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Estimates of the solutions of impulsive quasilinear functional differential equations

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RÉSUMÉ. — Dans le travail présenté des estimations pour les solutions des équations impulsives fonctionnelles-différentielles quasi-linéaires sont obtenues. On a prouvé l'existence des solutions de certaines classes d'équations non linéaires avec un temps de vie arbitrairement long pour des fonctions initiales suffisamment petites.

ABSTRACT. — In this paper estimates of the solutions of impulsive quasilinear functional differential equations are obtained and the existence of solutions of certain nonlinear equations with arbitrarily long lifespan for sufficiently small initial functions is proved.

1. Introduction

The mathematical theory of the impulsive differential equations (IDE) originates from the work of Mil'man and Myshkis [1] (1960). In a brief period of time the interest in this theory grew due to the numerous applications of IDE to science, technology and practice [2], [3].

A generalization of IDE are the impulsive functional differential equations (IFDE). They are an adequate mathematical model of many processes in various fields of sciences and technology.

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In the present paper some classes of integro-functional inequalities of Gronwall's type for piecewise continuous functions are investigated and by means of the results obtained for them estimates for the solutions of IFDE are found. As an application, the existence of solutions of certain nonlinear equations with arbitrarily long lifespan for sufficiently small initial functions is proved.

2. Preliminary notes

In the paper the following notation is used :

$\mathbb{R}^n$ is the n-dimensional Euclidean space
with elements $x = \mathrm{col}(x_1, \ldots, x_n)$ and norm $\|x\| = \left\{\sum_{i=1}^n x_i^2\right\}^{1/2}$;

$\|A\| = \sup_{x\in\mathbb{R}^n\setminus\{0\}}\left(\|Ax\|/\|x\|\right)$ is the norm of the $n\times n$-matrix A;

$\Omega = \left\{x \in \mathbb{R}^n \mid \|x\| < k\right\}$ for some $k > 0$;

$I = [\,t_0\,,\,T\,)$ for some T, $t_0 < T \le +\infty$;

$h > 0$, $I_h = [\,t_0 - h\,,\,T\,)$, $\tau = \min(T\,,\,t_0 + h)$.

Consider impulsive systems of FDE of the following forms

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-h)\,, & t \ne \tau_i \\ \Delta x\big|_{t=\tau_i} = p_i(x)\,, \end{cases} \tag{1}$$

$$\begin{cases} \dot{x}(t) = D\dot{x}(t-h) + Ax(t) + Bx(t-h)\,, & t \ne \tau_i \\ \Delta x\big|_{t=\tau_i} = p_i(x)\,, \end{cases} \tag{2}$$

and

$$\begin{cases} \dot{x}(t) = D\dot{x}(t-h) + Ax(t) + Bx(t-h) \\ \qquad\qquad +Q\big(x(t),\, x(t-h),\, \dot{x}(t-h)\big)\,, & t \ne \tau_i \\ \Delta x\big|_{t=\tau_i} = p_i(x)\,, \end{cases} \tag{3}$$

where $x \in \mathbb{R}^n$; A, B, D are constant $n\times n$ matrices: $\tau_i \in I$ $(i \in \mathbb{N})$ are fixed moments; $\Delta x\big|_{t=\tau_i} = x(\tau_i + 0) - x(\tau_i - 0)$, $\tau_{i+1} > \tau_i$ and if $T = +\infty$, then $\lim_{i\to+\infty}\tau_i = +\infty$; $p_i : \Omega \to \mathbb{R}^n$; $Q : \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$.

Let $\varphi_0 : [\,t_0 - h\,,\,t_0\,] \to \mathbb{R}^n$ be a continuous function.

DEFINITION 1.— *The function* $x(t\,;\varphi_0) : I_h \to \mathbb{R}^n$ *is said to be a solution of system (1) in the interval* I *if :*

1) for $t_0 \le t < T$, $t \ne \tau_i$ *it satisfies the system*

$$\dot{x}(t) = Ax(t) + Bx(t-h)\,,$$

2) for $t_0 \le t < T$, $t = \tau_i$ *it satisfies the condition*

$$\Delta x = p_i\big(x(t)\big)\,,$$

3) in the interval $[\,t_0 - h\,,\,t_0\,]$ *it coincides with the function* φ_0.

The definitions of solutions of system (2) and (3) are analogous to definition 1 but for them we shall require $\varphi_0 \in C^1[\,t_0 - h\,,\,t_0\,]$, and for $t = \tau_i + h$ we consider the absolute continuous part of v.

In order to obtain estimates for the solutions of system (1)-(3), we shall consider the corresponding systems of integral equations.

We shall say that conditions (H) are met if the following conditions hold:

H_1 The function $v(t) : I_h \to \mathbb{R}^n$ is piecewise continuous and piecewise differentiable in I and has points of discontinuity of the first kind $t = \tau_i$ which have no finite accumulation points.

H_2 $t_0 \le \tau_0 < \tau_1 < \ldots < \tau_i < \tau_{i+1} < \ldots$ and $\lim_{i\to+\infty} \tau_i = +\infty$ if $T = +\infty$.

H_3 $v(\tau_i - 0) = v(\tau_i)$, $i = 1, 2, \ldots$

H_4 For $t_0 - h \le t \le t_0$ the inequality $\big\|v(t)\big\| < \delta$ is valid, $\delta > 0$.

H_5 The functions $p_i : \Omega \to \mathbb{R}^n$, $i = 1, 2, \ldots$ are continuous in Ω.

H_6 A, B are constant matrices, $c = \text{const}$ is an n-dimensional vector.

Further on we shall use the following lemma.

LEMMA 1 [2].— *Let the following conditions hold:*

1) Condition H_2 *is valid.*

2) $m(t) : I \to [\,0\,,\,\infty\,)$ *is a piecewise continuous with points of discontinuity of the first kind* $t = \tau_i$ *at which it is continuous from the left.*

3) $p : [\,t_0\,,\,\infty\,) \to [\,0\,,\,\infty\,)$ *is a continuous function.*

4) For $t \in I$ the following inequality is valid

$$m(t) \le c_0 + \int_{t_0}^{t} p(s) m(s)\,\mathrm{d}s + \sum_{t_0 < \tau_i < t} \beta_i m(\tau_i)\,, \tag{4}$$

where $\beta_i \ge 0$ and c_0 is a constant.

Then,

$$m(t) \le c_0 \prod_{t_0 < \tau_i < t} (1 + \beta_i) \exp\left(\int_{t_0}^{t} p(\sigma)\,\mathrm{d}\sigma \right)\,, \quad t \in I\,. \tag{5}$$

Lemma 1 follows from the more general statement of lemma 2, as kindly suggested by the referee.

LEMMA 2.— *Assume that the piecewise continuous function $m(t) \ge 0$ satisfies the inequality $(\tau_0 = t_0)$*

$$m(t) \le c_0 + \sum_{t_0 < \tau_i < t} \beta_i \sup_{\tau_{i-1} < \tau < \tau_i} m(\tau)\,,$$

where $\beta_i > 0$ and c_0 is a constant.

Then,

$$m(t) \le c_0 \prod_{t_0 < \tau_i < t} (1 + \beta_i)\,.$$

Lemma 2 is proved by induction on $k = 0, 1, 2, \ldots$, where

$$\tau_k < t \le \tau_{k+1} \tag{6}$$

Now we shall show that lemma 2 $\Rightarrow$ lemma 1. Let (6) be valid for some k. Then (4) implies

$$\begin{aligned} m(t) \le c_0 &+ \int_{\tau_k}^{t} p(s) m(s)\,\mathrm{d}s + \\ &+ \sum_{i=1}^{k} \left(\beta_i + \int_{\tau_{i-1}}^{\tau_i} p(s)\,\mathrm{d}s \right) \sup_{\tau_{i-1} < \tau < \tau_i} m(\tau)\,. \end{aligned}$$

By Gronwall's lemma we have

$$m(t) \le \left\{ c_0 + \sum_{i=1}^{k} \left(\beta_i + \int_{\tau_{i-1}}^{\tau_i} p(s)\,\mathrm{d}s \right) \sup_{\tau_{i-1}<\tau<\tau_i} m(\tau) \right\} \times \\ \times \exp\left(\int_{\tau_k}^{t} p(s)\,\mathrm{d}s \right)$$

or

$$\begin{aligned} m(t) \exp\left(-\int_{\tau_k}^{t} p(s)\,\mathrm{d}s \right) &\le \\ &\le c_0 + \sum_{i=1}^{k} \left(\beta_i + \int_{\tau_{i-1}}^{\tau_i} p(s)\,\mathrm{d}s \right) \sup_{\tau_{i-1}<\tau<\tau_i} m(\tau) \\ &\le c_0 + \sum_{i=1}^{k} \left(\beta_i + \int_{\tau_{i-1}}^{\tau_i} p(s)\,\mathrm{d}s \right) \sup_{\tau_{i-1}<\tau<\tau_i} \left(m(\tau) \exp\left(-\int_{\tau_k}^{\tau} p(s)\,\mathrm{d}s \right) \right). \end{aligned}$$

Then by lemma 2 we have

$$\begin{aligned} m(t) \exp\left(-\int_{\tau_k}^{t} p(s)\,\mathrm{d}s \right) &\le c_0 \prod_{i=1}^{k} \left(1 + \beta_i + \int_{\tau_{i-1}}^{\tau_i} p(s)\,\mathrm{d}s \right) \\ &\le c_0 \prod_{i=1}^{k} (1+\beta_i) \left(1 + \int_{\tau_{i-1}}^{\tau_i} p(s)\,\mathrm{d}s \right) \\ &\le c_0 \prod_{i=1}^{k} (1+\beta_i) \exp\left(\int_{\tau_{i-1}}^{\tau_i} p(s)\,\mathrm{d}s \right) \\ &\le c_0 \prod_{i=1}^{k} (1+\beta_i) \exp\left(\int_{t_0}^{\tau_k} p(s)\,\mathrm{d}s \right), \end{aligned}$$

i.e.

$$m(t) \le c_0 \prod_{i=1}^{k} (1+\beta_i) \exp\left(\int_{t_0}^{t} p(s)\,\mathrm{d}s \right),$$

which is (5) for the case (6). □

3. Main results

THEOREM 1.— *Let the following conditions hold:*

1) Conditions (H) *are valid.*

2) For $t \in I$ the following equality is valid:

$$v(t) = c + \int_{t_0}^{t} \big[Av(s) + Bv(s-h)\big]\,\mathrm{d}s + \sum_{t_0<\tau_i<t} p_i\big(v(\tau_i)\big)\,, \quad h > 0\,.$$

3) There exist constants $\beta_i > 0$ such that the conditions $\big\|p_i(v)\big\| \le \beta_i\|v\|$, $i = 1, 2, \ldots$ hold.

Then for $t_0 < t < \tau = \min\{\tau, t_0 + h\}$ the following estimate is valid:

$$\big\|v(t)\big\| \le \delta\big(1 + \|B\|h\big) \prod_{t_0<\tau_i<t} (1+\beta_i)\, e^{\|A\|h}\,. \tag{7}$$

Proof.— From conditions 1) and 2) of theorem 1 it follows that for $t_0 < t < \tau$ the following inequality is valid

$$\begin{aligned}\big\|v(t)\big\| \le \|c\| + \|B\|h\delta + \|A\| \int_{t_0}^{t} \big\|v(s)\big\|\,\mathrm{d}s\,+ \\ + \sum_{t_0<\tau_i<t} \big\|p_i\big(v(\tau_i)\big)\big\|\,.\end{aligned} \tag{8}$$

Obviously, $c = v(t_0)$, thus $\|c\| < \delta$. From conditions 3) of theorem 1 there follows the validity of the inequality

$$\big\|v(t)\big\| < \delta\big(1 + \|B\|h\big) + \|A\| \int_{t_0}^{t} \big\|v(s)\big\|\,\mathrm{d}s + \sum_{t_0<\tau_i<t} \beta_i \big\|v(\tau_i)\big\|$$

to which we apply theorem 1 for $m(t) = \big\|v(t)\big\|$, $p = \|A\|$, $c_0 = \delta\big(1 + \|B\|h\big)$ and obtain inequality (7). □

Remark 1.— In particular, if $\delta > 0$ is small enough, from (7) we obtain $\big\|v(t)\big\| < k$, thus $T > t_0 + h$.

COROLLARY 1.— *Let the following conditions hold:*

1) Conditions (H) *are valid.*

2) Condition 2) of theorem 1 is valid.

3) There exists a constant $a > 0$ *such that*

$$\|p_i(v)\| \le a\|v\|, \quad i = 1, 2, \ldots$$

4) There exists a constant $\theta > 0$ *such that*

$$\tau_{i+1} - \tau_i \le \theta$$

for any $i = 1, 2, \ldots$.

Then for $t_0 < t < \tau$ *the following estimate is valid:*

$$\|v(t)\| < \delta(1 + \|B\|h) \exp\left\{\left[\|A\| + \frac{1}{\theta}\ln(1+a)\right]\right\}. \tag{9}$$

Proof.— Applying theorem 1 for $\beta_i = a$, we obtain the estimate

$$\|v(t)\| < \delta(1 + \|B\|h)(1+a)^{i\langle t_0,t\rangle} e^{\|A\|h},$$

where $i\langle t_0, t\rangle$ is the number of the points τ_i in the interval $[t_0, t)$, i.e., $i\langle t_0, t\rangle = i$ if $\tau_i < t \le \tau_{i+1}$.

Then condition 4) of corollary 1 implies the validity of (9). □

COROLLARY 2.— *Let the following conditions hold:*

1) Conditions (H) *are valid.*

2) Condition 2) of theorem 1 is valid.

3) There exists a constant $b > 0$ *such that*

$$\sum_{t_0 < \tau_i < t} \|p_i(v(\tau_i))\| \le b.$$

Then for $t_0 < t < \tau$ *the following estimate is valid:*

$$\|v(t)\| < (\delta + \|B\|h\delta + b)\, e^{\|A\|h}. \tag{10}$$

Proof.— From condition 1) and 2) of corollary 2 it follows that for $t_0 < t < \tau$ inequality (8) is valid. Condition 3) of corollary 2 implies the validity of the inequality

$$\|v(t)\| < \delta + \|B\|h\delta + B + \|A\| \int_{t_0}^{t} \|v(s)\| \,\mathrm{d}s$$

to which we apply Gronwall's lemma and obtain (10). □

THEOREM 2.— *Let the following conditions hold:*

1) *Conditions* (H) *are valid and $v(t)$ is absolutely continuous in* $[t_0 - h\,, t_0\,]$.
2) *Conditions 3) and 4) of Corollary 1 are valid.*
3) *D is a constant $n \times n$-matrix.*
4) *For $t \in I$ the following equality is valid:*

$$v(t) = c + D\big[v(t-h) - v(t_0 - h)\big] + \int_{t_0}^{t} \big[Av(s) + Bv(s-h)\big] \,\mathrm{d}s + \\ + \sum_{t_0 < \tau_i < t} p_i\big(v(\tau_i)\big)\,, \quad h > 0\,.$$

Then for $t_0 < t < \tau$ the following estimate is valid:

$$\|v(t)\| < \delta\big(1 + 2\|D\| + \|B\|h\big)\, e^{\left[\|A\| + (1/\theta)\ln(1+a)\right]h}\,. \tag{11}$$

Proof.— From condition 4) of theorem 2 it follows that for $t_0 < t < \tau$ the following inequality is valid:

$$\|v(t)\| < \delta\big(1 + 2\|D\| + \|B\|h\big) + \|A\| \int_{t_0}^{t} \|v(s)\| \,\mathrm{d}s + \\ + \sum_{t_0 < \tau_i < t} \big\|p_i\big(v(\tau_i)\big)\big\|\,.$$

Condition 3) of corollary 1 implies the validity of the inequality

$$\|v(t)\| < \delta\big(1 + 2\|D\| + \|B\|h\big) + \|A\| \int_{t_0}^{t} \|v(s)\| \,\mathrm{d}s + \\ + \sum_{t_0 < \tau_i < t} a\big\|v(\tau_i)\big\|$$

to which we apply lemma 1 for $\beta_i = a$, $m(t) = \|v(t)\|$, $p = \|A\|$, $c_0 = \delta(1+2\|D\|+\|B\|h)$ and condition 4) of corollary 1 and obtain estimate (11). □

Again we obtain that if $\delta > 0$ is small enough, then $T > t_0 + h$.

Remark 2.— The integral equation in condition 4) of theorem 2 is equivalent to system (2) only in the interval $[t_0, \tau]$. In general the integral equation corresponding to (2) is

$$v(t) = c + D\left[v(t-h) - \sum_{t_0<\tau_i<t-h} p_i(v(\tau_i)) - v(t_0-h)\right] + \\ + \int_{t_0}^{t} [Av(s) + Bv(s-h)]\, ds + \sum_{t_0<\tau_i<t} p_i(v(\tau_i)).$$

THEOREM 3.— *Let the following conditions hold:*

1) Conditions (H) *are valid.*

2) Condition 3) and 4) of Corollary 1 are valid.

3) The function $Q : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is such that*

$$\sup_{x\in\mathbb{R}^n,\, y\in\mathbb{R}^n,\, z\in\mathbb{R}^n} \{\|Q(x,y,z)\|\} \le \nu, \quad \nu > 0.$$

4) For $t_0 \le t < T$ *the following equality is valid:*

$$v(t) = c + D[v(t-h) - v(t_0-h)] + \\ + \int_{t_0}^{t} [Av(s) + Bv(s-h) + Q(v(s), v(s-h), \dot{v}(s-h))]\, ds + \\ + \sum_{t_0<\tau_i<t} p_i(v(\tau_i)), \quad h > 0. \tag{12}$$

Then for $t_0 < t < \tau$ *the following estimate is valid:*

$$\|v(t)\| \le [\delta(1 + 2\|D\| + \|B\|h) + \nu h]\, e^{[\|A\|+(1/\theta)\ln(1+a)]h}. \tag{13}$$

Proof.— From condition 1), 3) and 4) of theorem 3 it follows that for $t_0 < t < \tau$ the following inequality is valid:

$$\|v(t)\| \le \delta(1 + 2\|D\| + \|B\|h) + \|A\| \int_{t_0}^{t} \|v(s)\|\, ds + \\ + \nu h + \sum_{t_0<\tau_i<t} \|p_i(v(\tau_i))\|.$$

Condition 3) of corollary 1 implies the validity of the inequality

$$\|v(t)\| = \delta\big(1+2\|D\|+\|B\|h\big)+\nu h+\|A\|\int_{t_0}^{t}\|v(s)\|\,\mathrm{d}s+$$
$$+\sum_{t_0<\tau_i<t} a\|v(\tau_i)\|$$

to which we apply lemma 1 for $\beta_i = a$, $m(t) = \|v(t)\|$, $p = \|A\|$, $c_0 = \nu h + \delta\big(1+2\|D\|+\|B\|h\big)$ and condition 4) of corollary 1 and obtain (13). □

We shall note that for the integral equation (12) and system (3) an assertion analogous of this of remark 2 is valid.

COROLLARY 3.— *Let the following conditions hold:*

1) Conditions 1), 2) and 4) of theorem 3 are valid.

2) $\|\dot v(t)\| < \delta$ *for* $t_0 - h \le t \le t_0$.

3) The function $Q(x,y,z)$ *is defined in a neighbourhood of* 0 *in* $\mathbb{R}^{3n}$ *and satisfies* $\|Q(x,y,z)\| \le C\big(\|x\|^2+\|y\|^2+\|z\|^2\big)$.

4) $\|A\| < 1$, $a < 1$.

Then for each sufficiently small $\delta_1 > 0$ *there exists* $\delta > 0$ *such that* $\|v(t)\| < \delta_1 q\|\dot v(t)\| < \delta_1$ *for* $t_0 \le t \le t_0 + h$.

Proof.— Choose $\delta_1 < k$ so small that $Q(x,y,z)$ be defined in

$$\Big\{(x,y,z)\in\mathbb{R}^{3n} \mid \|x\|<\delta_1\,,\ \|y\|<\delta_1\,,\ \|z\|<\delta_1\Big\}.$$

Suppose that the assertion is not true and let

$$\tau = \sup\big\{t\in[\,t_0\,,\,t_0+h\,)\mid \|v(t)\|<\delta_1 q\|\dot v(t)\|<\delta_1 \text{ for } t_0\le t<\tau\big\}.$$

Then at least one of the following two inequalities is valid:

$$\|v(\tau+0)\| \ge \delta_1 \tag{14}$$

or

$$\|\dot v(\tau+0)\| \ge \delta_1\,. \tag{15}$$

Suppose that (14) holds. We apply theorem 3 with $\nu = C(2\delta^2 + \delta_1^2)$ and obtain

$$\|v(t)\| \le \left\{\delta\big(1 + 2\|D\| + \|B\|h\big) + Ch(2\delta^2 + \delta_1^2)\right\} \times \\ \times e^{[\|A\| + (1/\theta)\ln(1+a)]h} \tag{16}$$

for $t_0 \le t < \tau$.

If for any sufficiently small $\delta > 0$ we can make the right-hand side of (16) smaller than $\delta_1(1-a)q$ we get to a contradiction since the possible jump of $\|v(t)\|$ at the point τ does not exceed $a\|v(t)\| < a\delta_1$. Thus we wish to solve the inequality

$$2C\delta^2 + \delta\left(\frac{1 + 2\|D\|}{h} + \|B\|\right) + C\delta_1^2 - \frac{1-a}{he^{[\|A\| + (1/\theta)\ln(1+a)]h}}\,\delta_1 < 0. \tag{17}$$

If $\delta = 0$, then it takes the form

$$C\delta_1^2 < \frac{e^{-[\|A\| + (1/\theta)\ln(1+a)]h}}{h}\,\delta_1(1-a)$$

and is satisfied for

$$\delta_1 < \frac{(1-a)e^{-[\|A\| + (1/\theta)\ln(1+a)]h}}{Ch}.$$

Obviously inequality (17) will be satisfied too for any $\delta > 0$ small enough. Thus $\|v(\tau + 0)\| < \delta_1$. Now suppose that (15) holds.

On the other hand, from the equivalent in the interval $[t_0, \tau]$ to (12) system (3) we obtain

$$\|\dot v(t)\| \le \delta\big(\|D\| + \|B\|\big) + \|A\|\delta_1 + C(2\delta^2 + \delta_1^2) \text{ for } t_0 \le t \le \tau + 0. \tag{18}$$

If for any sufficiently small $\delta > 0$ we can make the right-hand side of (18) smaller than δ_1, we shall get to a contradiction and the assertion will be proved. Thus we want to solve the inequality

$$2C\delta^2 + \delta\big(\|D\| + \|B\|\big) + C\delta_1^2 - \big(1 - \|A\|\big)\delta_1 < 0. \tag{19}$$

If $\delta = 0$, then it takes the form

$$C\delta_1^2 < \big(1 - \|A\|\big)\delta_1$$

and is satisfied for

$$\delta_1 < \frac{(1 - \|A\|)}{C}.$$

Obviously inequality (19) will be satisfied too for any $\delta > 0$ small enough. □

COROLLARY 4.— *Let the following conditions hold:*

1) $v(t)$ is a solution of system (3) in I.

2) Conditions 1) and 2) of theorem 2 are valid.

3) Conditions 3) and 4) of corollary 3 are valid.

Then for $\delta > 0$ small enough the lifespan T of $v(t)$ can be chosen arbitrarily large.

Proof.— Let $T > 0$ be arbitrarily large and denote $n = T/h$. Integrate system (3) from $t_0 + nh$ to $t \leq T$ and obtain

$$\begin{aligned} v(t) = v(t_0 + nh + 0) + D\big(v(t-h)\big) + \\ - \sum_{t_0+(n-1)h<\tau_i<t-h} p_i\left(v(\tau_i) - v\big(t_0 + (n-1)h + 0\big)\right) + \\ + \int_{t_0+nh}^{t} \left[Av(s) + Bv(s-h) + Q\big(v(s),\, v(s-h),\, \dot v(s-h)\big)\right] \mathrm{d}s + \\ + \sum_{t_0+nh<\tau_i<t} p_i\big(v(\tau_i)\big)\,. \end{aligned}$$

Suppose that

$$\begin{gathered} \big\|v(t+0)\big\| < \delta_n\,, \quad \big\|\dot v(t+0)\big\| < \delta_n \\ \text{for } t_0 + (n-1)h \leq t \leq t_0 + nh\,. \end{gathered} \tag{20}$$

As in theorem 3 we obtain

$$\begin{aligned} \big\|v(t)\big\| \leq \delta_n \left[1 + \|D\| \left(2 + \frac{ha}{\theta}\right) + \|B\|h\right] + \|A\| \int_{t_0+nh}^{t} \big\|v(s)\big\| \,\mathrm{d}s + \\ + Ch\left(2\delta_n^2 + \sup_{t_0+nh<t<T} \big\|v(t)\big\|^2\right) . + \sum_{t_0+nh<\tau_i<t} a\big\|v(\tau_i)\big\| \end{aligned}$$

for $t_0 + nh < t < T$.

Further on, repeating some details of the proofs of theorem 3 and corollary 3, we prove that if $\delta_{n+1} < k$ is small enough, then there exists $\delta_n > 0$ such that

$$\big\|v(t)\big\| < \delta_{n+1}\,, \quad \big\|\dot v(t)\big\| < \delta_{n+1}$$

for $t_0 + nh \leq t \leq T$. Obviously we can choose $\delta_n < \delta_{n+1}$.

Analogously we prove that there exists sufficiently small $\delta_{n-1} < \delta_n$ such that if

$$\|v(t+0)\| < \delta_{n-1}\,, \quad \|\dot{v}(t+0)\| < \delta_{n-1}$$

for $t_0 + (n-2)h \leq t \leq t_0 + (n-1)h$, then (20) holds, etc. After a finite number of steps we find δ_1 and $\delta > 0$ so that the assertion of corollary 3 should hold. Going back, we obtain that if $\|v(t)\| < \delta$, $\|\dot{v}(t)\| < \delta$ for $t_0 - h \leq t \leq t_0$, then the solution is defined for $t < T$ since in view of what was proved above we do not leave the definition domains of $p_i(v)$ and Q. □

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