

J. DEPREZ

F. DILLEN

P. VERHEYEN

L. VERSTRAELEN

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CONDITIONS ON THE PROJECTIVE CURVATURE  
TENSOR OF HYPERSURFACES IN EUCLIDIAN SPACE

J. Deprez<sup>(\*)(1)</sup>, F. Dillen<sup>(\*)(2)</sup>, P. Verheyen<sup>(3)</sup> and L. Verstraelen<sup>(4)</sup>

(1)(2)(3)(4) *Katholieke Universiteit Leuven Dept. Wiskunde, Celestijnenlaan 200 B, 3030 Heverlee (Belgium)*

**Résumé :** Nous étudions les hypersurfaces d'un espace Euclidien satisfaisant à certaines conditions sur le tenseur de courbure projective et nous obtenons des caractérisations locales de certains types des immersions. Nous donnons une caractérisation de certaines hypersurfaces de révolution qui sont, dans le cas 3-dimensionnel, des généralisations de la caténoïde.

**Summary :** Hypersurfaces of a Euclidean space satisfying certain conditions on the projective curvature tensor are studied and local characterizations of certain types of immersions are obtained. A characterization is given of certain hypersurfaces of revolution that are, in the 3-dimensional case, generalizations of the catenoid.

I. - INTRODUCTION

In this paper we study hypersurfaces of a Euclidean space satisfying one of the conditions  $R \cdot P = 0$ ,  $P \cdot C = 0$ ,  $C \cdot P = 0$ ,  $P \cdot P = 0$ ,  $P \cdot R = 0$ ,  $P \cdot Q = 0$  or  $Q \cdot P = 0$ , where  $R$  denotes the Riemann-Christoffel curvature tensor,  $Q$  the Ricci endomorphism,  $C$  the Weyl conformal curvature

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(\*) *Research Assistant of the National Fund of Scientific Research of Belgium*

*Dedicated to Prof. Emer. Dr. A. Borgers.*

tensor and  $P$  the Weyl projective curvature tensor of the hypersurface and where the first tensor acts on the second as a derivation.

Riemannian manifolds and submanifolds satisfying similar conditions have been studied by various authors. For references one can consult [3] and [4].

We will prove the following theorems.

**THEOREM 1.** *Let  $f : (M^n, g) \rightarrow \mathbb{E}^{n+1}$  be an isometric immersion of an  $n$ -dimensional Riemannian manifold in  $\mathbb{E}^{n+1}$  ( $n > 2$ ). Then the following assertions are equivalent :*

- (i)  $(M^n, g)$  satisfies  $R \cdot P = 0$ ,
- (ii)  $(M^n, g)$  satisfies  $R \cdot R = 0$ ,
- (iii)  $f$  is
  - (a) congruent to the inclusion of an open part of a hypersphere  $S^n$  of  $\mathbb{E}^{n+1}$ , or
  - (b) congruent to the inclusion of an open part of an elliptic hypercone  $C^n$  of  $\mathbb{E}^{n+1}$ , or
  - (c) an immersion with type-number at most 2 in every point, or
  - (d) a locally extrinsic product of the inclusion of an  $n_1$ -sphere  $S^{n_1}$  in  $\mathbb{E}^{n_1+1}$  and the inclusion of an  $(n - n_1)$ -plane  $\mathbb{E}^{n-n_1}$  ( $n_1 \in \{3, \dots, n-1\}$ ), i.e.  $f(M)$  is an open part of a spherical hypercylinder, or
  - (e) a locally extrinsic product of the inclusion of an elliptic hypercone in  $\mathbb{E}^{n_1+1}$  and the inclusion of an  $(n - n_1)$ -plane  $\mathbb{E}^{n-n_1}$  ( $n_1 \in \{3, \dots, n-1\}$ ).

For the equivalence (ii)  $\Leftrightarrow$  (iii) and elliptic hypercones, see [3].

**THEOREM 2.** *Let  $f : (M^n, g) \rightarrow \mathbb{E}^{n+1}$  be an isometric immersion of an  $n$ -dimensional Riemannian manifold in  $\mathbb{E}^{n+1}$  ( $n > 3$ ). Then the following assertions are equivalent:*

- (i)  $(M^n, g)$  satisfies  $P \cdot C = 0$ ,
- (ii)  $(M^n, g)$  satisfies  $C \cdot P = 0$ ,
- (iii)  $(M^n, g)$  satisfies  $C \cdot R = 0$ ,
- (iv)  $(M^n, g)$  is conformally flat.

The equivalence (iii)  $\Leftrightarrow$  (iv) was shown in [4].

**THEOREM 3.** *Let  $f : (M^n, g) \rightarrow \mathbb{E}^{n+1}$  be an isometric immersion of an  $n$ -dimensional Riemannian manifold in  $\mathbb{E}^{n+1}$  ( $n > 2$ ). Then the following assertions are equivalent :*

- (i)  $(M^n, g)$  satisfies  $P \cdot R = 0$ ,
- (ii)  $(M^n, g)$  satisfies  $P \cdot P = 0$ ,
- (iii)  $(M^n, g)$  satisfies  $P \cdot Q = 0$ ,
- (iv)  $(M^n, g)$  satisfies  $P = 0$ ,
- (v)  $f$  is congruent to the inclusion of an open part of a hypersphere  $S^n$  of  $IE^{n+1}$  or  $f$  is a cylindrical immersion.

**THEOREM 4.** Let  $f : (M^n, g) \rightarrow IE^{n+1}$  be an isometric immersion ( $n > 2$ ). Then  $(M^n, g)$  satisfies  $Q \cdot P = 0$  if and only if

- (i)  $f$  is congruent to the inclusion of an open part of a hypersphere  $S^n$  of  $IE^{n+1}$ , or
- (ii) there exists an open dense subset  $U$  of  $M$  such that each restriction  $f_\alpha$  of  $f$  to a connected component  $U_\alpha$  of  $U$  is
  - (a) a cylindrical immersion, or
  - (b) an immersion which is locally congruent around each point in  $U_\alpha$  to the inclusion of a hypersurface of revolution  $K_c^n$  ( $c \in \mathbb{R}_0^+$ ).

For a description of the hypersurfaces  $K_c^n$ , see section 6. In particular, for  $n = 3$ , the hypersurfaces  $K_c^3$  of  $IE^4$  are hypercatenoids (in this respect, see also [1]).

## 2. - BASIC FORMULAS

Let  $(M^n, g)$  be a (connected)  $n$ -dimensional Riemannian manifold ( $n \geq 2$ ). In the following  $X, Y, Z$  denote vector fields which are tangent to  $M^n$ .  $\nabla$  is the Levi-Civita connection of  $(M^n, g)$  and  $R$  is the Riemann-Christoffel curvature tensor of  $(M^n, g)$ .  $Q$  is the  $(1,1)$ -tensor related to the Ricci tensor  $S$  of  $(M, g)$  by  $g(QX, Y) = S(X, Y)$  for all  $X$  and  $Y$ .  $\tau = \text{tr } Q$  is the scalar curvature of  $(M, g)$ .  $X \wedge Y$  is the  $(1,1)$ -tensor field defined by  $(X \wedge Y)(Z) := g(Z, Y)X - g(Z, X)Y$ . The *Weyl conformal curvature tensor* and the *Weyl projective curvature tensor* are defined by

$$(2.1) \quad C(X, Y) := R(X, Y) - \frac{1}{n-2} (QX \wedge Y + X \wedge QY) + \frac{\tau}{(n-1)(n-2)} X \wedge Y,$$

$$(2.2) \quad P(X, Y) := R(X, Y) - \frac{1}{n-1} (X \wedge Y) \circ Q.$$

Let  $f : (M^n, g) \rightarrow IE^{n+1}$  be an immersion of  $(M^n, g)$  in an  $(n+1)$ -dimensional Euclidean space. Let  $\xi$  be a local normal section on  $f$ . Then the *second fundamental form*  $h$  and the *second*

fundamental tensor  $A$  of  $f$  are defined by the formulas of Gauss and Weingarten :  $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi$  and  $\tilde{\nabla}_X \xi = -AX$  ( $\tilde{\nabla}$  is the standard connection of  $IE^{n+1}$ ).  $A$  is related to  $h$  by  $h(X, Y) = g(AX, Y)$ . We will not distinguish between  $A_p$  and its matrix ( $p \in M$ ). The *type-number* of  $f$  in  $p \in M$  is the rank of  $A_p$ . The equation of Codazzi is given by  $(\nabla_X A)Y = (\nabla_Y A)X$  and the equation of Gauss is given by

$$(2.3) \quad R(X, Y) = AX \wedge AY.$$

Let  $p \in M$ . In the following  $x, y, z$  denote vectors in  $T_p M$ . Let  $x \wedge y$  denote the endomorphism  $T_p M \rightarrow T_p M : z \mapsto g(z, y)x - g(z, x)y$ . Since  $A_p$  is symmetric, there exists an orthonormal basis  $\{e_1, \dots, e_n\}$  of  $(T_p M, g_p)$  consisting of eigenvectors of  $A_p$ , i.e. such that

$$(2.4) \quad Ae_i = \lambda_i e_i,$$

where  $\lambda_i \in \mathbb{R}$  for each  $i \in \{1, \dots, n\}$ .  $\lambda_1, \dots, \lambda_n$  are called the *principal curvatures* of  $f$  in  $p$ . (2.1), (2.2), (2.3) and (2.4) imply that

$$(2.5) \quad \begin{aligned} R(e_i, e_j) &= c_{ij} e_i \wedge e_j, \\ Q e_i &= \mu_i e_i, \\ C(e_i, e_j) &= a_{ij} e_i \wedge e_j, \\ P(e_i, e_j) e_k &= \left( c_{ij} - \frac{\mu_k}{n-1} \right) (\delta_{kj} e_i - \delta_{ki} e_j), \end{aligned}$$

where

$$\begin{aligned} c_{ij} &= \lambda_i \lambda_j, \\ \mu_i &= \lambda_i (\text{tr} A - \lambda_i), \\ a_{ij} &= c_{ij} - \frac{1}{n-2} (\mu_i + \mu_j) + \frac{(\text{tr} A)^2 - \text{tr} A^2}{(n-1)(n-2)} \end{aligned}$$

for all  $i, j$  and  $k$  in  $\{1, \dots, n\}$ .

Let  $\bar{\lambda}_1, \dots, \bar{\lambda}_r$  denote the mutually distinct eigenvalues of  $A_p$  with multiplicities  $s_1, \dots, s_r$  respectively. Denote by  $V_\alpha$  the space of eigenvectors with eigenvalue  $\bar{\lambda}_\alpha$  ( $\alpha \in \{1, \dots, r\}$ ). If  $e_i, e_k \in V_\alpha$  and  $e_j, e_\ell \in V_\beta$ , then  $c_{ij} = c_{k\ell}$ ,  $\mu_i = \mu_k$  and  $a_{ij} = a_{k\ell}$  ( $i, j, k, \ell \in \{1, \dots, n\}$  and  $\alpha, \beta \in \{1, \dots, r\}$ ). We define numbers  $\bar{c}_{\alpha\beta} := c_{ij}$ ,  $\bar{\mu}_\alpha := \mu_i$  and  $\bar{a}_{\alpha\beta} = a_{ij}$  where  $e_i \in V_\alpha$  and  $e_j \in V_\beta$  ( $i, j \in \{1, \dots, n\}$  and  $\alpha, \beta \in \{1, \dots, r\}$ ).

According to Lemma 2.1 in [8] there exist  $n$  continuous functions  $\lambda_1 \leq \dots \leq \lambda_n$  on the domain of  $\xi$  on  $M$  such that for each  $p$  in  $M$  the eigenvalues of  $A_p$  are given by  $\lambda_1(p), \dots, \lambda_n(p)$ .

It easily follows that the subsets  $M_r = \{ p \in M \mid \text{the number of distinct eigenvalues of } A_p \text{ is at least } r \}$  of  $M$  are open ( $i \in \{ 1, \dots, n \}$ ).  $U := M_n \cup \text{int}(M_{n-1} \setminus M_n) \cup \dots \cup \text{int}(M_1 \setminus M_2)$  is an open dense subset of  $M$  such that on each connected component of  $U$  the number of distinct eigenvalues is constant, the multiplicities of the eigenvalues are constant and the eigenvalue functions are differentiable (see [9]).

$(M^n, g)$  is called (locally) conformally flat if  $(M, g)$  is (locally) conformally equivalent to  $IE^n$ . It is well known that  $(M^n, g)$  is conformally flat if and only if  $C = 0$  for  $n \geq 4$ . We recall that every surface is conformally flat and that  $C = 0$  for every 3-dimensional Riemannian manifold. It is well known that  $(M^n, g)$  is locally projectively equivalent to  $IE^n$  (i.e. around each point of  $M^n$  there exists a mapping to  $IE^n$  preserving geodesics) if and only if  $P = 0$  for  $n \geq 3$ . Every surface satisfies  $P = 0$ .

$f$  is called totally umbilical if its second fundamental tensor is proportional to the identity map everywhere. It is well known that  $f$  is totally umbilical if and only if  $f$  is congruent to the inclusion of an open part of a hypersphere or a hyperplane [2].

$f$  is called quasi-umbilical if for each point  $p$  in  $M$   $A_p$  has an eigenvalue with multiplicity at least  $n-1$ . For  $n \geq 4$ , E. Cartan proved that  $f$  is quasi-umbilical if and only if  $(M^n, g)$  is conformally flat. We remark that  $C = 0$  in  $p$  if and only if  $A_p$  has an eigenvalue with multiplicity at least  $n-1$  if  $n \geq 4$  (i.e. also the «pointwise» version of Cartan's result holds).

$f$  is called cylindrical if  $\text{rank } A_p \leq 1$  for each  $p$  in  $M$ .  $f$  is cylindrical if and only if  $(M^n, g)$  is locally flat. A complete cylindrical immersion is a cylinder over a plane curve [5].

Concerning the notations  $P \cdot C = 0, C \cdot P = 0, P \cdot Q = 0, \dots$  we say for example that  $(M^n, g)$  satisfies  $P \cdot C = 0$  if and only if  $P(X, Y) \cdot C = 0$  for all vector fields  $X$  and  $Y$  tangent to  $M$ , where  $P(X, Y)$  acts as a derivation on the algebra of tensor fields on  $M$ , i.e.

$$(P(X, Y) \cdot C)(Z, W)V = P(X, Y)C(Z, W)V - C(P(X, Y)Z, W)V - C(Z, P(X, Y)W)V - C(Z, W)P(X, Y)V$$

for  $X, Y, Z, V, W$  tangent to  $M^n$ . The derivation  $R(X, Y)$  is the derivation  $\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$ .  $R(X, Y) \cdot g = 0$  and  $C(X, Y) \cdot g = 0$  for all vector fields  $X$  and  $Y$  while in general  $P(X, Y) \cdot g \neq 0$  and  $Q \cdot g \neq 0$ . We remark that  $P \cdot g = 0$  if and only if  $(M^n, g)$  is Einstein.

For any (1,3)-tensor field  $T$  on  $M$  we define  $(C_{1,4}T)(Y, Z) = \sum_{i=1}^n g(T(E_i, Y)Z, E_i)$  and  $(C_{1,3}T)(Y, Z) = \sum_{i=1}^n g(T(E_i, Y)E_i, Z)$  for all vector fields  $Y, Z$  and any local orthonormal frame field  $\{E_1, \dots, E_n\}$ . The following lemma shows that certain derivations commute with certain contractions.

LEMMA 2.1. Let  $B$  be a  $(1,1)$ -tensor field and  $T$  a  $(1,3)$ -tensor field on  $M$ . Then

- (i)  $C_{1,4}(B \cdot T) = B \cdot (C_{1,4}T)$ ,  
(ii)  $C_{1,3}(B \cdot T) = B \cdot (C_{1,3}T)$  if  $B$  is antisymmetric,  
(iii)  $(B \cdot P)(X, Y)Z = (B \cdot R)(X, Y)Z - \frac{1}{n-1} \{ (B \cdot S)(Z, Y)X - (B \cdot S)(Z, X)Y \}$  for all vector fields  $X, Y, Z$ .

*Proof.* (i) We have

$$\begin{aligned}
(C_{1,4}(B \cdot T))(Y, Z) &= \sum_{i=1}^n g((B \cdot T)(E_i, Y)Z, E_i) = \\
&= \sum_{i=1}^n \{ g(BT(E_i, Y)Z, E_i) - g(T(BE_i, Y)Z, E_i) \\
&\quad - g(T(E_i, BY)Z, E_i) - g(T(E_i, Y)BZ, E_i) \} \\
&= \sum_{i,j=1}^n g(BE_j, E_i) g(T(E_i, Y)Z, E_i) - \sum_{i,j=1}^n g(T(E_i, Y)Z, E_i) g(BE_j, E_i) \\
&\quad - (C_{1,4}T)(BY, Z) - (C_{1,4}T)(Y, BZ) \\
&= \sum_{i,j=1}^n g(BE_j, E_i) g(T(E_i, Y)Z, E_i) - \sum_{i,j=1}^n g(T(E_i, Y)Z, E_i) g(BE_j, E_i) \\
&\quad + (B \cdot (C_{1,4}T))(Y, Z) \\
&= (B \cdot (C_{1,4}T))(Y, Z)
\end{aligned}$$

for all vector fields  $Y, Z$ .

(ii) We have

$$\begin{aligned}
(C_{1,3}(B \cdot T))(Y, Z) &= \sum_{i=1}^n g((B \cdot T)(E_i, Y)E_i, Z) = \\
&= \sum_{i=1}^n \{ g(BT(E_i, Y)E_i, Z) - g(T(BE_i, Y)E_i, Z) \\
&\quad - g(T(E_i, BY)E_i, Z) - g(T(E_i, Y)BE_i, Z) \} \\
&= - \sum_{i=1}^n g(T(E_i, Y)E_i, BZ) - \sum_{i,j=1}^n g(T(E_i, Y)E_i, Z) g(BE_j, E_i) \\
&\quad - \sum_{i=1}^n g(T(E_i, BY)E_i, Z) - \sum_{i,j=1}^n g(T(E_i, Y)E_i, Z) g(BE_j, E_i)
\end{aligned}$$

$$\begin{aligned}
 &= - (C_{1,3}T)(Y,BZ) - \sum_{i,j=1}^n g(T(E_i,Y)E_j,Z)g(BE_j,E_i) \\
 &\quad - (C_{1,3}T)(BY,Z) - \sum_{i,j=1}^n g(T(E_i,Y)E_j,Z)g(BE_j,E_i) \\
 &= (B \cdot (C_{1,3}T))(Y,Z)
 \end{aligned}$$

for all vector fields  $Y,Z$  if  $B$  is antisymmetric.

(iii) This is proved by a straightforward computation. ■

### 3. - THE CONDITION $R \cdot P = 0$

The proof of the equivalence (ii)  $\Leftrightarrow$  (iii) in Theorem 1 was given in [3]. We show that each Riemannian manifold satisfying  $R \cdot R = 0$  also satisfies  $R \cdot P = 0$  and conversely (\*).

Suppose that a Riemannian manifold  $(M^n, g)$  satisfies  $R \cdot R = 0$ . By Lemma 2.1 (i) and (iii)  $(M^n, g)$  also satisfies  $R \cdot P = 0$  since  $C_{1,4}R = S$ . Conversely, assume that  $(M^n, g)$  is a Riemannian manifold with  $R \cdot P = 0$ . It is easily seen that  $C_{1,3}P = -\frac{n}{n-1}S + \frac{\tau}{n-1}g$ . By Lemma 2.1 (ii)  $(M^n, g)$  satisfies  $R \cdot (C_{1,3}P) = 0$ . Moreover, since  $R \cdot g = 0$ , this shows that  $R \cdot S = 0$ . Lemma 2.1 (iii) then implies that  $(M^n, g)$  satisfies  $R \cdot R = 0$ . This finishes the proof of Theorem 1. ■

### 4. - THE CONDITIONS $P \cdot C = 0$ AND $C \cdot P = 0$

The equivalence of (iii) and (iv) in Theorem 2 was shown in [4] and the implications (iv)  $\Rightarrow$  (i) and (iv)  $\Rightarrow$  (ii) are evident.

*Proof of (i)  $\Rightarrow$  (iv).* Let  $f : (M^n, g) \rightarrow \mathbb{E}^{n+1}$  be an isometric immersion of a Riemannian manifold satisfying  $P \cdot C = 0$ . We shall show that  $C = 0$ .

Let  $p \in M^n$  and choose a basis  $\{e_1, \dots, e_n\}$  of  $T_p M^n$  satisfying (2.4). Using the formulas (2.5), we find that

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(\*) We thank R. Deszcz for pointing this out to us.



$$\begin{aligned}
(P(e_i, e_j) \cdot C)(e_k, e_\ell) e_m &= \left\{ \delta_{jk} \delta_{\ell m} \left( c_{ij} - \frac{\mu_k}{n-1} \right) (a_{k\ell} - a_{i\ell}) \right. \\
&\quad \left. - \delta_{j\ell} \delta_{km} \left( c_{ij} - \frac{\mu_\ell}{n-1} \right) (a_{k\ell} - a_{ik}) \right\} e_i \\
&+ \left\{ -\delta_{ik} \delta_{\ell m} \left( c_{ij} - \frac{\mu_k}{n-1} \right) (a_{k\ell} - a_{ik}) \right. \\
&\quad \left. + \delta_{i\ell} \delta_{km} \left( c_{ij} - \frac{\mu_\ell}{n-1} \right) (a_{k\ell} - a_{jk}) \right\} e_j \\
&+ \left\{ -\delta_{i\ell} \delta_{jm} \left[ \left( c_{ij} - \frac{\mu_m}{n-1} \right) a_{k\ell} - \left( c_{ij} - \frac{\mu_\ell}{n-1} \right) a_{jk} \right] \right. \\
&\quad \left. + \delta_{im} \delta_{j\ell} \left[ \left( c_{ij} - \frac{\mu_m}{n-1} \right) a_{k\ell} - \left( c_{ij} - \frac{\mu_\ell}{n-1} \right) a_{ik} \right] \right\} e_k \\
&+ \left\{ \delta_{ik} \delta_{jm} \left[ \left( c_{ij} - \frac{\mu_m}{n-1} \right) a_{k\ell} - \left( c_{ij} - \frac{\mu_k}{n-1} \right) a_{j\ell} \right] \right. \\
&\quad \left. - \delta_{im} \delta_{jk} \left[ \left( c_{ij} - \frac{\mu_m}{n-1} \right) a_{k\ell} - \left( c_{ij} - \frac{\mu_k}{n-1} \right) a_{i\ell} \right] \right\} e_\ell
\end{aligned}$$

for all  $i, j, k$  and  $\ell$  in  $\{1, \dots, n\}$ . For mutually distinct  $i, j$  and  $k$  in  $\{1, \dots, n\}$ , we obtain from  $(P(e_i, e_j) \cdot C)(e_k, e_i) e_j = 0$ ,  $(P(e_i, e_j) \cdot C)(e_k, e_i) e_k = 0$  and  $(P(e_i, e_k) \cdot C)(e_i, e_k) e_i = 0$  that

$$(4.1) \quad (\mu_i - \mu_j) a_{ik} = 0,$$

$$(4.2) \quad ((n-1)c_{ij} - \mu_j)(a_{ik} - a_{jk}) = 0,$$

and

$$(4.3) \quad (\mu_i - \mu_k) a_{ik} = 0.$$

Now suppose  $C \neq 0$  in  $\mathfrak{p}$ . We shall then show that a contradiction follows. We may assume that  $a_{12} \neq 0$ . Taking  $i = 1$ ,  $k = 2$  and  $j \in \{3, \dots, n\}$  in (4.1) and (4.3), we obtain that  $\mu_1 = \mu_2 = \dots = \mu_n$ . This gives that

$$(4.4) \quad (\lambda_i - \lambda_j)(\text{tr} A - \lambda_i - \lambda_j) = 0$$

for all mutually distinct  $i$  and  $j$  in  $\{1, \dots, n\}$ .

Let  $\bar{\lambda}_1, \dots, \bar{\lambda}_r$  denote the mutually distinct eigenvalues of  $A$  in  $\mathfrak{p}$  and let  $s_1, \dots, s_r$  be their respective multiplicities.

Suppose  $r \geq 3$ . Take mutually distinct  $\alpha, \beta, \gamma$  in  $\{1, \dots, r\}$ . Then (4.4) implies that  $\text{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\beta = 0$  and that  $\text{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\gamma = 0$ . This gives a contradiction. Assume  $r = 2$ . (4.2) is equivalent to

$$(4.5) \quad \lambda_i(\lambda_i - \lambda_j)(\text{tr}A - \lambda_i - (n-1)\lambda_j)(\text{tr}A - \lambda_i - \lambda_j - (n-2)\lambda_k) = 0$$

for all mutually distinct  $i, j$  and  $k$  in  $\{1, \dots, n\}$ . We may suppose that  $s_2 > 1$ . Choosing mutually distinct  $i, j$  and  $k$  in  $\{1, \dots, n\}$  in (4.5) such that  $\lambda_i = \bar{\lambda}_2$  and  $\lambda_j = \lambda_k = \bar{\lambda}_1$ , we find that  $\bar{\lambda}_2 = 0$ . (4.4) now implies that  $s_1 = 1$ . From (2.5) it is easily seen that  $C = 0$  in  $p$  ( $A_p$  has an eigenvalue with multiplicity  $n-1$ ). This gives a contradiction.

If  $r = 1$ , (2.5) shows that  $C = 0$  in  $p$ , which again contradicts our initial assumption  $C \neq 0$  in  $p$ . This proves the implication.

*Proof of (ii)  $\Leftrightarrow$  (iii).* In the same way as in section 3 we can prove that  $(M^n, g)$  satisfies  $C \cdot R = 0$  if and only if it satisfies  $C \cdot P = 0$ . This finishes the proof of Theorem 2. ■

### 5. - THE CONDITIONS $P \cdot P = 0, P \cdot R = 0$ AND $P \cdot Q = 0$

First we will prove the following lemmas.

LEMMA 5.1. *Let  $f : (M^n, g) \rightarrow \mathbb{E}^{n+1}$  be an isometric immersion of an  $n$ -dimensional Riemannian manifold ( $n > 2$ ). Then the following statements are equivalent :*

- (i)  $(M^n, g)$  satisfies  $P \cdot R = 0$ ,
- (ii)  $(M^n, g)$  satisfies  $P \cdot P = 0$ ,
- (iii) for each  $p \in M^n$   $A_p$  is one of the following types :

(a)  $\lambda I_n$  with  $\lambda \in \mathbb{R}_0$ ,

(b)  $\left( \begin{array}{c|c} \lambda & 0 \\ \hline 0 & 0_{n-1} \end{array} \right)$  with  $\lambda \in \mathbb{R}$ .

*Proof.* It is easy to check that the implication (iii)  $\Rightarrow$  (i) holds : in fact  $P = 0$  if (iii) is true. Next we show that (i) implies (ii). Suppose that  $(M^n, g)$  satisfies  $P \cdot R = 0$ . By Lemma 2.1 (i) and (iii),  $(M^n, g)$  then also satisfies  $P \cdot P = 0$ . Finally, we prove that (ii) implies (iii). Suppose that  $(M^n, g)$  satisfies  $P \cdot P = 0$ .

Let  $p \in M^n$  and choose a basis  $\{e_1, \dots, e_n\}$  for  $T_p M$  satisfying (2.4). Using the formulas (2.5), we find that

$$\begin{aligned} (P(e_i, e_j) \cdot P)(e_k, e_i)e_\ell &= \delta_{k\ell} \left( c_{ij} - \frac{\mu_i}{n-1} \right) (c_{ik} - c_{jk}) e_j \\ &+ \delta_{j\ell} \left\{ \left( c_{ij} - \frac{\mu_i}{n-1} \right) \left( c_{jk} - \frac{\mu_\ell}{n-1} \right) - \left( c_{ij} - \frac{\mu_\ell}{n-1} \right) \left( c_{ik} - \frac{\mu_i}{n-1} \right) \right\} e_k \end{aligned}$$

for all mutually distinct  $i, j$  and  $k$  in  $\{1, \dots, n\}$  and all  $\ell$  in  $\{1, \dots, n\}$ . We obtain from  $(P(e_i, e_j) \cdot P)(e_k, e_i)e_j = 0$  and  $(P(e_i, e_j) \cdot P)(e_k, e_i)e_k = 0$  that

$$(5.1) \quad \lambda_i \lambda_j (\lambda_i - \lambda_j) (\text{tr} A - \lambda_i - \lambda_j - (n-2)\lambda_k) = 0$$

and

$$\lambda_i \lambda_k (\lambda_i - \lambda_j) (\text{tr} A - \lambda_i - (n-1)\lambda_j) = 0$$

for all mutually distinct  $i, j$  and  $k$  in  $\{1, \dots, n\}$ .  $\lambda_k (5.1) - \lambda_j (5.2)$  gives that

$$(5.3) \quad \lambda_i \lambda_j \lambda_k (\lambda_i - \lambda_j) (\lambda_k - \lambda_j) = 0$$

for all mutually distinct  $i, j$  and  $k$  in  $\{1, \dots, n\}$ .

Let  $\bar{\lambda}_1, \dots, \bar{\lambda}_r$  denote the mutually distinct eigenvalues of  $A$  in  $p$  and let  $s_1, \dots, s_r$  be their respective multiplicities.

Suppose  $r \geq 4$ . Then (5.3) yields  $\bar{\lambda}_\alpha \bar{\lambda}_\beta \bar{\lambda}_\gamma (\bar{\lambda}_\alpha - \bar{\lambda}_\beta) (\bar{\lambda}_\gamma - \bar{\lambda}_\beta) = 0$  for mutually distinct  $\alpha, \beta, \gamma$  and  $\delta$  in  $\{1, \dots, r\}$ . We may therefore assume that  $\bar{\lambda}_\alpha = 0$ . (5.3) now gives that  $\bar{\lambda}_\beta \bar{\lambda}_\gamma \bar{\lambda}_\delta (\bar{\lambda}_\beta - \bar{\lambda}_\gamma) (\bar{\lambda}_\delta - \bar{\lambda}_\gamma) = 0$ , which is impossible. We conclude that  $r \leq 3$ .

Assume  $r = 3$ . It then follows from (5.3) that  $\bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 (\bar{\lambda}_1 - \bar{\lambda}_2) (\bar{\lambda}_3 - \bar{\lambda}_2) = 0$  which implies that, for instance,  $\bar{\lambda}_1 = 0$ . Choosing  $i, j$  and  $k$  in  $\{1, \dots, n\}$  such that  $\lambda_i = \bar{\lambda}_2$ ,  $\lambda_j = \bar{\lambda}_3$  and  $\lambda_k = \bar{\lambda}_1$ , (5.1) gives that

$$(5.4) \quad (s_2 - 1)\bar{\lambda}_2 + (s_3 - 1)\bar{\lambda}_3 = 0.$$

Furthermore, for  $i, j$  and  $k$  in  $\{1, \dots, n\}$  such that  $\lambda_i = \bar{\lambda}_2$ ,  $\lambda_j = \bar{\lambda}_1$  and  $\lambda_k = \bar{\lambda}_3$ , (5.2) yields

$$(5.5) \quad (s_2 - 1)\bar{\lambda}_2 + s_3 \bar{\lambda}_3 = 0.$$

(5.4) contradicts (5.5). So  $r \neq 3$ .

Suppose  $r = 2$ . Then we may assume that  $s_2 \geq 2$ . Taking mutually distinct  $i, j$  and  $k$

in  $\{1, \dots, n\}$  such that  $\lambda_i = \lambda_k = \bar{\lambda}_2$  and  $\lambda_j = \bar{\lambda}_1$ , (5.3) gives that  $\bar{\lambda}_1 \bar{\lambda}_2^2 (\bar{\lambda}_2 - \bar{\lambda}_1)^2 = 0$ . We conclude that  $\bar{\lambda}_1 = 0$  or  $\bar{\lambda}_2 = 0$ . First we show that  $\bar{\lambda}_2 = 0$ . Suppose that  $\bar{\lambda}_2 \neq 0$ . Then  $\bar{\lambda}_1 = 0$ . It follows from (5.2), taking the same choice for the indices  $i, j$  and  $k$  as above, that  $\text{tr}A = \bar{\lambda}_2$ . This would mean that  $s_2 = 1$ . This contradicts one of our initial assumptions. Secondly, we show that  $s_1 = 1$ . Suppose  $s_1 \geq 2$ . Then we can choose mutually distinct  $i, j$  and  $k$  in  $\{1, \dots, n\}$  such that  $\lambda_i = \lambda_k = \bar{\lambda}_1$  and  $\lambda_j = \bar{\lambda}_2 = 0$ . Formula (5.2) now gives that  $\bar{\lambda}_1 = \text{tr}A$ , from which we conclude that  $s_1 = 1$ . This is in contradiction with the assumption  $s_1 \geq 2$ . This shows that the matrix of  $A_p$  in the basis  $\{e_1, \dots, e_n\}$  has one of the desired forms.

Finally, the case  $r = 1$  is trivial. This finishes the proof of Lemma 5.1. ■

LEMMA 5.2. Let  $f : (M^n, g) \rightarrow \mathbb{E}^{n+1}$  be an isometric immersion of an  $n$ -dimensional Riemannian manifold ( $n > 2$ ). Then  $(M^n, g)$  satisfies  $P \cdot Q = 0$  if and only if for each point  $p$  in  $M^n$   $A_p$  is one of the following types :

- (a)  $\lambda I_n$  with  $\lambda \in \mathbb{R}_0$ ,
- (b)  $\left( \begin{array}{c|c} (s_2-1)\lambda I_{s_1} & 0 \\ \hline 0 & -(s_1-1)\lambda I_{s_2} \end{array} \right)$  with  $\lambda \in \mathbb{R}_0, s_1, s_2 \in \mathbb{N} \setminus \{0, 1\}$   
and  $s_1 + s_2 = n$ .
- (c)  $\left( \begin{array}{c|c} \lambda & 0 \\ \hline 0 & 0_{n-1} \end{array} \right)$  with  $\lambda \in \mathbb{R}$ .

*Proof.* Let  $i : (M^n, g) \rightarrow \mathbb{R}^{n+1}$  be an isometric immersion of an  $n$ -dimensional Riemannian manifold. Let  $p$  be a point in  $M$  and choose a basis  $\{e_1, \dots, e_n\}$  for  $T_p M$  satisfying (2.4). Using the formulas (2.5), we find that  $(P(e_i, e_j) \cdot Q)e_k = (c_{ij} - \frac{\mu_k}{n-1}) \{(\mu_k - \mu_i)\delta_{jk}e_i - (\mu_k - \mu_j)\delta_{ik}e_j\}$  for all  $i, j$  and  $k$  in  $\{1, \dots, n\}$ . From this we learn that  $P \cdot Q = 0$  in  $p$  if and only if  $(P(e_i, e_j) \cdot Q)e_i = 0$  for all mutually distinct  $i$  and  $j$  in  $\{1, \dots, n\}$ . This implies that  $P \cdot Q = 0$  if and only if

$$(5.6) \quad \lambda_i(\lambda_i - \lambda_j)(\text{tr}A - \lambda_i - \lambda_j)(\text{tr}A - \lambda_i - (n-1)\lambda_j) = 0$$

for all mutually distinct  $i$  and  $j$  in  $\{1, \dots, n\}$ .

Let  $\bar{\lambda}_1, \dots, \bar{\lambda}_r$  denote the mutually distinct eigenvalues of  $A$  in  $p$  and let  $s_1, \dots, s_r$  be their respective multiplicities. We will show that  $P \cdot Q = 0$  in  $p$  if and only if

$$(5.7) \quad \text{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\beta = 0$$

for all distinct  $\alpha$  and  $\beta$  in  $\{1, \dots, r\}$ . For different  $\alpha$  and  $\beta$  in  $\{1, \dots, r\}$ , (5.6) gives that

$$(5.8) \quad \bar{\lambda}_\alpha(\text{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\beta)(\text{tr}A - \bar{\lambda}_\alpha - (n-1)\bar{\lambda}_\beta) = 0.$$

and

$$(5.9) \quad \bar{\lambda}_\beta(\text{tr}A - \bar{\lambda}_\beta - \bar{\lambda}_\alpha)(\text{tr}A - \bar{\lambda}_\beta - (n-1)\bar{\lambda}_\alpha) = 0.$$

Subtracting (5.9) from (5.8) gives  $(\bar{\lambda}_\alpha - \bar{\lambda}_\beta)(\text{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\beta)^2 = 0$ . So  $P \cdot Q = 0$  in  $\mathfrak{p}$  implies (5.7). The other implication is trivial.

Now it is easy to see that immersions for which all second fundamental tensors have the form described in the lemma are immersions of Riemannian manifolds satisfying  $P \cdot Q = 0$ . Next we show the converse.

Assume that  $r \geq 3$ . Choose mutually distinct  $\alpha, \beta$  and  $\gamma$  in  $\{1, \dots, r\}$ . Then, by (5.7) we have  $\text{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\beta = 0$  and  $\text{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\gamma = 0$ . This gives  $\bar{\lambda}_\beta = \bar{\lambda}_\gamma$ , which is impossible.

Suppose that  $r = 2$ . First we assume that  $s_1 \geq 2$  and  $s_2 \geq 2$ . (5.7) learns that  $P \cdot Q = 0$  if and only if  $A$  in  $\mathfrak{p}$  has the form (b) in the lemma. If, say,  $s_1 = 1$ , then  $\bar{\lambda}_2 = 0$  by (5.7). So  $A$  in  $\mathfrak{p}$  has the form (c) in the lemma.

If  $r = 1$ , then  $A$  in  $\mathfrak{p}$  has one of the desired forms. This proves the lemma. ■

Now we prove Theorem 3. Using Lemma 5.1 and Lemma 5.2 it is easy to see that (v)  $\Rightarrow$  (i), (i)  $\Rightarrow$  (ii) and (ii)  $\Rightarrow$  (iii) hold. The equivalence (iv)  $\Leftrightarrow$  (v) is well known. Thus, we only must show that (iii) implies (v).

Call

$$M_1 := \{ \mathfrak{p} \in M \mid A_{\mathfrak{p}} = \lambda(\mathfrak{p}) I_{T_{\mathfrak{p}}M} \text{ for some } \lambda(\mathfrak{p}) \in \mathbb{R}_0 \}$$

and

$$M_2 := \left\{ \mathfrak{p} \in M \mid \left. \begin{array}{l} A_{\mathfrak{p}} = \left( \begin{array}{c|c} (s_2(\mathfrak{p})-1)\lambda(\mathfrak{p}) I_{s_1(\mathfrak{p})} & \\ \hline & -(s_1(\mathfrak{p})-1)\lambda(\mathfrak{p}) I_{s_2(\mathfrak{p})} \end{array} \right) \\ \text{for some } s_1(\mathfrak{p}), s_2(\mathfrak{p}) \in \mathbb{N} \setminus \{0, 1\} \text{ and some } \lambda(\mathfrak{p}) \in \mathbb{R}_0. \end{array} \right\}$$

$M_1$  and  $M_2$  are open.

First, we show that  $M_2 = \emptyset$ . Suppose that  $M_2 \neq \emptyset$  and let  $W_2$  be a connected component of  $M_2$ . By Proposition 2.3 in [8], the distributions  $T_1 := \{ X \in TW_2 \mid A_X = (s_2-1)\lambda X \}$  and

$T_2 := \{ X \in TW_2 \mid AX = -(s_1-1)\lambda X \}$  are differentiable and involutive and  $\lambda$  is a constant function on  $W_2$ . We show that  $T_1$  and  $T_2$  are parallel. Let  $X_1$  and  $Y_1$  (resp.  $X_2$  and  $Y_2$ ) be vector fields with values in  $T_1$  (resp.  $T_2$ ). The equation of Codazzi  $(\nabla_{X_1} A)X_2 = (\nabla_{X_2} A)X_1$  then gives that  $(A + (s_1-1)\lambda)\nabla_{X_1} X_2 = (A - (s_2-1)\lambda)\nabla_{X_2} X_1$ . From this we obtain that  $(A + (s_1-1)\lambda)\nabla_{X_1} X_2 = 0$  and  $(A - (s_2-1)\lambda)\nabla_{X_2} X_1 = 0$ . Therefore,  $\nabla_{X_1} X_2$  has only values in  $T_2$  and  $\nabla_{X_2} X_1$  has only values in  $T_1$ .

Furthermore,  $0 = X_1 \langle Y_1, Z_2 \rangle = \langle \nabla_{X_1} Y_1, Z_2 \rangle + \langle Y_1, \nabla_{X_1} Z_2 \rangle = \langle \nabla_{X_1} Y_1, Z_2 \rangle$  for each vector field  $Z_2$  with values in  $T_2$ . This shows that  $\nabla_{X_1} Y_1$  always has only values in  $T_1$ . Similarly,  $\nabla_{X_2} Y_2$  always has only values in  $T_2$ . The equation of Gauss gives that

$$(5.10) \quad R(X_1, X_2) = -(s_1-1)(s_2-1)\lambda^2 X_1 \wedge X_2.$$

On the other hand,  $g(R(X_1, X_2)X, Y) = g(R(X, Y)X_1, X_2) = g(\nabla_X \nabla_Y X_1 - \nabla_Y \nabla_X X_1 - \nabla_{[X, Y]} X_1, X_2) = 0$  for all vector fields  $X$  and  $Y$  tangent to  $W_2$  since  $T_1$  is parallel. This gives a contradiction with (5.10). This proves that  $M_2 = \emptyset$ .

Suppose  $M_1 \neq \emptyset$ . Let  $W_1$  be a connected component of  $M_1$ .  $W_1$  is open.  $f|_{W_1}$  is totally umbilical. In particular,  $\lambda$  is a constant function on  $W_1$ .  $W_1$  is closed as well : since the eigenvalue functions of  $A$  can be chosen to be continuous functions (see [8]),  $A_q = \lambda I_{T_q M}$  (with  $\lambda \in \mathbb{R}_0$ ) for each  $q$  in  $\bar{W}_1$ , i.e.  $\bar{W}_1 \subset W_1$ . Since  $M^n$  is connected,  $W_1 = M^n$  and  $f$  is a totally umbilical immersion.

If  $M_1 = \emptyset$ ,  $f$  is a cylindrical immersion. This finishes the proof of Theorem 3. ■

## 6. - THE CONDITION Q.P = 0

A. LEMMA 6.1. *Let  $f : (M^n, g) \rightarrow \mathbb{E}^{n+1}$  be an isometric immersion of an  $n$ -dimensional Riemannian manifold.  $(M^n, g)$  satisfies Q.P = 0 if and only if for each  $p$  in  $M^n$   $A_p$  is one of the following types :*

- (a)  $\lambda I_n$  with  $\lambda \in \mathbb{R}_0$ ,
- (b)  $\left( \begin{array}{c|c} \lambda I_{n-1} & 0 \\ \hline 0 & (2-n)\lambda \end{array} \right)$  with  $\lambda \in \mathbb{R}_0$ ,
- (c)  $\left( \begin{array}{c|c} \lambda & 0 \\ \hline 0 & 0_{n-1} \end{array} \right)$  with  $\lambda \in \mathbb{R}$ .

*Proof.* Let  $f : (M^n, g) \rightarrow \mathbb{E}^{n+1}$  be an isometric immersion of an  $n$ -dimensional Riemannian manifold. Let  $p$  be a point in  $M^n$  and choose a basis  $\{e_1, \dots, e_n\}$  for  $T_p M^n$  satisfying (2.4). Using the formulas (2.5) we find that  $(Q \cdot P)(e_i, e_j)e_k = \left(\frac{\mu_k}{n-1} - c_{ij}\right) \{\delta_{jk}(\mu_j + \mu_k)e_i - \delta_{ik}(\mu_i + \mu_k)e_j\}$  for all  $i, j$  and  $k$  in  $\{1, \dots, n\}$ . From this we learn that  $Q \cdot P = 0$  in  $p$  if and only if  $(Q \cdot P)(e_i, e_j)e_i = 0$  for all distinct  $i$  and  $j$  in  $\{1, \dots, n\}$ . This implies that  $Q \cdot P = 0$  if and only if

$$(6.1) \quad \lambda_i(\text{tr}A - \lambda_i)(\text{tr}A - \lambda_i - (n-1)\lambda_j) = 0$$

for all different  $i$  and  $j$  in  $\{1, \dots, n\}$ . Let  $i, j$  and  $k$  be mutually distinct indices in  $\{1, \dots, n\}$ . Then  $\lambda_i(\text{tr}A - \lambda_i)(\text{tr}A - \lambda_i - (n-1)\lambda_j) = 0$  and  $\lambda_i(\text{tr}A - \lambda_i)(\text{tr}A - \lambda_i - (n-1)\lambda_k) = 0$ . Subtraction yields that

$$(6.2) \quad \lambda_i(\text{tr}A - \lambda_i)(\lambda_j - \lambda_k) = 0.$$

Conversely, (6.2) implies (6.1). Therefore,  $(M^n, g)$  satisfies  $Q \cdot P = 0$  if and only if (6.2) is fulfilled for all mutually distinct  $i, j$  and  $k$  in  $\{1, \dots, n\}$ . It is easy to see now that  $Q \cdot P = 0$  if all  $A_p$  have one of the forms described in the lemma. Next, we show the converse.

Let  $\bar{\lambda}_1, \dots, \bar{\lambda}_r$  denote the mutually distinct eigenvalues of  $A$  in  $p$  and let  $s_1, \dots, s_r$  be their respective multiplicities. First, suppose  $r \geq 3$ . Now, (6.2) implies that  $\bar{\lambda}_\alpha(\text{tr}A - \bar{\lambda}_\alpha) = 0$  for each  $\alpha \in \{1, \dots, r\}$ . This shows that  $A$  has at most two distinct eigenvalues. This contradicts our initial assumption.

Suppose that  $r = 2$ . If  $s_1 \geq 2$ , then (6.2) gives that  $\bar{\lambda}_1(\text{tr}A - \bar{\lambda}_1) = 0$  (take  $i$  and  $j$  with  $\lambda_i = \lambda_j = \bar{\lambda}_1$  and  $k$  with  $\lambda_k = \bar{\lambda}_2$ ). In the same way, if  $s_2 \geq 2$ , then  $\bar{\lambda}_2(\text{tr}A - \bar{\lambda}_2) = 0$ . If  $s_1 \geq 2$  and  $s_2 \geq 2$ , the only possibility is that, say,  $\bar{\lambda}_1 = 0$  and  $\bar{\lambda}_2 = \text{tr}A \neq 0$ . This is impossible as  $\text{tr}A = s_2 \bar{\lambda}_2 \neq \bar{\lambda}_2$ . Therefore, we may assume that for instance  $s_2 = 1$ . If  $\bar{\lambda}_1 = 0$ ,  $A_p$  has one of the forms described in the lemma. If  $\bar{\lambda}_1 \neq 0$ , then  $\bar{\lambda}_2 = (2-n)\bar{\lambda}_1$ .

The case  $r = 1$  is trivial. This proves the lemma. ■

### B. EXAMPLES

It is clear that  $(M^n, g)$  satisfies  $Q \cdot P = 0$  if  $f : (M^n, g) \rightarrow \mathbb{E}^{n+1}$  is a cylindrical immersion or a totally umbilical immersion, since in these cases  $(M^n, g)$  satisfies  $P = 0$ . Now we will give a non-trivial example of a hypersurface satisfying  $Q \cdot P = 0$ .

Let  $\gamma : I \rightarrow \mathbb{E}^{n+1} : u \mapsto (u, \varphi(u), 0, \dots, 0)$  be a plane curve in  $\mathbb{E}^{n+1}$  lying in the  $x_1 x_2$ -plane and suppose  $\varphi(u) > 0$  for all  $u$ . Let  $(M^n, g)$  be the hypersurface of revolution in  $\mathbb{E}^{n+1}$  obtained by rotation of  $\gamma$  around the  $x_1$ -axis, i.e.

$$M = \left\{ (u, \varphi(u) \cos \theta_2, \varphi(u) \sin \theta_2 \cos \theta_3, \dots, \varphi(u) \sin \theta_2 \sin \theta_3 \dots \cos \theta_n, \varphi(u) \sin \theta_2 \sin \theta_3 \dots \sin \theta_n) \mid \right. \\ \left. u \in I, \theta_2, \dots, \theta_n \in \mathbb{R} \right\}$$

with the induced differentiable and geometric structure. Let  $F$  be the obvious parametrization of  $M$  and call  $p = F(\bar{u}, \bar{\theta}_2, \dots, \bar{\theta}_n)$ , ( $\bar{u} \in I$  and  $\bar{\theta}_2, \dots, \bar{\theta}_n \in [0, 2\pi]$ ). Then  $T_p M$  is spanned by the vectors

$$\left( \frac{\partial F}{\partial u} (\bar{u}, \bar{\theta}_2, \dots, \bar{\theta}_n) \right)_p = (1, \varphi'(\bar{u}) \cos \bar{\theta}_2, \varphi'(\bar{u}) \sin \bar{\theta}_2 \cos \bar{\theta}_3, \dots, \\ \varphi'(\bar{u}) \sin \bar{\theta}_2 \sin \bar{\theta}_3 \dots \cos \bar{\theta}_n, \\ \varphi'(\bar{u}) \sin \bar{\theta}_2 \sin \bar{\theta}_3 \dots \sin \bar{\theta}_n)_p,$$

$$\left( \frac{\partial F}{\partial \theta_2} (\bar{u}, \bar{\theta}_2, \dots, \bar{\theta}_n) \right)_p = (0, -\varphi(\bar{u}) \sin \bar{\theta}_2, \varphi(\bar{u}) \cos \bar{\theta}_2 \cos \bar{\theta}_3, \dots, \\ \varphi(\bar{u}) \cos \bar{\theta}_2 \sin \bar{\theta}_3 \dots \cos \bar{\theta}_n, \\ \varphi(\bar{u}) \cos \bar{\theta}_2 \sin \bar{\theta}_3 \dots \sin \bar{\theta}_n)_p,$$

$$\left( \frac{\partial F}{\partial \theta_n} (\bar{u}, \bar{\theta}_2, \dots, \bar{\theta}_n) \right)_p = (0, 0, 0, \dots, -\varphi(\bar{u}) \sin \bar{\theta}_2 \sin \bar{\theta}_3 \dots \sin \bar{\theta}_n, \\ \varphi(\bar{u}) \sin \bar{\theta}_2 \sin \bar{\theta}_3 \dots \cos \bar{\theta}_n)_p$$

and

$$\xi_p := (-\varphi'(\bar{u}), \cos \bar{\theta}_2, \sin \bar{\theta}_2 \cos \bar{\theta}_3, \dots, \sin \bar{\theta}_2 \sin \bar{\theta}_3 \dots \sin \bar{\theta}_{n-1} \cos \bar{\theta}_{n-1}, \\ \sin \bar{\theta}_2 \sin \bar{\theta}_3 \dots \sin \bar{\theta}_{n-1} \sin \bar{\theta}_n)_p$$

is a normal vector in  $p$ . Then, if  $W(\bar{u}) > 0$  is defined by  $W^2(\bar{u}) := \|\xi_p\|^2 = 1 + \varphi'(\bar{u})^2$ ,  $U(p) = \frac{\xi_p}{W(\bar{u})}$

is a unit normal vector in  $p$ . We find that  $\left( \frac{\partial F}{\partial u} (\bar{u}, \bar{\theta}_2, \dots, \bar{\theta}_n) \right)_p$  is an eigenvector of  $A_p$  with eigenvalue  $\frac{\varphi''(\bar{u})}{W^3(\bar{u})}$  and that  $\left( \frac{\partial F}{\partial \theta_2} (\bar{u}, \bar{\theta}_2, \dots, \bar{\theta}_n) \right)_p, \dots, \left( \frac{\partial F}{\partial \theta_n} (\bar{u}, \bar{\theta}_2, \dots, \bar{\theta}_n) \right)_p$  are eigenvectors all with the same eigenvalue  $\frac{-1}{W(\bar{u})\varphi(\bar{u})}$ . Consequently,  $A_p$  has the form described in (b) of Lemma 6.1 if and

only if  $\varphi$  satisfies the following differential equation :

$$(*) \quad \varphi''\varphi = (n-2)(1+\varphi'^2).$$

Next, we describe the solutions of this differential equation.



Take  $c \in \mathbb{R}_0^+$  and let  $c' := c^{-1/n-2}$ . Consider the function  $h_{n,c}$  given by

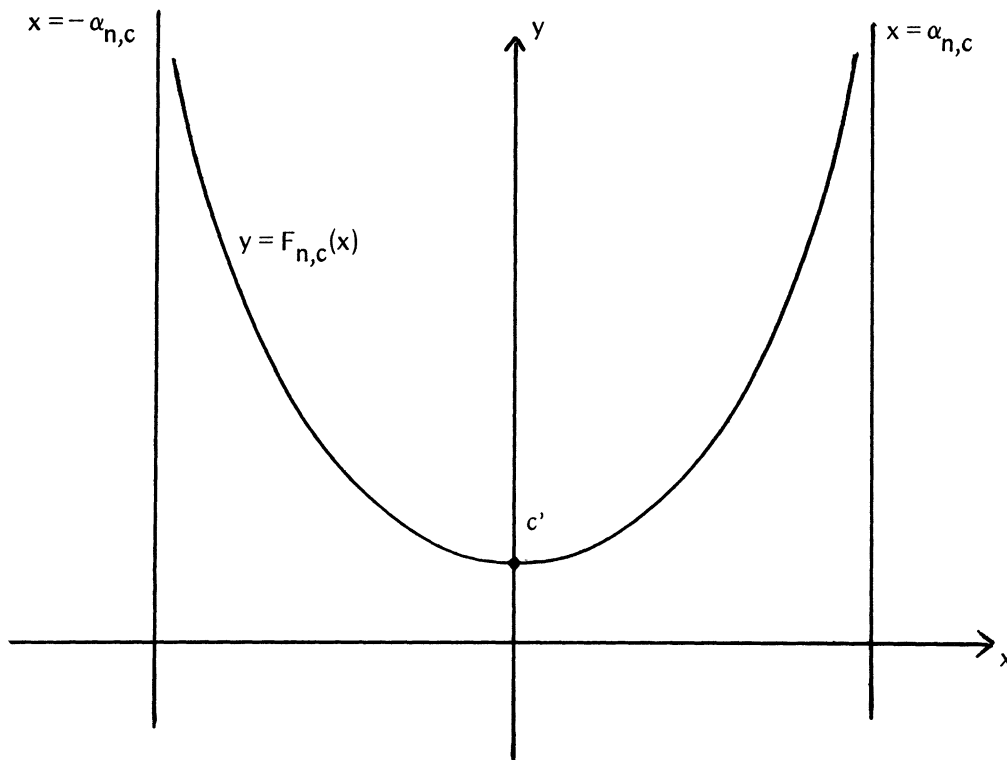
$$h_{n,c} : ]c', +\infty[ \rightarrow \mathbb{R} : x \mapsto \int_{c'}^x \frac{d\rho}{\sqrt{c^2 \rho^{2(n-2)} - 1}}.$$

Since  $h'_{n,c} \neq 0$  everywhere, we can define the inverse function  $g_{n,c} := h_{n,c}^{-1}$ . It can be shown that  $g_{n,c}$  is defined on  $]0, \alpha_{n,c}[$  with  $\alpha_{n,c} \in \mathbb{R}_0^+$  for all  $n > 3$  and  $\alpha_{3,c} = +\infty$ . Next we consider the function  $F_{n,c}$  given by

$$F_{n,c} : ]-\alpha_{n,c}; \alpha_{n,c}[ \rightarrow \mathbb{R}$$

$$x \longrightarrow \begin{cases} g_{n,c}(-x) & \text{if } x < 0 \\ c' & \text{if } x = 0 \\ g_{n,c}(x) & \text{if } x > 0 \end{cases}$$

For  $n > 3$  :



For each solution  $\varphi : I \rightarrow \mathbb{R}$  of the equation (\*) there exist numbers  $c \in \mathbb{R}_0^+$  and  $b \in \mathbb{R}$  such that  $\varphi(x) = F_{n,c}(x+b)$  for all  $x$  in  $I$ . We remark that  $F_{3,c}(x) = \frac{1}{c} \cosh cx$  for all  $x$  in  $\mathbb{R}$ , i.e.  $\gamma$  is a catenary.

Call  $K_c^n$  the hypersurface of revolution obtained by rotation of the curve  $\gamma_{n,c} : ]-\alpha_{n,c} ; \alpha_{n,c}[ \rightarrow \mathbb{E}^{n+1} : u \mapsto (u, F_{n,c}(u), 0, \dots, 0)$  around the  $x_1$ -axis. All hypersurfaces of revolution on  $\mathbb{E}^{n+1}$  such that all second fundamental tensors have the form described in Lemma 6.1 (b) are open parts of a  $K_c^n$ .

C. PROOF OF THEOREM 4.

It is clear from A and B that one of the implications holds. We now prove the other one. Suppose that  $(M^n, g)$  satisfies  $Q \cdot P = 0$ . The lemma determines the possible forms for the second fundamental tensors.

First, suppose that there is a point  $p$  in  $M$  with  $A_p$  a multiple of  $I_{T_p M}$ . In the same way as in the previous section, this implies that  $f$  is a totally umbilical immersion.

Next, we assume that  $M$  has no umbilical points. Call  $W = \{p \in M \mid \text{rank } A_p = n\}$ . Then  $W$  is open. Call  $U = W \cup \text{int}(M \setminus W)$ . Then  $U$  is an open dense subset of  $M$ . Take a connected component  $U_\alpha$  of  $U$ . If  $U_\alpha \subset \text{int}(M \setminus W)$  then  $f_\alpha := f|_{U_\alpha}$  is a cylindrical immersion. We next consider the case  $U_\alpha \subset W$ . We will need some more lemmas.

Define  $T_1 := \{x \in TU_\alpha \mid Ax = \lambda x\}$  and  $T_2 := \{x \in TU_\alpha \mid Ax = (2-n)\lambda x\}$ . By Proposition 2.3 in [8],  $T_1$  and  $T_2$  are differentiable involutive distributions and  $\lambda$  is constant along integral manifolds of  $T_1$ . Furthermore, for  $X_1$  a vector field with values in  $T_1$  and  $X_2$  a vector field with values in  $T_2$ , the equation  $(\nabla_{X_1} A)X_2 - (\nabla_{X_2} A)X_1 = 0$  of Codazzi implies that

$$(6.3) \quad \nabla_{X_2} X_1 \text{ takes its values in } T_1$$

and that  $(A - (2-n)\lambda)\nabla_{X_1} X_2 + (X_2 \cdot \lambda)X_1 = 0$ . If  $(\nabla_{X_1} X_2)_1$  denotes the component of  $\nabla_{X_1} X_2$  in  $T_1$ ,

$$(6.4) \quad (\nabla_{X_1} X_2)_1 = \frac{X_2 \cdot 1n\lambda}{1-n} X_1.$$

(6.3) implies that

$$(6.5) \quad \nabla_{X_2} Y_2 \text{ takes its values in } T_2$$

for each vector field  $Y_2$  with values in  $T_2$ .

For each  $p$  in  $U_\alpha$  we write  $M_1(p)$  for an integral manifold of  $T_1$  through  $p$  and  $\gamma_p : I \rightarrow M$  for an integral curve of  $T_2$  through  $p$ . We assume that  $\gamma_p(0) = p$  and that  $\gamma_p$  is parametrized by arclength. Around any  $p$  in  $U_\alpha$  we can choose a local orthonormal frame field  $\{E_1, \dots, E_n, E_{n+1}\}$  for  $IE^{n+1}$  which is adapted to  $f_\alpha$  and such that furthermore  $E_1, \dots, E_{n-1}$  span  $T_1$  and  $E_n$  spans  $T_2$ . (6.4) and (6.5) imply that

$$(6.6) \quad \nabla_{E_n} E_n = 0$$

and that

$$(6.7) \quad \nabla_{E_i} E_n = \frac{E_n \cdot 1n\lambda}{1-n} E_i.$$

In the following lemma we study the shape of the immersions  $(f_1)_p := f|_{M_1(p)}$ .

LEMMA 6.2. For each  $p$  in  $U_\alpha$ ,  $f(M_1(p))$  is an open part of an  $(n-1)$ -dimensional sphere in  $IE^{n+1}$  with radius  $\frac{1}{\sqrt{(\frac{E_n \cdot 1n\lambda}{n-1})^2 + \lambda^2}}$ . Consequently,  $(f_1)_p$  is local injective.

*Proof.* Let  $q \in M_1(p)$ . If  $\{E_1, \dots, E_n, E_{n+1}\}$  is a frame field around  $q$  as above, the normal bundle of  $(f_1)_p$  is spanned by  $E_n$  and  $E_{n+1}$ . Let  $A_{E_n}^i$  and  $A_{E_{n+1}}^i$  be the second fundamental tensors of  $f_1$  and denote by  $D'$  the normal connection of  $f_1$ . Then  $\tilde{\nabla}_{E_i} E_{n+1} = -AE_i = -\lambda E_i$ , ( $i \in \{1, \dots, n-1\}$ ). This yields that

$$(6.8) \quad A_{E_{n+1}}^i = \lambda I_{T_q}(M_1(p)).$$

We also have that  $\tilde{\nabla}_{E_i} E_n = \nabla_{E_i} E_n = \frac{E_n \cdot 1n\lambda}{1-n} E_i$ , ( $i \in \{1, \dots, n-1\}$ ), by (6.7). So

$$(6.9) \quad A_{E_n}^i = \frac{E_n \cdot 1n\lambda}{n-1} I_{T_q}(M_1(p)).$$

This proves the lemma. ■

Let  $IE^n(p)$  be the unique hyperplane of  $IE^{n+1}$  containing  $f(M_1(p))$ , call  $\nu_p$  the normal in this hyperplane on  $f(M_1(p))$  in  $p$  and let  $m(p)$  be the center of the sphere. Then

$$(6.10) \quad \nu_p = \frac{\lambda(p)E_{n+1}(p) + \frac{E_n(p) \cdot 1n\lambda}{n-1} E_n(p)}{\sqrt{(\frac{E_n(p) \cdot 1n\lambda}{n-1})^2 + \lambda(p)^2}}$$

and

$$(6.11) \quad m(p) = f(p) + \frac{\vec{\nu}_p}{\sqrt{\left(\frac{E_n(p) \cdot 1n\lambda}{n-1}\right)^2 + \lambda(p)^2}}.$$

$(\vec{\nu}_p)$  is the vectorpart of  $\nu_p$ ).

Next, we study the shape of the image  $f \circ \gamma_p$  of the integral curves.

LEMMA 6.3. For each  $p$  in  $U_\alpha$ ,  $f \circ \gamma_p$  is a plane curve with nowhere zero curvature.

*Proof.* Let  $q \in \text{im}\gamma_p$ . If  $\{E_1, \dots, E_n, E_{n+1}\}$  is a frame field around  $q$  as above, then

$$(6.12) \quad \begin{aligned} (f \circ \gamma_p)' &= E_n, \\ (f \circ \gamma_p)'' &= \tilde{\nabla}_{E_n} E_n = (2-n)\lambda E_{n+1}, \\ (f \circ \gamma_p)''' &= (2-n)(E_n \cdot \lambda)E_{n+1} - (2-n)^2 \lambda^2 E_n. \end{aligned}$$

Since  $(f \circ \gamma_p)' \wedge (f \circ \gamma_p)'' \wedge (f \circ \gamma_p)''' = 0$ ,  $f \circ \gamma_p$  is a plane curve. From (6.12) it is clear that the curvature of  $f \circ \gamma_p$  is nowhere zero. ■

Call  $IE^2(p)$  the unique plane in  $IE^{n+1}$  containing  $\text{im}(f \circ \gamma_p)$ .  $IE^2(p)$  is the plane through  $f(p)$  spanned by  $E_n(p)$  and  $E_{n+1}(p)$ . It is clear from (6.10) and (6.11) that  $m(p) \in IE^2(p)$ . We prove the following lemma concerning the position of the planes  $IE^2(p)$ .

LEMMA 6.4. Let  $p \in M$ . Then there is a line  $\ell(p)$  in  $IE^{n+1}$  such that  $\ell(p) = IE^2(p) \cap IE^2(q)$  for each  $q$  in  $M_1(p)$  which is distinct from  $p$  and for which  $f(q)$  is not the antipodal point of  $f(p)$ . Moreover,  $m(p) \in \ell(p)$  and  $\ell(p) \perp IE^n(p)$ .

*Proof.* Let  $q \in M_1(p)$  with  $q \neq p$  and  $f(q)$  not the antipodal point of  $f(p)$ . We prove that  $IE^2(p) \neq IE^2(q)$ .  $IE^2(p) \cap IE^n(p)$  contains  $f(p)$  and  $m(p)$ .  $IE^2(p) \not\subset IE^n(p)$  since the normal  $\eta_p$  on  $IE^n(p)$  lies in  $IE^2(p)$ . So  $IE^2(p) \cap IE^n(p)$  is the line  $f(p)m(p)$ . This line  $f(p)m(p)$  intersects  $f(M_1(p))$  in at most 2 points:  $f(p)$  and possibly the antipodal point of  $f(p)$ . Since  $f(q) \in f(M_1(p))$  and  $f(q)$  is neither of these points,  $f(q) \notin f(p)m(p)$ . As  $f(q) \in IE^n(p)$ , this shows that  $f(q) \notin IE^2(p)$ . In any case  $m(p) = m(q) \in IE^2(p) \cap IE^2(q)$  and  $\vec{\eta}_p = \vec{\eta}_q$  is a common direction of  $IE^2(p)$  and  $IE^2(q)$ . Therefore  $IE^2(p) \cap IE^2(q)$  is the line  $\ell(p)$  through  $m(p)$  in the direction  $\vec{\eta}_p$ . This line does not depend on  $q$ .

It is clear from the construction of  $\ell(p)$  that  $m(p) \in \ell(p)$  and that  $\ell(p) \perp IE^n(p)$ . ■

For  $p \in U_\alpha$  choose a coordinate system  $\mu : U \rightarrow ]-\epsilon, \epsilon[^n : q \mapsto (x^1(q), \dots, x^n(q))$  around  $p = \mu^{-1}(0, \dots, 0)$  such that for each choice of numbers  $a_1, \dots, a_n \in ]-\epsilon, \epsilon[$  the sets  $q \in U \mid x^n(q) = a^n$  are integral manifolds of  $T_1$  and the curves  $]-\epsilon, \epsilon[ \rightarrow U : t \mapsto \mu^{-1}(a_1, \dots, a_{n-1}, t)$  are integral curves of  $T_2$  (see [6] p. 182). We prove the following lemma concerning the position of the centers  $m(q)$  and the lines  $\ell(q)$ .

**LEMMA 6.5.** *Let  $p \in M$  and suppose that  $\mu : U \rightarrow ]-\epsilon, \epsilon[^n$  is a coordinate system around  $p$  as above. Then, for each  $q \in U$ ,  $\ell(q) = \ell(p)$ ,  $m(q) \in \ell(p)$  and  $\ell(p) \perp IE^n(q)$ .*

*Proof.* Suppose that  $\mu(q) = (c_1, \dots, c_n)$ . Call  $q' := \mu^{-1}(0, \dots, 0, c_n)$ ,  $q'' = \mu^{-1}(c_1, \dots, c_{n-1}, 0)$ . Then  $IE^2(q) = IE^2(q'')$  and  $IE^2(p) = IE^2(q')$ , which implies that  $\ell(p) = IE^2(p) \cap IE^2(q'') = IE^2(q'') \cap IE^2(q) = \ell(q)$ . The other statements in Lemma 6.5 now easily follow from Lemma 6.4. ■

Now, we can finish the proof of Theorem 4. Suppose  $p \in U_\alpha$  and let  $\mu : U \rightarrow ]-\epsilon, \epsilon[^n$  be a coordinate system around  $p$  as before. Call  $\gamma_p$  the curve  $\gamma_p : ]-\epsilon, \epsilon[ \rightarrow U_\alpha : t \mapsto \mu^{-1}(0, \dots, 0, t)$ . Determine the line  $\ell(p)$  in the way shown by Lemma 6.4. Call  $M'$  the hypersurface of  $IE^{n+1}$  obtained by rotation of  $f \circ \gamma_p$  around  $\ell(p)$ . We will show that  $f(U) \subset M'$ . Take  $q = \mu^{-1}(c_1, \dots, c_{n-1}, c_n) \in U$  and let  $q' = \mu^{-1}(0, \dots, 0, c_n)$ . Then  $f(M_1(q)) = f(M_1(q'))$  is an open part of a sphere in  $IE^n(q) \perp \ell(p)$  with center  $m(q) \in \ell(p)$  having the point  $f(q')$  in common with  $f \circ \gamma_p$ . This shows that  $f(q) \in M'$ . From the discussion in B it is clear that  $f|_{U_\alpha}$  is congruent to the inclusion of an open part of a  $K_c^n$ .

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