (a standard book on complex curves is [7]).

We are only interested in a domain $\overline{\Gamma}_{\mathrm{loc}}$ of the Riemann surface $\overline{\Gamma}$ wich contains $\Gamma_{\mathbb{R}}$ and the points 0^+ and 0^-. This Riemann surface $\overline{\Gamma}_{\mathrm{loc}}$ is parametrized by three coordinate charts A_-, A_t and A_+ with coordinates $x := x_1$, t and $y := y_1$ respectively. Then, by restriction to a small enough domain, it is always possible to get a Riemann surface $\overline{\Gamma}_{\mathrm{loc}}$ such that the only singularities of the NVE on it are 0^+ and 0^-.

Let γ be the closed simple path in $\overline{\Gamma}_{\mathrm{loc}}$ surrounding $\Gamma_{\mathbb{R}}$. If we denote by m_γ the corresponding monodromy matrix of the NVE, then by the double covering $t \to x$ of the above section, we have $m_\gamma = m_\sigma^2$. Hence, by the Theorem 4, the following result.

PROPOSITION 2.— *The matrix R in Theorem 3 is a rotation if, and only if, $m_\gamma = \mathbf{1}$.*

In order to obtain the connection with the Galois Theory we need to make some analysis on the algebraic groups of $\mathrm{SL}(2,\mathbb{C})$ generated by hyperbolic elements.

LEMMA 1.— *Let M be a subgroup of $\mathrm{SL}(2,\mathbb{C})$ generated by k elements $m_1, m_2, \ldots, m_k$, such that each m_i has eigenvalues $(\lambda_i, \lambda_i^{-1})$ with $|\lambda_i| \neq 1$, $i = 1, 2, \ldots, k$. Then the closed group $\overline{M}$ (in the Zariski topology) must be one of the groups 4), 6) or 7) of the Proposition 1.*

Proof.— As $\overline{M}$ is an algebraic subgroup of $\mathrm{SL}(2,\mathbf{C})$, it is one of the groups 1)-7) in the Proposition 1. We are going to analyze each of the cases. The group M is not a finite group as m_i has infinite order. Also m_i is not contained in the triangular groups of types 2) or 3) of Proposition 1, because the eigenvalues of all the elements of these groups have eigenvalues on the unit circle.

Finally, if we are in case 5), necessarily $m_i \in G_0$ (because the eigenvalues of $G \setminus G_0$ are in the unit circle). But then $M \subset G_0$ (G_0 is a group) and $\overline{M} \neq G$ ($\overline{M}$ is the smallest algebraic group wich contains M, and furthermore G_0 is an algebraic group). □

As we shall show, this elementary result is central in our analysis.

Now we come back to the local homoclinic complex orbit Γ_{loc} with the two singularities 0^+, 0^- coming from the equilibrium point, and with monodromy matrices m_+, m_-. Let $m_\gamma = m_+ m_-$ be the monodromy around the two singular points as in the analysis in the first part of this section. Let G_{loc} be the Galois group of the NVE restricted to Γ_{loc} (this is a linear differential equation with meromorphic coefficients over the simply connected domain of the complex plane $\overline{\Gamma}_{\text{loc}}$ obtained by adding to Γ_{loc} the two singular points 0^+ and 0^-). As a direct consequence of the lemma above we get the following result.

PROPOSITION 3.— *The monodromy matrix m_γ is equal to the identity if, and only if, the identity component $(G_{\text{loc}})_0$ is abelian. Furthermore, in this case, the Galois group is of the type 4) of the Proposition 1.*

Proof.— If $m_+m_- = \mathbf{1}$ it is clear that the monodromy group M is abelian, for the monodromy group is generated by a single element (for instance, m_+). As the equation is of Fuchs type, then $\overline{M} = G_{\text{loc}}$ is abelian and of the type 4) of the Proposition 1 (as the reader can verify, the Zariski closure of the group generated by a diagonal matrix of infinite order in $\mathrm{SL}(2, \mathbb{C})$ is always of the type 4)).

Reciprocally, we know that the monodromy group has two generators m_+, m_- with eigenvalues inverses one of the another and liying outside of the unit circle. By the lemma, if the identity component $(G_{\text{loc}})_0$ is abelian, then as $\overline{M} = G_{\text{loc}}$, G is of the type 4). Furthermore, from the fact that the two matrices m_+, m_- have inverses eigenvalues ($\lambda_+ = \lambda_-^{-1}$), being $(\lambda_+, \lambda_+^{-1})$ and $(\lambda_-, \lambda_-^{-1})$ the eigenvalues of m_+ and m_- respectively, we get the desired result. □

In this way, we have proved that two unrelated first order obstructions to integrability, are in fact the same (under the suitable assumptions of analyticity). The first one given by the condition in the Lerman's theorem has been formulated in terms of real dynamics (the existence of transversal homoclinic orbits). The second one is formulated in an algebraic way (Theorem 1) and only has a meaning in the complex setting. Summarizing, we have obtained the following differential Galois interpretation of the Lerman and Grotta-Ragazzo results.

THEOREM 5.— *If the identity component $(G_{\text{loc}})_0$ is not abelian, then there does not exist an additional meromorphic first integral in a neigbourghood of Γ_{loc} and the invariant manifolds of the Lyapounov orbits must intersect transversally.*

6. Example

We shall to apply the above to a two degrees of freedom potential with a NVE of Lamé type.

We take in (3)

$$\varphi(x_1) = -\frac{1}{2e_1e_2}\, x_1^2(x_1 - e_1)(x_1 - e_2)\,,$$
$$\alpha(x_1) = \frac{3}{16\, e_1e_2}\, x_1^2 + ax_1 + b\,,$$

where we normalize $\nu = 1$ and $b := \omega^2$. So, the system depends on four (real) parameters e_1, e_2, a, b, being $e_1 \neq e_2$ and $b > 0$.

We are going to explicite for this example $\Gamma_{\mathbb{R}}$, C, Γ, and $\overline{\Gamma}$, introduced in the Sections 3 and 4 (from these constructions we get Γ_{loc} and $\overline{\Gamma}_{\text{loc}}$ as in Section 5).

The (complex) analytical curve C is $y_1^2 + 2\varphi(x_1) = 0$ $(x_2 = y_2 = 0)$. Without loss of generality we assume $e_1 > 0$ and then either $0 < e_1 < e_2$ or $e_2 < 0 < e_1$. In both cases we can take $\Gamma_{\mathbb{R}}$ as the unique real homoclinic orbit contained in C, $0 < x_1 \leq e_1$ (the canonical change $x_1 \to -x_1$, $y_1 \to -y_1$, reduces all the possibilities to the above one). The complex orbit Γ is C minus the origin (we recall that the temporal parameter t is a local parameter on Γ).

The desingularized curve $\overline{\Gamma}$ is the projective line $\mathbf{P}^1$. In fact, by the standard birrational change $x_1 = \widetilde{x}$, $y_1 = \widetilde{x}\widetilde{y}/\sqrt{e_1e_2}$, we get the genus zero curve $\widetilde{y}^2 = (\widetilde{x} - e_1)(\widetilde{x} - e_2)$. Now, with the change $\widetilde{y} = (1/2)(e_1 - e_2)\widehat{y}$, $\widetilde{x} = (1/2)(e_1 - e_2)\widehat{x} + (1/2)(e_1 + e_2)$, one obtain the curve $\widehat{x}^2 - \widehat{y}^2 = 1$. This last curve is parametrized by the rational functions

$$\widehat{x} = \frac{r^2+1}{r^2-1}\,, \quad \widehat{y} = 2\frac{r}{r^2-1}\,.$$

If we compose all these changes we obtain a rational r-parametrization of $\overline{\Gamma}$. So, $\overline{\Gamma} = \Gamma \cup \{r = \pm\sqrt{e_2/e_1}\,,\ r = \pm 1\}$, being $r = \pm\sqrt{e_2/e_1}$ the two points corresponding to the origin $x_1 = y_1 = 0$, and $r = \pm 1$ the two points at the infinity. It is interesting to express $\Gamma_{\mathbb{R}}$ in this parametrization:

$$\Gamma_{\mathbb{R}} = \begin{cases} \left\{ r \in \mathbb{C} \;\middle|\; \Re(r) = 0, |r| > \sqrt{-\dfrac{e_2}{e_1}} \right\} \cup \{\infty\} & \text{if } e_1e_2 < 0\,, \\ \left\{ r \in \mathbb{C} \;\middle|\; \Im(r) = 0\,,\ |r| > \sqrt{\dfrac{e_2}{e_1}} \right\} \cup \{\infty\} & \text{if } e_1e_2 > 0\,. \end{cases}$$

Then, the NVE in these coordinates is

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}r^2}+P\frac{\mathrm{d}\eta}{\mathrm{d}r}+Q\eta=0\,, \tag{6}$$

where

$$P=2\frac{re_1}{e_1r^2-e_2}\quad\text{and}\quad Q=\frac{Cr^4-Dr^2+E}{4\left(e_1r^2-e_2\right)^2\left(r^2-1\right)^2}\,,$$

with

$$C=16\,ae_1^2e_2+16\,be_1e_2+3e_1^2\,,\quad D=16\,ae_1^2e_=2+32\,be_1e_2+16\,ae_1e_2^2\,,$$
$$E=16\,be_1e_2+3e_2^2+16\,ae_1e_2^2\,.$$

We know from the general theory that this equation is symplectic. It is easy to check this in a direct way, so $P=(\mathrm{d}/\mathrm{d}r)\big(\log(e_1r^2-e_2)\big)$ (see at the end of Section 3). Furthermore, their singularities are $r=\pm1$ (with difference of exponents $1/2$) and $r=\pm\sqrt{e_2/e_1}$ (with exponents $\pm i\sqrt{b}$). From this, and from the symmetry in r it follows that it is possible to reduce this equation to a Lamé differential equation if we take r^2 as the new independent variable. But we prefer to make this reduction in another more standard way.

So, by the covering $\overline{\Gamma}=\mathbf{P}^1\to\mathbf{P}^1$ ($r\mapsto x=x_1$), as in Section 4, we obtain the algebraic NVE (5),

$$\begin{aligned}\frac{\mathrm{d}^2\xi}{\mathrm{d}x^2}&+\left(\frac{1}{x}+\frac{1/2}{x-e_1}+\frac{1/2}{x-e_2}\right)\frac{\mathrm{d}\xi}{\mathrm{d}x}+\\&+\frac{(3/4)x^2+4e_1e_2ax+4e_1e_2b}{4x^2(x-e_1)(x-e_2)}\,\xi=0\,.\end{aligned} \tag{7}$$

In order to show that this equation is of Lamé type, it is necessary to make some transformations. First, if we take $z=1/x$, we get

$$\begin{aligned}\frac{\mathrm{d}^2\xi}{\mathrm{d}z^2}&+\left(\frac{1/2}{z-s_1}+\frac{1/2}{z-s_2}\right)\frac{\mathrm{d}\xi}{\mathrm{d}z}+\\&+\left(\frac{3/16}{z^2}+\frac{(b-3/16)z+a-(3/16)(s_1+s_2)}{z(z-s_1)(z-s_2)}\right)\xi=0\,,\end{aligned} \tag{8}$$

where $s_i=1/e_i$, $i=1,\,2$.

The next reduction is obtained by the change (cf. [14, p. 78])

$$\xi(z) = z^{1/4}\eta(z)\,. \tag{9}$$

By the above change, (8) is transformed into

$$\begin{aligned}\frac{d^2\eta}{dz^2} + \frac{1}{2}\left(\frac{1}{z} + \frac{1}{z-s_1} + \frac{1}{z-s_2}\right)\frac{d\eta}{dz} + \\ + \frac{(4b+1/4)z + 4a + (1/4)(s_1+s_2)}{4z(z-s_1)(z-s_2)}\,\eta = 0\,.\end{aligned} \tag{10}$$

With the change of the independent variable

$$p = z - \frac{1}{3}(s_1+s_2) \tag{11}$$

(10) becomes into the standard algebraic form of the Lamé equation ([14], [16])

$$\frac{d^2\eta}{dp^2} + \frac{f'(p)}{2f(p)}\frac{d\eta}{dp} - \frac{Ap+B}{f(p)}\,\eta = 0\,, \tag{12}$$

where $f(p) = 4p^3 - g_2 p - g_3$, with

$$g_2 = \frac{4}{3}(s_1+s_2)^3 - s_1 s_2\,, \quad g_3 = -\frac{4}{27}(s_1+s_2)(s_2-2s_1)(s1-2s_2)$$

and

$$A = -\left(4b + \frac{1}{4}\right)\,, \quad B = -\left(4b+\frac{1}{4}\right)\frac{s_1+s_2}{3} - 4a - \frac{s_1+s_2}{4}\,.$$

Finally, with the well known change $p = \mathcal{P}(\tau)$, we get the Weierstrass form of the Lamé equation

$$\frac{d^2\eta}{d\tau^2} - (A\mathcal{P}(\tau) + B)\eta = 0\,, \tag{13}$$

being $\mathcal{P}$ the elliptic Weierstrass function.

This equation is defined in a torus Π (genus one Riemann surface) with only one singular point at the origin. Let $2w_1$, $2w_3$ be the real and imaginary periods of the Weierstrass function $\mathcal{P}$ and $\mathbf{g}_1$, $\mathbf{g}_2$ their corresponding monodromies in the above equation. If $\mathbf{g}_*$ represents the monodromy around the singular point, then $\mathbf{g}_* = [\mathbf{g}_1\,,\,\mathbf{g}_2]$ ([16], [14]).

It is easy to see that $\Gamma_{\mathbb{R}}$ corresponds, by the global change $r \mapsto \tau$, to the real segment between the origin and $2w_1$ in the plane τ. Hence, the monodromy $m_\gamma = m_\sigma^2$ (Sect. 5) is equal to $\mathbf{g}_1^2$. (In reference [14, Chap. IX], it is studied the relation between the monodromy groups of the equations like (12) and (13), by the covering $\Pi \to \mathbf{P}^1$, $\tau \mapsto p$. For a modern account, see [3]).

We shall need the following elementary property of the Lamé equation.

LEMMA 2.— *Let*

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} - \big(A\mathcal{P}(\tau) + B\big)\eta = 0\,, \tag{14}$$

be a Lamé type equation. If $\mathbf{g}_1^2 = \mathbf{1}$ *(or if* $\mathbf{g}_2^2 = \mathbf{1}$*), then the monodromy (and the Galois) group is abelian.*

Proof.— From $\mathbf{g}_1^2 = \mathbf{1}$ it follows $\mathbf{g}_1 = \mathbf{1}$ or $\mathbf{g}_1 = -\mathbf{1}$ (because $\mathbf{g}_1$ is in $\mathrm{SL}(2,\mathbb{C})$). If $\mathbf{g}_1 = \mathbf{1}$, it is clear that $\mathbf{g}_* = [\,\mathbf{g}_1\,, \mathbf{g}_2\,] = \mathbf{1}$ (the case $\mathbf{g}_1 = -\mathbf{1}$ is analogous). As the monodromy group is generated by $\mathbf{g}_1$ and $\mathbf{g}_2$, the necessary and sufficient condition in order to have an abelian monodromy group is $\mathbf{g}_* = \mathbf{1}$. The Galois group is also abelian because for a fuchsian equation, it is the Zariski adherence of the monodromy group. □

We remark that $\mathbf{g}_1^2 = \mathbf{1}$ (or $\mathbf{g}_2^2 = \mathbf{1}$) corresponds to the so called Lamé's solutions ([14], [16]).

Now if, as usual, we write $A = n(n+1)$, being n a new parameter, the condition $\mathbf{g}_* = \mathbf{1}$ is equivalent to be n an integer (this follows easily from the roots n, $-(n+1)$ of the indicial equation at the singular point).

We come back to our example. As $A = -(4b + 1/4)$ with $b > 0$, n is not an integer (the roots of the indicial equation are $-1/2 \pm 2i\sqrt{b}$) and $\mathbf{g}_1^2 \neq \mathbf{1}$, equivalent (by the Proposition 3) to $\left(G_{\mathrm{loc}}\right)_0$ not abelian. Therefore we have obtained the following non-integrability result.

PROPOSITION 4.— *Let*

$$H = \frac{1}{2}\,(y_1^2 + y_2^2) + \varphi(x_1) + \frac{1}{2}\,\alpha(x_1)x_2^2 + \text{h.o.t.}\,(x_2)\,, \tag{15}$$

be a Hamiltonian, where

$$\varphi(x_1) = -\frac{1}{2e_1e_2}\,x_1^2(x_1 - e_1)(x_1 - e_2)\,,$$

$$\alpha(x_1) = \frac{3}{16\,e_1e_2}\,x_1^2 + ax_1 + b\,,$$

(with real parameters $b > 0$, $e_1 \neq e_2$, a). Then the invariant manifolds of the Lyapounov orbits around the origin of the above Hamiltonian system must intersect transversally, and there does not exist an additional global meromorphic first integral.

We note that the (global) Galois group G of the NVE is either of type 6) or 7) of Proposition 1, because by Proposition 3 and Lemma 1, the local Galois group G_{loc} is already of this type and $G_{\text{loc}} \subset G$ (since $\overline{\Gamma}_{\text{loc}} \subset \overline{\Gamma}$, Sect. 2). We shall go to prove that G is $\mathrm{SL}(2,\mathbb{C})$.

We need the following result of [11].

LEMMA 3.— *Let*

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = A(x)\xi\,, \tag{16}$$

be a linear differential equation, being the entries of $A(x)$ meromorphic functions on a Riemann surface Y. Let $\pi : X \to Y$ be a finite branched covering between Riemann surfaces (i.e., a change of variables $z \mapsto x$). Let

$$\frac{\mathrm{d}\xi}{\mathrm{d}z} = B(z)\xi\,, \tag{17}$$

be the resulting differential equation on X. Then, the identity component of the Galois groups of (16) and (17) are isomorphics.

The above result is proved by Katz for compact Riemann surfaces [6, Prop. 4.3]. And, for fuchsian differential equations (only regular singularities), this result is obtained also in [1, Prop. 4.7].

Now, the relation between the Galois groups of the initial NVE (defined over $\overline{\Gamma}$) and of the equation (13) is given by the following lemma.

LEMMA 4.— *The identity component of the Galois groups of the NVE (eq. (6)) and of the equation (13) are the same.*

Proof.— First, the identity component G_0 of the Galois group of each of the above equations (7), (8), (10), (12) and (13) is the same. In fact, it is clear the equivalence between (7) and (8), and between (10) and (12). On the other hand, in the change (9) we only introduce algebraic functions which do not affect to the identity component. Furthermore, by the Lemma 3 the coverings $\mathbf{P}^1 = \overline{\Gamma} \to \mathbf{P}^1$ $(r \mapsto x)$ and $\Pi \to \mathbf{P}^1$ $(\tau \mapsto p)$ preserve the identity component of the Galois group. Hence, the identity component of the Galois group of the equation (6) is the same that the identity component of the Galois group of the algebraic NVE (eq. (7)). □

So, we shall compute the identity component of the Galois group of the equation (13).

We recall that the roots of the indicial equation at the origin are $-1/2 = \pm 2i\sqrt{b}$. The eigenvalues of the corresponding monodromy matrix $\mathbf{g}_*$ are not in the unit circle and cases 1), 2) and 3) of Proposition 1 are not possible. As n is not an integer, the abelian case 4) is impossible too. We can not fall in case 5), for this metaabelian case do not appears in the Lamé equation ([12, pp. 160-161]). Finally, if we are in case 6), the commutator of the monodromy matrices along the periods, $\mathbf{g}_* = [\,\mathbf{g}_1\,,\,\mathbf{g}_2\,]$, has eigenvalues equal to 1. Necessarily we are in case 7), $G = G_0 = \mathrm{SL}(2,\mathbb{C})$. Then by the above lemma, the identity component for the NVE (eq. (8)) is also $\mathrm{SL}(2,\mathbb{C})$ and its Galois group must be $\mathrm{SL}(2,\mathbb{C})$.

Finally, we remark that the family of (complex) Hamiltonians

$$H = -\frac{1}{2}\,(y_1^2 + y_2^2) + \varphi(x_1) + \frac{1}{2}\,\alpha(x_1)x_2^2 + \text{h.o.t.}\,(x_2)\,, \tag{18}$$

is obtained from (15) by the symplectic transformation $y \mapsto iy$, $t \mapsto it$. Hence, the above family and the family (15) represents the same Hamiltonian system, and the Proposition 4 is true for the both families (it is implied that in the family (18) the phase space is given by the coordinates (iy_1, iy_2, x_1, x_2), with y_1, y_2, x_1, x_2 reals).

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