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Fluid flow in macromolecular systems and related perturbation problems

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RÉSUMÉ. — On prouve l'existence et l'unicité des solutions faibles du problème aux valeurs initiales et limites décrit dans l'équation (1) de l'introduction. Cette équation modélise le flux d'un fluide visqueux et incompressible à travers un objet élastique et perméable.

ABSTRACT. — We establish existence and uniqueness of weak solutions to the initial-boundary value problem presented in equations (1) of the introduction, which describes the flow of a viscous, incompressible fluid through an elastic permeable medium.

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1. Introduction

We are interested in the model proposed by Wiegel in [13], section 28, to describe the flow of a viscous, incompressible fluid through an elastic, permeable medium (typically, a macromolecular system), at low values of the Reynolds number. This model plays an important role in many branches of applied chemical physics and biophysics; it is described by the equations

$$\begin{cases} \epsilon\rho_0 v_t = -\nabla p + \epsilon\eta_0\Delta v + \delta\eta_0(u_t - v) \\ \operatorname{div} v = 0 \\ \rho_1 u_{tt} = \mu\nabla u + \lambda\nabla \operatorname{div} u - \delta\eta_0(u_t - v) \end{cases} \tag{1}$$

where v and p are the "macroscopic" velocity and pressure of the fluid and u is the displacement of the medium; the positive constants $\epsilon, \ldots, \lambda$

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will be described later. Equations (1) are obtained combining the familiar linear Navier-Stokes equations for incompressible fluid with Darcy's law, and coupling them with the equations of elasticity, describing the deformations of the medium through which the fluid flows. Indeed, quite often the porous medium is represented as a rigid object, such as a cookie or a piece of chalk; in some instances, however, the fluid may cause deformations in the medium, which can then reasonably be assumed to respond and behave like an elastic body (Schneiddeger, [12]). Some remarks are in order in relation to equations (1): we follow closely Wiegel's discussion of [13], sections 1 to 5, to which we refer for most details. We first consider the complete Navier-Stokes equations for incompressible fluids

$$\begin{cases} \rho_0\big(v_t + (v\cdot\nabla)v\big) = -\nabla p + \eta_0 \Delta v \\ \operatorname{div} v = 0\,, \end{cases} \tag{2}$$

where v and p are the "microscopic" velocity and pressure of the fluid, ρ_0 its mass density and η_0 its viscosity. In many conditions of practical interest, the nonlinear term $\rho_0(v\cdot\nabla)v$ can be neglected when compared to the linear term $\eta_0\Delta v$, since the ratio of these terms is of the order of the Reynolds number R. For instance, for a macromolecular coil in water, $R \cong 10^{-6}$ (Wiegel, [13]): for such small values of R, the nonlinear equations (2) can be approximated by their linear version

$$\begin{cases} \rho_0 v_t = -\nabla p + \eta_0 \Delta v \\ \operatorname{div} v = 0\,. \end{cases} \tag{3}$$

Next, we note that the fields in (2) and (3) usually display a rapidly oscillating behavior in space and time; however, linearity in (3) allows averaging, and the corresponding "macroscopic" quantities, which are then slowly varying functions of space and time, satisfy similar equations. For example, the macroscopic velocity is defined by

$$v(x,t) = |\omega|^{-1}|\tau|^{-1}\int_{\omega\times\tau} \widehat{v}(x',t')\,\mathrm{d}x'\,\mathrm{d}t'\,,$$

where $\widehat{v}$ is the microscopic velocity, ω a ball will center at x, τ an interval with center at t; the sizes for ω and τ are conveniently chosen, relative to the characteristic macroscopic dimensions of the system and to the typical sizes of its repeating units. Thus, we consider all quantities in equations (3) to be macroscopic, and proceed to a more specific exam of the interaction of the fluid with the medium through which it flows. The fluid is subject to two forces, the resultant of which should be inserted in equation (3.1): a force $-\nabla p$ due to the macroscopic pressure in the fluid, and a frictional force F exerted on the fluid by the medium. According to Darcy's law, the first force is proportional to the macroscopic velocity of the fluid:

$$-\nabla p = \delta \eta_0 v \,, \tag{4}$$

where $\delta > 0$ is the reciprocal of the hydrodynamic permeability, a physical quantity of dimension [length]^2. If the medium is at rest, the two forces cancel, so that

$$F - \nabla p = 0 \,, \tag{5}$$

and combination with Darcy's law (4) yields

$$F = -\delta \eta_0 v \,; \tag{6}$$

if the medium is not at rest, but suffers deformations, causing its particles to move with (small) velocities u_t, then the frictional force (6) is modified into

$$F = -\delta \eta_0 (v - u_t) \,. \tag{7}$$

If we assume the response of the medium to be that of an elastic body, subject to Hooke's law, the equations of elasticity describing its displacement read

$$\rho_1 u_{tt} = \mu \Delta u + \lambda \nabla \operatorname{div} u + F \,, \tag{8}$$

where ρ_1 is the mass density of the medium and λ and μ are the Lame's constants. The explicit form of the total force F exerted on the medium will in general depend on its rheological properties, but will in any case contain an additive term like (7), due to the interaction with the fluid. Combination of the elasticity equations (8) with Darcy's lav (7), coupled with equations (3), yields systems (1), where we have added the constant $\epsilon > 0$ to measure the effect of the inertial forces in the fluid. We propose to study system (1), subject to initial data

$$v(x,0) = v_0(x) \,, \quad u(x,0) = u_0(x) \,, \quad u_t(x,0) = u_1(x) \tag{9}$$

and to homogeneous boundary conditions, either of Dirichlet type or of tangential or normal type; for definiteness, we shall impose conditions

$$n \times v = 0 \,, \quad u\Big|_{\partial\Omega} = 0 \,. \tag{10}$$

We remark that Darcy's modified law (5) + (7), that is

$$-\nabla p = \delta\eta_0(v - u_t)\,, \tag{11}$$

can be formally obtained from equation (1.a) when $\epsilon = 0$: the question then naturally arises, whether Darcy's law is actually a consequence of equations (1) when inertial forces are negligible. Also, in many situations one typically has, other than $\epsilon \ll 1$, either $\eta_0 \ll 1$ (small viscosity) or $\delta \ll 1$ (large permeability), so that it is of some interest to study the behavior of the solutions of (1) when these quantities vanish. We shall see that indeed the solutions of (1) possess, at the vanishing of ϵ, δ, η_0, suitable limits which are solutions of the formal limit equations, obtained from (1) by setting $\epsilon = 0$, $\delta = 0$ and $\eta_0 = 0$ respectively, so that equations (1) can be regarded as perturbations of these limit equations. Perturbation processes of this type arise in quite a few physical situations; a similar study for two related nonlinear one-dimensional problems is carried out in [8] and [9]; see [10] for an analogous problem concerning the quasilinear Maxwell equations. For a general treatment of this type of perturbation problems, see Lions [6]; see also [11] for a partial extension to smooth solutions of nonlinear problems.

2. Functional spaces

We shall consider some Sobolev type functional spaces, which have turned out to be quite convenient for the study of Navier-Stokes equations; we recall their definition and main properties from [1], [3] and [5].

Given $\Omega \subseteq \mathbb{R}^3$, a simply connected (for simplicity; indeed, [3] and [7] deal with multiply connected domains) bounded open set with smooth boundary $\partial\Omega$, having (again for simplicity) only one connected component, we set $L^2 = L^2(\Omega)$, $\mathcal{L}^2 = (L^2)^3$, $H^1 = H^1(\Omega)$, etc.; in general, if E is a space, we set $\mathcal{E} = E^3$. We define the spaces

$$H(\mathrm{curl}) = \{u \in \mathcal{L}^2 \mid \mathrm{curl}\, u \in \mathcal{L}^2\}$$
$$H(\mathrm{div}) = \{u \in \mathcal{L}^2 \mid \mathrm{div}\, u \in L^2\}$$

which are Hilbert spaces with respect to the norms

$$||u|| + ||\,\mathrm{curl}\, u|| \quad \text{and} \quad ||u|| + ||\,\mathrm{div}\, u||$$

respectively (here, and in the sequel, $||\cdot||$ denotes the usual L^2 norm). These spaces are dense in $\mathcal{L}^2$, because $\mathcal{D}(\overline{\Omega})^3$ is dense in them ([2], chapter

VII, sections 4, 5); this also allows us to define tangential and normal components on $\partial\Omega$ for elements in $H(\mathrm{curl})$ and $H(\mathrm{div})$ respectivley, as elements of $\mathcal{H}^{-1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$, so that the following integration by parts formulas hold:

PROPOSITION 1.— *$\forall\ u \in H(\mathrm{curl})$, $\forall\ v \in H(\mathrm{div})$, $\forall\ \varphi \in \mathcal{H}^1$, $\forall\ \psi \in H^1$:*

$$\langle n \times u\,,\ \varphi\rangle = (\mathrm{curl}\, u\,,\ \varphi) - (u\,,\ \mathrm{curl}\, \varphi) \tag{2.1}$$

$$\langle n \cdot v\,,\ \psi\rangle = (\mathrm{div}\, v\,,\ \psi) + (v\,,\ \nabla\psi) \tag{2.2}$$

where $(\cdot\,,\,\cdot)$ is the L^2 scalar product, and $\langle\cdot\,,\,\cdot\rangle$ is the duality between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$.

We shall actually indicate by $\langle\cdot\,,\,\cdot\rangle$ the duality product between a space and its dual, specifying these spaces whenever they are not clear in the context.

Of particular interest in the sequel will be the spaces

$$H_0(\mathrm{curl}) = \{u \in H(\mathrm{curl}) \mid n \times u = 0\}\,,$$
$$H_0(\mathrm{div}) = \{u \in H(\mathrm{div}) \mid n \cdot u = 0\}\,.$$

As a matter of notation, if E and F are two Hilbert spaces, we write $E \hookrightarrow F$ to mean that E is densely and continuously embedded in F; if $T > 0$, we abbreviate $L^2(0,T;E)$ into $L^2(E)$, and similarly for $L^\infty(0,T;E)$, $C(0,T;E)$; finally, if $E \subseteq \mathcal{L}^2$, we set

$$E_d = \{u \in E \mid \mathrm{div}\, u = 0\}\,.$$

We shall consider the spaces

$$\begin{aligned} H &= \{u \in \mathcal{L}^2 \mid \mathrm{div}\, u = 0\,,\ n \cdot u = 0\} \\ G &= \{u \in \mathcal{L}^2 \mid \mathrm{curl}\, u = 0\} \\ R &= \{u \in \mathcal{L}^2 \mid \mathrm{curl}\, u = 0\,,\ n \times u = 0\} \\ Y &= \{u \in \mathcal{L}^2 \mid \mathrm{curl}\, u \in \mathcal{L}^2\,,\ \mathrm{div}\, u = 0\,,\ n \times u = 0\} \end{aligned}$$

of which we recall the following properties:

PROPOSITION 2

a) $\mathcal{L}^2 = H \oplus G = \mathcal{L}^2_d \oplus R$;

b) $G = \{u \in \mathcal{L}^2 \mid \exists\ p \in H^1\,,\ u = \nabla p\}$;

c) $R = \left\{ u \in \mathcal{L}^2 \mid \exists\, p \in H^1 ,\ p\big|_{\partial\Omega} = \text{const} ,\ u = \nabla p \right\}$*;*

d) Y *is a closed subspace of* $\mathcal{H}^1_d$*, on which* $\|u\|_Y = \|\operatorname{curl} u\|$ *is a norm, equivalent to the one induced by* $\mathcal{H}^1$*;*

e) $Y \hookrightarrow \mathcal{L}^2_d \hookrightarrow Y'$.

Proposition 3

a) $\forall\ u \in H_0(\mathrm{curl})$, $\operatorname{curl} u \in H$*;*

b) $\forall\ u \in \mathcal{L}^2$, $\operatorname{curl} u \in \left(H_0(\mathrm{curl})\right)' \equiv H^{-1}(\mathrm{curl})$*;*

c) $\forall\ u, v \in H_0(\mathrm{curl})$, $\langle \operatorname{curl}^2 u\,,\ v \rangle = (\operatorname{curl} u, \operatorname{curl} v) = \langle u\,,\ \operatorname{curl}^2 v \rangle$ *(duality between* $H_0(\mathrm{curl})$ *and* $H^{-1}(\mathrm{curl})$*);*

d) $\operatorname{curl}^2 : Y \to Y'$ *is an isomorphism.*

We postpone the proof of these propositions to section 7 at the end of the paper.

3. Statement of results

We now consider system (1) + (9) + (10), and set $\rho_0 = \rho_1 = \mu = 1$ for simplicity. We are interested in two classes of weak solutions, in relation to the regularity assumed of the initial data (9). At first we assume

$$u_0 \in \mathcal{H}^1_0 , \quad u_1 \in \mathcal{L}^2 ; \quad v_0 \in Y \tag{3.1}$$

and consider the following

Problem I.— find $u \in L^2(\mathcal{H}^1_0)$ with $u_t \in L^2(\mathcal{L}^2)$, $v \in L^2(Y)$ with $v_t \in L^2(\mathcal{L}^2_d)$, $\operatorname{curl}^2 v \in L^2(\mathcal{L}^2)$ such that

i) $u(0) = u_0$, $v(0) = v_0$;

ii) $\forall\ \xi \in L^2(\mathcal{L}^2_d)$,

$$\int_0^T (\epsilon v_t + \delta\eta_0 v + \epsilon\eta_0 \operatorname{curl}^2 v - \delta\eta_0 u_t\,,\ \xi) = 0\,; \tag{3.2}$$

iii) $\forall\ \varphi \in T \equiv \left\{ u \in L^2(\mathcal{H}^1_0) \mid u_t \in L^2(\mathcal{L}^2) ,\ u(T) = 0 \right\}$,

$$\begin{cases} \displaystyle\int_0^T \left\{ (u_t\,,\ \delta\eta_0\varphi - \varphi_t) + (\operatorname{curl} u\,,\ \operatorname{curl}\varphi) + (1+\lambda)(\operatorname{div} u\,,\ \operatorname{div}\varphi) \right\} = \\ \qquad\qquad = \delta\eta_0 \displaystyle\int_0^T (v, \varphi) + \left(u_1, \varphi(0)\right) . \end{cases} \tag{3.3}$$

We note that problem I is well stated : indeed, $\mathcal{H}_0^1 \hookrightarrow \mathcal{L}^2$ and $Y \hookrightarrow \mathcal{L}_d^2$ (by e) of proposition 2), so that theorem 3.1 of [7], chapter I, applies and guarantees that, in particular, $\varphi \in C(\mathcal{L}^2)$ if $\varphi \in L^2(\mathcal{H}_0^2)$ and $\varphi_t \in L^2(\mathcal{L}^2)$; analogously for u and v. Moreover, if u and v are solutions of problem I, (3.2) implies that for almost all $t \in [\,0\,,\,T\,]$, setting

$$F(t) \equiv (\epsilon v_t + \delta\eta_0 v + \epsilon\eta_0 \operatorname{curl}^2 v - \delta\eta_0 u_t)(t)\,,$$

$F(t) \in \mathcal{L}^2$ and $F(t) \perp \mathcal{L}_d^2$. By a) and c) of proposition 2 then, $F(t) \in R$ so that $F(t) = -\nabla p(t)$ for some $p \in L^2(H^1)$: it is in this sense that equation (1) holds. Finally, note that if $u \in \mathcal{H}^2$, so that $\operatorname{curl} u$ and $\operatorname{div} u$ are defined in $\mathcal{H}^{1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ respectively, then from proposition 1 we would have, since $\varphi\big|_{\partial\Omega} = 0$:

$$\begin{cases} (\operatorname{curl}^2 u\,,\,\varphi) = (\operatorname{curl} u\,,\,\operatorname{curl}\varphi) - \langle n \times \varphi\,,\,\operatorname{curl} u\rangle = (\operatorname{curl} u\,,\,\operatorname{curl}\varphi)\,, \\ -(\nabla \operatorname{div} u\,,\,\varphi) = (\operatorname{div} u\,,\,\operatorname{div}\varphi) - \langle n\cdot\varphi\,,\,\operatorname{div} u\rangle = (\operatorname{div} u\,,\,\operatorname{div}\varphi)\,. \end{cases} \tag{3.4}$$

We claim then:

THEOREM 1.— *There exist unique $u \in L^\infty(\mathcal{H}_0^1)$, $v \in L^\infty(Y)$, solutions of problem I. Moreover, $u_t \in L^\infty(\mathcal{L}^2)$ and $v_t \in L^\infty(\mathcal{L}_d^2)$.*

Proof.— See section 4.

We now consider a weaker class of solutions, assuming only

$$u_0 \in \mathcal{L}^2\,,\quad u_1 \in \mathcal{H}^1\,;\quad v_0 \in \mathcal{L}_d^2\;: \tag{3.5}$$

we state the following

Problem II.— Find $u \in L^2(\mathcal{L}^2)$ and $v \in L^2(Y)$ such that

i) $\forall\,\varphi \in T_u \equiv \{u \in L^2(\mathcal{H}^2 \cap \mathcal{H}_0^1) \mid u_{tt} \in L^2(\mathcal{L}^2)\,,\; u(T) = u_t(T) = 0\}$,

$$\begin{cases} \displaystyle\int_0^T \big(u\,,\,\varphi_{tt} - \delta\eta_0\varphi_t + \operatorname{curl}^2\varphi - (1+\lambda)\nabla\operatorname{div}\varphi\big) = \\ \displaystyle\qquad = \delta\eta_0 \int_0^T (v,\varphi) + \langle u_1\,,\,\varphi(0)\rangle + \big(u_0\,,\,\delta\eta_0\varphi(0) - \varphi_t(0)\big)\,; \end{cases} \tag{3.6}$$

ii) $\forall\,\xi \in T_v \equiv \{u \in L^2(Y) \mid u_t \in L^2(\mathcal{L}_d^2)\,,\; u(T) = 0\}$,

$$\begin{cases} \displaystyle\int_0^T \{(v\,,\,\delta\eta_0\xi - \epsilon\xi_t) + \epsilon\eta_0(\operatorname{curl} v\,,\,\operatorname{curl}\xi)\} = \\ \displaystyle\qquad = -\delta\eta_0 \int_0^T (u,\xi_t) + \big(\epsilon v_0 - \delta\eta_0 u_0\,,\,\xi(0)\big)\,. \end{cases} \tag{3.7}$$

Again, we note that problem II is well stated; in particular, $\varphi \in C(\mathcal{H}_0^1)$ and $\varphi_t \in L^2(\mathcal{H}_0^1) \cap \mathcal{C}(\mathcal{L}^2)$ if $\varphi \in L^2(\mathcal{H}^2 \cap \mathcal{H}_0^1)$ and $\varphi_{tt} \in L^2(\mathcal{L}^2)$; analogously for ξ. Also, note that now $\varphi \in \mathcal{H}^2$, so that if $u \in \mathcal{H}_0^1$, (3.4) could be continued into

$$\begin{cases} (\operatorname{curl} u\,,\, \operatorname{curl} \varphi) = (\operatorname{curl}^2 \varphi\,,\, u) + \langle n \times u\,,\, \operatorname{curl} \varphi \rangle = (\operatorname{curl}^2 \varphi\,,\, u)\,, \\ (\operatorname{div} u\,,\, \operatorname{div} \varphi) = -(\nabla \operatorname{div} \varphi\,,\, u) + \langle n \cdot u\,,\, \operatorname{div} \varphi \rangle = -(\nabla \operatorname{div} \varphi\,,\, u)\,. \end{cases} \tag{3.8}$$

Analogously for v: if $\operatorname{curl}^2 v \in \mathcal{L}^2$, then $\operatorname{curl} v \in \mathcal{H}^1$ because of a) of proposition 3 and proposition 1.4 of [3]; hence, $\operatorname{curl} v\big|_{\partial\Omega}$ could be defined in $\mathcal{H}^{1/2}(\partial\Omega)$ and from (2.1) we would have, for $\xi \in Y$:

$$(\operatorname{curl}^2 v\,,\, \xi) = (\operatorname{curl} v\,,\, \operatorname{curl} \xi) - \langle n \times \xi\,,\, \operatorname{curl} v \rangle = (\operatorname{curl} v\,,\, \operatorname{curl} \xi)\,. \tag{3.9}$$

On the other hand, we do not have as straightforward an interpretation of equation (1) as we did for problem I; however, note that since $\operatorname{div} \xi = \operatorname{div} \xi_t = 0$ if $\xi \in T_v$, by proposition 1.3 of [3] there exists $\psi \in \mathcal{H}^1$ such that $\xi = \operatorname{curl} \psi$ (we could actually show, as we did for v in (3.9), that $\psi \in \mathcal{H}^2 \cap \mathcal{H}_0^1$ since $\xi \in Y$). Thus, (3.7) implies that $\operatorname{curl} F = 0$ as a distribution, so that F can be interpreted as a gradient. Also, we shall see that the solution to problem II can be found as a limit of sequences of solutions to problem I, for which equation (1) had sense; in either case, this shows that F is a gradient whenever it is in $\mathcal{L}^2$. We claim:

THEOREM 2.— *There exist unique $u \in L^\infty(\mathcal{L}^2)$, $v \in L^\infty(\mathcal{L}_d^2) \cap L^2(Y)$, solutions of problem II. Moreover, $u_t \in L^\infty(\mathcal{H}^{-1})$ and $v_t \in L^2(\mathcal{L}_d^2)$.*

Proof.— See section 5.

We remark that the additional space regularity enjoyed by v with respect to u, in both problem I and II, is due to the fact that equation (1) is essentially parobolic in v, while (3) is essentially hyperbolic in u.

4. Proof of theorem 1

We proceed as in [7], chapter III, section 8, using a Faedo-Galerkin approximation: let $\{w_n\}$ and $\{y_n\}$ be total bases for $\mathcal{H}_0^1$ and Y respectively, and set, $\forall\, n \in \mathbb{N}$, $V_n = \operatorname{span}[w_1, \ldots, w_n]$, $Y_n = \operatorname{span}[y_1, \ldots, y_n]$. We choose sequences u_0^n, u_1^n in V_n, and $v_0^n \in Y_n$, such that

$$u_0^n \to u_0 \text{ in } \mathcal{H}_0^1\,, \quad u_1^n \to u_1 \text{ in } \mathcal{L}^2\,; \quad v_0^n \to v_0 \text{ in } Y \tag{4.1}$$

as $n \to \infty$, and state for all n

Problem III.— Find $u^n = u^n(t) \in V_n$ and $v^n = v^n(t) \in Y_n$ such that for all $j = 1, \ldots, n$:

$$\begin{cases} \epsilon(v_t^n, w_j) + \delta\eta_0(v^n, w_j) + \epsilon\eta_0(\operatorname{curl} v^n, \operatorname{curl} w_j) = \delta\eta_0(u_t^n, w_j) \\ \langle u_{tt}^n, y_j\rangle + \delta\eta_0(u_t^n, y_j) + (\operatorname{curl} u^n, \operatorname{curl} y_j) + \\ \qquad\qquad + \ell(\operatorname{div} u^n, \operatorname{div} y_j) = \delta\eta_0(v^n, y_j) \\ u^n(0) = u_0^n, \quad u_t^n(0) = u_1^n; \quad v^n(0) = v_0^n, \end{cases} \tag{4.2}$$

where $\ell = 1 + \lambda$. Problem (4.2) is then equivalent to a system of ordinary differential equations in the unknowns $\{\alpha_1(t), \ldots, \alpha_n(t)\}$ and $\{\beta_1(t), \ldots, \beta_n(t)\}$, as defined by $u^n(t) = \sum_{j=1}^n \alpha_j(t) w_j$ and $v^n(t) = \sum_{j=1}^n \beta_j(t) y_j$. We proceed now to obtain *a priori* estimates on the (local in time) solutions of (4.2). At first, we multiply equation (4.2a) by $2\alpha_j(t)$, and equation (4.2b) by $2\beta_j'(t)$: summing for $j = 1, \ldots, n$, and integrating by parts, as is admissible (compare to (3.4)), we obtain (we omit the index n for convenience):

$$\begin{cases} \epsilon \dfrac{\mathrm{d}}{\mathrm{d}t} \|v\|^2 + 2\delta\eta_0\|v\|^2 + 2\epsilon\eta_0\|\operatorname{curl} v\|^2 = \\ \qquad = 2\delta\eta_0(u_t, v) \leq \delta\eta_0\|u_t\|^2 + \delta\eta_0\|v\|^2, \end{cases} \tag{4.3}$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \|u_t\|^2 + 2\delta\eta_0\|u_t\|^2 + \dfrac{\mathrm{d}}{\mathrm{d}t} \left(\|\operatorname{curl} u\|^2 + \ell\|\operatorname{div} u\|^2\right) = \\ \qquad = 2\delta\eta_0(v, u_t) \leq \delta\eta_0\|v\|^2 + \delta\eta_0\|u_t\|^2; \end{cases} \tag{4.4}$$

from which, summing and integrating in $[0, t]$, $t \in]0, T]$:

$$\begin{cases} \|u_t\|^2 + \|\operatorname{curl} u\|^2 + \ell\|\operatorname{div} u\|^2 + \epsilon\|v\|^2 + \epsilon\eta_0 \displaystyle\int_0^t \|\operatorname{curl} v\|^2 \leq \\ \qquad \leq \|u_1\|^2 + \|\operatorname{curl} u_0\|^2 + \ell\|\operatorname{div} u_0\|^2 + \epsilon\|v_0\|^2. \end{cases} \tag{4.5}$$

Because of (4.1) and (3.1), we deduce that, as $n \to \infty$,

$$\begin{cases} u = u^n \in \text{bounded set of } L^\infty(\mathcal{H}_0^1), \\ u_t = u_t^n \in \text{bounded set of } L^\infty(\mathcal{L}^2), \\ v = v^n \in \text{bounded set of } L^\infty(L^2) \cap L^2(Y). \end{cases} \tag{4.6}$$

Next, we specialize the choise of the basis $\{w_n\}$ to that of the eigenvalues of curl^2, i.e. such that $\forall\, n$, $\operatorname{curl}^2 w_n = \lambda_n w_n$; we multiply (4.2a) by $2\beta_j'(t)$

and then, with w_j so replaced by $\operatorname{curl}^2 w_j$, by $2\beta_j(t)$ again: summing for $j = 1, \ldots, n$, we obtain

$$\begin{cases} 2\epsilon \displaystyle\int_0^t \|v_t\|^2 + \delta\eta_0\|v\|^2 + \epsilon\eta_0\|\operatorname{curl} v\|^2 + \epsilon\|\operatorname{curl} v\|^2 + \\ \qquad\qquad + 2\delta\eta_0 \displaystyle\int_0^t \|\operatorname{curl} v\|^2 + 2\epsilon\eta_0\|\operatorname{curl}^2 v\|^2 = \\ = \delta\eta_0\|v_0\|^2 + \epsilon\eta_0\|\operatorname{curl} v_0\|^2 + \epsilon\|\operatorname{curl} v_0\|^2 + 2\delta\eta_0 \displaystyle\int_0^t (u_t\,,\, v_t + \operatorname{curl}^2 v)\,; \end{cases} \tag{4.7}$$

splitting then

$$\begin{cases} 2\epsilon\eta_0 \displaystyle\int_0^t (u_t\,,\, v_t + \operatorname{curl}^2 v) \le \\ \qquad \le \dfrac{(\delta\eta_0)^2 + \delta^2\eta_0}{\epsilon} \displaystyle\int_0^t \|u_t\|^2 + \epsilon \int_0^t \|v_t\|^2 + \epsilon\eta_0 \int_0^t \|\operatorname{curl}^2 v\|^2\,, \end{cases}$$

we deduce from (4.7), because of (4.1), (4.2) and (4.6b), that

$$\begin{cases} v_t = v_t^n \in \text{bounded set of } L^\infty(\mathcal{L}_d^2)\,, \\ v = v^n \in \text{bounded set of } L^\infty(Y)\,, \\ \operatorname{curl}^2 v = \operatorname{curl}^2 v^n \in \text{bounded set of } L^2(\mathcal{L}^2)\,. \end{cases} \tag{4.8}$$

Estimates (4.6) and (4.8) are sufficient to conclude the proof of theorem 1; there exist in fact fields u and v such that, for suitable subsequences u^n, v^n,

$$\begin{gathered} u^n \to u \text{ in } L^\infty(\mathcal{H}_0^1) \text{ weak}^* \\ u_t^n \to u_t \text{ in } L^\infty(\mathcal{L}^2) \text{ weak}^* \\ v^n \to v \text{ in } L^\infty(Y) \text{ weak}^* \\ v_t^n \to v_t \text{ in } L^\infty(\mathcal{L}_d^2) \text{ weak}^* \\ \operatorname{curl}^2 v^n \to \operatorname{curl}^2 v \text{ in } L^2(\mathcal{L}^2) \text{ weak.} \end{gathered}$$

Let now $\xi \in L^2(\mathcal{L}_d^2)$: since $\overline{\bigcup_{n\ge 1} Y_n} \hookrightarrow Y \hookrightarrow \mathcal{L}_d^2$, there exists an approximating sequence $\{\xi_n\}$ such that $\forall\, n$, $\xi_n \in Y_n$ and $\xi_n \to \xi$ in $L^2(\mathcal{L}_d^2)$ strongly; in particular, $\xi_n(t) = \sum_{j=1}^n a_j(t) y_j$, for suitable functions $a_j \in C\big([\,0\,,T\,]\big)$. Multiplying then (4.2a) by $a_j(t)$, summing for $j = 1, \ldots, n$, integrating by parts and letting $n \to \infty$, we obtain equation (3.2). We act analogously for equation (3.3), approximating any $\varphi \in T$ by a sequence $\{\varphi_n\}$, where $\forall\, n$, $\varphi_n(t) = \sum_{j=1}^n b_j(t) w_j \in W_n$, and the b_j are suitable

$C^1([0, T])$ functions such that $\beta_j(T) = 0$. In particular then, u and v solve the equations

$$\begin{cases} \epsilon v_t + \delta\eta_0 v + \epsilon\eta_0 \operatorname{curl}^2 v = \delta\eta_0 u_t \\ u_{tt} + \delta\eta_0 u_t + \operatorname{curl}^2 u - \ell\nabla \operatorname{div} u = \delta\eta_0 v \\ u(0) = u_0 , \quad u_t(0) = u_1 , \quad v(0) = v_0 \end{cases} \tag{4.9}$$

in distributional sense. That these solutions are unique, can be seen as in [7], loc. cit.; it would be an immediate consequence of estimate (4.5), with the initial data replaced by zero, provided a cutting off the value of $u(T)$ be previously made; we omit the details. □

5. Proof of theorem 2

We need a preliminary result:

PROPOSITION 4.— *Consider the operator* $A : u \mapsto \operatorname{curl}^2 u - \ell\nabla \operatorname{div} u$. *Then*

a) $A : \mathcal{H}_0^1 \to \mathcal{H}^1$ *is bijective,*

b) $\forall\ u \in \mathcal{H}_0^1$, $||Au||_{-1} \leq 2||u||_1 \leq 2||Au||_{-1}$.

In b), we have written $||\cdot||_{-1}$ *the norm in* $\mathcal{H}^{-1}$, *and* $||\cdot||_1$ *is the norm in* $\mathcal{H}_0^1$ *defined by*

$$||u||_1^2 = ||\operatorname{curl} u||^2 + \ell||\operatorname{div} u||^2 ;$$

this norm is equivalent to the usual one, as shown in [2], chapter VII, section 6.

Proof

i) Let $u \in \mathcal{H}_0^1$ and $\varphi \in \mathcal{D}(\Omega)^3$: then

$$\begin{aligned} \langle Au , \varphi\rangle &= (u , \operatorname{curl}^2 \varphi - \ell\nabla \operatorname{div} \varphi) \\ &= (\operatorname{curl} u , \operatorname{curl} \varphi) + \ell(\operatorname{div} u , \operatorname{div} \varphi) \leq 2\|u\|_1 \|\varphi\|_1 ; \end{aligned}$$

since $\mathcal{D}(\Omega)^3 \hookrightarrow \mathcal{H}_0^1$, this shows that A is linear continuous on $\mathcal{H}_0^1$.

ii) Given $f \in \mathcal{H}^{-1}$, the problem

$$\begin{cases} \text{find } u \in \mathcal{H}_0^1 \text{ such that } \forall\ v \in \mathcal{H}_0^1 \\ (\operatorname{curl} u , \operatorname{curl} v) + \ell(\operatorname{div} u , \operatorname{div} v) = \langle f , v\rangle \end{cases} \tag{5.1}$$

is coercive on $\mathcal{H}_0^1$ and, therefore, produces a unique solution to the equation $Au = f$; hence, A is bijective.

iii) Given $u \in \mathcal{H}_0^1$, let $f = Au$: then if $v \in \mathcal{H}_0^1$ and $||v||_1 \leq 1$, we see from (5.1) that, as in i) above,

$$\left|\langle f\,,\,v\rangle\right| \leq 2\left\|u\right\|_1\left\|v\right\|_1 \leq 2\left\|u\right\|_1\,, \quad \text{i.e. } \left\|f\right\|_{-1} \leq 2\left\|u\right\|_1\,;$$

but also, letting $v = u$ in (5.1)

$$\left\|u\right\|_1^2 = \langle f\,,\,v\rangle \leq \left\|f\right\|_{-1}\left\|u\right\|_1\,, \quad \text{i.e. } \left\|u\right\|_1 \leq \left\|f\right\|_{-1}\,.\,\square$$

To prove theorem 2, we proceed as in [7], chapter III, section 9. Since $\mathcal{H}_0^1 \hookrightarrow \mathcal{L}^2 \hookrightarrow \mathcal{H}^1$ and $Y \hookrightarrow \mathcal{L}_d^2$, we can select sequences u_0^n, u_1^n and v_0^n such that

$$\begin{cases} H_0^1 \ni u_0^n \to u_0 \text{ in } \mathcal{L}^2 \\ \mathcal{L}^2 \ni u_1^n \to u_1 \text{ in } \mathcal{H}^1 \\ Y \ni v_0^n \to v_0 \text{ in } \mathcal{L}_d^2\,. \end{cases} \tag{5.2}$$

By theorem 1 we can find, for each $n \in \mathbb{N}$, solutions u^n, v^n of problem I, corresponding to the initial data u_0^n, u_1^n, v_0^n: we refer to these as solutions to problems I_n. We can obtain estimates independent of n, considering sequences $w^n \in L^\infty(H_0^1)$, $z^n \in L^\infty(Y)$ such that

$$Aw^n = \operatorname{curl}^2 w^n - \ell\nabla \operatorname{div} w^n = u_t^n\,, \quad \operatorname{curl}^2 z^n = w^n\,; \tag{5.3}$$

this is possible by proposition 4 and d) of proposition 3, since in particular $u_t^n(t) \in \mathcal{H}^{-1}$ and $v^n(t) \in Y'$ for almost all t. We multiply then the equations in (4.9), where now $u = u^n$, $v = v^n$ (the solutions to problems I_n), by $2z^n$ and $2w^n$ respectively (this is rather ambiguous, since now the index n is used with a different meaning; in fact, there are two limit processes involved: one, to establish the existence of solutions to problem I_n; the other, to establish the existence of a solution to problem II. Since the estimates are the same, however, no confusion should arise). Noting that, for instance,

$$\begin{aligned} \langle u_{tt}^n\,,\,w^n\rangle &= \langle Aw_t^n\,,\,w^n\rangle \\ &= \left(\operatorname{curl} w_t^n\,,\,\operatorname{curl} w^n\right) + \ell\left(\operatorname{div} w_t^n\,,\,\operatorname{div} w^n\right) \\ &= \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\left\|w^n\right\|_1^2\,, \end{aligned}$$

$$\begin{cases} \langle \operatorname{curl}^2 u^n - \ell\nabla \operatorname{div} u^n\,,\,w^n\rangle = \left(\operatorname{curl} u^n\,,\,\operatorname{curl} w^n\right) + \ell\left(\operatorname{div} u^n\,,\,\operatorname{div} w^n\right) = \\ \qquad = \langle u^n\,,\,\operatorname{curl}^2 w^n - \ell\nabla \operatorname{div} w^n\rangle = \langle u^n\,,\,u_t^n\rangle = \dfrac{1}{2}\,\dfrac{\mathrm{d}}{\mathrm{d}t}\left\|u^n\right\|^2\,, \end{cases}$$

and analogously for v, we obtain

$$\begin{cases} \epsilon \dfrac{\mathrm{d}}{\mathrm{d}t} \left\| \operatorname{curl} z^n \right\|^2 + 2\delta_0 \left\| \operatorname{curl} z^n \right\|^2 + 2\epsilon\eta_0 \left\| v^n \right\|^2 = 2\delta\eta_0 \left(u_t^n , z^n \right) = \\ \qquad = 2\delta\eta_0 \left(\operatorname{curl} w^n , \operatorname{curl} z^n \right) \le \delta\eta_0 \left\| \operatorname{curl} w^n \right\|^2 + \delta\eta_0 \left\| \operatorname{curl} z^n \right\|^2 , \end{cases} \tag{5.4}$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} \left\| w^n \right\|_1^2 + 2\delta\eta_0 \left\| w^n \right\|_1^2 + \frac{\mathrm{d}}{\mathrm{d}t} \left\| u^n \right\|^2 = 2\delta\eta_0 \left(v^n , w^n \right) = \\ \qquad = 2\delta\eta_0 \left(\operatorname{curl} z^n , \operatorname{curl} w^n \right) \le \delta\eta_0 \left\| \operatorname{curl} z^n \right\|^2 + \delta\eta_0 \left\| \operatorname{curl} w^n \right\|^2 , \end{cases} \tag{5.5}$$

from which, summing and integrating in $[\,0\,,t\,]$:

$$\begin{cases} \epsilon \left\| \operatorname{curl} z^n \right\|^2 + 2\epsilon\eta_0 \displaystyle\int_0^t \left\| v^n \right\|^2 + \left\| w^n \right\|_1^2 + \left\| u^n \right\|^2 \le \\ \qquad \le \epsilon \left\| \operatorname{curl} z^n(0) \right\|^2 + \left\| \operatorname{curl} w^n(0) \right\|^2 + \left\| u^n(0) \right\|^2 . \end{cases} \tag{5.6}$$

Because of parts b) and d) of propositions 4 and 3, we have that

$$\left\| u_t^n \right\|_{-1} \le 2 \left\| w_n \right\|_1 \le 2 \left\| u_t^n \right\|_{-1} , \qquad \left\| \operatorname{curl} z^n \right\| = \left\| v^n \right\|_{Y'} \le \left\| v^n \right\| ; \tag{5.7}$$

hence, we obtain from (5.6)

$$\begin{cases} \epsilon \left\| v^n \right\|_{Y'}^2 + 2\epsilon\eta_0 \displaystyle\int_0^t \left\| v^n \right\|^2 + \left\| u_t^n \right\|_{-1}^2 + \left\| u^n \right\|^2 \le \\ \qquad \le \epsilon \left\| v_0^n \right\|^2 + \left\| v_1^n \right\|_{-1}^2 + \left\| v_0^n \right\|^2 ; \end{cases} \tag{5.8}$$

because of (5.2), the right side of (5.8) is uniformly bounded with respect to n, so that as $n \to \infty$ we have

$$\begin{cases} u^n \in \text{ bounded set of } L^\infty(\mathcal{L}^2) \\ u_t^n \in \text{ bounded set of } L^\infty(\mathcal{H}^{-1}) \\ v^n \in \text{ bounded set of } L^\infty(Y') \cap L^2(\mathcal{L}_d^2) . \end{cases} \tag{5.9}$$

Recalling then (5.7) and that $\operatorname{div} v^n = 0 \ \forall\, n$, we can modify (the analogous of) estimate (4.3) as follows:

$$\begin{cases} \epsilon \dfrac{\mathrm{d}}{\mathrm{d}t} \left\| v^n \right\|^2 + 2\delta\eta_0 \left\| v^n \right\|^2 + 2\epsilon\eta_0 \left\| \operatorname{curl} v^n \right\|^2 = 2\delta\eta_0 \left(A w^n , v^n \right) = \\ \qquad = 2\delta\eta_0 \left(\operatorname{curl} w^n , \operatorname{curl} v^n \right) \le \\ \qquad \le \dfrac{\delta^2\eta_0}{\epsilon} \left\| \operatorname{curl} w^n \right\|^2 + \epsilon\eta_0 \left\| \operatorname{curl} v^n \right\|^2 \le \\ \qquad \le \dfrac{\delta^2\eta_0}{\epsilon} \left\| u_t^n \right\|_{-1}^2 + \epsilon\eta_0 \left\| \operatorname{curl} v^n \right\|^2 \end{cases} \tag{5.10}$$

so that, integrating in $[0, t]$ and using (5,9b), we have that

$$v^n \in \text{ bounded set of } L^\infty(\mathcal{L}^2_d) \cap L^2(Y), \qquad \text{as } n \to \infty. \tag{5.11}$$

Finally, we multiply (4.9a) by $2z_t^n$ and integrate on $[0, t]$, obtaining, as before

$$\left\{\begin{aligned} & 2\epsilon \int_0^t \|\operatorname{curl} z_t^n\|^2 + \delta\eta_0 \|\operatorname{curl} z^n\|^2 + \epsilon\eta_0\|v^n\|^2 \leq \\ & \qquad \leq \delta\eta_0 \|\operatorname{curl} z^n(0)\|^2 + \epsilon\eta_0\|v_0^n\|^2 + 2\delta\eta_0 \int_0^t \langle u_t^n, 2z_t^n\rangle \leq \\ & \qquad \leq \delta\eta_0 \|v_0^n\|^2_{Y'} + \epsilon\eta_0\|v_0^n\|^2 + 2\delta\eta_0 \int_0^t \|u_t^n\|_{-1}\|\operatorname{curl} z^n\| \leq \\ & \qquad \leq (\delta+\epsilon)\eta_0\|v_0^n\|^2 + \frac{(\delta\eta_0)^2}{2}\int_0^t \|u_t^n\|^2_{-1} + \epsilon\int_0^t \|\operatorname{curl} z^n\|^2 \end{aligned}\right. \tag{5.12}$$

from which again, as $n \to \infty$

$$\left\{\begin{array}{l} v^n \in \text{ bounded set of } L^\infty(\mathcal{L}^2_d) \cap L^2(Y'), \\ v_t^n \in \text{ bounded set of } L^2(Y'). \end{array}\right. \tag{5.13}$$

There exist therefore subsequences u^m, v^m, and fields u, v such that, as $m \to \infty$

$$\left\{\begin{array}{ll} u^m \to u & \text{in } L^\infty(\mathcal{L}^2) \text{ weak}^* \\ u_t^m \to u_t & \text{in } L^\infty(\mathcal{H}^{-1}) \text{ weak}^* \end{array}\right. \tag{5.14}$$

$$\left\{\begin{array}{ll} u^m \to v & \text{in } L^\infty(\mathcal{L}^2_d) \text{ weak}^* \cap L^2(Y) \text{ weak} \\ v_t^m \to v_t & \text{in } L^2(Y') \text{ weak.} \end{array}\right. \tag{5.15}$$

To prove that u and v are solutions of problem II, we specialize in (3.2) and (3.3) of problems I_n, $\varphi \in T_u \subseteq T$ and $\xi \in T_v \subseteq L^2(\mathcal{L}^2_d)$: (3.6) and (3.7) are then easily obtained from (5.14) and (5.15) after integration by parts and letting $m \to +\infty$. As for uniqueness, we repeat the remark we made for problem I.

6. Perturbations processes

We now consider the solutions of (1) as dependent of the parameters δ, η_0 and ϵ, and address the question of their convergence, at the vanishing of these quantities, to the solutions of the corresponding limit problems

$$\left\{\begin{array}{lll} \epsilon(v_t - \eta_0\Delta u) = -\nabla p, & \operatorname{div} v = 0, & v(0) = v_0 \\ u_{tt} = \Delta u + \lambda\nabla \operatorname{div} u, & u(0) = u_0, & u_t(0) = u_1 \end{array}\right. \tag{6.1}$$

when $\delta \searrow 0$ and ϵ, η_0 are kept fixed;

$$\begin{cases} \epsilon v_t = -\nabla p\,, & \operatorname{div} v = 0, \quad v(0) = v_0 \\ u_{tt} = \Delta u + \lambda \nabla \operatorname{div} u\,, & u(0) = u_0, \quad u_t(0) = u_1 \end{cases} \tag{6.2}$$

when $\eta_0 \searrow 0$ and δ, ϵ are kept fixed;

$$\begin{cases} \delta\eta_0(u_t - v) = \nabla p\,, & \operatorname{div} v = 0 \\ u_{tt} = \Delta u + \lambda \nabla \operatorname{div} u\,, & u(0) = u_0, \quad u_t(0) = u_1 \end{cases} \tag{6.3}$$

when $\epsilon \searrow 0$ and δ, η_0 are kept fixed. We note that a consequence of these limit processes is the uncoupling of the elasticity and the fluid dynamics equations in (6.1) and (6.2); also, convergence in (6.3) will be singular in time for v, due to the loss of the corresponding initial condition. Finally, note that (6.3a) shows that we recover Darcy's law (11) when the internal forces become negligible. Of course, we interpret equations (6.1), (6.2) and (6.3) in analogy to problems I and II, when we formally set δ, η_0 and ϵ equal to zero in equations (3.2), (3.3) or (3.6), (3.7).

We start with the more regular solutions : assuming (3.1), we claim:

THEOREM 3

a) *Let $u = u^\delta$ and $v = v^\delta$ solve problem I for $\delta > 0$. There exist u and v such that, as $\delta \searrow 0$:*

$$u^\delta \to u \text{ in } L^\infty(\mathcal{H}_0^1) \text{ weak}^*, \quad u_t^\delta \to u_t \text{ in } L^\infty(\mathcal{L}^2) \text{ weak}^*;$$
$$v^\delta \to v \text{ in } L^\infty(Y) \text{ weak}^*, \quad v_t^\delta \to v_t \text{ in } L^\infty(\mathcal{L}_d^2) \text{ weak}^*;$$

moreover, u and v solve problem I with $\delta = 0$.

b) *Analogous statement for $u = u^\eta$ and $v = v^\eta$, when $\eta = \eta_0 \searrow 0$.*

THEOREM 4.— *Let $u = u^\epsilon$ and $v = v^\epsilon$ solve problem I for $\epsilon > 0$. There exist u and v such that, as $\epsilon \searrow 0$:*

$$u^\epsilon \to u \text{ in } L^\infty(\mathcal{H}_0^1) \text{ weak}^*, \quad u_t^\epsilon \to u_t \text{ in } L^\infty(\mathcal{L}^2) \text{ weak}^*;$$
$$v^\epsilon \to v \text{ in } L^\infty(\mathcal{L}_d^2);$$

moreover, u and v solve problem I with $\epsilon = 0$.

Proof.— Theorem 3 is an immediate consequence of estimates (4.5) and (4.7), which are independent of δ and η_0 if $\delta \le 1$ and $\eta_0 \le 1$; more specifically, these estimates show that, as $\delta \searrow 0$ and $\eta_0 \searrow 0$,

$$\begin{cases} u^\delta,\, u^\eta \in \text{ bounded set of } L^\infty(\mathcal{H}_0^1) \\ u_t^\delta,\, u_t^\eta \in \text{ bounded set of } L^\infty(\mathcal{L}^2)\,, \end{cases} \tag{6.4}$$

$$\begin{cases} v^\delta,\ v^\eta \in \text{ bounded set of } L^\infty(Y)\,, \\ v_t^\delta,\ v_t^\eta \in \text{ bounded set of } L^\infty(\mathcal{L}_d^2)\,; \end{cases} \tag{6.5}$$

hence, the conclusion follows from (6.4) and (6.5), letting $\delta \searrow 0$ and $\eta_0 \searrow 0$, in (3.2) and (3.3). As for theorem 4, these same estimates are not sufficient to pass to the limit when $\epsilon \searrow 0$; yet, note that (4.5) are independent of ϵ if $\epsilon \leq 1$, so that in particular, when $\epsilon \searrow 0$,

$$\begin{cases} u^\epsilon \in \text{ bounded set of } L^2(\mathcal{H}_0^1)\,, \\ u_t^\epsilon \in \text{ bounded set of } L^\infty(\mathcal{L}^2)\,, \\ \sqrt{\epsilon}v^\epsilon \in \text{ bounded set of } L^\infty(\mathcal{L}_d^2) \cap L^2(Y)\,; \end{cases} \tag{6.6}$$

using then (6.6b) back in (4.3), we see that, also,

$$v^\epsilon \in \text{ bounded set of } L^2(\mathcal{L}_d^2)\,. \tag{6.7}$$

Given then $\zeta \in L^2(\mathcal{L}_d^2)$, let $\zeta^m \in T_v$ be such that $\zeta^m \to \zeta$ in $L^2(\mathcal{L}_d^2)$: specialize $\zeta = \zeta^m$ in (3.2), integrate by parts in time and space, then let $\epsilon \searrow 0$ to obtain, $\forall\ m$:

$$\delta\eta_0 \int_0^T (v - u_t\,,\, \zeta_m) = 0\,, \tag{6.8}$$

where

$$v = w - \lim_{\epsilon \searrow 0} v^\epsilon\,, \quad u_t = w - \lim_{\epsilon \searrow 0} u_t^\epsilon\,,$$

as from (6.6a) and (6.7).

The conclusion then follows from (6.8) letting $m \to \infty$. □

We new turn to the weaker solutions: assuming (3.5) only, we claim:

THEOREM 5

a) Let $u = u^\delta$ and $v = v^\delta$ solve problem II for $\delta > 0$. There exist u and v such that, as $\delta \searrow 0$:

$$u^\delta \to u \text{ in } L^\infty(\mathcal{L}^2) \text{ weak}^*\,, \quad u_t^\delta \to u_t \text{ in } L^\infty(\mathcal{H}^{-1}) \text{ weak}^*\,;$$
$$v^\delta \to v \text{ in } L^\infty(Y') \text{ weak}^*\,;$$

moreover, u and v solve problem II with $\delta = 0$ and, in (3.7), (v, ζ) replaced by $\langle v, \zeta\rangle$, duality between Y' and Y.

b) Analogous statement for $u = u^\eta$, $v = v^\eta$, *when* $\eta = \eta_0 \searrow 0$.

c) Analogous statement for $u = u^\epsilon$, $v = v^\epsilon$, *when* $\epsilon \searrow 0$.

Proof. — We note that estimates (5.8) are independent of δ, η_0 and ϵ (if $\epsilon \leq 1$); more precisely, we have that, when δ, η_0 and ϵ vanish,

$$\begin{cases} \sqrt{\epsilon} v \in \text{ bounded set of } L^\infty(Y'), \\ u_t \in \text{ bounded set of } L^\infty(\mathcal{H}^{-1}), \\ u \in \text{ bounded set of } L^\infty(\mathcal{L}^2). \end{cases} \tag{6.9}$$

But then obtain from (5.4), because of (6.9b) and recalling (5.9), that

$$\epsilon \frac{\mathrm{d}}{\mathrm{d}t} \|v\|^2_{Y'} + \delta\eta_0 \|v\|^2_{Y'} \leq \delta\eta_0 \|u_t\|^2_{-1} \leq \text{const},$$

from which we deduce that, as δ, η and ϵ vanish,

$$v \in \text{ bounded set of } L^\infty(Y'). \tag{6.10}$$

Given then $\zeta \in T_v$, choose ζ^m such that $\operatorname{curl}^2 \zeta^m \in L^2(\mathcal{L}^2)$ and $\zeta^m \to \zeta$ in T_v; specialize $\zeta = \zeta^m$ in (3.7) and integrate by parts in space; let then $\delta \searrow 0$ or $\eta_0 \searrow 0$ in (3.6) and in the transformed equation, and use (6.9), and (6.10) when $\epsilon \searrow 0$: the conclusion then follows letting $m \to +\infty$. □

7. Proof of propositions 2 and 3

Proof of proposition 2

a), b), c) are proved in sections 1.1, 1.2 and 1.3 of [3]. From [2], chapter VII, theorem 6.1, we know that

$$\{u \in \mathcal{H}^1 \mid n \cdot u = 0\} = \{u \in H(\operatorname{curl}) \cap H(\operatorname{div}) \mid n \cdot u = 0\},$$

and that there exists $c_1 = c_1(\Omega)$ such that, for all u in such space,

$$\|u\|_{\mathcal{H}^1} \leq c_1 \left(\|u\| + \|\operatorname{curl} u\| + \|\operatorname{div} u\|\right). \tag{7.1}$$

A similar argument holds for the boundary condition $n \times u$: thus, d) is a consequence of (7.1) and of Friedrichs' inequality ([4])

$$\|u\| \leq c_2 \left(\|\operatorname{curl} u\| + \|\operatorname{div} u\|\right) \tag{7.2}$$

where $c_2 = c_2(\Omega)$ and, again, u is in either space above.

Finally, we deduce from [2], sections 4 and 5, that

$$\overline{\mathcal{D}_d} \equiv \left\{u\big(\mathcal{D}(\overline{\Omega})\big)^3 \mid \operatorname{div} u = 0\right\}$$

is dense in $\mathcal{L}^2_d$, so that $Y \hookrightarrow \mathcal{L}^2_d \hookrightarrow Y'$ since $\overline{\mathcal{D}_d} \subseteq Y \subseteq \mathcal{L}^2_d$.

Proof of proposition 3

a) We first note that since $\operatorname{div}\operatorname{curl} u = 0$, $\operatorname{curl} u \in H(\operatorname{div})$ so that $n \cdot \operatorname{curl} u$ is defined in $H^{-1/2}(\partial\Omega)$. Given then any $\psi \in \mathcal{D}(\partial\Omega)$, let $\varphi \in H^2(\Omega)$ be such that $\varphi\big|_{\partial\Omega} = \psi$ (for instance, let φ solve $\left\{\begin{array}{ll} -\Delta\varphi = 0 & \text{on } \Omega\,, \\ \varphi - \psi = 0 & \text{on } \partial\Omega \end{array}\right)$ then by proposition 1 we have

$$\begin{aligned} \big\langle n \cdot \operatorname{curl} u\,,\, \psi\big\rangle &= (\operatorname{curl} u\,,\, \nabla\varphi) \\ &= \big\langle n \times u\,,\, \nabla\varphi\big\rangle + (u\,,\, \operatorname{curl}\nabla\varphi) = 0\,. \end{aligned}$$

Thus, $n \cdot \operatorname{curl} u = 0$ in $\mathcal{D}'(\partial\Omega)$ and hence in $H^{-1/2}(\partial\Omega)$, since $\mathcal{D}(\partial\Omega) \hookrightarrow H^{1/2}(\partial\Omega)$ ([7], chapter I, section 7.3).

b) We know from [2], chapter VII, remark 4.2, that $\big(\mathcal{D}(\Omega)\big)^3 \hookrightarrow H_0(\operatorname{curl})$; if $u \in \mathcal{L}^2$ and $\varphi \in \big(\mathcal{D}(\partial\Omega)\big)^2$, we have

$$\begin{aligned} \big\langle \operatorname{curl} u\,,\, \varphi\big\rangle = \big\langle u\,,\, \operatorname{curl}\varphi\big\rangle &= (u\,,\, \operatorname{curl}\varphi) \le \\ &\le \|u\|\,\|\operatorname{curl}\varphi\| \le \|u\|\,\|\varphi\|_{H_0(\operatorname{curl})} \end{aligned}$$

so that $\operatorname{curl} u \in H^{-1}(\operatorname{curl})$.

c) If u, $v \in H_0(\operatorname{curl})$, by b) $\operatorname{curl}^2 u$ and $\operatorname{curl}^2 v \in H^{-1}(\operatorname{curl})$, and

$$\big\langle \operatorname{curl}^2 u\,,\, v\big\rangle = (\operatorname{curl} u\,,\, \operatorname{curl} v) = \big\langle \operatorname{curl}^2 v\,,\, u\big\rangle\,.$$

d) Since $Y = H_0(\operatorname{curl}) \cap \mathcal{L}^2_d \subseteq H_0(\operatorname{curl})$, $H^{-1}(\operatorname{curl}) \subseteq Y'$ so that by b) $\operatorname{curl}^2 u \in Y'$ if $u \in Y$. To verify that $\operatorname{curl}^2 : Y \to Y'$ is one-to-one, assume that $\operatorname{curl}^2 u = 0$ and let $v = \operatorname{curl} u$: then since $\operatorname{curl} v = 0$, $\operatorname{div} v = 0$ and $n \cdot v = 0$ (by a)), Friedrichs' inequality (7.2) implies that $v = 0$; then, $\operatorname{curl} u = 0$, $\operatorname{div} u = 0$ and $n \times u = 0$ imply $u = 0$ as well. To verify that $\operatorname{curl}^2 : Y \to Y'$ is onto, let $f \in Y'$ and consider the problem

$$\left\{\begin{array}{l} \text{find } u \in Y \text{ such that } \forall\ v \in Y \\ (\operatorname{curl} u\,,\, \operatorname{curl} v) = \langle f, v\rangle\,; \end{array}\right.$$

by d) of proposition 2, this problem is coercive in Y and, by c), its unique solution u is such that $\operatorname{curl}^2 u = f$.

Finally, to verify that $\|\operatorname{curl}^2 u\|_{Y'} = \|u\|_Y = \|\operatorname{curl} u\|$, let $f = \operatorname{curl}^2 u$: if $v \in Y$ and $\|v\|_Y = \|\operatorname{curl} v\| \leq 1$, from c) we have

$$\big|\langle f, v\rangle\big| = \big|(\operatorname{curl} u\,,\,\operatorname{curl} v)\big| \leq \|\operatorname{curl} u\|\,,$$

so that $\|f\|_{Y'} \leq \|u\|_Y$; but also

$$\begin{aligned}\|\operatorname{curl} u\|^2 &= (\operatorname{curl} u\,,\,\operatorname{curl} u) = \langle f, u\rangle \\ &\leq \|f\|_{Y'}\,\|u\|_Y = \|f\|_{Y'}\,\|\operatorname{curl} u\|\,,\end{aligned}$$

so that $\|u\|_Y \leq \|f\|_{Y'}$. □

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