MARTA GARCÍA-HUIDOBRO
RAÚL MANÁSEVICH

Jumping nonlinearities for fourth order B.V.P.

Jumping nonlinearities for fourth order B.V.P.[(*)]

Marta García-Huidobro[(1)] and Raúl Manásevich[(2)]

Résumé. — Dans cet article, nous étudions le problème avec conditions aux bords

$$(P) \qquad \begin{cases} u^{(iv)} = f(u) + H(t) \\ u^{(i)}(0) = 0, \ i \in I \ ; \quad u^{(j)}(T) = 0, \ j \in J, \end{cases}$$

où $I = \{i_1, i_2\}$, $i_1 < i_2$ et $J = \{j_1, j_2\}$, $j_1 < j_2$ sont deux sous-ensembles arbitraires de $\{0, 1, 2, 3\}$ tels que la première valeur propre du problème linéaire associé

$$(L) \qquad \begin{cases} y^{(iv)} = \lambda y \\ y^{(i)}(0) = 0, \ i \in I \ ; \quad y^{(j)}(T) = 0, \ j \in J, \end{cases}$$

est strictement positive. Soit ϕ_1 une fonction propre non négative correspondant à λ_1. On suppose $H(t) = h(t) - s\phi_1(t)$, $s \in \mathbb{R}^+$. Alors sous la condition que f "croise" les k premières valeurs propres de (L), on montre l'existence d'au moins $2k$ solutions du problème (P) pour s suffisamment grand et positif.

Abstract. — In this paper we study the boundary value problem

$$(P) \qquad \begin{cases} u^{(iv)} = f(u) + H(t) \\ u^{(i)}(0) = 0, \ i \in I \ ; \quad u^{(j)}(T) = 0, \ j \in J, \end{cases}$$

where $I = \{i_1, i_2\}$, $i_1 < i_2$ and $J = \{j_1, j_2\}$, $j_1 < j_2$ are two arbitrary sets of integers from $\{0, 1, 2, 3\}$ which are such that the associated linear eigenvalue problem

$$(L) \qquad \begin{cases} y^{(iv)} = \lambda y \\ y^{(i)}(0) = 0, \ i \in I \ ; \quad y^{(j)}(T) = 0, \ j \in J, \end{cases}$$

satisfies $\lambda_1 > 0$. Let ϕ_1 be a nonnegative eigenfunction corresponding to λ_1. Suppose $H(t) = h(t) - s\phi_1(t)$, $s \in \mathbb{R}^+$. Then, under the assumption that f "jumps" over the first k eigenvalues of problem (L), we prove the existence of at least $2k$ solutions to problem (P) for large positive s.

[(1)] Facultad de Matemáticas, Universidad Católica de Chile, Casilla 6177 Correo 22, Santiago, Chile

[(2)] Departamento de Matemáticas, F.C.F.M., Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile

Marta García–Huidobro and Raúl Manásevich

1. Introduction

In this paper we consider the boundary value problem

$$u^{(iv)} = f(u) + H(t) \tag{1.1}$$

$$u^{(i)}(0) = 0, \quad i \in I; \quad u^{(j)}(T) = 0, \quad j \in J, \tag{1.2}$$

where $I = \{i_1, i_2\}$, $i_1 < i_2$ and $J = \{j_1, j_2\}$, $j_1 < j_2$ are two arbitrary but fixed sets of integers from $\{0, 1, 2, 3\}$, $H \in C[0, T]$, and $f \in C^1(\mathbf{R})$ satisfies that the limits

$$\lambda_- := \lim_{x \to -\infty} f(x)/x \quad \text{and} \quad \lambda_+ := \lim_{x \to +\infty} f(x)/x \tag{1.3}$$

exist as finite real numbers. In (1.1)-(1.2) and from now on, $u^{(k)} = (\mathrm{d}^k/\mathrm{d}t^k)u$, $k \in \mathbb{N}$.

We associate to (1.1)-(1.2) the eigenvalue problem

$$\begin{aligned} &y^{(iv)} = \lambda y \\ &y^{(i)}(0) = 0, \quad i \in I; \quad y^{(j)}(T) = 0, \quad j \in J. \end{aligned} \tag{1.4}$$

It is known, see [4], that the eigenvalues of this problem form an infinite increasing sequence $\{\lambda_i\}_{i=1}^{\infty} \subseteq \mathbb{R}_0^+$ with no finite accumulation points. We will assume the sets I and J to be such that $\lambda = 0$ is not an eigenvalue of (1.4). A necessary and sufficient condition for this to hold will be given in section 2.

Let ϕ_1 be an eigenfunction corresponding to the first eigenvalue λ_1 of (1.4). We will consider the case when function H in (1.1) is of the form

$$H(t) = h(t) - s\phi_1(t), \quad s \in \mathbb{R}^+. \tag{1.5}$$

Also, we will assume that $\lambda_- > 0$ and that the interval (λ_-, λ_+) contains exactly the first k eigenvalues of problem (1.4). That is, we will assume that

$$0 < \lambda_- < \lambda_1 \le \lambda_k < \lambda_+ < \lambda_{k+1} \tag{1.6}$$

for some $k \in \mathbb{N}$. Under these conditions, we will prove in section 4 that for large values of the parameter s in (1.5), the boundary value problem (1.1)-(1.2) has at least $2k$ solutions.

A nonlinearity f satisfying (1.3) is known in the literature as a *jumping nonlinearity*.

Boundary value problems with jumping nonlinearities have been studied mainly for the second order case under Dirichlet and Neumann boundary conditions, see for example [6], [7], [8] and [9]. In [5], some results for n-th order equations with a jumping nonlinearity were obtained.

In this paper we are dealing with an important extension of the results in [5] for the fourth order case. To make this point clear, let us consider the boundary value problem

$$
\begin{aligned}
& u^{(iv)} = \widetilde{f}(u) + \widetilde{H}(t) \\
& y^{(i)}(0) = 0 \text{ for } i \in I_r \equiv \{0, 1, 2, 3\} \setminus \{r\}; \quad y^{(r_0)}(T) = 0 .
\end{aligned}
\tag{1.7}
$$

where $\widetilde{H} \in C[\,0\,,\,T\,]$ and $\widetilde{f} \in C^1(\mathbf{R})$ satisfies that the limits

$$
\widetilde{\lambda}_- := \lim_{x \to -\infty} \widetilde{f}(x)/x \quad \text{and} \quad \widetilde{\lambda}_+ := \lim_{x \to +\infty} \widetilde{f}(x)/x
\tag{1.8}
$$

exist and are finite. Also, let the integers r_0 and r in (1.7) be such that $0 \leq r_0 \leq r \leq 3$. This last condition ensures that $\lambda = 0$ is not an eigenvalue of the associated linear problem

$$
\begin{aligned}
& y^{(iv)} = \lambda y \\
& y^{(i)}(0) = 0 , \quad i \in I_r ; \quad y^{(r_0)}(T) = 0 .
\end{aligned}
\tag{1.9}
$$

Setting $\widetilde{H}(t) = s\phi_1(t) + h(t)$, it follows from [5] that (1.7) has at least $2k$ solutions for large values of s when the interval $(\widetilde{\lambda}_+, \widetilde{\lambda}_-)$, $\widetilde{\lambda}_- < 0$, contains the first k eigenvalues of (1.9).

The purpose of the present paper is to treat all the remaining open cases for the fourth order case, for which the corresponding linear problem does not have $\lambda = 0$ as an eigenvalue and the nonlinearity f jumps over the first k eigenvalues. These cases are represented by (1.1)-(1.2), and they form a class of twenty boundary value problems, ten of which are nonselfadjoint.

Let $I^c = \{i_3, i_4\}$, $i_3 < i_4$ and $J^c = \{j_3, j_4\}$, $j_3 < j_4$ be, respectively, the complementary sets of I and J with respect to $\{0, 1, 2, 3\}$ and set

$I^* = \{3 - i_4\,,\, 3 - i_3\}$ and $J^* = \{3 - j_4\,,\, 3 - j_3\}$. It can be easily verified that the adjoint problem to (1.4) is

$$
\begin{aligned}
y^{(iv)} &= \mu y \\
y^{(i)}(0) &= 0\,, \quad i \in I^*\,; \quad y^{(j)}(T) = 0\,, \quad j \in J^*\,.
\end{aligned}
\tag{1.10}
$$

This adjoint problem will play a fundamental role in the proof of some of our intermediate results.

For $s > 0$, let us make the substitution $u = sy$ in (1.1)-(1.2). For s large, and because of conditions (1.3) and (1.5), the resulting problem could be studied by considering the limiting equation as $s \to \infty$

$$
y^{(iv)} = \lambda_+ y^+ - \lambda_- y^- - \phi_1
\tag{1.11}
$$

or its linearised version

$$
y^{(iv)} = \lambda_+ \left(y + \frac{\phi_1}{\lambda_+ - 1} \right)^+ - \lambda_- \left(y + \frac{\phi_1}{\lambda_+ - 1} \right)^- - \lambda_+ \phi_1\,,
\tag{1.11$_h$}
$$

where as usual, $y^+ = \max\{y, 0\}$ and $y = y^+ - y^-$, together with the boundary conditions (1.2). For some technical reasons that will become clear later, it will be convenient to consider a slight modification of $(1.11)_h$, namely, the boundary value problem

$$
\begin{aligned}
y^{(iv)} &= \lambda_+ (y + z_n)^+ - \lambda_- (y + z_n)^- - \lambda_+ z_n \\
y^{(i)}(0) &= 0\,, \quad i \in I\,; \quad y^{(j)}(T) = 0\,, \quad j \in J\,,
\end{aligned}
\tag{1.12}
$$

where z_n is a constant multiple of $\phi_{1,n}$, a positive eigenfunction corresponding to the first eigenvalue $\lambda_{1,n}$ of (1.4) with T replaced by $T + 1/n$, $n \in \mathbb{N}$.

The problem of searching solutions to (1.12) or to (1.1)-(1.2) can be thought as a *two parameter* dependant initial value problem. Thus, in section 2, after listing down some known facts concerning the linear problem (1.4) and its corresponding adjoint, we will prove some results concerning (1.12) and its related initial value problem. These results will make it possible to work with a one parameter boundary value problem and thus, to use shooting techniques to prove our multiplicity result. In section 3 we first prove the existence of two specific solutions to (1.1)-(1.2) and then use the results of section 2 to prove some basic lemmas that will be used in section 4 to prove our main multiplicity result.

Finally, we want to remark that our results will still hold true if we replace the differential operator $Lu \equiv u^{(iv)}$ in (1.1) by the general fourth order disconjugate linear differential operator.

2. Preliminary results

We begin this section with a summary of the properties of the solutions to the linear problem (1.4) that we will use.

A direct computation shows that a necessary and sufficient condition that $\lambda = 0$ is not an eigenvalue of (1.4) is that the elements of I^c and J satisfy

$$j_1 \leq i_3 \quad \text{and} \quad j_2 \leq i_4. \tag{2.1}$$

Similarly, $\mu = 0$ is not an eigenvalue of the adjoint problem (1.10) if and only if

$$3 - j_3 \leq 3 - i_1 \quad \text{and} \quad 3 - j_4 \leq 3 - i_2. \tag{2.2}$$

Henceforth in this paper we assume that (2.1) and (2.2) are satisfied. In fact, it is not difficult to show that (2.1) holds if and only if (2.2) holds.

It is known that the eigenvalues of (1.4) (resp. (1.10)) are simple. Moreover, an eigenfunction ϕ_i (resp. ϕ_i^*) corresponding to the i-th eigenvalue λ_i (resp. μ_i) has exactly $i-1$ zeros on $(0, T)$, all of which are simple. In particular, ϕ_1 (resp. ϕ_1^*) can be taken strictly positive on $(0, T)$. Also, the only derivatives of ϕ_i (resp. ϕ_i^*) which vanish at the endpoints 0 or T are those appearing in (1.2).

The following proposition may be proved by induction on m.

PROPOSITION 2.1. — *The sequence of eigenvalues $\{\lambda_m\}_{m=1}^{\infty}$ of (1.4) and $\{\mu_m\}_{m=1}^{\infty}$ of (1.10) satisfy $\lambda_m = \mu_m$ for every $m \in \mathbb{N}$.*

The following three results are particular cases of the more general case studied by Elías in [2] and [3].

PROPOSITION 2.2. — *Let $S(u)$ denote the number of changes of sign of u on $(0, T)$. Then, for $i = 1, 2, 3$,*

$$S\big(\phi_m^{(i)}\big) = m - 1 + N_i(\phi_m) - i, \tag{2.3}$$

where $N_i(\phi_m)$ denotes the number of boundary conditions imposed to ϕ_m in (1.2) among $\phi_m, \phi_m', \ldots, \phi_m^{(i-1)}$. Moreover, these changes of sign are the only zeros of $\phi_m^{(i)}$ and they are all simple.

PROPOSITION 2.3. — *The eigenvalues λ_m, $m \in \mathbb{N}$ of (1.4), regarded as functions of the endpoint T, are continuous and strictly decreasing.*

PROPOSITION 2.4. — *If $\widetilde{I} = \{\widetilde{\imath}_1, \widetilde{\imath}_2\}$, $\widetilde{J} = \{\widetilde{\jmath}_1, \widetilde{\jmath}_2\}$ are two sets of different integers such that $\widetilde{\imath}_k \leq i_k$, and $\widetilde{\jmath}_k \leq j_k$ for $k = 1$, 2 and at least one of these inequalities is strict, then $\widetilde{\lambda}_m > \lambda_m$ for all $m \in \mathbb{N}$ and if only one of the inequalities is strict, then $\lambda_{m+1} > \widetilde{\lambda}_m > \lambda_m$, where $\widetilde{\lambda}_m$, $m \in \mathbb{N}$ denotes the m-th eigenvalue of (1.4) with I and J replaced by $\widetilde{I}$ and $\widetilde{J}$.*

Let us consider now the semilinear boundary value problem

$$y^{(iv)} = \lambda_+ y^+ - \lambda_- y^- - \phi_1 \tag{2.4}$$

$$y^{(i)}(0) = 0, \quad i \in I; \quad y^{(j)}(T) = 0, \quad i \in J \tag{2.5}$$

where $y^+ = \max\{y, 0\}$, $y^- = y^+ - y$ and

$$0 < \lambda_- < \lambda_1 \leq \lambda_k < \lambda_+ < \lambda_{k+1} \tag{2.6}$$

for some $k \in \mathbb{N}$. Then, problem (2.4)-(2.5) has the two trivial solutions

$$y_1(t) = \frac{\phi_1(t)}{\lambda_+ - \lambda_1}, \tag{2.7}$$

$$y_2(t) = \frac{\phi_1(t)}{\lambda_- - \lambda_1}. \tag{2.8}$$

It is clear that y_1 is strictly positive on $(0, T)$, y_2 is strictly negative on $(0, T)$ and that these are the only solutions of (2.4)-(2.5) which do not change sign on $(0, T)$.

By means of the substitution

$$w = y - y_1, \tag{2.9}$$

we see that y is a solution to (2.5) if and only if w is a solution to

$$w^{(iv)} = g(w + y_1) - g(y_1)$$
$$w^{(i)}(0) = 0, \quad i \in I; \quad w^{(j)}(T) = 0, \quad j \in J, \tag{2.10}$$

where $g(\xi) = \lambda_+ \xi^+ - \lambda_- \xi^-$. Next, let us consider the Initial Value Problem

$$w^{(iv)} = g(w + x) - g(x)$$
$$w^{(i)}(0) = 0, \quad i \in I; \quad w^{(i_3)}(0) = c; \quad w^{(i_4)}(0) = d, \tag{2.11}$$

where $x \in C(\mathbb{R})$. We recall that $I \cup \{i_3, i_4\} = \{0, 1, 2, 3\}$. Let $w(\cdot, c, d)$ be the solution to this problem.

PROPOSITION 2.5. — *Given any $t_0 > 0$ and any $c \in \mathbb{R}$, there exists a unique $d \in \mathbb{R}$ such that $w^{(j_2)}(t_0, c, d) = 0$.*

Proof. — Let the real numbers c_1, c_2, d_1, d_2 be such that $c_1 \geq c_2$ and $d_1 \geq d_2$, with at least one of the inequalities being strict. On setting $w_i(t) := w(t, c_i, d_i)$, $i = 1, 2$, we see that $\overline{w} = w_1 - w_2$ satisfies the initial value problem

$$w^{(iv)} = g(w + w_2 + x) - g(w_2 + x)$$
$$w^{(i)}(0) = 0, \ i \in I; \quad w^{(i_3)}(0) = c_1 - c_2; \quad w^{(i_4)}(0) = d_1 - d_2. \tag{2.12}$$

From our assumptions on c_i and d_i, there exists $\delta > 0$ such that $\overline{w}(t) > 0$ for $t \in (0, \delta)$. Suppose there exists $t_1 > 0$ such that $\overline{w}(t) > 0$ for $t \in (0, t_1)$ and $\overline{w}(t_1) = 0$. Since g is an increasing function, we see from the first of (2.12) that $\overline{w}^{(iv)}(t) > 0$ for $t \in (0, t_1)$ and thus an application of Taylor's Theorem yields

$$\overline{w}(t) \geq (c_1 - c_2)\frac{t^{i_3}}{i_3!} + (d_1 - d_2)\frac{t^{i_4}}{i_4!}, \quad t \in (0, t_1) \tag{2.13}$$

implying the contradiction $\overline{w}(t_1) > 0$. Hence, $\overline{w}(t) > 0$ for every $t > 0$. Similarly,

$$\overline{w}^{(i)}(t) > \alpha_i(c_1 - c_2)\frac{t^{i_3 - i}}{(i_3 - i)!} + (d_1 - d_2)\frac{t^{i_4 - i}}{(i_4 - i)!}, \quad t > 0, \ i \leq j_2 \tag{2.14}$$

where

$$\alpha_i = \begin{cases} 1 & \text{if } i \leq i_3 \\ 0 & \text{if } i > i_3. \end{cases} \tag{2.15}$$

Now, for fixed $t_0 > 0$ and $c \in \mathbb{R}$, consider the sequences $\{\omega_n\}_{n=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ defined by $\omega_n = w^{(j_2)}(t_0, c, n)$ and $\nu_n = w^{(j_2)}(t_0, c, -n)$. By setting $i = j_2$ in (2.14), we obtain that

$$\omega_{n+1} - \omega_n > \frac{t_0^{i_4 - j_2}}{(i_4 - j_2)!} > 0 \tag{2.16}$$

and

$$\nu_n - \nu_{n+1} > \frac{t_0^{i_4 - j_2}}{(i_4 - j_2)!} > 0 \tag{2.17}$$

implying that

$$\lim_{n \to \infty} \omega_n = +\infty \quad \text{and} \quad \lim_{n \to \infty} \nu_n = -\infty \,.$$

Hence, from the continuity of solutions to (2.11) on the initial conditions and from (2.13)-(2.14), there exists a unique $d \in \mathbb{R}$ such that

$$w^{(j_2)}(t_0, c, d) = 0 \,.$$

Hence the proposition. □

The following corollary is a direct consequence of (2.14).

COROLLARY 2.6. — *Let $t_0 > 0$ be fixed. Then, the unique real number d given by proposition 2.5 is a uniformly continuous decreasing function of c.*

In what follows we will denote by $\widetilde{w}(\,\cdot\,, x, c)$ the solution of (2.11) which satisfies the conditions $w^{(i)}(0) = 0$, $i \in I$, $w^{(i_3)}(0) = c$ and $w^{(j_2)}(T) = 0$.

Let $\phi_{1,n}$ denote a positive eigenfunction corresponding to the first eigenvalue $\lambda_{1,n}$ of (1.4) with $[\,0\,, T\,]$ replaced by $[\,0\,, T + 1/n\,]$ and set

$$z_n(t) = \frac{\phi_{1,n}(t)}{\lambda_+ - \lambda_1} \,, \qquad (2.18)$$

where n is chosen large enough, say $n \geq n_1$ (see proposition 2.3), to have

$$\lambda_- < \lambda_{1,n} \quad \text{and} \quad \lambda_{k,n} < \lambda_+ < \lambda_{k+1,n} \,. \qquad (2.19)$$

Let $x(t) = z_n(t)$ in (2.11). From now on, we will refer to problem (2.11) with x replaced by z_n as $(2.11)_n$. Set $\widetilde{w}_n(\,\cdot\,, c) \equiv \widetilde{w}(\,\cdot\,, z_n, c)$ and $\widetilde{w}(\,\cdot\,, c) \equiv \widetilde{w}(\,\cdot\,, y_1, c)$. Clearly, as $n \to \infty$, $\widetilde{w}_n^{(i)}(\,\cdot\,, c)$ converges, uniformly on bounded intervals, to $\widetilde{w}^{(i)}(\,\cdot\,, c)$ for $i = 0, 1, 2, 3$.

We will now construct two different solutions to $(2.11)_n$. To this effect, we first prove the following lemma.

LEMMA 2.7. — *For any $m \in \mathbb{N}$ and $\lambda \in (\lambda_m, \lambda_{m+1})$ every nontrivial solution $y(t, \lambda)$ of*

$$\begin{aligned} y^{(iv)} &= \lambda y \\ y^{(i)}(0) &= 0 \,, \quad i \in I \,; \quad y^{(j_2)}(T) = 0 \end{aligned} \qquad (2.20)$$

is such that $y^{(j_1)}$ has exactly m zeros on $(0, T)$, all of which are simple. In particular, it follows that any nontrivial solution of (2.20) with $\lambda = \lambda_+$ is such that $y^{(j_1)}$ has exactly k zeros on $(0, T)$ and that these zeros are simple.

Proof.— Suppose on the contrary that there exist $m_0 \in \mathbb{N}$ and $\lambda_0 \in (\lambda_{m_0}, \lambda_{m_0+1})$ such that $y^{(j_1)}(\,\cdot\,, \lambda)$ has less than m_0 zeros on $(0, T)$, and consider the set

$$A = \{\lambda \leq \lambda_{m_0+1} \mid y^{(j_1)}(\,\cdot\,, \alpha) \text{ has at least}$$

$$m_0 \text{ zeros on } (0, T), \ \forall \ \alpha \in [\,\lambda, \lambda_{m_0+1}\,]\}.$$

From proposition 2.2 above, and since $j_1 \leq i_3$ and $j_1 < j_2$, we obtain that $\phi_{m_0+1}^{(j_1)}$ has exactly m_0 simple zeros on $(0, T)$ and therefore $\lambda_{m_0+1} \in A$. Also, from our assumption, A is bounded below by λ_0 and $\lambda_0 \leq \inf A \equiv \lambda^*$.

We claim that $y^{(j_1)}(\,\cdot\,, \lambda^*)$ has exactly $m_0 - 1$ zeros on $(0, T)$ and $y^{(j_1)}(T, \lambda^*) = 0$. Indeed, if $y^{(j_1)}(\,\cdot\,, \lambda^*)$ has less than m_0 zeros on $(0, T)$, then $y^{(j_1)}(\,\cdot\,, \lambda^* + \epsilon)$ would have less than m_0 zeros on $(0, T)$ for all small enough $\epsilon > 0$. Also, if $y^{(j_1)}(\,\cdot\,, \lambda^*)$ has more than $m_0 - 1$ zeros on $(0, T)$, then for sufficiently small $\epsilon > 0$, $y^{(j_1)}(\,\cdot\,, \lambda^* - \epsilon)$ would have at least m_0 zeros on $(0, T)$. Since both alternatives contradict the definition of λ^*, our claim follows and $\lambda^* = \lambda_{m_0}$. Since $\lambda_0 \in (\lambda_{m_0}, \lambda_{m_0+1})$ this is also a contradiction. Therefore, $y^{(j_1)}(\,\cdot\,, \lambda_0)$ has at least m_0 zeros on $(0, T)$. A similar reasoning shows that $y^{(j_1)}(\,\cdot\,, \lambda_0)$ can have at most m_0 zeros on $(0, T)$ and the lemma follows. $\square$

PROPOSITION 2.8.— *For each* $n \in \mathbb{N}$, *there exist two real numbers* $c_{k,n} > 0$ *and* $c_{-k,n} < 0$, *such that the solution* $\widetilde{w}_n(\,\cdot\,, c_{\pm k,n})$ *of equation* $(2.11)_n$ *is such that* $\widetilde{w}_n^{(j_1)}(\,\cdot\,, c_{\pm k,n})$ *has exactly* k *zeros on* $(0, T)$ *and all of these zeros are simple.*

Proof.— Let $\epsilon_n > 0$ be such that the solution $y(\,\cdot\,, \lambda_+)$ of (2.20) for which $y^{(i_3)}(0) = 1$ satisfies

$$\pm\epsilon_n \, y(t, \lambda_+) + z_n(t) \geq 0 \quad \text{for } t \in [\,0\,, T\,].$$

Then, the functions

$$w_{k,n}(t) = \epsilon_n \, y(t, \lambda_+) \quad \text{and} \quad w_{-k,n}(t) = -\epsilon_n \, y(t, \lambda_+) \qquad (2.21)$$

satisfy equation $(2.11)_n$ on $[\,0\,, T\,]$, $w_{\pm k,n}^{(i)}(0) = 0$ for $i \in I$, $w_{k,n}^{(i_3)}(0) = \epsilon_n$, $w_{-k,n}^{(i_3)}(0) = -\epsilon_n$ and $w_{\pm k,n}^{(j_2)}(T) = 0$. Thus $w_{\pm k,n}(t) = \widetilde{w}_n(t, c_{\pm k,n})$ for $t \in [\,0\,, T\,]$, where $c_{\pm k,n} = \pm\epsilon_n$. $\square$

Consider now the problem

$$w^{(iv)} = \lambda_+ w^+ - \lambda_- w^-$$
$$w^{(i)}(0) = 0, \ i \in I; \quad w^{(i_2)}(0) = c; \quad w^{(j_2)}(T) = 0. \qquad (2.22)$$

The conditions on λ_+ and λ_- imply that the solution $w(\cdot, c)$ of this problem satisfies $w^{(j_1)}(T, c) \neq 0$. Actually, under the additional assumption that $\lambda_+ > \lambda_2$, we have the following result.

THEOREM 2.9. — *Let $\lambda_- < \lambda_1$ and $\lambda_+ > \lambda_2$. Then the solution $w(\cdot, c)$, $c \in \{-1, 1\}$ of problem (2.22) is such that:*

i) $w^{(j_1)}(\cdot, 1)$ changes sign exactly once on $(0, T)$;

ii) $w^{(j_1)}(\cdot, -1)$ does not change sign on $(0, T)$.

Remark. — From the point of view of our multiplicity result, the assumption $\lambda_+ > \lambda_2$ is not a restrictive one, as it will be seen later.

Proof. — Let ϕ_1^* be the positive eigenfunction corresponding to the first eigenvalue $\mu_1 = \lambda_1$ of (1.10), and multiply equation (2.22) by ϕ_1^*. We see then that w satisfies the equation

$$\left(\phi_1^* w''' - \phi_1^{*\prime} w'' + \phi_1^{*\prime\prime} w' - \phi_1^{*\prime\prime\prime} w\right)(t) =$$
$$= (\lambda_+ - \lambda_1) \int_0^t w^+(\tau)\phi_1^*(\tau)\,d\tau + (\lambda_1 - \lambda_-)\int_0^t w^-(\tau)\phi_1^*(\tau)\,d\tau \equiv$$
$$\equiv G(t)\,.$$
$$(2.23)$$

Since $\lambda_- < \lambda_1 < \lambda_+$, $G(t) > 0$ for every $t > 0$ and therefore, evaluating at $t = T$ and observing that the left hand side of (2.23) can be written in the two ways

$$\sum_{m=1}^4 (-1)^{3-j_m}\phi_1^{*(3-j_m)}(t) w^{(j_m)}(t) = \sum_{m=1}^4 (-1)^{3-i_m}\phi_1^{*(3-i_m)}(t) w^{(i_m)}(t)\,,$$
$$(2.24)$$

we obtain that

$$(-1)^{3-j_1}\phi_1^{*(3-j_1)}(T) w^{(j_1)}(T) > 0\,.$$
$$(2.25)$$

A direct computation shows that $(-1)^{3-j_1}\phi_1^{*(3-j_1)}(T) < 0$ for all possible choices of j_1, and therefore, we have that $w^{(j_1)}(T) < 0$, and we conclude that the number of sign changes of $w^{(j_1)}(\cdot, c)$ is odd when $c = 1$ and it is even when $c = -1$.

We will now show the same holds true for the number of sign changes of $w(\cdot, c)$. This is clearly true if $j_1 = 0$. If $j_1 \neq 0$, let $\widetilde{I} = I$, $\widetilde{J} = \{0, j_2\}$, and denote by $\widetilde{\lambda}_1$ the first eigenvalue of (1.4) with I, J replaced by $\widetilde{I}$, $\widetilde{J}$. From

proposition 2.4, we have that $\lambda_1 < \widetilde{\lambda}_1 < \lambda_2$ and therefore, on multiplying equation (2.22) by $\widetilde{\phi}_1^*$, the positive eigenfunction corresponding to the first eigenvalue of (1.10) with I^*, J^* replaced by $\widetilde{I}^*$, $\widetilde{J}^*$, we obtain that

$$-\widetilde{\phi}_1^{*\prime\prime\prime}(T)w(T) > 0. \qquad (2.26)$$

Since $\widetilde{\phi}_1^{*\prime\prime\prime}(T) > 0$, (2.26) implies that $w(T) < 0$.

Now, for $t \in (0,T)$, let $w = \phi_1^* v$ on (2.23). Then v satisfies the differential equation

$$\left(\phi_1^{*2}v''\right)' - 2\left[\left(\phi_1^{*\prime}\right)^2 - 2\phi_1^*\phi_1^{*\prime\prime}\right]v' = G. \qquad (2.27)$$

Since from proposition 2.2 the function $\left(\phi_1^{*\prime}\right)^2 - 2\phi_1^*\phi_1^{*\prime\prime}$ is strictly positive on $(0,T)$, we conclude that if $v'(t) \geq 0$ for $t \in (\alpha,\beta)$, then

$$\phi_1^{*2}(\beta)v''(\beta) > \phi_1^{*2}(\alpha)v''(\alpha). \qquad (2.28)$$

Now, let $c = 1$ and suppose that $w(\,\cdot\,,1)$ changes sign at three different points on $(0,T)$. Then, since $\phi_1^*(t) > 0$ for every $t \in (0,T)$, so does v at the same three points. Let us denote these points by t_i, $i = 1, 2, 3$ with $0 < t_1 < t_2 < t_3 < T$, and assume that $v(t) \leq 0$ for $t \in (t_1,t_2)$ and $v(t) \geq 0$ for $t \in (t_2,t_3)$. Then, it follows the existence of an interval $(\alpha,\beta) \subset (t_1,t_2)$ such that $v''(\alpha) \geq 0$, $v''(\beta) \leq 0$ and $v'(t) \geq 0$ for $t \in (\alpha,\beta)$ contradicting (2.28). Thus, $w(\,\cdot\,,1)$ changes sign exactly once on $(0,T)$.

For the case $c = -1$, the same argument as above shows that $w(t,-1) < 0$ for $t \in (0,T)$ when the integers i_1, i_2 are nonconsecutive. Indeed, in this case, either $v(0) = 0$ or $v'(0) = 0$ and it follows from (2.28) that $w(\,\cdot\,,-1)$ changes sign at most once on $(0,T)$. Thus, we conclude from (2.25) and (2.26) that in fact $w(t,-1) < 0$ for every $t \in (0,T)$. When $i_2 = i_1 + 1$, this result follows by considering the solution w_j, $j \in \{0, 1, 2, 3\}$ of the linear problem

$$
\begin{aligned}
w^{(iv)} &= \lambda_- w \\
w^{(i)}(0) &= 0, \ i \in \{i_1, i_1 + 1\}; \quad w^{(i_3)}(0) = -1; \quad w^{(j)}(T) = 0.
\end{aligned} \qquad (2.29)
$$

It can be easily verified that the function $z_{j,k} := w_j - w_k$ satisfies the first of (2.29) and does not change sign on $(0,+\infty)$. Also, $z_{j,k}(0) = 0$ and $z_{j,k}^{(j)}(T) = -w_k^{(j)}(T)$ for every choice of the integers j, $k \in \{0, 1, 2, 3\}$. Therefore, since $w_0(t) < 0$ for every $t \in (0,T)$ and $w_0'(T) > 0$, by using a

boot strap argument, we conclude that $w_j(t) < 0$ for $t \in (0, T)$. Finally, an application of Rolle's Theorem and the boundary conditions $w^{(i)}(0) = 0$, $i \in \{i_1, i_2\}$ and $w^{(j_2)}(T) = 0$ yield the result of the theorem. $\square$

COROLLARY 2.10. — *Under the assumptions of theorem 2.9, there exists two real numbers $c_1 > 0$ and $c_0 < 0$ such that the solutions $\widetilde{w}_n(\cdot, c_i)$, $i = 0, 1$ of $(2.11)_n$ satisfy:*

i) $\widetilde{w}_n^{(j_1)}(\cdot, c_1)$ changes sign exactly once on $(0, T)$;

ii) $\widetilde{w}_n^{(j_1)}(\cdot, c_0)$ does not change sign on $(0, T)$.

Proof. — The corollary follows easily from the fact that the solution $w_{n,\delta}(\cdot, c)$ of

$$w^{(iv)} = g(w + \delta z_n) - g(\delta z_n)$$
$$w^{(i)}(0) = 0, \, i \in I; \quad w^{(i_3)}(0) = c; \quad w^{(j_2)}(T) = 0 \tag{2.30}$$

converges in $C^4[0, T + (1/n)]$ to the solution $w(\cdot, c)$ of (2.22) as δ goes to 0^+. $\square$

We finish this section with a result concerning the multiplicity of the zeros of a solution to $(2.11)_n$. Since its proof is very similar to that of Lemma 3.3 in [5], we will omit it.

PROPOSITION 2.11. — *If $c \neq 0$, then the solution $\widetilde{w}_n(\cdot, c)$ of $(2.11)_n$ and its derivatives may have only simple zeros on $(0, T)$.*

3. Basic lemmas

Let $n \in \mathbb{N}$, $n \geq n_1$ and consider the family of boundary value problems

$$u^{(iv)} = f(u) - s\phi_{1,n} + h \tag{3.1}_n$$

$$u^{(i)}(0) = 0, \quad i \in I; \quad u^{(j)}\left(T + \frac{1}{n}\right) = 0, \quad j \in J. \tag{3.2}_n$$

Herein we will refer to $(3.1)_\infty$-$(3.2)_\infty$ when speaking of the limiting problem (1.1)-(1.2) with H satisfying condition (1.5).

Let W_n (W_∞) be the Banach space of functions $f \in C^3[0, T + (1/n)]$ $(f \in C^3[0, T])$ which satisfy $(3.2)_n$ $((3.2)_\infty)$ provided with the usual norm, and let

$$z_{1,n} = \frac{\phi_{1,n}}{\lambda_+ - \lambda_{1,n}}, \quad z_{2,n} = \frac{\phi_{1,n}}{\lambda_- - \lambda_{1,n}}. \tag{3.3}$$

Our first result in this section is similar to the corresponding one in [6], and thus we will only sketch the proof.

LEMMA 3.1.— *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, there exists an $s_0 > 0$ such that for $s \geq s_0$, the boundary value problem $(3.1)_n$-$(3.2)_n$ possesses a unique nonnegative solution $u_{n,s}$ and a unique nonpositive solution $\widetilde{u}_{n,s}$ satisfying*

$$\left\| \frac{u_{n,s}}{s} - z_{1,n} \right\|_{W_n} < \varepsilon \quad and \quad \left\| \frac{\widetilde{u}_{n,s}}{s} - z_{2,n} \right\|_{W_n} < \varepsilon. \qquad (3.4)$$

Moreover, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, neither ε_0 nor s_0 depend on n, and for $s \geq s_0$,

$$\left\| \frac{u_{n,s}}{s} - y_1 \right\|_{C^3[0,T]} < 2\varepsilon \quad for \ each \ n \geq n_0, \qquad (3.5)$$

where y_1 is the function defined in (2.7).

Proof.— The proof is based on the Banach Fixed Point Theorem applied to each of the mappings

$$F_{n,s} : W_n \to W_n$$

$$(F_{n,s}u)(t) = \frac{\phi_{1,n}(t)}{\lambda_+ - \lambda_{1,n}} + \int_0^T G_n(t,\tau) \left[\frac{f(su(\tau)) + h(\tau)}{s} - \lambda_+ u(\tau) \right] d\tau,$$

$$(3.6)$$

where $G_n(t,\tau)$ is the Green's function corresponding to the problem

$$u^{(iv)} - \lambda_+ u = h$$
$$u^{(i)}(0) = 0, \ i \in I; \quad u^{(j)}\left(T + \frac{1}{n}\right) = 0, \ j \in J. \qquad (3.7)$$

In a first step, the positive numbers $\varepsilon_{0,n}$ and $\widetilde{s}_{0,n}$ are determined by the requirement that the restriction $F_{n,s}/B_\varepsilon(z_{1,n})$ be a contraction for $\varepsilon \in (0, \varepsilon_{0,n}]$ and $s \geq \widetilde{s}_{0,n}$, and in a second step, $s_{0,n}$ is determined by the requirement that for $s \geq s_{0,n}$ and for all $\varepsilon \in (0, \varepsilon_{0,n}]$, $F_{n,s}(B_\varepsilon(z_{1,n})) \subseteq B_\varepsilon(z_{1,n})$. (Here, for $f \in W_n$, $B_r(f) = \left\{ g \in W_n \mid \|g - f\|_{W_n} < r \right\}$.)

We will show that there exists $n_0 \geq n_1$ such that for $n \geq n_0$, $\varepsilon_{0,n}$ and $s_{0,n}$ may be taken independent of n.

Indeed, Let M_1 be a uniform upper bound for $\left|(\partial^i/\partial t^i)G_n(t,\tau)\right|$ on $\left[0, T+(1/n)\right]^2$ and let M_2 be such that $\left|f'(\xi)-\lambda_+\right| \leq M_2$ for $\xi \in [0, \infty)$. Also, let $\delta > 0$ be such that $\delta < T/2$, and $\delta < 1/9\,M_1 M_2$. Now set $m_0 = \inf\{y_1(t) \mid t \in [\delta/2, T-\delta/2]\}$ and choose $\varepsilon_{0,\infty} > 0$ such that if $u \in B_{\varepsilon_{0,\infty}}(y_1)$, then $u(t) > 0$ on $(0,T)$ and $u(t) \geq m_0/2$ on $[\delta/2, T-\delta/2]$.

Let $n_0 \in \mathbb{N}$ be such that

$$\frac{|\lambda_{1,n}-\lambda_1|}{|\lambda_+-\lambda_1|}\,\|z_{1,n}\|_{W_n} < \frac{\varepsilon_0}{6} \quad \text{for } n \geq n_0\,,$$

$$\left(1+\frac{1}{nT}\right)^4 < \frac{3}{2}, \quad n_0 \geq n_1 \quad \text{and} \quad \frac{1}{n} < \delta\,. \tag{3.8}$$

Then, if $n \geq n_0$, and $w \in W_n$ is such that

$$\left\|w - \frac{\phi_{1,n}}{\lambda_+-\lambda_{1,n}}\right\|_{W_n} \leq \frac{\varepsilon_{0,\infty}}{2}\,, \tag{3.9}$$

we have, by setting

$$u_n(t) = w\left(\frac{T+(1/n)}{T}\,t\right)\,,$$

that $u_n \in W_\infty$ and

$$u_n^{(i)}(t) - y_1^{(i)}(t) = \left(1+\frac{1}{nT}\right)^i\left[w^{(i)}\left(\frac{T+(1/n)}{T}\,t\right) - \frac{\phi_{1,n}^{(i)}\left(\frac{T+(1/n)}{T}\,t\right)}{\lambda_+-\lambda_{1,n}}\right]$$

$$+ \left(1+\frac{1}{nT}\right)^i\left(\frac{\lambda_{1,n}-\lambda_1}{\lambda_+-\lambda_1}\right)\frac{\phi_{1,n}^{(i)}\left(\frac{T+(1/n)}{T}\,t\right)}{\lambda_+-\lambda_{1,n}} \tag{3.10}$$

for $i = 0, 1, 2, 3$. Therefore,

$$\|u_n - y_1\|_{W_\infty} \leq \left(1+\frac{1}{nT}\right)^4\|w-z_{1,n}\|_{W_n} +$$

$$+ \left(1+\frac{1}{nT}\right)^4\frac{|\lambda_{1,n}-\lambda_1|}{\lambda_+-\lambda_1}\,\|z_{1,n}\|_{W_n} \tag{3.11}$$

$$< \frac{3\varepsilon_{0,\infty}}{4} + \frac{3\varepsilon_{0,\infty}}{12} = \varepsilon_{0,\infty}\,,$$

which implies that $u_n(t) > 0$ for all $t \in (0,T)$ and $u_n(t) \geq m_0/2$ for $t \in [\delta/2, T-\delta/2]$.

Hence, $w(t) > 0$ for every $t \in \big(0, T + (1/n)\big)$ and $w(t) \geq m_0/2$ for all $t \in [\delta, T - \delta]$.

Let us set $\varepsilon_0 = \varepsilon_{0,\infty}/2$. Since $\lim_{\xi \to +\infty} f'(\xi) = \lambda_+$, there exists $R > 0$ such that

$$\left| f'(\xi) - \lambda_+ \right| < \frac{1}{3}(T+1)M_1 \quad \text{for } \xi \geq R.$$

Now, for $\varepsilon \in (0, \varepsilon_0)$, choose $s_0 > 0$ such that

$$s_0 \geq \max\left\{ \frac{2R}{m_0}, \frac{2\|h\|_\infty M_1(T+1)}{\varepsilon} \right\}, \tag{3.12}$$

and such that for $s \geq s_0$,

$$\left| \frac{f(su)}{s} - \lambda_+ u \right| \leq \frac{\varepsilon}{2M_1(T+1)} \quad \text{for } 0 \leq u \leq 2\|y_1\|_{W_\infty} + \varepsilon_0. \tag{3.13}$$

Then s_0 depends only on ε and h, and it can be easily verified that for each $n \geq n_0$ and $s \geq s_0$, $F_{n,s}/B_\varepsilon(z_{1,n})$ is a contraction and $F_{n,s}\big(B_\varepsilon(z_{1,n})\big) \subseteq B_\varepsilon(z_{1,n})$. Thus, the existence of a unique solution $u_{n,s}$ to $(3.1)_n$-$(3.2)_n$ satisfying the first of (3.4) follows by setting $u_{n,s} = sz_{n,s}$ where $z_{n,s}$ is the unique fixed point of $F_{n,s}$ in $B_\varepsilon(z_{1,n})$.

The existence of $\widetilde{u}_{n,s}$ is proved similarly. $\square$

Let $v = su - z_{n,s}$ in $(3.1)_n$, $g_0 = f - g$, and consider the boundary value problem

$$v^{(iv)} = g(v + z_{n,s}) - g(z_{n,s}) + \frac{g_0\big(s(v + z_{n,s})\big) - g_0(sz_{n,s})}{s} \tag{$3.14)_n$}$$

$$v^{(i)}(0) = 0, \ i \in I; \quad v^{(i_3)}(0) = c; \quad v^{(j_2)}(T) = 0. \tag{3.15}$$

For $d \in \mathbb{R}$, let $w_n(\cdot, c, d)$ be the solution to $(2.11)_n$, and denote by $v_{n,s}(\cdot, c, d)$ the solution to $(3.14)_n$ which satisfy $v^{(i)}(0) = 0$, $i \in I$, $v^{(i_3)}(0) = c$, $v^{(i_4)}(0) = d$.

It can be easily verified that there exists some constant $M_0 > 0$ independent of s and n, such that for $i = 0, 1, 2, 3$ and $t \in \big[0, T + (1/n)\big]$,

$$\left| v_{n,s}^{(i)}(t, c, d) \right| \leq M_0\big(|c| + |d|\big). \tag{3.16}$$

Now let c_0 and c_1 be as in corollary 2.10, set $c_0^* = \max\{-c_0, c_1\}$ and let $d_0^* > 0$ be such that for every $n \geq n_0$ and $|c| \leq c_0^*$,

$$w_n^{(j_2)}(T, c, d_0^*) > 1 \quad \text{and} \quad w_n^{(j_2)}(T, c, -d_0^*) < -1. \tag{3.17}$$

Let also the constant M_1 be such that

$$\left| z_{n,s}(t) \right| \leq M_1 \quad \text{for } t \in \left[0, T + \frac{1}{n} \right] \text{ and } n \geq n_0, \qquad (3.18)$$

and choose $s_1 \geq s_0$ such that for

$$s \geq s_1, \quad \xi \in \left[-M_0(c_0^* + d_0^*) - M_1, \, M_0(c_0^* + d_0^*) + M_1 \right]$$

and $n > n_0$, it holds that

$$\left| \frac{g_0(s\xi)}{s} \right| \leq \varepsilon \, \frac{e^{-4(T+1)^4 \lambda_+}}{6(T+1)} \qquad (3.19)$$

and

$$\left| z_{n,s}(t) - z_n(t) \right| \leq \varepsilon \, \frac{e^{-4(T+1)^4 \lambda_+}}{48 \, (T+1)^5 \lambda_+}. \qquad (3.20)$$

Then, for $s \geq s_1$, $n \geq n_0$, $|c| \leq c_0^*$ and $|d_i| \leq d_0^*$, $i = 1, 2$, we have

$$\left| v_{n,s}^{(j)}(t, c, d_1) - w_n^{(j)}(t, c, d_2) \right| \leq \frac{\varepsilon}{2} + 4|d_1 - d_2|(T+1)^3 \, e^{4(T+1)^4 \lambda_+}. \quad (3.21)$$

On setting $d_1 = d_2$, $|d_i| = d_0^*$, $i = 1, 2$ in (3.21), we conclude that for each $c \in [-c_0^*, c_0^*]$, $(3.14)_n$-(3.15) has at least one solution $\overline{v}_{n,s}$. Also, from (2.13) and the identity

$$\left| \overline{v}_{n,s}^{(j_2)}(T) - w_n^{(j_2)}\big(T, c, \overline{v}_{n,s}^{(i_4)}(0)\big) \right| = \left| w_n^{(j_2)}(T, c) - w_n^{(j_2)}\big(T, c, \overline{v}_{n,s}^{(i_4)}(0)\big) \right|, \qquad (3.22)$$

we obtain that for every $n \geq n_0$ and $s \geq s_1$,

$$\frac{T^{i_4 - j_2}}{(i_4 - j_2)!} \left| w_n^{(i_4)}(0) - \overline{v}_{n,s}^{(i_4)}(0) \right| < \frac{\varepsilon}{2}. \qquad (3.23)$$

Let now $c_{\pm k, n}$ be as in proposition 2.8.

LEMMA 3.2.— *There exists $s_2 \geq s_1$, such that for all $n \geq n_0$ and $s \geq s_2$ any solution of $(3.14)_n$-(3.15) with $c = c_{\pm k, n}$ satisfies that its j_1-th order derivative has exactly k zeros on $[0, T]$, all of which are simple and contained on $(0, T)$.*

Proof. — Let $\bar{\varepsilon} \in (0, \varepsilon_0]$ be such that if $v \in C^3[0, T + (1/n)]$ satisfies $v^{(i)}(0) = 0$, $i \in I$, $v^{(j_2)}(T) = 0$ and $\|v - w_n(\cdot, c_{\pm k,n})\|_{C^3[0, T+(1/n)]} < \bar{\varepsilon}$, then $v^{(j_1)}$ has exactly k zeros on $[0, T]$, all of them simple. Let $s_2 \geq s_1$ be chosen so that any solution to $(3.14)_n$ with $c = c_{\pm k,n}$ satisfies (3.23) for

$$\varepsilon = \bar{\varepsilon} \, \frac{e^{-4(T+1)^4 \lambda_+}}{16 \, (T+1)^3} \, \frac{(i_4 - j_2)!}{T^{i_4 - j_2}} \tag{3.24}$$

and (3.19)-(3.20) are satisfied for $\varepsilon = \bar{\varepsilon}$. Let $v_{n,s}$ denote any such a solution, and for $s \geq s_2$ define the function

$$\theta_s(t) = \max \left\{ \sup_{\tau \in [0,t]} \left| v_{n,s}^{(i)}(\tau) - w_n^{(i)}(\tau, c_{\pm k,n}) \right| \; \middle| \; i = 0, 1, 2, 3 \right\}. \tag{3.25}$$

Then, it can be easily verified by using Gronwall's Lemma, that

$$\theta_s(t) \leq \frac{3\bar{\varepsilon}}{4} \quad \text{for every } t \in \left[0, T + \frac{1}{n}\right] \tag{3.26}$$

and the result follows from the choice of $\bar{\varepsilon}$. □

For c_0 and c_1 as in corollary 2.10, the following results can be proved similarly.

LEMMA 3.3. — *There exists $s_3 \geq s_2$ such that for $s \geq s_3$ any solution of problem $(3.14)_n$-(3.5) is such that:*

i) for $c = c_0$, its j_1-th order derivative is strictly negative in $(0, T + (1/n)]$;

ii) for $c = c_1$, its j_1-th order derivative has exactly one zero on $(0, T + (1/n))$ and this zero is simple.

LEMMA 3.4. — *There exists $s_4 \geq s_3$ such that for $s \geq s_4$ and c in compact intervals not containing 0, any solution of $(3.14)_n$-(3.5) and its derivatives may have only simple zeros on $(0, T + (1/n))$.*

4. The multiplicity result

We are now in a position to prove our multiplicity result. As we mentioned before, this will be done by using shooting techniques. To this end, for $s \geq s_4$ and $j = 1, 2, \ldots, k$, we define the sets

$$P_{j,n} = \big\{ x > 0 \mid \text{any solution to } (3.14)_n\text{-}(3.15) \text{ is such}$$
$$\text{that its } j_1\text{-th order derivative has at} \qquad (4.1)$$
$$\text{least } j \text{ zeros on } (0, T) \text{ for each } c \in [\, c_{k,n}\,,\, x\,]\big\}$$

and

$$N_{j,n} = \big\{ x < 0 \mid \text{any solution to } (3.14)_n\text{-}(3.15) \text{ is such}$$
$$\text{that its } j_1\text{-th order derivative has at} \qquad (4.2)$$
$$\text{least } j \text{ zeros on } (0, T) \text{ for each } c \in [\, x\,,\, c_{-k,n}\,]\big\}.$$

From lemma 3.2, both sets are nonempty for all j. Also, from lemma 3.3, $P_{j,n}$ is bounded for $j = 2, \ldots, k$ and $N_{j,n}$ is bounded for $j = 1, 2, \ldots, k$ so that we can set

$$c_{j,n} = \sup P_{j,n}, \quad j = 2, \ldots, k \qquad (4.3)$$

$$c_{-j,n} = \inf N_{j,n}, \quad j = 1, 2, \ldots, k. \qquad (4.4)$$

We will first prove the following result.

LEMMA 4.1.— *Any solution $v_{\pm j,n,s}$ to (3.14)$_n$-(3.15) with $c = c_{\pm j,n}$ satisfies the boundary conditions (1.2).*

Proof. — We will prove that any such a solution has at most $j - 1$ simple zeros on $(0, T)$ and at least j zeros on $(0, T]$. Indeed, suppose for example that $v_{j,n,s}^{(j_1)}$ has more than $j - 1$ zeros on $(0, T)$. Then, due to the fact that the zeros of $v_{j,n,s}^{(j_1)}$ are simple as well as to the continuous dependence of $v_{j,n,s}^{(j_1)}$ on t and on c (the latter one being uniform with respect to $t \in [0, T]$), we conclude that any solution $\bar{v}_{j,n,s}$ of (3.14)$_n$-(3.15) with $c = c_{j,n} + \varepsilon$ will satisfy that its j_1-th order derivative has at least j zeros on $(0, T)$ for every sufficiently small $\varepsilon > 0$ in contradiction with the definition of $c_{j,n}$. A similar argument shows that $v_{j,n,s}^{(j_1)}$ has at least j zeros on $(0, T]$ and we conclude that $v_{j,n,s}^{(j_1)}(T) = 0$ and $v_{j,n,s}^{(j_1)}$ has exactly $j - 1$ zeros on $(0, T)$. $\square$

Now, since $0 < c_{j,n} < c_1$ for $j = 2, \ldots, k$ and $0 > c_{-j,n} > c_0$ for $j = 1, 2, \ldots, k$, we may assume, by considering subsequences if necessary, that the sequences $\{c_{j,n}\}_{n=n_0}^{\infty}$ and $\{c_{-j,n}\}_{n=n_0}^{\infty}$ are convergent. Let us set

$$c_j = \lim_{n\to\infty} c_{j,n}, \quad j = 2, \ldots, k \qquad (4.5)$$

$$c_{-j} = \lim_{n\to\infty} c_{-j,n}, \quad j = 1, 2, \ldots, k \qquad (4.6)$$

We will now state and prove our main result.

THEOREM 4.2. — *Let $f \in C^1(\mathbf{R})$ satisfy condition (1.3) and suppose that for some $k \in \mathbb{N}$,*

$$0 < \lambda_- < \lambda_1 \le \lambda_k < \lambda_+ < \lambda_{k+1}. \qquad (4.7)$$

Then, for any $h \in C[0, T]$, there exists $s^* > 0$ such that for $s \ge s^*$, the boundary value problem

$$
\begin{aligned}
u^{(iv)} &= f(u) + h(t) - s\phi_1(t), \quad 0 < t < T \\
u^{(i)}(0) &= 0, \ i \in I; \quad u^{(j)}(T) = 0, \ j \in J
\end{aligned}
\qquad \text{(B.V.P.)}
$$

has at least $2k$ distinct solutions.

Proof. — Let us first consider the case $k = 1$. In this case, we set $s^* = s_0$ and we have that the functions $u_{\infty,s}$ and $\widetilde{u}_{\infty,s}$ given by lemma 3.1 are two different solutions to (B.V.P.) and our assertion follows.

Let now $k > 1$. Letting $s^* = s_4$, it is clear that the solutions $v_{j,s}$, $j = 2, 3, \ldots, k$ and $v_{-j,s}$, $j = 1, 2, \ldots, k$ of $(3.14)_{\infty}$-(3.15) with $c = c_j$ and $c = c_{-j}$ respectively, satisfy the boundary conditions (1.2), and that their j_1-th order derivative can have at most $j - 1$ zeros on $(0, T)$. We will show that in fact they have exactly $j - 1$ zeros on $(0, T)$. Suppose for example that $v_{j,s}^{(j_1)}$ has less than $j-1$ zeros on $(0, T)$. Then at least one of the zeros of the j_1-th order derivative of a solution $v_{j,n,s}$ of $(3.14)_n$-(3.15) with $c = c_{j,n}$ is leaving the interval $(0, T)$ through one of its endpoints as $n \to \infty$. Thus, the number of derivatives of $v_{j,s}$ which vanish at that endpoint is at least 3, in contradiction with proposition 2.10. Similarly, $v_{-j,s}^{(j_1)}$ has exactly $j - 1$ zeros on $(0, T)$ and the theorem follows by setting

$$u_{j,s} = s(v_{j,s} + z_{\infty,s}), \quad j = 2, \ldots, k$$

$$u_{-j,s} = s(v_{-j,s} + z_{\infty,s}), \quad j = 1, 2, \ldots, k$$

and $u_{\infty,s} = sz_{\infty,s}$. $\square$

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