Solutions for Cooperative Games with Restricted Coalition Formation and Almost Core Allocations

Rong Zou

UNIVERSITY OF TWENTE.

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DISSERTATION

to obtain
the degree of doctor at the University of Twente,
on the authority of the rector magnificus,
prof. dr. ir. A. Veldkamp,
on account of the decision of the Doctorate Board,
to be publicly defended
on Friday 3 March 2023 at 10.45 hours

by

Rong Zou

born on the 5th of January 1994
in Ankang, China

This dissertation has been approved by the supervisors
prof. dr. M.J. Uetz and prof. dr. G. Xu

The research reported in this thesis has been carried out within the framework of the MEMORANDUM OF AGREEMENT FOR A DOUBLE DOCTORATE DEGREE BETWEEN NORTHWESTERN POLYTECHNICAL UNIVERSITY, PEOPLE'S REPUBLIC OF CHINA AND THE UNIVERSITY OF TWENTE, THE NETHERLANDS.

UNIVERSITY OF TWENTE. | DIGITAL SOCIETY INSTITUTE

DSI Ph.D. Thesis Series No. 23-002
Digital Society Institute
P.O. Box 217, 7500 AE Enschede,
The Netherlands.

ISBN (print): 978-90-365-5530-2
ISBN (digital): 978-90-365-5531-9
ISSN: 2589-7721 (DSI Ph.D. thesis Series No. 23-002)
DOI: 10.3990/1.9789036555319
Available online at `https://doi.org/10.3990/1.9789036555319`

Typeset with LaTeX
Printed by Ipskamp Printing, Enschede
Cover design by Rong Zou

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Preface

The thesis contains six chapters. In addition to the introductory Chapters 1 and 2, there are four research chapters. Chapters 3 and 5 are mainly based on research that was done while the author was working at Northwestern Polytechnical University in Xi'an, China. Chapters 4 and 6 are based on research of the author at the University of Twente, the Netherlands.

Papers underlying this thesis

[1] Zou R, Xu G, Li W, Hu X. A coalitional compromised solution for cooperative games. *Social Choice and Welfare* (2020) 55(4): 735-758. https://doi.org/10.1007/s00355-020-01262-2

(Chapter 3)

[2] Zou R, Li W, Uetz M, Xu G. Two-step Shapley-solidarity value for cooperative games with coalition structure. *OR Spectrum* (2023) 45(1): 1-25. https://doi.org/10.1007/s00291-022-00694-9

(Chapter 4)

[3] Zou R, Xu G, Hou D. Efficient extensions of the Myerson value based on endogenous claims from players. To appear in *Annals of Operations Research* (2023). https://doi.org/10.1007/s10479-023-05221-9

(Chapter 5)

[4] Zou R, Lin B, Uetz M, Walter M. Algorithmic solutions for maximizing shareable costs. Submitted.

(Chapter 6)

Other relevant papers by the author

[5] Li W, Xu G, Zou R, Hou D. The allocation of marginal surplus for cooperative games with transferable utility. *International Journal of Game Theory* (2022) 51(2): 353-377. https://doi.org/10.1007/s00182-021-00795-9

[6] Zou R, Xu G. Coalitional gap desirability and the equal allocation of non-separable contributions value. *Journal of Systems Science and Mathematical Sciences* (2022) 42(4): 780-790. https://doi.org/10.12341/jssms21518T

[7] Congestion resource network problem and related games. (Manuscript). (with A. Skopalik and M. Uetz).

Contents

Chapter 1

Introduction

Game theory provides an approach to analyse strategic interactions among multiple rational players through mathematical models. Since the groundbreaking book, *Theory of Games and Economic Behavior* authored by John von Neumann and Oskar Morgenstern [117] published in 1944, game theory has developed rapidly into a rich and mature research area, which is increasingly applied in other areas such as social science, economics and computer science.

Game theory distinguishes between situations in which players act independently from all other players, and situations in which players can coordinate their actions by binding commitments. These different settings are dealt within non-cooperative and cooperative game theory respectively. While non-cooperative game theory has the purpose to predict individuals' strategic decisions and find out what happens in a society when each player aims to maximize his or her own utility, cooperative game theory focuses on analysing which coalitions will form, and how worth can be distributed to stabilize or improve the collective welfare. So the object of interest in cooperative game theory is an *allocation* of worth to the individual players. Although the two areas have a different scope of research, they are also related, as cooperative games can be analysed also through the lens of non-cooperative game theory. For example, solutions for cooperative games can be implemented by a non-cooperative mechanism. The object of interest of this thesis is cooperative games, or more precisely, cooperative games with transferable utility (TU-games for short), which means that the worth of any coalition of players is just a single number representing the total amount (like money) that can be arbitrarily distributed among its members.

As mentioned, there are two core questions that are addressed in cooperative game theory: coalition formation and distribution of coalitions' collective worth. With respect to the former question, it is assumed in standard TU-games that every subset of the player set can form a coalition. However, it should be clear that the assumption of free cooperation between any subset of players is sometimes not realistic. In many real-life situations, cooperation among players can be affected by various factors such as hierarchical and social relationships. It implies that, due to these restrictions, only partial cooperation is possible.

For representing restricted cooperation among players accurately, more sophisticated models appear in the literature. Aumann and Drèze [9] firstly considered coalition structures in which the player set is partitioned into disjoint and independent unions, and there are no side payments between different unions. The possibility of cooperation between different unions is taken into account by Owen [94] who interpreted the unions as "bargaining blocks". Winter [118] introduced a more general restricted structure called the level structure, which applies a nested sequence of coalition structures to further indicate the possible relationship between players within each union.

Myerson [89] introduced so-called communication structures. Note that it is neither a special case, nor a generalization of the coalition structure suggested by Owen [94]. Generally, a communication structure is employed to analyse cooperative situations in which the communication among players is restricted. These restrictions can be modelled and visualized by (undirected) graphs. Here, cooperation of a set of players is possible if they are connected in the induced subgraph. Moreover, Myerson [90] also considered a setting with restricted communication that is described by a hypergraph, called conference structures.

There are many other forms of restricted cooperation that appear in the literature which are however not directly related to the topics of this thesis. We briefly mention some of them here. Restrictions in cooperation arising from hierarchies are captured also by so-called permission structures [54, 55], where players need permissions or approval from their superior players before they are allowed to cooperate. Faigle and Kern [48] considered restricted structures modelling precedence constraints where the set of players is (partially) ordered by some precedence relations. Algaba et al. [3] developed the so-called antimatroid which generalizes the concept of permission structures, and Béal et al. [18] developed priority structures which are mathematically identical to the acyclic conjunctive permission structures in [55] and the structure of precedence constraints. Furthermore, several other combinatorial structures are incorporated into general TU-games, such as union stable systems [2], convex geometries [23, 24], augmenting systems [25], regular set systems [78], union closed systems [113], accessible union stable network structures [4], voting structures [1], intersection closed system [19], etc.

Clearly, all these restrictions of cooperative structures require prior information about the possible cooperative behavior of the players, and given that information as an input, it is usually straightforward to figure out which coalitions are feasible, and could potentially form. The question that is being studied in cooperative game theory is how to distribute the total worth of the grand coalition among all players in a fair and reasonable way? In other words, there is the silent assumption that the grand coalition will form, and one asks for the allocation of the worth of the grand coalition among all players. This question is indeed at the heart of cooperative game theory, as otherwise, if the grand coalition does not form, one has to answer the same question for each coalition that does form.

If coalition formation is restricted by any of the above structures, that leads to the main topic of this thesis, namely, solutions for cooperative games with restricted cooperation. Specifically, we focus on solutions for cooperative games with two fundamental and widely studied structures for restricted cooperation, which are known as coalition structures and communication structures. We propose new solutions for these cooperative games and justify their reasonableness by making use of various methods employed in this literature, known under terms like axiomatization, the potential approach, implementation (by bidding mechanisms) and the procedural approach.

Another topic of the thesis investigates the stability of the grand coalition for standard TU-games. When studying solutions for TU-games, with or without restricted cooperation, there is usually the assumption that the grand coalition is being formed. An ultimate notion of stability for this to actually happen is to assume that no coalition of players would be willing to deviate from the grand coalition. When would a coalition want to deviate? This could happen when the worth that it could obtain through deviating exceeds the worth that it could get from distributing the total worth of the grand coalition. Hence, the stability of the grand coalition can be assumed to be guaranteed for any allocation of worth which provides each coalition a payoff that can not be improved upon by deviating from the grand coalition and acting on its own. Such allocations are said to satisfy coalitional rationality, or coalitional stability.

This is akin to the core as introduced by Gillies [56]. It is defined as the set of allocations satisfying coalitional rationality, as well as efficiency. Efficiency requires the total payoff that is allocated to all players should be equal to the worth of the grand coalition. However, the core is not a perfect candidate to ensure the stability of grand coalition, since it may be empty for many important classes of games. When dropping the requirement of efficiency, while focusing on coalitional rationality, this results in a problem that always has a feasible solution. We refer to the set of allocations satisfying coalitional rationality as the almost core. Especially, we investigate the problem to find optimal almost core allocations, that is, allocations in which the total amount being distributed over all players is maximal (or minimal) while maintaining the stability of the grand coalition.

To sum up, the thesis includes two research topics: Solutions for TU-games with restricted cooperation, and extremal solutions that generate coalitional stability. In the remainder of this chapter, we give an overview of chapters within this thesis.

1.1 Overview of thesis

Except for this introductory chapter, the thesis consists of five chapters. Chapter 2 introduces basic notations and definitions. Chapters 3 and 4 study two new solutions for cooperative games with coalition structures. Chapter 5 investigates efficient extensions of the Myerson value for cooperative games with communication structures. In Chapter 6, we shift our attention to the standard TU-games and analyze the stability of the grand coalition based on the concept of the almost core. We proceed by outlining the main results of Chapters 3 to 6.

In Chapter 3, we introduce a compromise solution, called α-egalitarian Owen value, for cooperative games with coalition structures. It aims to integrate egalitarianism and marginalism while taking the unions' differences into consideration. In cooperative games with coalition structures, the Owen value [94] puts emphasis on the individuals' marginal contributions. In contrast, the equal coalitional division value (ECD-value) gives priority to egalitarianism. Through introducing the guarantee coefficient α, we define an α-egalitarian Owen value as the corresponding convex combination of the Owen value and the ECD-value. This solution reduces to the corresponding α-egalitarian Shapley value [71] when the coalition structure is trivial. As main results, we characterize the α-egalitarian Owen value by three approaches, including axiomatization, potential function and implementation through a bidding mechanism.

Chapter 4 studies an alternative for the two-step Shapley value proposed by Kamijo [72] for cooperative games with coalition structures. The value is based on the idea that within a union of players, worth should be distributed based on the solidarity principle. Specifically, we propose a two-step Shapley-solidarity value, in which the surplus of a union's Shapley value [102] in the quotient game is distributed equally among the union's members, and players obtain the solidarity value [92] of the respective subgame within their union. An intuitive procedural characterization is given for this value, and three axiomatizations are provided to pinpoint the differences to comparable values.

Chapter 5 deals with the problem of efficient extensions of the Myerson value [89] introduced by Myerson for cooperative games with communication structures. We resort to the bankruptcy rules to extend the Myerson value efficiently. Firstly, we identify a graph-induced bankruptcy problem by viewing the Shapley value of the original game as endogenous claims of players, and the surplus is viewed as the endowment that needs to be distributed. Then, two classical bankruptcy rules, namely the constrained equal awards rule and the constrained equal losses rule introduced by Aumann and Maschler [10], are employed to achieve efficient extensions of the Myerson value. Correspondingly, the two extensions yield the efficient constrained equal awards Myerson value and the efficient constrained equal losses Myerson value. We characterize these two efficient graph game values by axiomatizations.

In Chapter 6, we study the stability of the grand coalition by addressing an optimization problem defined over the almost core for cost TU-games. The optimization problem maximizes the total amount that can be distributed over the individual players, and asks for cost allocations, namely "almost core" allocations, satisfying all core constraints except for being efficient, so that no proper subset of the players would prefer to deviate. The problem is well defined for games with both empty and non-empty cores. For games with an empty core, we show it is equivalent to several core relaxations proposed earlier in the literature. For games with a non-empty core, the problem has hardly been considered. We derive several complexity theoretic results for the computation of almost core allocations. For the class of minimum cost spanning tree games, we show that computing an optimal (non-negative) almost core allocation is NP-hard and derive a tight 2-approximation algorithm for the corresponding restricted problem with non-negative allocations.

Chapter 2

Preliminaries

2.1 Cooperative games

A cooperative game with transferable utility (or TU-game) is a pair (N, v) consisting of a nonempty and finite set of players N and the characteristic function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. Denote by $\mathcal{G}^N$ the family of all TU-games over N, and $\mathcal{G}$ the family of all TU-games. An element $i \in N$ and a subset S of N are called a player and a coalition respectively. Especially, N is called the grand coalition. For each $S \subseteq N$, $v(S)$ represents the worth of coalition S, which can be interpreted as either (nonnegative) revenues or (nonpositive) costs for coalition S. The cardinality of S is denoted by the corresponding lower case letter s or $|S|$. With some abuse of notation, we omit the braces for singletons. Thus we write $S \cup i$ for $S \cup \{i\}$, $S \setminus i$ for $S \setminus \{i\}$ etc.

The following definitions are important ingredients for the analysis of TU-games and the characterization of solutions for TU-games.

A TU-game $(N, v) \in \mathcal{G}$ is said to be

- *non-negative* if $v(S) \geq 0$ for all $S \subseteq N$;
- *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$, $S \cap T = \emptyset$;
- *strictly superadditive* if $v(S) + v(T) < v(S \cup T)$ for all $S, T \subseteq N$, $S \cap T = \emptyset$;
- *monotonic* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$;

- *zero-monotonic* if $v(S)+\sum_{i\in T\setminus S} v(i) \leq v(T)$ for all $S \subseteq T \subseteq N$.
- *balanced* if $\sum_{S\in 2^N\setminus\{\emptyset\}} \lambda_S v(S) \leq v(N)$ for every balanced collection of weights λ, where a vector $\lambda = (\lambda_S)_{S\in 2^N\setminus\{\emptyset\}}$ with $\lambda_S \in [0,1]$ for $S \in 2^N \setminus \{\emptyset\}$ is called a balanced collection of weights if $\sum_{S\in 2^N\setminus\{\emptyset\}: i\in S} \lambda_S = 1$ for $i \in N$. The set of all balanced collections of weights is denoted by $\mathscr{B}(N)$.

Definition 2.1. Given $(N,v) \in \mathscr{G}$ and $T \subseteq N$, the *subgame* of (N,v) with respect to T is $(T, v_{|T}) \in \mathscr{G}^T$, where for any $S \subseteq T$,

$$v_{|T}(S) = v(S).$$

For any two TU-games $(N,v),(N,w) \in \mathscr{G}^N$, $\alpha \in \mathbb{R}$, the characteristic functions of TU-games $(N, v+w)$ and $(N, \alpha v) \in \mathscr{G}^N$ are respectively given by $(v+w)(S) = v(S)+w(S)$ and $(\alpha v)(S) = \alpha v(S)$ for all $S \subseteq N$.

Definition 2.2. Given $T \in 2^N \setminus \emptyset$, the TU-game $(N, u_T) \in \mathscr{G}^N$ is called a *unanimity game* where

$$u_T(S) = \begin{cases} 1, & T \subseteq S \\ 0, & \text{otherwise.} \end{cases}$$

Harsanyi [62] showed that the collection of TU-games $\{(N,u_T)\}_{T\in 2^N\setminus\emptyset}$ is a basis for $\mathscr{G}^N$. Hence, it follows that any TU-game $(N,v) \in \mathscr{G}^N$ can be uniquely represented by,

$$v = \sum_{T\in 2^N\setminus\emptyset} c_T u_T, \tag{2.1}$$

where $c_T = \sum_{S\subseteq T} (-1)^{t-s} v(S)$. The latter numbers are known as the Harsanyi dividends.

2.1.1 Solutions for cooperative games

Solutions for cooperative TU-games provide possible allocation schemes to distribute the worth of a set of players who cooperate. As discussed earlier, it refers to distributing the worth of the grand coalition.

Given a TU-game $(N,v) \in \mathscr{G}$, a *payoff vector* is an n-dimensional vector $x = (x_i)_{i\in N} \in \mathbb{R}^N$ with x_i being the payoff allocated to player $i \in N$. For convenience, we write $x(S) = \sum_{i \in S} x_i$. A *solution* is a function defined on $\mathcal{G}$ that assigns to every $(N, v) \in \mathcal{G}$ a set of payoff vectors. A single-point solution is often associated with the unique element of the singleton solution, and is also called a *value*.

One well-known set-valued solution for TU-games is the *core* [56].

Definition 2.3. For any $(N, v) \in \mathcal{G}$, the core is given by

$$C(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(S) \geq v(S), \forall S \subsetneq N\}.$$

Each core element distributes the total worth of the grand coalition while making each proper subset of players $S \subsetneq N$ obtain at least what it could gain on its own. Hence, no subgroup of players can be better off by deviating from the grand coalition.[1]

The *Shapley value* [102] is probably the best known value. It offers each player his expected marginal contribution when assuming all possible $n!$ orders of the n players occur with the same probability.

Definition 2.4. For any $(N, v) \in \mathcal{G}$, the Shapley value is given by

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \left[v(S \cup i) - v(S)\right],\ i \in N.$$

The *equal division value* equally distribute the worth of the grand coalition over all players, which is totally independent of players' marginal contributions.

Definition 2.5. For any $(N, v) \in \mathcal{G}$, the equal division value is given by

$$ED_i(N, v) = \frac{v(N)}{n},\ i \in N.$$

The *α-egalitarian Shapley value* [71] is a convex combination of the Shapley value and the equal division value.

[1]In cost setting, if we use $(N, c) \in \mathcal{G}$ where $c : 2^N \to \mathbb{R}_{\geq 0}$ to denote a cost sharing TU-game, $c(S)$ is the cost that players in S achieve when they cooperate among themselves. The core of (N, c) is then formulated by $C(N, c) = \{x \in \mathbb{R}^n \mid x(N) = c(N),\ x(S) \leq c(S), \forall S \subsetneq N\}$. Note that it is also termed as the anti-core in some references such as [42] and [88]. For simplicity, we stick with the core in this thesis. In this setting, core allocations distribute the total cost of the grand coalition while no subset of players can achieve a smaller cost by choosing to act on its own.

Definition 2.6. Given $\alpha \in [0,1]$, for any $(N,v) \in \mathcal{G}$, the α-egalitarian Shapley value is given by

$$ESh_i^{\alpha}(N,v) = (1-\alpha)Sh_i(N,v) + \alpha ED_i(N,v),\ i \in N.$$

The *solidarity value* introduced by Nowak & Radzik [92], employs the *average marginal contribution* instead of the marginal contribution as in the Shapley value. Hence, it allocates each player his expected average marginal contribution under the assumption that all possible $n!$ orders of the n players occur with the same probability.

Definition 2.7. For any $(N,v) \in \mathcal{G}$, the solidarity value is given by

$$Sol_i(N,v) = \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} \left[\frac{1}{s} \sum_{j \in S} (v(S) - v(S \setminus j)) \right],\ i \in N.$$

2.1.2 Axiomatizations of values

Let φ be a value on $\mathcal{G}$. The characteristic properties of φ can be described by some axioms. Moreover, a value can be uniquely identified by a set of axioms. Such an approach to characterize a value is usually referred to as axiomatization which is a classic way in cooperative game theory to justify the fairness and reasonableness of a value.

Let us first recall some kinds of players that are relevant to classical axioms. Given a TU-game $(N,v) \in \mathcal{G}$, a player $i \in N$ is called a (an)

- *null player* in (N,v) if $v(S \cup i) - v(S) = 0$ for all $S \subseteq N \setminus i$;
- *dummy player* in (N,v) if $v(S \cup i) - v(S) = v(i)$ for all $S \subseteq N \setminus i$;
- *A-null player* in (N,v) if $\frac{1}{s}\sum_{j \in S}(v(S) - v(S \setminus j)) = 0$ for all $S \subseteq N$ with $i \in S$.

While a null player makes no marginal contribution to any coalition containing him, a dummy player always brings a marginal contribution of his singleton worth to coalitions. If the average marginal contribution of player i to any coalition $S \ni i$ is zero, then i is an A-null player.

Two players $i, j \in N$ are *symmetric* in $(N, v) \in \mathcal{G}$ if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$.

Next, we recall some axioms which are used to axiomatize the Shapley value and the solidarity value. The value φ is said to satisfy

- *efficiency* (**E**) if for all $(N, v) \in \mathcal{G}$, $\sum_{i \in N} \varphi_i(N, v) = v(N)$;
- *symmetry* (**S**) if for all $(N, v) \in \mathcal{G}$ and two symmetric players $\{i, j\} \subseteq N$, $\varphi_i(N, v) = \varphi_j(N, v)$;
- *additivity* (**A**) if for all $(N, v), (N, w) \in \mathcal{G}$, $\varphi_i(N, v + w) = \varphi_i(N, v) + \varphi_i(N, w)$;
- *the null player axiom* (**NP**) if for all $(N, v) \in \mathcal{G}$ and null player $i \in N$, $\varphi_i(N, v) = 0$;
- *the A-null player axiom* (**ANP**) if for all $(N, v) \in \mathcal{G}$ and A-null player $i \in N$, $\varphi_i(N, v) = 0$.

Efficiency requires that the total amount that is distributed over all players should be equal to the worth of the grand coalition. *Symmetry* requires that two players who contribute equally to all coalitions excluding them should be treated equally. *Additivity* states that the sum of payoffs of players in two separate games is the same as this player's payoff in the sum game of the two involved games. The *null player axiom* and the *A-null player axiom* require a zero payoff to be allocated to a null-player and an A-null player respectively.

The first four axioms above give rise to a standard axiomatization for the Shapley value [102], meaning that a value is the Shapley value if and only if it satisfies efficiency, additivity, symmetry and the null player axiom. When replacing **NP** with **ANP**, an axiomatization of the solidarity value is obtained [92].

Myerson [90] also characterized the Shapley value by the so-called balanced contributions axiom and efficiency. The value φ is said to satisfy

- *balanced contributions* (**BC**) if for all $(N, v) \in \mathcal{G}$ and $\{i, j\} \subseteq N$,

$$\varphi_i(N, v) - \varphi_i(N \setminus j, v) = \varphi_j(N, v) - \varphi_j(N \setminus i, v).$$

It requires that any two players must have the same impacts on mutual payoff when one of them departs from the game. Subsequently, Xu et al. [120] introduced quasi-balanced contributions axiom. The value φ is said to satisfy

- *quasi-balanced contributions* (**QBC**) if for all $(N,v) \in \mathscr{G}$ and $\{i,j\} \subseteq N$,

$$\varphi_i(N,v) - \varphi_i(N \setminus j, v) + \frac{1}{n} v(N \setminus j) = \varphi_j(N,v) - \varphi_j(N \setminus i, v) + \frac{1}{n} v(N \setminus i).$$

It is verified in [120] that the solidarity value is the unique efficient value satisfying this property.

2.2 Cooperative games with coalition structures

In real-life situations, players may be partitioned into subgroups in a cooperative scenario, due to physical characteristics such as geographic location, or because players actively organize themselves into subgroups in order to improve their bargaining position, such as cartels and syndicates. Such partition of players is usually called a coalition structure. The precise interpretation of what a coalition structure actually means for cooperation, or rather in restricting possible cooperation, is different in the literature. Some of these options will be discussed in detail later.

Definition 2.8. Given a finite set of players N, $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ is a *coalition structure* over N if

(1) $C_h \neq \emptyset$ for all $h \in M$, where $M = \{1, 2, \ldots, m\}$;

(2) $\bigcup_{h \in M} C_h = N$;

(3) $C_h \cap C_r = \emptyset$ when $h \neq r$.

We call an element $C_h \in \mathscr{C}$ a union. Note that M is the set of unions, and an element $h \in M$ represents the corresponding player when regarding the union $C_h \in \mathscr{C}$ as an entity. There are two trivial coalition structures, namely $\mathscr{C}_N = \{N\}$ and $\mathscr{C}_n = \{\{i\} \mid i \in N\}$. That means that the grand coalition forms in $\mathscr{C}_N$ and each union is a singleton in $\mathscr{C}_n$. Denote by $\mathscr{C}^N$ the set of all possible coalition structures over N. For any $S \subseteq N$, we denote the restriction of $\mathscr{C}$ on the player set S as $\mathscr{C}_{|S}$, i.e., $\mathscr{C}_{|S} = \{C_h \cap S | C_h \in \mathscr{C}$ and $C_h \cap S \neq \emptyset\}$. For $\mathscr{C} \in \mathscr{C}^N$ and each non-empty $T \subseteq N$, denote by $D_T = \{h \in M \mid C_h \cap T \neq \emptyset\}$ the set of unions that have a nonempty intersection with T.

A permutation on N is a bijective mapping $\pi : N \to N$ that associates every player i with a position $\pi(i)$. Let Π_N denote the collection of all $n!$ permutations on N. Given $\pi \in \Pi_N$, the predecessors of player $i \in N$ is denoted by $P^\pi(N,i) = \{j \in N \mid \pi(j) \leq \pi(i)\}$, which includes all players that are in positions before player i, and player i himself. For a given coalition structure $\mathscr{C} \in \mathscr{C}^N$, a permutation $\pi \in \Pi_N$ is *consistent* with $\mathscr{C}$ if $i \in C_h \in \mathscr{C}$ and $j \in C_h \in \mathscr{C}$ and $k \in N$, $\pi(i) < \pi(k) < \pi(j)$ implies that the player k also belongs to union C_h. The set of all the permutations on N that are consistent with $\mathscr{C}$ is denoted by $\Pi_{N,\mathscr{C}}$.

A *cooperative game with a coalition structure* is a triple $(N, v, \mathscr{C})$ where (N, v) is a TU-game and $\mathscr{C}$ is a coalition structure over N. We denote by $\mathscr{C}\mathscr{G}^N$ the collection of all TU-games with coalition structures over player set N, and by $\mathscr{C}\mathscr{G}$ the collection of all TU-games with coalition structures. Given a non-empty coalition S, denote the restriction of $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}^N$ to S as the TU-game with a coalition structure $(S, v_{|S}, \mathscr{C}_{|S})$.

TU-games with coalition structures were first considered by Aumann and Drèze [9]. In their model, unions are considered to be independent from each other and there are no side payments among the set of unions. The cooperation among unions begins with Owen [94] who assumed that the coalition of all players is being formed, and hence, the worth of the grand coalition is distributed and the unions are interpreted as "bargaining blocks". The difference between the two models can be illustrated by the following example.

Example 2.1. Let $N = \{1, 2, 3, 4, 5\}$ be the player set. Consider the TU-game (N, v) where the characteristic function v is given by $v(\{1, 2\}) = v(\{4, 5\}) = 6$, $v(\{1, 2, 3\})$=8, $v(N) = 20$ and $v(S) = 0$ otherwise, and the coalition structure $\mathscr{C} = \{C_I, C_{II}\}$ where $C_I = \{1, 2, 3\}$ and $C_{II} = \{4, 5\}$.

According to Aumann and Drèze [9], C_I and C_{II} are independent from each other, and the set of all feasible coalitions is

$$\mathscr{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{1, 2, 3\}\}.$$

Since there are no side payments between the two unions, the TU-game with a coalition structure $(N, v, \mathscr{C})$ can be viewed as two separate subgames $(C_I, v_{|C_I})$ and $(C_{II}, v_{|C_{II}})$. In contrast, in Owen's model, the cooperative possibility exists between the two unions and the grand coalition is being formed. The set of all feasible coalitions becomes

$$\begin{aligned}\mathscr{F} =&\{\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},\{1,3\},\{2,3\},\{4,5\},\{1,2,3\},\\&\{1,4,5\},\{2,4,5\},\{3,4,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\},\\&\{1,2,3,4\},\{1,2,3,5\},\{1,2,3,4,5\}\}.\end{aligned}$$

Here, a coalition is feasible if it is either a subset of one union, or a coalition that is the "union" of a subset of one union and the other complete union. Then, the TU-game with a coalition structure $(N, v, \mathscr{C})$ is a restricted game with the player set N and only a feasible coalition can earn its coalitional worth. Hereafter, we refer to cooperative games with coalition structures as Owen's model.

A *coalitional value* for cooperative games with coalition structures is a function ψ: $\mathscr{CG} \to \mathbb{R}^N$ that assigns to each cooperative game with a coalition structure $(N, v, \mathscr{C})$ a payoff vector.

The Owen value [94] is established by a two-step approach. In the first step, the unions play a quotient game in which each union is regarded as a player and obtains their own Shapley value. Here, given a TU-game with a coalition structure $(N, v, \mathscr{C}) \in \mathscr{CG}$, the *quotient game* $(M, v^{\mathscr{C}})$ is defined as

$$v^{\mathscr{C}}(Q) = v(\cup_{h\in Q} C_h),\ Q \subseteq M.$$

In the second step, for each $C_k \in \mathscr{C}$, an induced internal game (C_k, v_{C_k}) is constructed to determine the distribution of the payoff received by union C_k in the first step, among players within this union. It is worth noting that $v_{C_k} \neq v_{|C_k}$. Given a coalition $S \subseteq C_k \in \mathscr{C}$, denote the coalition's complement in C_k by $\bar{S} = C_k \setminus S$. Then, the characteristic function v_{C_k} is given by

$$v_{C_k}(S) = Sh_k(M, (v_{|N\setminus\bar{S}})^{\mathscr{C}_{|N\setminus\bar{S}}}),$$

for all $\emptyset \neq S \subseteq C_k$ and $v_{C_k}(\emptyset) = 0$. The worth of a non-empty coalition S in the induced internal game is specified by the Shapley value of the quotient game when replacing the union C_k with coalition S. The Owen value of the original game coincides with the Shapley value of the induced internal game defined for each of unions.

Definition 2.9. For any $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$, the Owen value is given by

$$Ow_i(N,v,\mathscr{C}) = Sh_i(C_k, v_{C_k}),\ i \in C_k \in \mathscr{C}.$$

In contrast to the Owen value, the Shapley-solidarity (SS) value [33] employs different rules between guiding cooperation among the players within a union and interaction among unions. Specifically, a player gets his payoff according to the solidarity value [92] instead of the Shapley value of the induced internal game.

Definition 2.10. For any $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$, the Shapley-solidarity value is given by

$$SS_i(N,v,\mathscr{C}) = Sol_i(C_k, v_{C_k}),\ i \in C_k \in \mathscr{C}.$$

Unlike the two-step procedure above, Kamijo [72] introduced the two-step Shapley value with a different, arguably simpler, two-step approach. To be more specific, one considers the *surplus* $Sh_k(M,v^{\mathscr{C}}) - v(C_k)$ in the first step, which is equally divided among the players in C_k. Then, there is no need to resort to a new worth to assess a coalition's power when considering the intra-union bargaining. For $C_k \in \mathscr{C}$, one can just consider the corresponding subgame $(C_k, v_{|C_k})$ and Kamijo proposed to use the Shapley value to distribute $v(C_k)$.

Definition 2.11. For any $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$, the two-step Shapley value is given by

$$TSh_i(N,v,\mathscr{C}) = Sh_i(C_k, v_{|C_k}) + \frac{Sh_k(M,v^{\mathscr{C}}) - v(C_k)}{|C_k|},\ i \in C_k \in \mathscr{C}.$$

We proceed with recalling some axioms in the coalition structure setting. Note that a union $C_h \in \mathscr{C}$ is called a *null coalition* in $(N,v,\mathscr{C})$ if h is a null player in $(M,v^{\mathscr{C}})$, and two unions $C_h, C_r \in \mathscr{C}$ are *symmetric coalitions* in $(N,v,\mathscr{C})$ if h and r are symmetric players in $(M,v^{\mathscr{C}})$. The coalitional value ψ is said to satisfy

- *efficiency* (**E**) if for all $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, $\sum_{i \in N} \psi_i(N, v, \mathscr{C}) = v(N)$;
- *additivity* (**A**) if for all $(N, v, \mathscr{C}), (N, w, \mathscr{C}) \in \mathscr{C}\mathscr{G}$,

$$\psi_i(N, v + w, \mathscr{C}) = \psi_i(N, v, \mathscr{C}) + \psi_i(N, w, \mathscr{C});$$

- *coalitional symmetry* (**CS**) if for all $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, and two symmetric coalitions C_h and C_r in $(N, v, \mathscr{C})$,

$$\sum_{i \in C_h} \psi_i(N, v, \mathscr{C}) = \sum_{i \in C_r} \psi_i(N, v, \mathscr{C});$$

- *intracoalitional symmetry* (**IS**) if for all $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, and two symmetric players $i, j \in C_h \in \mathscr{C}$ in (N, v),

$$\psi_i(N, v, \mathscr{C}) = \psi_j(N, v, \mathscr{C});$$

- *the null player axiom* if for all $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, and null player i in (N, v), $\psi_i(N, v, \mathscr{C}) = 0$.

Efficiency, additivity and the null player axiom for TU-games with coalition structures are identical to those for standard TU-games. Coalitional symmetry requires to treat symmetric unions equally, and intracoalitional symmetry requires that two players who are symmetric in the corresponding standard TU-games should get equal payoff. Owen [94] used the above five axioms to axiomatize the Owen value.

Calvo et al. [31] gave another axiomatization of the Owen value by employing efficiency and two axioms related to balanced contributions, called coalitional balanced contributions axiom and intracoalitional balanced contributions axiom. The two axioms can be obtained by generalizing the balanced contributions axiom introduced by Myerson [90] to cooperative games with coalition structures. Firstly, the coalitional balanced contributions axiom is formulated from the perspective of unions. The coalitional value ψ is said to satisfy

• *coalitional balanced contributions* (**CBC**) if for all $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, and $C_h, C_r \in C$ with $r \neq h$,

$$\sum_{i \in C_h} \psi_i(N, v, \mathscr{C}) - \sum_{i \in C_h} \psi_i(N \backslash C_r, v_{|N \backslash C_r}, \mathscr{C}_{|N \backslash C_r})$$
$$= \sum_{i \in C_r} \psi_i(N, v, \mathscr{C}) - \sum_{i \in C_r} \psi_i(N \backslash C_h, v_{|N \backslash C_h}, \mathscr{C}_{|N \backslash C_h}).$$

Coalitional balanced contributions axiom requires that two unions are equally affected in the sense that equal gains or losses are borne by them when the other leaves the game. Correspondingly, the intracoalitional balanced contributions axiom is introduced to measure the mutual influence of two players within the same union. The coalitional value ψ is said to satisfy

• *intracoalitional balanced contributions* (**IBC**) if for all $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, and $i, j \in C_h \in \mathscr{C}$ with $i \neq j$,

$$\psi_i(N, v, \mathscr{C}) - \psi_i(N \backslash j, v_{|N \backslash j}, \mathscr{C}_{|N \backslash j}) = \psi_j(N, v, \mathscr{C}) - \psi_j(N \backslash i, v_{|N \backslash i}, \mathscr{C}_{|N \backslash i}).$$

This axiom means that, given two players in the same union, the amounts that both players gain or lose when the other leaves the game should be equal.

2.3 Cooperative games with communication structures

Implicitly, it is assumed that cooperation among players occurs when they can communicate with each other. There are many cases in which players' communication is limited. For example, when players are connected via supply routes, computer networks or web links etc., some players may not be able to communicate with those players who are isolated from them. Cooperative games with communication restrictions were firstly analysed by Myerson [90]. In Myerson's model, communication restrictions are mathematically represented by a graph, and a coalition is called feasible if and only if its members are connected directly or indirectly through their links in the induced subgraph.

Given the player set N, a *communication graph* on N is an undirected graph Γ, where $\Gamma \subseteq \Gamma^N = \{\{i, j\} : i, j \in N, i \neq j\}$. An edge $\{i, j\}$ of Γ represents a communication link between players $i, j \in N$, and it is written as ij for simplicity. We denote by $\mathscr{L}^N$ the set of all communication graphs on N.

Given $S \subseteq N$, $\Gamma_S = \{ij \in \Gamma : i, j \in S\}$ is the *subgraph* of Γ induced by player set S. Players $i, j \in S$ are *connected* if there exists a path from i to j in Γ_S. If every pair of players belonging to S is connected in Γ_S, we call S connected. A subset $T \subseteq S$ is a *component* of S if T induces a connected subgraph of Γ_S. We denote by S/Γ_S the set of all components of S and $C(i)$ the element of N/Γ containing i. Note that N/Γ is just the set of connected components of Γ.

A *cooperative game with a communication structure* (graph game for short) is a triple (N, v, Γ) which consists of a TU-game $(N, v) \in \mathscr{G}^N$ and a communication graph $\Gamma \in \mathscr{L}^N$. Particularly, we call (N, v, Γ) a connected graph game when Γ is connected. Denote the sets of all graph games and all connected graph games by $\mathscr{GL}$ and $\mathscr{GL}_C$ respectively.

Let $(N, v, \Gamma) \in \mathscr{GL}$ be a graph game. A *graph restricted game* for (N, v, Γ) is denoted by $(N, v^\Gamma) \in \mathscr{G}^N$ where the corresponding characteristic function is defined as

$$v^\Gamma(S) = \sum_{T \in S/\Gamma_S} v(T),\ S \subseteq N.$$

The worth of coalition S in the graph restricted game is the sum of the worths generated by its components.

Example 2.2. Consider the TU-game with a communication structure (N, v, Γ) where (N, v) is the TU-game in Example 2.1 and the communication graph on N is given by $\Gamma = \{\{1, 2\}, \{2, 3\}, \{4, 5\}\}$, as shown in Figure 2.1.

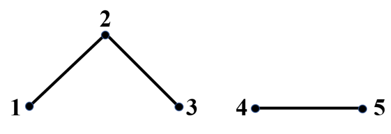

Figure 2.1: Communication graph on N

For this graph game, the set of all feasible coalitions is

$$\mathscr{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{2, 3\}, \{4, 5\}, \{1, 2, 3\}\}.$$

As for the graph-restricted game (N, v^Γ), we have

$$v^\Gamma(N) = v(\{1,2,3\}) + v(\{4,5\}) = 14,$$
$$v^\Gamma(\{1,2,4,5\}) = v(\{1,2\}) + v(\{4,5\}) = 12,$$
$$v^\Gamma(\{1,2,3,4\}) = v^\Gamma(\{1,2,3,5\}) = v^\Gamma(\{1,2,3\}) = 8,$$
$$v^\Gamma(\{1,3,4,5\}) = v^\Gamma(\{2,3,4,5\}) = v^\Gamma(\{1,2,4\}) = v^\Gamma(\{1,2,5\}) =$$
$$v^\Gamma(\{1,4,5\}) = v^\Gamma(\{2,4,5\}) = v^\Gamma(\{3,4,5\}) = v^\Gamma(\{4,5\}) = v^\Gamma(\{1,2\}) = 6,$$
$$v^\Gamma(S) = 0, \text{otherwise.}$$

A *graph game value* is a mapping that assigns a payoff vector to every graph game. We use f as our generic notation for a graph game value, then for each $(N, v, \Gamma) \in \mathcal{GL}$, we have $f(N, v, \Gamma) = (f_i(N, v, \Gamma))_{i \in N} \in \mathbb{R}^N$. The graph game value f is said to satisfy

- *efficiency* (**E**) if for all $(N, v, \Gamma) \in \mathcal{GL}$, $\sum_{i \in N} f_i(N, v, \Gamma) = v(N)$;
- *component efficiency (**CE**)* if for all $(N, v, \Gamma) \in \mathcal{GL}$ and all connected components $S \in N/\Gamma$, $\sum_{i \in S} f_i(N, v, \Gamma) = v(S)$.
- *fairness (**F**)* if for all $(N, v, \Gamma) \in \mathcal{GL}$, and $ij \in \Gamma$,

$$f_i(N, v, \Gamma) - f_i(N, v, \Gamma \setminus ij) = f_j(N, v, \Gamma) - f_j(N, v, \Gamma \setminus ij);$$

- *component decomposability (**CD**)* if for all $(N, v, \Gamma) \in \mathcal{GL}$, and $i \in C(i) \in N/\Gamma$, $f_i(N, v, \Gamma) = f_i(C(i), v_{|C(i)}, \Gamma_{C(i)})$.

Component efficiency requires that the sum of the payoffs to the players in a component is precisely the worth of that component. Since there is only one component for connected graph games, i.e. the grand coalition, component efficiency coincides with efficiency in this case. Fairness states that for every link in the graph, the incident players lose or gain the same amount from breaking this link. Component decomposability states that distribution of payoffs within a component is not affected by the players outside that component.

The Myerson value [89] for a graph game $(N, v, \Gamma) \in \mathcal{GL}$ is established as the Shapley value of the corresponding graph restricted game.

Definition 2.12. For any $(N, v, \Gamma) \in \mathscr{GL}$, the Myerson value is given by

$$\mu_i(N, v, \Gamma) := Sh_i(N, v^\Gamma), \quad \text{for all } i \in N.$$

In Myerson's seminal paper [89], the Myerson value is firstly characterized by only two axioms, namely component efficiency and fairness. The Myerson value is generally said to satisfy component efficiency instead of efficiency because the Myerson value violates efficiency for games with unconnected graphs. Besides, it has been verified that the Myerson value satisfies the component decomposability axiom [115].

The following axiom, called coherence with the Myerson value for connected graphs [15], plays an important role in characterizing efficient extensions of the Myerson value. The graph game value f is said to satisfy

- coherence with the Myerson value for connected graphs **(CMC)** if for all $(N, v, \Gamma) \in \mathscr{GL}_C$, $f_i(N, v, \Gamma) = \mu_i(N, v, \Gamma)$.

CMC requires a graph game value to coincide with the Myerson value for connected graph games. This axiom has been employed to characterize several efficient extensions of the Myerson value, such as the efficient egalitarian Myerson value [15], the efficient two-step surplus Myerson value [67] and the efficient α-proportional Myerson value [106] etc.

2.4 Minimum cost spanning tree games

Chapter 6 addresses a well-known class of cost TU-games known as minimum cost spanning tree games. These games can be motivated by a scenario in which a group of players with different geographical locations want a particular service provided by a common supplier, called the source. Players can be served by directly connecting to the source at a certain cost, however they are indifferent between being connected directly or indirectly to the source, and players may also connect to other players, at certain costs. That means that, for the whole group to be connected to the source, it is sufficient to establish a spanning tree. The problem to compute an overall cheapest solution is known as the minimum cost spanning tree problem.

Let us formally define minimum cost spanning tree problems. Let $N = \{1, \cdots, n\}$ be a finite set of n players, and let 0 be a source to which players need to be connected. For simplicity, denote by $N_0 = N \cup 0$ all players including the source. Consider a complete graph whose nodes are elements of the set N_0. A cost matrix on N_0, $W = (w_{ij})_{i,j \in N_0}$ represents the costs of the direct links between any pair of nodes. It is assumed that $w_{ij} = w_{ji} \geq 0$ for each $i, j \in N_0$ and $w_{ii} = 0$ for each $i \in N_0$. Since $w_{ij} = w_{ji}$, that means that we work with an undirected graph. Denote by $\mathscr{W}^N$ the set of all cost matrices on N_0. Given $W \in \mathscr{W}^N$ and $S \subseteq N$, the restriction of the cost matrix W to the set S is denoted by $W_{|S} = (w_{ij})_{i,j \in S_0}$.

A *minimum cost spanning tree (mcst) problem* is described by a pair (N_0, W) where N_0 includes the set of all players N and the source 0, and $W \in \mathscr{W}^N$. Given a subset $S \subseteq N$, we denote by $(S_0, W_{|S})$ the restriction of problem (N_0, W) to the subset of players S.

A network g on vertex set N_0 is a subset of $\{\{i, j\} : i, j \in N_0\}$. The elements of g are called edges. Given a network g and a pair of nodes $i, j \in N_0$, a *path* from i to j is a sequence of distinct edges $\{\{i_{k-1}, i_k\}\}_{k=1}^{l}$ satisfying $\{i_{k-1}, i_k\} \in g$ for all $k \in \{1, 2, \cdots, l\}$, $i = i_0$, and $j = i_l$. For each $S \subseteq N$, a *spanning tree* over $S_0 = S \cup 0$ is a network such that for all $i, j \in S_0$ there exists a unique path from player i to j. Denote by $\mathscr{T}(S)$ the set of all spanning trees over S_0. For each $T \in \mathscr{T}(S)$, the cost associated with T is defined as $w(T) = \sum_{\{i,j\} \in T} w_{ij}$. A minimum cost spanning tree for $(S_0, W_{|S})$ is a spanning tree whose cost is minimum among all spanning trees in $\mathscr{T}(S)$.

Algorithms have been developed in the literature to compute a minimum cost spanning tree, such as Prim's algorithm [98] and Kruskal's algorithm [77]. In Prim's algorithm, players are sequentially connected, either directly or indirectly to the source. Hence, the algorithm consists of n steps. At each step, one of the cheapest edges between the connected and the unconnected players is added:

Prim's algorithm: Let $T_\emptyset = \emptyset$. Assume at step $s = |S|$, a minimum spanning tree T_S has been constructed for players within S, i.e., $T_S \in \mathscr{T}(S)$. At step $s + 1$, choose $i^* \in S \cup 0$ and $j^* \in N \setminus S$ such that $w_{i^* j^*}$ is minimum among $\{w_{ij} : i \in S \cup 0, j \in N \setminus S\}$. Then, add $\{i^*, j^*\}$ to T_S. It turns out that $T_S \cup \{i^*, j^*\} \in \mathscr{T}(S \cup i^*)$. The algorithm ends with all players being connected, and a minimum cost spanning tree for all the players is obtained.

A minimum cost spanning tree may not be unique (when the minimizer arc is not unique), but each minimum cost spanning tree can be obtained via Prim's algorithm via tie breaking. Once a minimum cost spanning tree is constructed, an issue studied in the game theory literature is the distribution of the total cost of minimum cost spanning trees over all players.

One approach to address this problem is associating an mcst problem with a cooperative game at first, and then analysing solutions for this associated cooperative game. Generally, we denote by $c : 2^N \to \mathbb{R}_{\geq 0}$ the characteristic function which assigns to every coalition S the worth $c(S)$ representing the cost of an "outside option", that is, the minimum total cost that the players in S can achieve if they cooperate among themselves. Let us denote a cost TU-game by (N, c).

Given an mcst problem (N_0, W), the associated minimum cost spanning tree game is given by (N, c) where the characteristic function is defined as

$$c(S) = \min_{T \in \mathcal{T}(S)} w(T),$$

for all $S \subseteq N$. That is, the cost of the "outside option" of any subset of players $S \subseteq N$ is defined as the cost of any of the minimum cost spanning trees over S_0. That means that in the mcst game, the worth of a coalition S is obtained while assuming that the players in $N \setminus S$ are not available if player set S decides to cooperate among themselves. This mcst game is therefore also called a private game. Following [58], the associated "monotonized" minimum cost spanning tree game $(N, \bar{c})$ is obtained by defining the characteristic function using the monotonized cost function

$$\bar{c}(S) := \min_{R \supseteq S} c(R).$$

It means that in evaluating the cost of a set of players S, they can decide to also involve players outside S. Indeed, note that $\bar{c}(S) \leq \bar{c}(R)$ for $S \subseteq R$, and for the associated cores of these two games, we have that $C(N, \bar{c}) \subseteq C(N, c)$. Moreover, it is well known that the core of both games is non-empty, and a core allocation $x \in C(N, \bar{c})$ is obtained in polynomial time by just one minimum cost spanning tree computation: if T is an mcst, let $e_v \in T$ be the edge incident with

v on the unique path from v to the source node 0 in T, then letting

$$x_v := w(e_v),$$

one gets an element x in the core of the monotonized minimum cost spanning tree game $(N, \bar{c})$ [58], and hence also a core element for the game (N, c). One convenient way of thinking about this core allocation is a run of Prim's algorithm to compute minimum cost spanning trees [98]: starting to build the tree with vertex 0, whenever a new vertex v is added to the spanning tree constructed so far, v gets charged the cost of the edge e_v that connects v. This allocation is also known as Bird's rule [26].

Chapter 3

Egalitarian Owen Values for Cooperative Games with Coalition Structures

Solutions for cooperative games provide possible allocation schemes for distributing the worth generated by all players when they cooperate. These solutions manifest different distribution principles in their own way. Two representative distribution principles are marginalism and egalitarianism. In this chapter, we propose a solution for cooperative games with coalition structures, which expects to coordinate the two distribution principles and provide a revenue allocation scheme with a guarantee of basic interests for players.

3.1 Introduction

Players expect to obtain some benefits via cooperation, and their main concern is how to distribute the collective revenue generated by all players. There are usually two typical pay modes for revenue distribution. One could be called a high stability mode which is nearly independent from players' efforts, while the other could be called a flexible mode which mainly depends on the players' performance. In fact, the two distribution modes are the embodiment of two typical distribution principles, i.e., egalitarianism and marginalism respectively. While egalitarianism is in favour of the equal allocation disregarding the difference among the players, marginalism supports an allocation method based on a player's performance.

In cooperative game theory, marginalism is deeply etched in the Shapley value [102], one of the most influential values, which provides each player with a payoff prescribed by his expected marginal contribution with all possible entrance orders of the players happening with the same probability. This kind of performance-based evaluation results in giving little attention to the protection of players' basic interests, because a player is likely to get nothing. Not surprisingly, it can lead to an allocation system with lacking of guarantee mechanism and thereby become unattractive to players.

It seems that a harmonizing pay mode turns out to be more appealing in the real world. For example, a considerable number of companies' salary structures include not merely a basic salary but also a performance related salary, which provides employees with both the basic living security and the incentive to work harder. In fact, the idea of this kind of combination can be found in a class of solutions for cooperative games, namely the egalitarian Shapley values introduced by Joosten [71]. This class of values is represented by the convex combination of the Shapley value and the equal division value, which assigns every player the average value of the worth of the grand coalition. In this case, the basic interests implying the equality and stability can be identified by the equal division value which obviously embodies the egalitarianism.

Though the egalitarian Shapley values coordinate the two rather extreme distribution principles to some extent, it is worth noting that all players have the same basic benefits specified by the equal division value. Is this reasonable for a cooperative situation in which players have group differences? For example, in a firm, all employees may be divided into different groups according to the value of the post. It is unlikely that a clerk and an executive have the similar basic salary. Instead, only those who are in the same group generally have the equivalent basic wage. Due to this fact, an issue is raised: how can we achieve a revenue distribution system which reaches a compromise between performance-based payoff and a stable payoff evaluated by the basic salary for a cooperative setting with group differences?

To address the issue, we aim to study a kind of revenue distribution system with basic guarantee for cooperative games with coalition structures. Group differences among players are embedded in a coalition structure by dividing players into different unions. Since Owen assumed that cooperative possibilities can also happen in the unions' level, various solutions for TU-games with coalition structures have been proposed and axiomatized, such as [5, 6, 69, 111]. Particularly, as an extension of the Shapley value, the Owen value [94] assigns each player a payoff from a marginalism view as well.

Inspired by Joosten [71], we introduce the guarantee coefficient α which means that the α percent of the overall revenue will be used to guarantee players' basic interests.[1] Then, a compromise solution, called the α-egalitarian Owen value, is proposed for TU-games with coalition structures. The coalitional solution emerges in the form of a convex combination of the Owen value and the equal coalitional division value (ECD-value). Especially, the ECD-value is used to evaluate the basic interests that players within different unions can obtain regardless of their performance. Moreover, we characterize the α-egalitarian Owen value in three approaches: axiomatization, potential function and non-cooperative implementation.

The rest of this chapter is organized as follows. In Section 3.2, we define the α-egalitarian Owen value. In Section 3.3, two axiomatizations of the coalitional value are provided. In Section 3.4, we characterize the α-egalitarian Owen value by the potential approach. A revised bidding mechanism is used to implement the α-egalitarian Owen value in Section 3.5, and conclusions are given in Section 3.6.

3.2 Egalitarian Owen values

As may be seen by checking the definition of the Owen value, it has a strong flavour to marginalism as the Shapley value does. Corresponding to the equal division value for standard TU-games, there is a coalitional value for cooperative games with coalition structures, called the equal coalitional division value

[1] Joosten [71] also interpreted α as a level of solidarity or egalitarianism when defining the α-egalitarian Shapley value.

(ECD-value for short), which favors egalitarianism as well. The equal coalitional division value puts egalitarianism into practice by allocating the worth of the grand coalition equally both among unions and players within the same union.

For $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, the equal coalitional division value is given by

$$ECD_i(N, v, \mathscr{C}) = \frac{v(N)}{|\mathscr{C}||C_h|}, \quad \text{for all } i \in C_h \in \mathscr{C}.$$

This coalitional value requires an equal distribution of the worth of the grand coalition both in inner part of a coalition and inter-negotiation among coalitions. Compared to the equal division value, the ECD-value embodies intergroup and intragroup egalitarianism, which not only inherits the spirit of egalitarianism but also takes group differences into consideration.

Next, we define a coalitional compromise solution for TU-games with coalition structures, which combines the idea of marginalism and egalitarianism based on a guarantee coefficient α.

Definition 3.1. Given $\alpha \in [0, 1]$, for any TU-game with a coalition structure $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, where $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$, $i \in C_h \in \mathscr{C}$, the α-egalitarian Owen value is given by

$$EO_i^{\alpha}(N, v, \mathscr{C}) = (1 - \alpha) Ow_i(N, v, \mathscr{C}) + \alpha ECD_i(N, v, \mathscr{C}). \tag{3.1}$$

This value is established as the convex combination of the Owen value and the equal coalitional division value with coefficient α. Every player $i \in C_h \in \mathscr{C}$ will be guaranteed the basic interests of $\alpha ECD_i(N, v, \mathscr{C})$. The guarantee coefficient α reflects the extent to which the basic interests of players are guaranteed. At two extremes, when the coefficient α takes 0 and 1, the α-egalitarian Owen value reduces to the Owen value and the ECD-value respectively. Moreover, if the coalition structure is trivial, i.e., $\mathscr{C} = \mathscr{C}_N = \{N\}$ or $\mathscr{C} = \mathscr{C}_n = \{\{i\} | i \in N\}$, then the α-egalitarian Owen value reduces to the α-egalitarian Shapley value in [71].

3.3 Axiomatizations

As the most typical approach, axiomatization is devoted to finding solutions for TU-games by a set of desirable axioms. These axioms formalize certain attractive properties which a solution for the distribution of gains by cooperation should possess, and on which all agents would be willing to agree. Our axiomatizations follow the spirit of the axioms used to characterize the Owen value. There are diverse characterizations for the Owen value in the literature, and the reader is referred to [8,57,76,81–83]. In this section, we focus on two representative ones on which the axiomatizations for α-egalitarian Owen value are based. One of the two is given by Owen [94] and the other is due to Calvo et al. [31].

3.3.1 α-Indemnificatory null player axiom

The Owen value [94] is characterized by efficiency, additivity, null player axiom, coalitional symmetry and intracoalitional symmetry.[2] In particular, the null player axiom requires a coalitional value to allocate a zero payoff to a player with no marginal contribution to any coalition. However, it is also reasonable to provide such players with some payoffs to guarantee for their basic interests. For example, women employees are entitled to maternity leave with basic salary in most cases, even if they may not create revenues for the company during their absence.

With this idea in mind, we propose a variation of the null player axiom, called α-indemnificatory null player axiom. The coalitional value ψ is said to satisfy

- *the α-indemnificatory null player axiom* if for $\alpha \in [0,1]$, $(N,v,\mathscr{C}) \in \mathscr{CG}$, and null player $i \in C_h \in \mathscr{C}$ in (N,v), $\psi_i(N,v,\mathscr{C}) = \alpha\frac{v(N)}{|\mathscr{C}||C_h|}$.

This axioms requires that a null player $i \in C_h$ should be guaranteed a basic payoff of $\alpha\frac{v(N)}{|\mathscr{C}||C_h|}$. It is common that the basic interest is usually provided in a way that is irrelevant to players' performance, which is actually consistent with the spirit of egalitarianism. However, in a coalition structure setting, it seems more natural to follow the spirit of intergroup and intragroup egalitarianism. Hence, we hereby

[2]The reader is referred to Chapter 2 for the definitions of the mentioned axioms.

employ the equal coalitional division value to become a measurement of basic interests which can be adjusted by the guarantee coefficient α. With the aid of the α-indemnificatory null player axiom, we use a parallel way to carry out an axiomatization for the α-egalitarian Owen value.

Theorem 3.1. *Given $\alpha \in [0,1]$, the α-egalitarian Owen value is the unique coalitional value over $\mathscr{CG}$ that satisfies efficiency, additivity, intracoalitional symmetry, coalitional symmetry and the α-indemnificatory null player axiom.*

Proof. Existence. It is straightforward to verify that the α-egalitarian Owen value satisfies the above five axioms.

Uniqueness. The class of unanimity TU-games consists of a basis of $\mathscr{G}^N$. Due to additivity and Eq. (2.1), it is sufficient to validate that, for any unanimity game with a coalition structure, a coalitional value can be uniquely determined by efficiency, the α-indemnificatory null player axiom, intracoalitional symmetry and coalitional symmetry.

Given $T \in 2^N \setminus \emptyset$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$, we denote by $T_h = T \cap C_h$, $h \in M$. Recall that $D_T = \{h \in M \mid T_h \neq \emptyset\}$. For each $(N, u_T, \mathscr{C}) \in \mathscr{CG}^N$, the corresponding quotient game is given by $(M, u_T^{\mathscr{C}})$ where

$$u_T^{\mathscr{C}}(R) = u_T(\cup_{h \in R} C_h) = u_{D_T}(R) = \begin{cases} 1 & D_T \subseteq R \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

for all $R \subseteq M$.

Therefore, by efficiency, coalitional symmetry and the α-indemnificatory null player axiom, we have

$$\sum_{i \in C_h} \psi_i(N, u_T, \mathscr{C}) = \begin{cases} \frac{1-\alpha}{|D_T|} + \frac{\alpha}{m} & h \in D_T \\ \frac{\alpha}{m} & h \in M \setminus D_T, \end{cases}$$

where $m = |M|$.

Any player $i \in N \setminus T$ is a null player. And by intracoalitional symmetry, there is

$$\psi_i(N,u_T,\mathscr{C}) = \begin{cases} \frac{1-\alpha}{|D_T||T_h|} + \frac{\alpha}{m|C_h|} & i \in T_h \\ \frac{\alpha}{m|C_h|} & i \in C_h \backslash T_h, \end{cases}$$

for $i \in C_h \in \mathscr{C}$, as desired. This completes the proof.

□

The only difference between our axiomatization and the one for the Owen value is that null players are treated differently. That is, we replace the null player axiom of the Owen value with the α-indemnificatory null player axiom. In fact, the axioms associated with null players play an important role in axiomatizing solutions with additivity, see [108, 112] etc. In our axiomatization, precisely on the different point, the basic interests of a null player can be guaranteed, which is an illustration of a kind of solidarity among all players.

3.3.2 Coalitional quasi-balanced contributions

The balanced contributions axiom was firstly proposed by Myerson [90], which requires that any two players must have the same impacts on mutual payoff when one of them departs from the game. Calvo et al. [31] generalized this axiom to a more general domain of TU-games with level structures and characterized the level structure value [118] together with efficiency. Since coalition structures are special cases of level structures, their characterization gives rise to an axiomatization for the Owen value as well. In the setting of TU-games with level structures, the condition of balanced contributions is imposed on two unions that belong to the same union on a higher level. It can be translated into two axioms for TU-games with coalition structures, namely the so called intracoalitional and coalitional balanced contributions axioms.

The idea behind the axioms related to balanced contributions is that a reasonable solution should be balanced in the sense that one object, like a player or a union, affects another one's payoff in the same manner that the latter affects the former's payoff. From the perspective of players, we remove the basic benefits that are irrelevant to personal performance when evaluating players' interactive influence. This gives rise to the following modified intracoalitional balanced contributions axiom, called the intracoalitional quasi-balanced contributions axiom which requires that any two players within the same coalition have the same impacts on mutual performance-based payoff when one of them leaves the game. The coalitional value ψ is said to satisfy

- *intracoalitional quasi-balanced contributions with respect to* α if for $\alpha \in [0,1]$, $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$ and $i,j \in C_h \in \mathscr{C}$ with $i \neq j$, there is

$$\left[\psi_i(N,v,\mathscr{C}) - \frac{\alpha v(N)}{|\mathscr{C}||C_h|}\right] - \left[\psi_i(N\setminus j, v_{|N\setminus j}, \mathscr{C}_{|N\setminus j}) - \frac{\alpha v(N\setminus j)}{|\mathscr{C}|(|C_h|-1)}\right]$$
$$= \left[\psi_j(N,v,\mathscr{C}) - \frac{\alpha v(N)}{|\mathscr{C}||C_h|}\right] - \left[\psi_j(N\setminus i, v_{|N\setminus i}, \mathscr{C}_{|N\setminus j}) - \frac{\alpha v(N\setminus i)}{|\mathscr{C}|(|C_h|-1)}\right]$$

or in a simplified form,

$$\begin{aligned}&\psi_i(N,v,\mathscr{C}) - \psi_i(N\setminus j, v_{|N\setminus i}, \mathscr{C}_{|N\setminus j}) + \frac{\alpha v(N\setminus j)}{|\mathscr{C}|(|C_h|-1)}\\ =\ &\psi_j(N,v,\mathscr{C}) - \psi_j(N\setminus i, v_{|N\setminus i}, \mathscr{C}_{|N\setminus j}) + \frac{\alpha v(N\setminus i)}{|\mathscr{C}|(|C_h|-1)}.\end{aligned} \tag{3.3}$$

Within the context of coalition structures, when we evaluate the interactive influence of two unions, we introduce the following axiom on the basis of similar considerations. The coalitional value ψ is said to satisfy

- *coalitional quasi-balanced contributions with respect to* α if for $\alpha \in [0,1]$, $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$ and $C_h, C_r \in \mathscr{C}$ with $h \neq r$, there is

$$\begin{aligned}&\sum_{k\in C_h}\psi_k(N,v,\mathscr{C}) - \sum_{k\in C_h}\psi_k(N\setminus C_r, v_{|N\setminus C_r}, \mathscr{C}_{|N\setminus C_r}) + \frac{\alpha v(N\setminus C_r)}{|\mathscr{C}|-1}\\ =&\sum_{k\in C_r}\psi_k(N,v,\mathscr{C}) - \sum_{k\in C_r}\psi_k(N\setminus C_h, v_{|N\setminus C_h}, \mathscr{C}_{|N\setminus C_h}) + \frac{\alpha v(N\setminus C_h)}{|\mathscr{C}|-1}.\end{aligned} \tag{3.4}$$

The intracoalitional and coalitional quasi-balanced contributions with respect to α coincide with intracoalitional and coalitional balanced contributions respectively when $\alpha = 0$.

Proposition 3.1. *Given $\alpha \in [0,1]$, the α-egalitarian Owen value satisfies the intracoalitional quasi-balanced contributions and coalitional quasi-balanced contributions with respect to α.*

Proof. By Theorem 2 in Calvo et al. [31], the Owen value satisfies the intracoalitional balanced contributions and coalitional balanced contributions. Along with the definition of the α-egalitarian Owen value, it is straightforward to check that this coalitional value satisfies the two axioms mentioned in the above proposition. □

With the help of the two adjusted axioms, we provide an axiomatization for the α-egalitarian Owen value.

Theorem 3.2. *Given $\alpha \in [0,1]$, a coalitional value ψ satisfies efficiency, the intracoalitional quasi-balanced contributions and the coalitional quasi-balanced contributions with respect to α if and only if $\psi(N,v,\mathscr{C}) = EO^{\alpha}(N,v,\mathscr{C})$ for all $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$.*

Proof. By Proposition 3.1, it is left to show the uniqueness. Assume that there exists an efficient coalitional value ψ satisfying the intracoalitional and coalitional quasi-balanced contributions with respect to α, we show that ψ is identical with the α-egalitarian Owen value, i.e.,

$$\psi_i(N,v,\mathscr{C}) = EO_i^{\alpha}(N,v,\mathscr{C}), \quad \text{for all } i \in N. \tag{3.5}$$

Firstly, we show that

$$\sum_{k \in C_h} \psi_k(N,v,\mathscr{C}) = \sum_{k \in C_h} EO_k^{\alpha}(N,v,\mathscr{C}), \quad \text{for all } C_h \in \mathscr{C}, \tag{3.6}$$

by induction.

For any game $(N,v,\mathscr{C})$ with $|\mathscr{C}| = 1$, assuming that $\mathscr{C} = \{C_h\}$, we have $\sum_{k \in C_h} \psi_k(N,v,\mathscr{C}) = v(C_h) = \sum_{k \in C_h} EO_k^{\alpha}(N,v,\mathscr{C})$ according to efficiency. Assume that Eq. (3.6) holds for all games with $|\mathscr{C}| \leq m-1$, then we consider games with $|\mathscr{C}| = m$. Without loss of generality, let $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$, there is

$$\begin{aligned}\sum_{k\in C_h}\psi_k(N\setminus C_r, v_{|N\setminus C_r}, \mathscr{C}_{|N\setminus C_r}) = \sum_{k\in C_h} EO_k^{\alpha}(N\setminus C_r, v_{|N\setminus C_r}, \mathscr{C}_{|N\setminus C_r})\\ \sum_{k\in C_r}\psi_k(N\setminus C_h, v_{|N\setminus C_h}, \mathscr{C}_{|N\setminus C_h}) = \sum_{k\in C_r} EO_k^{\alpha}(N\setminus C_h, v_{|N\setminus C_h}, \mathscr{C}_{|N\setminus C_h})\end{aligned} \tag{3.7}$$

where $C_h, C_r \in \mathscr{C}$. Therefore, together with the coalitional quasi-balanced contributions with respect to α, taking Eq. (3.4) for the two values and making a subtraction, we have

$$\begin{aligned}&\sum_{k\in C_h}\psi_k(N,v,\mathscr{C}) - \sum_{k\in C_h} EO_k^{\alpha}(N,v,\mathscr{C})\\ =&\sum_{k\in C_r}\psi_k(N,v,\mathscr{C}) - \sum_{k\in C_r} EO_k^{\alpha}(N,v,\mathscr{C}),\end{aligned} \tag{3.8}$$

then fixing h, summing over r, there is

$$\begin{aligned}&\sum_{r\in M}\left[\sum_{k\in C_h}\psi_k(N,v,\mathscr{C}) - \sum_{k\in C_h} EO_k^{\alpha}(N,v,\mathscr{C})\right]\\ =&\sum_{r\in M}\left[\sum_{k\in C_r}\psi_k(N,v,\mathscr{C}) - \sum_{k\in C_r} EO_k^{\alpha}(N,v,\mathscr{C})\right]\\ =&v(N) - v(N) = 0.\end{aligned} \tag{3.9}$$

The penultimate equality is due to efficiency. Therefore, Eq. (3.6) holds for each $C_h \in \mathscr{C}$. Now, we prove that, for every $i \in N$, there is $\psi_i(N,v,C) = EO_i^{\alpha}(N,v,C)$.

For any $C_h \in \mathscr{C}$ with $|C_h| = 1$, denote by $i \in C_h$. By Eq. (3.6), it follows that the Eq. (3.5) holds.

Assume that Eq. (3.5) holds for all TU-games with $|C_h| \leq k-1$, then for any game $(N,v,\mathscr{C})$ with a coalition structure $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ where $|C_h| \leq$

$k-1$ $(h \in M)$ and any two player $i, j \in C_h \in \mathscr{C}$,

$$\psi_i(N\backslash j, v_{|N\backslash j}, \mathscr{C}_{|N\backslash j}) = EO_i^{\alpha}(N \setminus j, v_{|N\backslash j}, \mathscr{C}_{|N\backslash j})$$
$$\psi_j(N \setminus i, v_{|N\backslash i}, \mathscr{C}_{|N\backslash i}) = EO_j^{\alpha}(N \setminus i, v_{|N\backslash i}, \mathscr{C}_{|N\backslash i}).$$

Therefore, for games with $|C_h| = k$, since both ψ and the α-egalitarian Owen value satisfy the intracoalitional quasi-balanced contributions with respect to α, by Eq. (3.3), we have

$$\psi_i(N, v, \mathscr{C}) - EO_i^{\alpha}(N, v, \mathscr{C}) = \psi_j(N, v, \mathscr{C}) - EO_j^{\alpha}(N, v, \mathscr{C}).$$

Similarly, fixing i of the left part in the above equation and summing over $j \in C_h$ of the right, we have

$$\sum_{j \in C_h} \left[\psi_i(N, v, \mathscr{C}) - EO_i^{\alpha}(N, v, \mathscr{C})\right] = \sum_{j \in C_h} \left[\psi_j(N, v, \mathscr{C}) - EO_j^{\alpha}(N, v, \mathscr{C})\right],$$

then,

$$|C_h| \left[\psi_i(N, v, \mathscr{C}) - EO_i^{\alpha}(N, v, \mathscr{C})\right] = \sum_{j \in C_h} \psi_j(N, v, \mathscr{C}) - \sum_{j \in C_h} EO_j^{\alpha}(N, v, \mathscr{C}).$$

By Eq. (3.6), there is $\psi_i(N, v, \mathscr{C}) = EO_i^{\alpha}(N, v, \mathscr{C})$ for each $i \in C_h$. Combining with the random selection of C_h, this completes the proof. □

3.4 The guarantee potential function

The concept of a potential function was introduced into cooperative game theory by Hart and Mas-Collel [63] to characterize the Shapley value. A potential function here is a map which assigns every TU-game a real number. The total amount of all players' marginal contributions (according to the potential function) is the worth of the grand coalition. Moreover, the marginal contribution of each player mentioned is in accordance with his Shapley payoff. Joosten [71] introduced a family of potentials depending on a tuple (a, b, α), and associated with an (a, b, α)-potential a unique efficient and linear value. Naumova [91] provided a generalized potential with respect to the family of functions that determines the efficient consistent value. Driessen and Radzik [45] developed a unified pseudo-potential approach for efficient values. Recently, Xu et al. [120] proposed the A-potential function to characterize the solidarity value.

Winter [119] generalized the idea of potential function to games with coalition structures, fulfilling the characterization of the Owen value. In order to characterize the α-egalitarian Owen value, we propose the α-guarantee potential function with an adjustment to the one proposed by Winter.

Let $P : \mathscr{C}\mathscr{G} \to \mathbb{R}^m$ be a function. It maps every TU-game with a coalition structure $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$ to a vector whose dimension is identical to the number of corresponding unions contained in the coalition structure. Then for every $C_h \in \mathscr{C}$ and $i \in C_h$, the marginal contribution of player i (according to the potential) is

$$D^i P(N, v, \mathscr{C}) = P_h(N, v, \mathscr{C}) - P_h(N \setminus i, v_{|N\setminus i}, \mathscr{C}_{|N\setminus i}). \tag{3.10}$$

The function P, with $P_h(N \setminus C_h, v_{|N\setminus C_h}, \mathscr{C}_{|N\setminus C_h}) = 0$, is considered as a *potential function* for games with coalition structures [119] if it satisfies

$$\sum_{i\in N} D^i P(N, v, \mathscr{C}) = v(N),$$
$$\sum_{i\in C_h} D^i P(N, v, \mathscr{C}) = D^h P(M, v^{\mathscr{C}}),$$

where $D^h P(M, v^{\mathscr{C}})$ is the player h's marginal contribution (according to the potential defined by Hart and Mas-Collel [63]) to the quotient game $(M, v^{\mathscr{C}})$, and the second equation holds for every $C_h \in \mathscr{C}$.

Obviously, the potential function approach is designed to concentrate on the performance-based evaluation. Since the distribution system with guarantee coefficient α argues that the α percent of overall revenue will be used as basic interests for players, it is the remaining worth $(1 - \alpha)v(N)$ that can be regulated among players based on their performance. When the distribution process reaches the point of distributing coalitional payoff among players within unions, there is only $(1-\alpha)D^h P(M, v^{\mathscr{C}})$ left to coalitional members for performance-based allocation. Then, we introduce a modified potential approach as follows.

Definition 3.2. Given $\alpha \in [0,1]$, for any $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, a function P^* : $\mathscr{C}\mathscr{G} \to \mathbb{R}^m$ is an α-guarantee potential function if it satisfies

$$\sum_{i \in N} D^i P^*(N, v, \mathscr{C}) = (1-\alpha)v(N), \tag{3.11}$$

$$\sum_{i \in C_h} D^i P^*(N, v, \mathscr{C}) = (1-\alpha)D^h P(M, v^{\mathscr{C}}), \tag{3.12}$$

where Eq. (3.12) holds for every $C_h \in \mathscr{C}$.

From the above definition, for any $C_h \in \mathscr{C}$ and $T \in \{(N \setminus C_h) \cup S | S \subseteq C_h\}$, the subgame $(T, v_{|T}, \mathscr{C}_{|T})$ satisfies

$$\sum_{i \in T \cap C_h} D^i P^*(T, v_{|T}, \mathscr{C}_{|T}) = (1-\alpha)D^h P(M, v^{\mathscr{C}_{|T}}).$$

By Eq. (3.10), we have

$$\sum_{i \in T \cap C_h} [P_h^*(T, v_{|T}, \mathscr{C}_{|T}) - P_h^*(T \setminus i, v_{|T\setminus i}, \mathscr{C}_{|T \setminus i})] = (1-\alpha)D^h P(M, v^{\mathscr{C}_{|T}}).$$

Therefore, we can get a recursive definition:

$$P_h^*(T, v_{|T}, \mathscr{C}_{|T}) = \frac{1}{|T \cap C_h|}[(1-\alpha)D^h P(M, v^{\mathscr{C}_{|T}}) + \sum_{i \in T \cap C_h} P_h^*(T \setminus i, v_{|T\setminus i}, \mathscr{C}_{|T \setminus i})] \tag{3.13}$$

for all $T \in \{(N \setminus C_h) \cup S | S \subseteq C_h\}$, and $P_h^*(N \setminus C_h, v_{|N\setminus C_h}, \mathscr{C}_{|N \setminus C_h}) = 0$.

Proposition 3.2. *Given $\alpha \in [0,1]$, for any $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$ and $C_h \in \mathscr{C}$, then*

$$\begin{aligned} P_h^*(N, v, \mathscr{C}) = &\sum_{Q \subseteq M \setminus h} \sum_{\emptyset \neq S \subseteq C_h} (1-\alpha) a_m^q b_{|C_h|}^{|S|} v(\cup_{t \in Q} C_t \cup S) - \\ &\sum_{Q \subseteq M \setminus h} \sum_{p=1}^{|C_h|} \frac{(1-\alpha)}{p} a_m^q v(\cup_{t \in Q} C_t) \end{aligned} \tag{3.14}$$

where $m = |M|$, $q = |Q|$, *and* $a_m^q = \frac{q!(m-q-1)!}{m!}$, $b_{|C_h|}^{|S|} = \frac{(|S|-1)!(|C_h|-|S|)!}{|C_h|!}$.

Proof. Let us prove it by induction. It is obvious that $P_h^*(N\backslash C_h, v_{|N\backslash C_h}, \mathscr{C}_{|N\backslash C_h}) = 0$ holds for all $C_h \in \mathscr{C}$. Assume that for $i \in C_h \in \mathscr{C}$,

$$P_h^*(N \setminus i, v_{|N\setminus i}, \mathscr{C}_{|N\setminus i}) = \sum_{Q\subseteq M\backslash h}\sum_{\emptyset\neq S\subseteq\{C_h\backslash i\}} (1-\alpha)a_m^q b_{|C_h|-1}^{|S|} v(\cup_{t\in Q}C_t \cup S) - \sum_{Q\subseteq M\backslash h}\sum_{p=1}^{|C_h|-1} \frac{(1-\alpha)}{p} a_m^q v(\cup_{t\in Q}C_t).$$

By Eq. (3.13) and the fact that $D^h P(M, v^{\mathscr{C}}) = Sh_h(M, v^{\mathscr{C}})$, we have

$$\begin{aligned}
&P_h^*(N, v, \mathscr{C})\\
=&\frac{1}{|C_h|}\left[(1-\alpha)D^h P(M, v^{\mathscr{C}}) + \sum_{i\in C_h} P_h^*(N\setminus i, v_{|N\setminus i}, \mathscr{C}_{|N\setminus i})\right]\\
=&\frac{1}{|C_h|}\sum_{Q\subseteq M\backslash h}(1-\alpha)a_m^q\left[v(\cup_{t\in Q}C_t\cup C_h) - v(\cup_{t\in Q}C_t)\right] + \frac{1}{|C_h|}\sum_{i\in C_h}\Bigg[\sum_{Q\subseteq M\backslash h}\\
&\sum_{\emptyset\neq S\subseteq\{C_h\backslash i\}}(1-\alpha)a_m^q b_{|C_h|-1}^{|S|}v(\cup_{t\in Q}C_t\cup S) - \sum_{Q\subseteq M\backslash h}\sum_{p=1}^{|C_h|-1}\frac{(1-\alpha)}{p}a_m^q v(\cup_{t\in Q}C_t)\Bigg]\\
=&\sum_{Q\subseteq M\backslash h}(1-\alpha)a_m^q\left[\frac{1}{|C_h|}v(\cup_{t\in Q}C_t\cup C_h) + \sum_{i\in C_h}\sum_{\emptyset\neq S\subseteq\{C_h\backslash i\}}\frac{1}{|C_h|}b_{|C_h|-1}^{|S|}v(\cup_{t\in Q}C_t\cup S)\right]\\
&-\sum_{Q\subseteq M\backslash h}(1-\alpha)a_m^q\left\{\frac{1}{|C_h|} + \frac{1}{|C_h|}\sum_{i\in C_h}\sum_{p=1}^{|C_h|-1}\frac{1}{p}\right\}v(\cup_{t\in Q}C_t)\\
=&\sum_{Q\subseteq M\backslash h}(1-\alpha)a_m^q\Bigg[b_{|C_h|}^{|C_h|}v(\cup_{t\in Q}C_t\cup C_h) + \sum_{0\neq|S|\leq|C_h|-1}\frac{1}{|C_h|}b_{|C_h|-1}^{|S|}(|C_h|-|S|)\\
&v(\cup_{t\in Q}C_t\cup S)\Bigg] - \sum_{Q\subseteq M\backslash h}(1-\alpha)a_m^q\sum_{p=1}^{|C_h|}\frac{1}{p}v(\cup_{t\in Q}C_t)
\end{aligned}$$

$$= \sum_{Q\subseteq M\setminus h}\sum_{\emptyset\neq S\subseteq C_h}(1-\alpha)a_m^q b_{|C_h|}^{|S|} v(\cup_{t\in Q}C_t\cup S) - \sum_{Q\subseteq M\setminus h}\sum_{p=1}^{|C_h|}\frac{(1-\alpha)}{p}a_m^q v(\cup_{t\in Q}C_t).$$

This completes the proof. □

From the aspect of marginal contributions, the modified potential reflects the payoff deviation brought by player i's performance. We call the sum of player i's marginal contribution according to the α-guarantee potential function and the fixed basic interests a supplementary marginal contribution for player i. Denote by $S^iP^*(N,v,\mathscr{C})$ the supplementary marginal contribution for $i\in C_h\in\mathscr{C}$, which is given by

$$S^iP^*(N,v,\mathscr{C}) = D^iP^*(N,v,\mathscr{C}) + \frac{\alpha v(N)}{|\mathscr{C}||C_h|}. \tag{3.15}$$

Next, we show that the vector of supplementary marginal contributions coincides with the α-egalitarian Owen value.

Theorem 3.3. *Given $\alpha\in[0,1]$, there exists a unique α-guarantee potential function P^* on $\mathscr{C}\mathscr{G}$. The payoff vector $SP^*(N,v,\mathscr{C}) = (S_iP^*(N,v,\mathscr{C}))_{i\in N}$ identifies with the α-egalitarian Owen value $EO^\alpha(N,v,\mathscr{C})$ for all $(N,v,\mathscr{C})\in\mathscr{C}\mathscr{G}$.*

Proof. The existence and uniqueness of the α-guarantee potential function can be easily checked by Eq. (3.14). Therefore, in order to prove $SP^*(N,v,\mathscr{C}) = EO^\alpha(N,v,\mathscr{C})$, it is only left to verify that all axioms satisfied by the α-egalitarian Owen value are also met by SP^*.

Firstly, efficiency of SP^* is just derived from the definition. As for additivity, it is easily obtained by Eq. (3.14). Because there is $P^*(N,v_1+v_2,\mathscr{C}) = P^*(N,v_1,\mathscr{C}) + P^*(N,v_2,\mathscr{C})$ for $(N,v_1,\mathscr{C}),(N,v_2,\mathscr{C})\in\mathscr{C}\mathscr{G}$. The result is straightforward, and we omit details of its proof.

Then, given two intracoalitional symmetric players $i,j\in C_h\in\mathscr{C}$, we show that there is $P_h^*(T\setminus i, v_{|T\setminus i},\mathscr{C}_{|T\setminus i}) = P_h^*(T\setminus j, v_{|T\setminus j},\mathscr{C}_{|T\setminus j})$ for every $T\subseteq N$ and $i,j\in T$. When $T=\{i,j\}$, by Eq. (3.11) and Eq. (3.12), the conclusion holds.

Now we assume that it also applies to all subgames with $2 \leq |T| \leq n-1$, i.e.,

$$\begin{aligned}
&P_h^*(N \setminus i, v_{|N\setminus i}, \mathscr{C}_{|N\setminus i}) - P_h^*(N \setminus j, v_{|N\setminus j}, \mathscr{C}_{|N\setminus j}) \\
&= \frac{1}{|C_h|-1}\left[(1-\alpha)D^h P(M, v^{\mathscr{C}_{|N\setminus i}}) + \sum_{k\in C_h\setminus i} P_h^*(N \setminus \{i,k\}, v_{|N\setminus\{i,k\}}, \mathscr{C}_{|N\setminus\{i,k\}})\right] \\
&- \frac{1}{|C_h|-1}\left[(1-\alpha)D^h P(M, v^{\mathscr{C}_{|N\setminus j}}) + \sum_{k\in C_h\setminus j} P_h^*(N \setminus \{j,k\}, v_{|N\setminus\{j,k\}}, \mathscr{C}_{|N\setminus\{j,k\}})\right] \\
&= \frac{(1-\alpha)}{|C_h|-1}\left[D^h P(M, v^{\mathscr{C}_{|N\setminus i}}) - D^h P(M, v^{\mathscr{C}_{|N\setminus j}})\right] = 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
S^i P^*(N, v, \mathscr{C}) =& D^i P^*(N, v, \mathscr{C}) + \frac{\alpha v(N)}{|\mathscr{C}||C_h|} \\
=& D^j P^*(N, v, \mathscr{C}) + \frac{\alpha v(N)}{|\mathscr{C}||C_h|} \\
=& S^j P^*(N, v, \mathscr{C}).
\end{aligned}$$

So the intracoalitional symmetry axiom is satisfied by SP^* as well.

Now, we pay attention to the α-indemnificatory null player axiom. We show that this axiom holds for SP^* by induction on the cardinality of C_h containing null player $i \in N$. Given a coalition structure $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ and null player $i \in C_h \subseteq N$, if $|C_h| = 1$, we have

$$S^i P^*(\{N \setminus C_h\} \cup i, v_{|\{N\setminus C_h\}\cup i}, \mathscr{C}_{|\{N\setminus C_h\}\cup i}) = \frac{\alpha v(\{N \setminus C_h\} \cup i)}{|\mathscr{C}|}$$

due to $P_h^*(N \setminus C_h, v_{|N\setminus C_h}, \mathscr{C}_{|N\setminus C_h}) = 0$. Assume that $S^i P^*(T, v_{|T}, \mathscr{C}_{|T}) = \frac{\alpha v(T)}{|\mathscr{C}||T\cap C_h|}$ applies to all subgames $(T, v_{|T}, \mathscr{C}_{|T})$ where $T \in \{(N \setminus C_h) \cup S | S \subsetneq C_h\}$. Thus, for all $j \in C_h \setminus i$, there is

$$P_h^*(N \setminus j, v_{|N\setminus j}, \mathscr{C}_{|N\setminus j}) - P_h^*(N \setminus \{i,j\}, v_{|N\setminus\{i,j\}}, \mathscr{C}_{|N\setminus\{i,j\}}) = 0,$$

then by Eq. (3.15), we have

$$
\begin{aligned}
&|C_h|S^iP^*(N,v,\mathscr{C})\\
=&|C_h|\left[P_h^*(N,v,\mathscr{C})-P_h^*\left(N\setminus i,v_{|N\setminus i},\mathscr{C}_{|N\setminus i}\right)+\frac{\alpha v(N)}{|\mathscr{C}||C_h|}\right]\\
=&|C_h|P_h^*(N,v,\mathscr{C})-(|C_h|-1)P_h^*\left(N\setminus i,v_{|N\setminus i},\mathscr{C}_{|N\setminus i}\right)\\
&-P_h^*\left(N\setminus i,v_{|N\setminus i},\mathscr{C}_{|N\setminus i}\right)+\frac{\alpha v(N)}{|\mathscr{C}|}\\
=&(1-\alpha)D^hP(M,v^{\mathscr{C}})+\sum_{k\in C_h}P_h^*\left(N\setminus k,v_{|N\setminus k},\mathscr{C}_{|N\setminus k}\right)-[(1-\alpha)D^hP(M,v^{\mathscr{C}_{|N\setminus i}})\\
&+\sum_{k\in C_h\setminus i}P_h^*\left(N\setminus\{i,k\},v_{|N\setminus\{i,k\}},\mathscr{C}_{|N\setminus\{i,k\}}\right)]\\
&-P_h^*\left(N\setminus i,v_{|N\setminus i},\mathscr{C}_{|N\setminus i}\right)+\frac{\alpha v(N)}{|\mathscr{C}|}\\
=&(1-\alpha)\left(D^hP(M,v^{\mathscr{C}})-D^hP(M,v^{\mathscr{C}_{|N\setminus i}})\right)+\sum_{k\in C_h\setminus i}\left[P_h^*(N\setminus k,v_{|N\setminus k},\mathscr{C}_{|N\setminus k})\right.\\
&\left.-P_h^*\left(N\setminus\{i,k\},v_{|N\setminus\{i,k\}},\mathscr{C}_{|N\setminus\{i,k\}}\right)\right]+\frac{\alpha v(N)}{|\mathscr{C}|}\\
=&(1-\alpha)\left(D^hP(M,v^{\mathscr{C}})-D^hP(M,v^{\mathscr{C}_{|N\setminus i}})\right)+\frac{\alpha v(N)}{|\mathscr{C}|}\\
=&\frac{\alpha v(N)}{|\mathscr{C}|}.
\end{aligned}
$$

Hence, there is $S^iP^*(N,v,\mathscr{C})=\frac{\alpha v(N)}{|\mathscr{C}||C_h|}$.

Ultimately, as for coalitional symmetry of SP^*, assume that two coalitions $C_h,C_r\in\mathscr{C}$ are symmetric coalitions, then player h and r are two symmetric players in the quotient game $(M,v^{\mathscr{C}})$. Then, by the symmetry axiom of the Shapley value and Eq. (3.12), we have

$$
\begin{aligned}
&\sum_{i\in C_h}S^iP^*(N,v,\mathscr{C})-\sum_{j\in C_r}S^jP^*(N,v,\mathscr{C})\\
=&\sum_{i\in C_h}\left[D^iP^*(N,v,\mathscr{C})+\frac{\alpha v(N)}{|\mathscr{C}||C_h|}\right]-\sum_{j\in C_r}\left[D^jP^*(N,v,\mathscr{C})+\frac{\alpha v(N)}{|\mathscr{C}||C_r|}\right]\\
=&(1-\alpha)D^hP(M,v^{\mathscr{C}})-(1-\alpha)D^rP(M,v^{\mathscr{C}})=0.
\end{aligned}
$$

We here conclude that $SP^*(N,v,\mathscr{C})$ is in accordance with the α-egalitarian Owen value. This completes the proof. □

3.5 The punishment-reward bidding mechanism

Implementation is an efficient way to associate cooperative games with the non-cooperative theory. Specifically, it uses a non-cooperative approach to characterize solutions for cooperative games. Several kinds of implementations have been proposed to characterize the Shapley value, see [60, 64, 97]. Especially, the bidding mechanism introduced by Pérez-Castrillo and Wettstein [97] is widely generalized. For example, van den Brink et al. [110] realized the implementation of the α-egalitarian Shapley value, and Vidal-Puga and Bergantiños [116] implemented the Owen value by applying this mechanism to games with coalition structures. In this section, we embed a punishment-reward bidding principle to the one suggested by Vidal-Puga and Bergantiños [116], by which we implement the α-egalitarian Owen value.

Firstly, we focus on a characteristic satisfied by the α-egalitarian Owen value. It mainly reflects how the total payoff of a union will change when one player of this union leaves the game.

Proposition 3.3. *Given $\alpha \in [0,1]$ and a strictly superadditive game $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$ such that $j \in C_h \in \mathscr{C}$, $\{j\} \neq C_h$, then*

$$\sum_{i\in C_h} EO_i^{\alpha}(N,v,\mathscr{C}) > \sum_{i\in C_h\setminus j} EO_i^{\alpha}(N\setminus j, v_{|N\setminus j}, \mathscr{C}_{|N\setminus j}) + \alpha v(j). \tag{3.16}$$

Proof. The Owen value satisfies the following inequality (see the Proposition 1 in [116]),

$$\sum_{i\in C_h} Ow_i(N,v,\mathscr{C}) \geq \sum_{i\in C_h\setminus j} Ow_i(N\setminus j, v_{|N\setminus j}, \mathscr{C}_{|N\setminus j}) + v(j). \tag{3.17}$$

Obviously,

$$\sum_{i\in C_h} EO_i^{\alpha}(N,v,\mathscr{C}) = \sum_{i\in C_h}(1-\alpha)Ow_i(N,v,\mathscr{C}) + \alpha\frac{v(N)}{|\mathscr{C}|}$$

$$\sum_{i\in C_h\setminus j} EO_i^{\alpha}(N\setminus j, v_{|N\setminus j}, \mathscr{C}_{|N\setminus j}) = \sum_{i\in C_h\setminus j}(1-\alpha)Ow_i(N\setminus j, v_{|N\setminus j}, \mathscr{C}_{|N\setminus j}) + \alpha\frac{v(N\setminus j)}{|\mathscr{C}|}.$$

With the strict superadditivity of v, we have $\alpha\frac{v(N)}{|\mathscr{C}|} > \alpha\frac{v(N\setminus j)}{|\mathscr{C}|}$, the Eq. (3.16) holds. This completes the proof. □

Next, we elaborate the bidding mechanism being designed to implement the α-egalitarian Owen value. It is noteworthy that we perform our analysis within the class of strictly superadditive games.

The punishment-reward bidding mechanism. When the player set consists of only one player, he receives $v(i)$. When there is more than one player, the mechanism is defined recursively. Now, we assume that the regulations of the mechanism which is played by at most $n-1$ players have been known. Then given a set of players $N = \{1, 2, \cdots, n\}$ and a coalition structure $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$, the modified bidding mechanism with respect to a given guarantee coefficient α proceeds as follows:

(1) **Round 1.** In this round, players of any union $C_h \in \mathscr{C}$ play the bidding mechanism in order to obtain the resources of C_h. Generally, when there is only one player in this union, then he owns his resources. Assume that the rules for a union with $|C_h| - 1$ players have been known, then we consider rules in the case that the union has $|C_h|$ players.

Stage 1 Each player $i \in C_h$ makes bids $b_j^i \in \mathbb{R}$ to every player $j \in C_h \setminus i$. The net bid is defined by $B^i = \sum_{j\in C_h\setminus i}(b_j^i - b_i^j)$. Denote by $\delta_h = \arg\max_{i\in C_h}\{B^i\}$. The player with the maximum net bid will be the proposer in the next stage (*if there are several players having the maximum net bid, the proposer will be selected with equal probability among them*). Then the selected proposer must pay his bid to each player $j \in C_h \setminus \delta_h$.

Stage 2 As a result of being chosen as the proposer, the player δ_h gives the amount $a_j^{\delta_h} = \frac{\alpha(v(N\backslash j)-v(N\backslash\{\delta_h,j\}))}{|\mathscr{C}|(|C_h|-1)}$ to each player $j \in C_h \setminus \{\delta_h\}$ in return. In addition, the proposer makes offers $p_j^{\delta_h} \in \mathbb{R}$ to every player $j \in C_h \setminus \delta_h$.

Stage 3 Each player in $C_h \setminus \delta_h$ determines whether or not to accept the offer sequently. The offer is accepted only when all of the union's players accept it while it is rejected if someone rejects it.

With no confusion, we assume that the unions of $\mathscr{C}$ play in turn in the order $C_1, C_2, \ldots, C_m$ until we encounter a union C_{h_0} whose proposer's offer is rejected or the offer of δ_h is accepted for every $C_h \in \mathscr{C}$.

- For the union C_{h_0}, since the offer proposed by δ_{h_0} is rejected, the proposer is punished by the amount $(1-\alpha)v(\delta_{h_0})$ and leaves the game with the worth $\alpha v(\delta_{h_0}) - \sum_{j\in C_{h_0}\backslash\delta_{h_0}}(b_j^{\delta_{h_0}} + a_j^{\delta_{h_0}})$. All players other than δ_{h_0} proceed again the mechanism with $(N\backslash\delta_{h_0}, v_{|N\backslash\delta_{h_0}}, \mathscr{C}_{|N\backslash\delta_{h_0}})$.
- If for any $C_h \in \mathscr{C}$ the offer of δ_h is accepted, then the proposer δ_h has to pay each player $j \in C_h\backslash\delta_h$ his promised offer $p_j^{\delta_h}$ and steps into Round 2 with the available resources of $C_h \in \mathscr{C}$ as the union's representative. Any other player $j \in C_h\backslash\delta_h$ gets the final payoff $a_j^{\delta_h} + b_j^{\delta_h} + p_j^{\delta_h}$ and leaves the game. The payoff of δ_h in this round is denoted by $x^1_{\delta_h}$.

At the end of Round 1, we can find a representative for each union $C_h \in \mathscr{C}$, denoted by r_h. Moreover, notice that some proposers in the previous bidding may be removed because of the rejection to their offer. We denote by C_h^r the set of players whose resources are obtained by r_h. Then, we have $C_h^r \subseteq C_h$ and $r_h \in C_h$.

(2) **Round 2.** The representatives play the bidding mechanism introduced by van den Brink et al. [110] to implement the α-egalitarian Shapley value of the game (N^r, v^r) where $N^r = \{r_1, r_2, \cdots, r_m\}$ and $v^r(S) = v(\bigcup_{r_h\in S} C_h^r)$ for all $S \subseteq N^r$. Similarly, we denote by $x^2_{r_h}$ the payoff obtained by the representative r_h. Therefore, the representative r_h's final payoff is the sum of payoffs obtained both in Rounds 1 and 2, i.e., $x^1_{\delta_h} + x^2_{r_h}$.

Theorem 3.4. *Given $\alpha \in [0,1]$ and a strictly superadditive game $(N, v, \mathscr{C}) \in \mathscr{CG}$, the outcome in any subgame perfect equilibrium of the punishment-reward bidding mechanism coincides with the payoff vector $EO^{\alpha}(N, v, \mathscr{C})$.*

Proof. The proof is similar to Theorem 1 in [116]. We restate the strategies (actions) and types of equilibrium that are different. We proceed by induction on the number of players. It is easy to check that the theorem holds for $n = 1$. Assume that it holds for at most $n-1$ players, and we show that it is satisfied by n players.

Firstly, we show that the α-egalitarian Owen value is a payoff of a SPE outcome. Let us consider the following strategies.

- **Round 1**. We firstly pay attention to the strategies in the bidding mechanism linked with each $C_h \in \mathscr{C}$.
  - ◇ At Stage 1. Each player $i \in C_h$ bids $b_j^i = EO_j^{\alpha}(N, v, \mathscr{C}) - EO_j^{\alpha}(N \setminus i, v_{|N\setminus i}, \mathscr{C}_{|N\setminus i}) - \frac{\alpha v(N\setminus j)}{|\mathscr{C}|(|C_h|-1)} + \frac{\alpha v(N\setminus\{ij\})}{|\mathscr{C}|(|C_h|-1)}$.
  - ◇ At Stage 2. As for the return part, it is a common promissory information to all players playing the game. So the proposer must pay the amount defined before. In addition, the proposer δ_h offers $p_j^{\delta_h} = EO_j^{\alpha}(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$ to every player $j \in C_h \setminus \delta_h$.
  - ◇ At Stage 3. Each player $j \in C_h \setminus \delta_h$, accepts any offer $p_j^{\delta_h} \geq EO_j^{\alpha}(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$ and rejects the offer otherwise.
- **Round 2**. We assume that the strategies of players in N^r are in accordance with the ones introduced by van den Brink et al. [110]. These strategies constitute a SPE and the corresponding outcome coincides with the α-egalitarian Shapley value of the game (N^r, v^r).

Based on the strategies above, the game ends in the acceptance at Stage 3 of Round 1 for each union $C_h \in \mathscr{C}$. Accordingly, so is Round 2. Therefore, for any player who is not the proposer at Round 1, denoted by $j \in C_h \in \mathscr{C}$, he will get his own α-egalitarian Owen value due to the amount $a_j^{\delta_h} + b_j^{\delta_h} + p_j^{\delta_h} = EO_j^{\alpha}(N, v, \mathscr{C})$. For the proposer δ_h who becomes the representative r_h

at Round 2, by van den Brink et al. [110], the payoff $x^2_{r_h}$ coincides with his α-egalitarian Shapley value of the game (N^r, v^r). Then the player δ_h also gets his α-egalitarian Owen value because

$$\begin{aligned}
x^1_{\delta_h} + x^2_{r_h} &= \sum_{j\in C_h\setminus\delta_h} -(a_j^{\delta_h} + b_j^{\delta_h} + p_j^{\delta_h}) + Sh^\alpha_h(N^r, v^r) \\
&= \sum_{j\in C_h\setminus\delta_h} -EO^\alpha_j(N, v, \mathscr{C}) + \left((1-\alpha)Sh_h(N^r, v^r) + \alpha\frac{v^r(N^r)}{|N^r|}\right) \\
&= \sum_{j\in C_h\setminus\delta_h} -\left((1-\alpha)Ow_j(N, v, \mathscr{C}) + \alpha\frac{v(N)}{m|C_h|}\right) \\
&\quad + \left((1-\alpha)\sum_{j\in C_h} Ow_j(N, v, \mathscr{C}) + \alpha\frac{v(N)}{m}\right) \\
&=(1-\alpha)Ow_{\delta_h}(N, v, \mathscr{C}) + \alpha\frac{v(N)}{m|C_h|} \\
&=EO^\alpha_{\delta_h}(N, v, \mathscr{C}).
\end{aligned}$$

Now we prove that the previous strategies constitute a SPE. By the strict superadditivity of game (N, v), the game (N^r, v^r) is strictly zero-monotonic. Therefore, according to van den Brink et al. [110], we draw the conclusion that the strategies of the subgame obtained after Round 2 induce a SPE. It remains only to prove that strategies in Round 1 also induce a SPE. For this purpose, it is sufficient to show the strategies of the corresponding players at each stage of Round 1 are their best responses.

Firstly, notice that the strategies at Stage 3 are best responses. In case of rejection, assume that player $i \in C_h \setminus \delta_h$ rejects the offer, then the remaining players $N \setminus \delta_h$ continuously play the uncompleted bidding mechanism. By the induction hypothesis, we have the α-egalitarian Owen value as the outcome of the subgame $(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$. Therefore, each player $j \in C_h \setminus \delta_h$ accepts any offer that is larger than or equal to $EO^\alpha_j(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$ and rejects the offer otherwise.

Then we verify that the strategies at Stage 2 are best responses. According to the strategies of players at Stage 3, if the proposer δ_h wants his proposal to be accepted, he will make an offer which is larger than or equal to $EO_j^\alpha(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$. Hence, when the proposer makes the offer $p_j^{\delta_h} = EO_j^\alpha(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$, he will be accepted at Stage 3. Then he can get a surplus at this stage which is

$$\begin{aligned} & x_{r_h}^2 - \sum_{j\in C_h\setminus\delta_h} p_j^{\delta_h} - \sum_{j\in C_h\setminus\delta_h} a_j^{\delta_h} \\ & = Sh_h^\alpha(N^r, v^r) - \sum_{j\in C_h\setminus\delta_h} EO_j^\alpha(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h}) - \sum_{j\in C_h\setminus\delta_h} a_j^{\delta_h} \\ & = \sum_{j\in C_h} EO_j^\alpha(N, v, \mathscr{C}) - \sum_{j\in C_h\setminus\delta_h} EO_j^\alpha(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h}) - \sum_{j\in C_h\setminus\delta_h} a_j^{\delta_h}. \end{aligned} \tag{3.18}$$

If he makes an offer $p_j^{\delta_h}$ that is less than $EO_j^\alpha(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$ to a player $j \in C_h \setminus \delta_h$, he will be rejected and obtain the worth $\alpha v(\delta_h) - \sum_{j\in C_h\setminus\delta_h} a_j^{\delta_h}$, which is strictly worse by Eq. (3.16). However, if he provides an offer such that $p_j^{\delta_h} > EO_j^\alpha(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$ to every player $j \in C_h \setminus \delta_h$, he will be accepted definitely and obtain the resources of this union, but his final payoff will decrease. Therefore, the referred strategies at Stage 2 are best responses.

As for the strategies at Stage 1, note that all net bids are zero by the intra-coalitional quasi-balanced contributions with respect to α of the α-egalitarian Owen value.

$$\begin{aligned} B^i = & \sum_{j\in C_h\setminus i} (b_j^i - b_i^j) \\ = & \sum_{j\in C_h\setminus i} \Bigg[\left(EO_j^\alpha(N, v, \mathscr{C}) - EO_j^\alpha(N \setminus i, v_{|N\setminus i}, \mathscr{C}_{|N\setminus i}) - \frac{\alpha v(N \setminus j)}{|\mathscr{C}|(|C_h| - 1)} \right) \\ & - \left(EO_i^\alpha(N, v, \mathscr{C}) - EO_i^\alpha(N \setminus j, v_{|N\setminus j}, \mathscr{C}_{|N\setminus j}) - \frac{\alpha v(N \setminus i)}{|\mathscr{C}|(|C_h| - 1)} \right) \Bigg] \\ = & 0. \end{aligned} \tag{3.19}$$

If the player δ_h increases his bid to other players, he will become the proposer. However, he must pay for more, which results in the decrease of his final payoff.

If he reduces his total bid, the proposer will be another player and his final payoff will not be improved.

As for the reverse part, we prove that any SPE yields the α-egalitarian Owen value by a series of claims which have been proved to be similar to those in [116]. We restate here for clarity.

- Claim 1. In any SPE, for each $C_h \in \mathscr{C}$, each player $j \in C_h \setminus \delta_h$ accepts the offer $p_j^{\delta_h} > EO_j^{\alpha}(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$. Otherwise, the offer $p_j^{\delta_h} < EO_j^{\alpha}(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$ will be rejected.

  Without loss of generality, we assume that the decision node reaches players of the union C_h in the sequence $\{i_1, i_2, \cdots, i_{|C_h|}\}$. In the case of rejection, every player $j \neq \delta_h$ in the union gets $EO_j^{\alpha}(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$. Now, assume that it is the last player $i_{|C_h|}$'s turn to make a decision. There is no doubt that his optimal strategy refers to accepting an offer $p_{i_{|C_h|}}^{\delta_h} > EO_{i_{|C_h|}}^{\alpha}(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$ and rejecting an offer $p_{i_{|C_h|}}^{\delta_h} < EO_{i_{|C_h|}}^{\alpha}(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$. As for the penultimate player $i_{|C_h|-1}$, he makes his decision with anticipating the last player's strategies. If $p_{i_{|C_h|-1}}^{\delta_h} > EO_{i_{|C_h|-1}}^{\alpha}(N\setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$, he will accept the offer whatever the last player's strategy is. Because if $p_{i_{|C_h|}}^{\delta_h} > EO_{i_{|C_h|}}^{\alpha}(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$, then the last player will accept the offer, and he will accept the offer as well. If the offer $p_{i_{|C_h|}}^{\delta_h} < EO_{i_{|C_h|}}^{\alpha}(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$, the last player will reject the offer definitely, so whether he accepts it or not, makes no difference to the final result. Similarly, the player $i_{|C_h|-1}$ will reject an offer $p_{i_{|C_h|-1}}^{\delta_h} < EO_{i_{|C_h|-1}}^{\alpha}(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$. Iterating the same argument backward, Claim 1 is true.

- Claim 2. In any SPE, every player $j \in C_h \setminus \delta_h$ accepts the offer of proposer δ_h.

  To the contrary, assume that there is one player $k \in C_h \setminus \delta_h$ who rejects the offer $p_k^{\delta_h}$ at a SPE outcome. Then, the proposer δ_h gets the payoff $\alpha v(\delta_h)$. However, given $\varepsilon > 0$, if the proposer δ_h makes an offer $q_j^{\delta_h} = EO_j^{\alpha}(N \setminus \delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h}) + \varepsilon$ to every player $j \in C_h \setminus \delta_h$, by Claim 1, player j will accept $q_j^{\delta_h}$. Thus, the proposer δ_h will become the representative of union C_h and continue the mechanism at Round 2. In Round 2, representatives of all unions will play a SPE of the bidding mechanism with respect to (N^r, v^r), by van den Brink et al. [110], the payoff of δ_h obtained in Round 2 is

$$(1-\alpha)Sh_h(N^r, v^r) + \alpha\frac{v^r(M)}{m} = \sum_{i\in C_h} EO_i^\alpha(N, v, \mathscr{C}).$$

So after he pays his offer to every $j \in C_h \setminus \delta_h$, he obtains

$$\begin{aligned}&\sum_{j\in C_h} EO_j^\alpha(N, v, \mathscr{C}) - \sum_{j\in C_h\setminus\delta_h} \left[EO_j^\alpha(N\setminus\delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h}) + \varepsilon\right]\\ =&\sum_{j\in C_h} EO_j^\alpha(N, v, \mathscr{C}) - \sum_{j\in C_h\setminus\delta_h} EO_j^\alpha(N\setminus\delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h}) - (|C_h|-1)\varepsilon.\end{aligned}$$

By Eq. (3.16), we have

$$\beta = \sum_{j\in C_h} EO_j^\alpha(N, v, \mathscr{C}) - \sum_{j\in C_h\setminus\delta_h} EO_j^\alpha(N\setminus\delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h}) - \alpha v(\delta_h) > 0.$$

Hence, we allow $0 < \varepsilon < \frac{\beta}{|C_h|-1}$, then the payoff of the proposer δ_h will be improved by offering $q_j^{\delta_h}$ to every player $j \in C_h\setminus\delta_h$, which contradicts to the SPE strategy of proposer δ_h, i.e., p^{δ_h}.

- Claim 3. In any SPE, for every $C_h \in \mathscr{C}$, the offer of proposer δ_h is $EO_j^\alpha(N\setminus\delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$ for every player $j \in C_h \setminus \delta_h$.

  At a SPE, given p^{δ_h} be an offer of the proposer δ_h, then by Claim 1 and Claim 2, the offer p^{δ_h} must be accepted by every player $j \in C_h \setminus \delta_h$, and there is $p_j^{\delta_h} \geq EO_j^\alpha(N\setminus\delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$. If there exists one player j_0, the proposer δ_h provides him with the offer $p_{j_0}^{\delta_h} > EO_{j_0}^\alpha(N\setminus\delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$. For convenience, let $\gamma = p_{j_0}^{\delta_h} - EO_{j_0}^\alpha(N\setminus\delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h})$. Then the proposer can adjust slightly his offer to enhance his payoff, but his offer remains to be accepted. That is, he can increase his offer by an increment $\frac{\gamma}{|C_h|}$ to every player $j \in C_h \setminus \delta_h$. Then, his offer becomes $q_j^{\delta_h} = EO_j^\alpha(N\setminus\delta_h, v_{|N\setminus\delta_h}, \mathscr{C}_{|N\setminus\delta_h}) + \frac{\gamma}{|C_h|}$, by Claim 1, other players in this union will accept this offer as well. But there is $\sum_{j\in C_h\setminus\delta_h} q_j^{\delta_h} < \sum_{j\in C_h\setminus\delta_h} p_j^{\delta_h}$. That is to say, the offer p^{δ_h} can not be a SPE strategy of proposer δ_h, which is a contradiction.

- Claim 4. In any SPE, at Stage 1 of Round 1, for every $C_h \in \mathscr{C}$, $B^i = B^j$ for all $i, j \in C_h$, and B^i=0 for each $i \in C_h$.

  By convention, the set of players with maximum net bid is recorded as $\Omega = \{i \in C_h \mid B^i = \max_j B^j\}$. If $\Omega = C_h$, the claim holds because $\sum_{i \in C_h} B^i = 0$. Otherwise, we can randomly select a player $j \in C_h \backslash \Omega$ and a player $i \in \Omega$. Denote $0 < \beta = B^i - B^j$. Then assume that the player i makes a new bid $\hat{b}$ to other players in $C_h \backslash i$, i.e., $\hat{b}^i_k = b^i_k + \delta$ for $k \in \Omega \backslash i$, $\hat{b}^i_j = b^i_j - |\Omega|\delta$ and $\hat{b}^i_k = b^i_k$ for $k \in C_h \backslash \{\Omega \cup j\}$. Correspondingly, the new net bids become $\hat{B}^k = B^k - \delta$ for $k \in \Omega$, $\hat{B}^j = B^j + |\Omega|\delta$ and $\hat{B}^k = B^k$ for $k \in C_h \backslash \{\Omega \cup j\}$. Because $\beta > 0$, there exists a real number $0 < \delta < \beta$ satisfying $B^j + |\Omega|\delta < B^i - \delta$. However, it turns out that Ω remains the same. That is, the player i has the same probability of being the representative, but with a higher expected payoff. This is impossible at a SPE.

- Claim 5. In any SPE, at Stage 1 of Round 1, and for every $C_h \in \mathscr{C}$, the final payoff of each player $i \in C_h$ is the same no matter who is chosen as the proposer.

  By Claim 4, all net bids B^i are equal to zero. If player $i \in C_h$ has a strong desire to become the proposer, he can change the payoff by increasing slightly his bid b^i_j. In addition, if player i would strictly prefer that the proposer will be someone else in this union, he can decrease his bid b^i_j. However, both cases will not happen in a SPE.

- Claim 6. In any SPE, each player's final payoff is in accordance with his α-egalitarian Owen value.

  For every player $i \in C_h \subseteq N$, his final payoff is denoted by x_i. On the one hand, if player i is the proposer, his final payoff is

$$\sum_{j \in C_h} EO^\alpha_j(N, v, \mathscr{C}) - \sum_{j \in C_h \backslash i} EO^\alpha_j(N \setminus i, v_{|N \backslash i}, \mathscr{C}_{|N \backslash i}) - \sum_{j \in C_h \backslash i} \left(a^i_j + b^i_j \right).$$

On the other hand, if the proposer is $j \in C_h \setminus i$, then his payoff is given by $a_i^j + b_i^j + EO_i^\alpha(N \setminus j, v_{|N\setminus j}, \mathscr{C}_{|N\setminus j})$. By Claim 4 and Claim 5, we have

$$
\begin{aligned}
|C_h|x_i = & \sum_{j\in C_h} EO_j^\alpha(N,v,\mathscr{C}) - \sum_{j\in C_h\setminus i} EO_j^\alpha(N\setminus i, v_{|N\setminus i}, \mathscr{C}_{|N\setminus i}) - \\
& \sum_{j\in C_h\setminus i}\left(a_j^i + b_j^i\right) + \sum_{j\in C_h\setminus i}\left(a_i^j + b_i^j + EO_i^\alpha(N\setminus j, v_{|N\setminus j}, \mathscr{C}_{|N\setminus j})\right) \\
= & \sum_{j\in C_h} EO_j^\alpha(N,v,\mathscr{C}) + \sum_{j\in C_h\setminus i} (EO_i^\alpha(N\setminus j, v_{|N\setminus j}, \mathscr{C}_{|N\setminus j}) - \\
& EO_j^\alpha(N\setminus i, v_{|N\setminus i}, \mathscr{C}_{|N\setminus i}) - \frac{\alpha v(N\setminus j)}{|\mathscr{C}|(|C_h|-1)} + \frac{\alpha v(N\setminus i)}{|\mathscr{C}|(|C_h|-1)}) \\
= & \sum_{j\in C_h} EO_j^\alpha(N,v,\mathscr{C}) + \sum_{j\in C_h\setminus i}\left[EO_i^\alpha(N,v,\mathscr{C}) - EO_j^\alpha(N,v,\mathscr{C})\right] \\
= & |C_h| EO_i^\alpha(N,v,\mathscr{C}).
\end{aligned}
$$

The penultimate equation is due to the intracoalitional quasi-balanced contributions with respect to α of the α-egalitarian Owen value. Thus, there is $x_i = EO_i^\alpha(N,v,\mathscr{C})$ for all $i \in N$. This completes the proof.

□

The main differences between the punishment-reward bidding mechanism and the mechanism suggested by Vidal-Puga and Bergantiños [116] lie in three key points. Firstly, our mechanism gives each player a reward offered by the corresponding proposer and this is a common knowledge. Secondly, at Round 1, one proposer i whose offer is rejected by his union's members will be punished by $1-\alpha$ percent of his singleton worth $v(i)$, and thereby he will receive $\alpha v(i)$. Finally, the bidding mechanism that we use at Round 2 is the one which implements the α-egalitarian Shapley value instead of the Shapley value.

3.6 Conclusions

We use convex combinations of the Owen value and the coalitional equal division value to reach a compromise between marginalism and egalitarianism. In our two axiomatizations for the α-egalitarian Owen value, the guarantee coefficient α is involved in the axioms introduced in this chapter. It would be interesting to come up with other axioms which do not depend on α. Especially, for the axiomatization with additivity, we introduce the α-indemnificatory null player axiom which requires to assign "certain" payoffs to null players. Note that Casajus and Huettner [36] suggested the so-called null player in a productive environment (NPE) axiom which requires a null player to obtain a non-negative payoff whenever the worth generated by the grand coalition is non-negative, and provided an alternative axiomatization for the α-egalitarian Shapley value. Hence, it could be worthwhile to consider an axiom following the spirit of NPE for coalitional values, which could become less demanding compared to the α-indemnificatory null player axiom and yield another axiomatization for the α-egalitarian Owen value.

Chapter 4

Two-step Shapley-solidarity Value for Cooperative Games with Coalition Structures

Pre-defined subgroups in a coalition structure entail that the cooperation among players can happen within a given subgroup, whilst cooperation outside the subgroups happens on the level of the subgroups themselves. In view of such a two-level cooperation structure, solutions for cooperative games with coalition structures are usually defined in two steps, as well. In this chapter, following a two-step approach suggested by Kamijo [72], we provide a new coalitional value, called the two-step Shapley-solidarity value. We will show its similarities and differences to other comparable values proposed earlier in the literature, in particular, the two-step Shapley value [72].

4.1 Introduction

Aumann and Drèze [9] assumed that there are no side payments between unions in a coalition structure, and defined the *Aumann-Drèze* (AD) value [9] which assigns every player his Shapley value of the subgame that he is playing within his union. Instead, Owen [94] interpreted the unions as "bargaining blocks" to distribute the worth of the grand coalition. The Owen value [94] is defined by taking two levels of interaction among players into account, first among unions and then within each union: First the unions get the Shapley value of the game played in the so-called quotient game, which is the game where the unions are the players. Then, to distribute the each union's Shapley payoff over its players, Owen defined an *induced internal game* in which he considered the worth of a coalition $S \subseteq C_k$ to be the union's Shapley value of the quotient game where the union C_k is replaced with S. The payoff is then again distributed using the Shapley value. Following the Owen procedure, several other values have been extended to TU-games with coalition structures, including the Banzhaf value [14], the τ-value [107], the equal division surplus value [44], etc. We refer to [95], [37] and [7] for these.

In this chapter, along the lines of the previously mentioned papers, we suggest a new value for cooperative games with coalition structures. This value is closely related to another value for cooperative games with coalition structures that has been suggested by Kamijo [72], the so-called two-step Shapley value. That value exhibits a certain conceptual simplicity e.g. when compared to Owen's value: In the first step, all players of a union equally share the *Shapley net surplus* of the union containing them, i.e., there is an equal distribution of the difference between the Shapley value obtained by this union in the quotient game, and the worth of it. That means the union is left with its worth, which is again distributed using the Shapley value. However we believe that it lies in the nature of games with coalition structures that players within one union should exhibit a higher degree of solidarity, which is not captured by using the Shapley value for the game within unions. In the following, we elaborate a bit more on Kamijo's value, as well as other closely related values from the literature, in order to motivate our proposal.

Kamijo's two-step Shapley value actually establishes an "interpolation" between the approaches suggested by Owen and Aumann and Drèze. On the one hand, it affirms Owen's assumption that the grand coalition is being formed, and unions first play the quotient game to distribute the worth of the grand coalition among them. This is the same as in the first step of Owen's approach. On the other hand, it also retains the idea of separation between unions, because cooperation of the players within a union is simply modelled by the corresponding subgame restricted to the players of a union, as by Aumann and Drèze. In the second step, every player is just assigned the Shapley value of the respective subgame, and the sum of the two parts gives the two-step Shapley value [72]. With a change to the weighted Shapley value [101] in the first step, this was generalized even further to the so-called collective value [73].

Observe that *solidarity* among players *within* one union is embedded in Kamijo's approach only from the perspective of interaction among unions, as the union's Shapley surplus is equally divided among its members in the first step. However, solidarity is not really reflected by using the Shapley value for the games *within* the unions, because the Shapley value is known to be a purely performance-based value, which has been formalized elegantly by an axiomatization based on marginality as defined by Young [121]. Here, marginality refers to the fact that a player's payoff *only* depends on his own marginal contributions.

Almost parallel to our work, Hu [66] also suggested to address this issue, i.e., incorporating a higher degree of solidarity among the players within a union, by using the equal division value for the subgames per union, which then gives rise to the so-called weighted Shapley-egalitarian value [66]. The equal division value, however, is totally independent of players' performance. Another value that suggests itself in this context is the solidarity value by Nowak & Radzik [92]. Unlike equal division, the solidarity value takes individuals' differences into consideration, yet implements the solidarity principle as well: It employs the *average* marginal contribution instead of the marginal contribution as in the Shapley value, and in this way, the value is equipped with the feature of solidarity by providing support to "weaker" players: a player who contributes less than the average marginal contribution is supported by stronger partners.

That said, it should be mentioned that the solidarity value has previously been adopted into games with coalition structures by Calvo and Gutiérrez [33]. They also use the Shapley value for the quotient game of all unions, but following Owen's procedure, consider the induced internal games to distribute the unions' Shapley payoffs among the players, and this is based on the solidarity value. The resulting value is the Shapley-solidarity value.

In this chapter, inspired by the conceptual simplicity of Kamijo's two-step approach, and the idea to incorporate a realistic level of solidarity among the players of a given union, we suggest to marry Kamijo's two-step approach with the idea to use the solidarity value for the game played by players within any given union. Note that this is different from Owen's value as well as Calvo and Gutiérrez's Shapley-solidarity value, as we follow Kamijo's approach and only distribute the *net surplus* of the unions' Shapley payoffs in the first step. Hence for the second step, there is no need to revert to Owen's induced internal game, and we simply use the solidarity value to distribute the union's worth among its players. Arguably, this is conceptually simpler.

In lack of a better name and to avoid confusion with the values proposed earlier, we call this new value the *two-step Shapley-solidarity* value. Our main contributions are an intuitive procedural interpretation for this new value, and to give three axiomatizations that highlight the differences between this value and specifically the two-step Shapley value. As to the technical contribution of the paper, in order to get our axiomatizations done, we use an axiom that we call the coalitional A-null player axiom, and moreover, we have to revert to a new basis of the space of all games.

The rest of this chapter is organized as follows. After introducing the two-step Shapley-solidarity value in Section 4.2, we provide a procedural interpretation for it in Section 4.3. Finally, three axiomatic characterizations are presented in Section 4.4. Section 4.5 gives some final conclusions.

4.2 Two-step Shapley-solidarity value

Similar to the two-step Shapley value and the collective value, the two-step Shapley-solidarity value proposed here also distributes the worth of the grand coalition in two steps. Firstly, players within one union act collectively to bargain with other unions, all unions play the quotient game and obtain a payoff prescribed by the Shapley value. The surplus of the difference between the obtained payoff and the worth of the union is distributed equally among union members. Then, players within one union negotiate the worth that they can guarantee on their own, namely the worth of the union they belong to, and they obtain the solidarity value for the subgame restricted on the corresponding union.

Definition 4.1. For any $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, the two-step Shapley-solidarity value is given by

$$TSS_i(N, v, \mathscr{C}) = Sol_i(C_k, v_{|C_k}) + \frac{Sh_k(M, v^{\mathscr{C}}) - v(C_k)}{|C_k|}, \; i \in C_k \in \mathscr{C}. \quad (4.1)$$

Clearly, the so defined two-step Shapley-solidarity value will reduce to the Shapley value, respectively the solidarity value in the two extreme cases when the coalition structure is either all singleton players, or the grand coalition.

Remark 4.1. For $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$ and if $\mathscr{C} = \mathscr{C}_N$, $TSS(N, v, \mathscr{C}) = Sol(N, v)$, and for $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$ and $\mathscr{C} = \mathscr{C}_n$, $TSS(N, v, \mathscr{C}) = Sh(N, v)$.

In that sense, the level of solidarity increases with the more players joining unions. The same property is shared by the Shapley-solidarity value of Calvo and Gutiérrez [33]. Except for the equal distribution of the surplus of a union which is not present there, the main difference lies in another intra-union game, i.e., Owen's induced internal game. In this game the unions' internal behavior is actually re-assessed from a "non-solidarity" perspective. Intuitively speaking, a coalition S contained in one union C_k takes into consideration that the remaining members $C_k \setminus S$ might defect. Hence, they re-evaluate their worth to be what they can earn in the quotient game while assuming the remaining members are breaking away from their union.

Compared with the two close relatives, the two-step Shapley value and the Shapley-solidarity value, the two-step Shapley-solidarity value embeds more of a solidarity principle in the intra-union game, as it avoids the possible divergence among union members for the evaluation of their internal cooperation, and as it uses the solidarity value instead of the Shapley value. In this sense, the outcome should reflect a larger level of solidarity within unions. This can also be illustrated with the following, simple example of a four players game.

Example 4.1. Consider player set $N = \{1, 2, 3, 4\}$ and TU-game (N, v) where the characteristic function v is given by $v(\{3\}) = v(\{1, 2, 3\}) = 1$, $v(\{4\}) = v(\{3, 4\}) = v(\{1, 2, 4\}) = v(\{1, 3, 4\}) = v(\{2, 3, 4\}) = v(N) = L$ $(L \in \mathbb{R}_+)$, and $v(S) = 0$ otherwise. Now, players 1, 2 and 3 form the union C_I and player 4 remains alone, which gives rise to the coalition structure $\mathscr{C} = \{C_I = \{1, 2, 3\}, C_{II} = \{4\}\}$.

The above description gives rise to the TU-game with the coalition structure $(N, v, \mathscr{C})$. Note that players 1 and 2 can be considered the "weak" players for union C_I, because they can not generate any worth on their own, no matter if they choose to act alone or cooperate. The corresponding quotient game $(M, v^{\mathscr{C}})$ is a two-person TU-game where $M = \{I, II\}$. Hence, it is easy to get that

$$Sh_I(M, v^{\mathscr{C}}) = \frac{1}{2}, \; Sh_{II}(M, v^{\mathscr{C}}) = L - \frac{1}{2}.$$

Following Owen's procedure, we get the induced internal game for union C_I, namely (C_I, v_{C_I}), where $v_{C_I}(\{1\}) = v_{C_I}(\{2\}) = -L/2$, $v_{C_I}(\{3\}) = 1/2$, $v_{C_I}(\{1,2\}) = v_{C_I}(\{1,3\}) = v_{C_I}(\{2,3\}) = 0$ and finally, $v_{C_I}(\{1,2,3\}) = 1/2$. Obviously, player 1 and 2 are symmetric in (C_I, v_{C_I}). For Kamijo's two-step approach, players 1, 2 and 3 bargain with their union worth based on the restricted subgame $(C_I, v_{|C_I})$, and the symmetric relationship between player 1 and 2 holds in this subgame as well. Meanwhile, there is no need to consider the intra-bargaining for union C_{II} since it only contains player 4. Then, we can compute the three coalitional values for the TU-game with the coalition structure in Example 4.1 as shown in Table 4.1.

Table 4.1 Three payoff vectors for Example 4.1

Values	payoffs
Two-step Shapley value [72]	$(0, 0, \frac{1}{2}, L - \frac{1}{2})$
Shapley-solidarity value [33]	$(\frac{1}{8} - \frac{L}{24}, \frac{1}{8} - \frac{L}{24}, \frac{1}{4} + \frac{L}{12}, L - \frac{1}{2})$
Two-step Shapley-solidarity value	$(\frac{1}{12}, \frac{1}{12}, \frac{1}{3}, L - \frac{1}{2})$

Player 4 obtains the Shapley value in the quotient game as the final payoff since he forms a union alone, and there exists no difference in his payoff assigned by the three coalitional values. We focus on the payoffs of players in union C_I: The symmetry of players 1 and 2 accounts for their same payoff in all three coalitional values. Hence, the payoff difference between them and player 3 directly reflects the level of solidarity of union C_I. For this example, the difference is $1/4$ within players of union C_I for the two-step Shapley-solidarity value, compared to $1/2$ for the two-step Shapley value. We see the same effect also when compared to the Shapley-solidarity value, as long as $L > 1$. Moreover, it turns out that the Shapley-solidarity value has a payoff difference of $(L+1)/8$ within the players of union C_I which grows linearly in L, even though the subgame within union C_I has worths 0 and 1 only.

4.3 Procedural characterization

Along the lines of Shapley's procedural characterization of the Shapley value via average marginal contributions for all $n!$ permutations of players, we here provide a corresponding characterization of the two-step Shapley-solidarity value. First, with the restriction of coalition structures, it is assumed that the grand coalition forms in a consistent permutation, which indicates that the players within the same union enter the grand coalition consecutively. For each $\pi_c \in \Pi_{N,\mathscr{C}}$ and $i \in C_k$, we denote by $p^{\pi_c}(N,i)$ and $p^{\pi_c}(C_k,i)$ the sets of predecessors of player i with respect to N and C_k respectively, i.e., $p^{\pi_c}(N,i) = \{j \in N \,|\, \pi_c(j) \leq \pi_c(i)\}$, $p^{\pi_c}(C_k,i) = \{j \in C_k \,|\, \pi_c(j) \leq \pi_c(i)\}$. The set of all predecessors of a union $C_k \in \mathscr{C}$ is denoted by $p^{\pi_c}(N,C_k) = \{j \in N \,|\, \pi_c(j) < \min_{i \in C_k} \pi_c(i)\}$. In the following, we present a procedure in which the allocation scenario is envisaged to generate the two-step Shapley-solidarity value.

Given a TU-game with a coalition structure $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$, the procedure consists of the following steps:

Step 1 The players enter the grand coalition in a consistent permutation, and all consistent permutations have the same probability.

Step 2 Every entering player $i \in C_k \in \mathscr{C}$ joins in and forms the new coalition $p^{\pi_c}(N,i)$. The player brings the marginal contribution $M_i^{\pi_c}(N) := v(p^{\pi_c}(N,i)) - v(p^{\pi_c}(N,i) \setminus i)$. With a near-sighted union solidarity principle in mind, the player takes his marginal contribution with respect to the union he belongs to, namely $M_i^{\pi_c}(C_k) := v(p^{\pi_c}(C_k,i)) - v(p^{\pi_c}(C_k,i) \setminus i)$, and splits it equally among his union predecessors $p^{\pi_c}(C_k,i)$.

Step 3 The residual (negative or positive) brought by player i's joining, $M_i^{\pi_c}(N) - M_i^{\pi_c}(C_k)$, is equally shared by the union successors of player i, namely $s^{\pi_c}(C_k, i) = \{j \in C_k \mid \pi_c(j) > \pi_c(i)\}$.

Step 4 The last player of a union $i \in C_k \in \mathscr{C}$, so when $|p^{\pi_c}(C_k, i)| = |C_k|$, is then to be treated in a special way, and obtains a residual of $M_i^{\pi_c}(N) - M_i^{\pi_c}(C_k) - v(p^{\pi_c}(N, C_k))$, which is denoted by $\gamma_i^{\pi_c}$.

As shown in the procedure, each player focuses only on the corresponding union members under the restriction of the coalition structure. Either one player's marginal contribution or the residual is shared only by the players who are in the same union. This is exactly an embodiment of solidarity within a union. Besides, with the last union member joining in, this union is complete and the last player of the union thereby affords a payment to the union's predecessors to prevent their coalition's worth from being infringed. In view of the fact that the last union player has no union successors, a residual of $M_i^{\pi_c}(N) - M_i^{\pi_c}(C_k) - v(p^{\pi_c}(N, C_k))$ is shared "by himself".

In order to show that this procedural description coincides with the two-step Shapley-solidarity value, note that for each $(N, v, \mathscr{C}) \in \mathscr{CG}$ and each $\pi_c \in \Pi_{N,\mathscr{C}}$, Steps 1-4 determine a payoff $\psi_i^{\pi_c}(N, v, \mathscr{C})$ for each $i \in C_k \in \mathscr{C}$ as follows:

$$\psi_i^{\pi_c}(N, v, \mathscr{C}) = \begin{cases} M_i^{\pi_c}(C_k) + \beta_i^{\pi_c}, & \pi_c(i) = |p^{\pi_c}(N, C_k)| + 1; \\ \frac{M_i^{\pi_c}(C_k)}{|p^{\pi_c}(C_k,i)|} + \beta_i^{\pi_c} + \alpha_i^{\pi_c}, & 1 < \pi_c(i) - |p^{\pi_c}(N, C_k)| < |C_k|; \\ \frac{M_i^{\pi_c}(C_k)}{|p^{\pi_c}(C_k,i)|} + \alpha_i^{\pi_c} + \gamma_i^{\pi_c}, & \pi_c(i) = |p^{\pi_c}(N, C_k)| + |C_k|, \end{cases} \tag{4.2}$$

where

$$\alpha_i^{\pi_c} = \sum_{r=|p^{\pi_c}(N,C_k)|+1}^{\pi_c(i)-1} \frac{M_{\pi_c^{-1}(r)}^{\pi_c}(N) - M_{\pi_c^{-1}(r)}^{\pi_c}(C_k)}{|p^{\pi_c}(N, C_k)| + |C_k| - r},$$

and

$$\beta_i^{\pi_c} = \sum_{z=\pi_c(i)+1}^{|p^{\pi_c}(N,C_k)|+|C_k|} \frac{M_{\pi_c^{-1}(z)}^{\pi_c}(C_k)}{|p^{\pi_c}(C_k, \pi_c^{-1}(z))|}.$$

Then, for $i \in C_k \in \mathscr{C}$ the *procedural outcome* is given by

$$\psi_i(N,v,\mathscr{C}) := \frac{1}{|\Pi_{N,\mathscr{C}}|} \sum_{\pi_c \in \Pi_{N,\mathscr{C}}} \psi_i^{\pi_c}(N,v,\mathscr{C}). \tag{4.3}$$

Next, we will show the two-step Shapley-solidarity value and this procedural outcome coincide.

Theorem 4.1. *For each TU-game with a coalition structure* $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$*, the procedural outcome given by Eq.* (4.3)*,* $\psi(N,v,\mathscr{C})$ *coincides with the two-step Shapley-solidarity value* $TSS(N,v,\mathscr{C})$*.*

Proof. For each $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$, $i \in C_k \in \mathscr{C}$, it follows from Eq. (4.2) and Eq. (4.3) that,

$$\begin{aligned}
&\psi_i(N,v,\mathscr{C}) \\
=&\frac{1}{|\Pi_{N,\mathscr{C}}|} \sum_{\pi_c \in \Pi_{N,\mathscr{C}}} \psi_i^{\pi_c}(N,v,\mathscr{C}) \\
=&\frac{1}{|\Pi_{N,\mathscr{C}}|} \left(\sum_{\pi_c \in \Pi_{N,\mathscr{C}}} \frac{M_i^{\pi_c}(C_k)}{|p^{\pi_c}(C_k,i)|} + \sum_{\substack{\pi_c \in \Pi_{N,\mathscr{C}}: \\ \pi_c(i) \neq |p^{\pi_c}(N,C_k)|+|C_k|}} \beta_i^{\pi_c} + \sum_{\substack{\pi_c \in \Pi_{N,\mathscr{C}}: \\ \pi_c(i) \neq |p^{\pi_c}(N,C_k)|+1}} \alpha_i^{\pi_c} + \sum_{\substack{\pi_c \in \Pi_{N,\mathscr{C}}: \\ \pi_c(i) = |p^{\pi_c}(N,C_k)|+|C_k|}} \gamma_i^{\pi_c} \right) \\
=&\underbrace{\frac{1}{|\Pi_{N,\mathscr{C}}|} \sum_{\pi_c \in \Pi_{N,\mathscr{C}}} \frac{M_i^{\pi_c}(C_k)}{|p^{\pi_c}(C_k,i)|} + \sum_{\substack{\pi_c \in \Pi_{N,\mathscr{C}}: \\ \pi_c(i) \neq |p^{\pi_c}(N,C_k)|+|C_k|}} \frac{1}{|\Pi_{N,\mathscr{C}}|} \sum_{z=\pi_c(i)+1}^{|p^{\pi_c}(N,C_k)|+|C_k|} \frac{M^{\pi_c}_{\pi_c^{-1}(z)}(C_k)}{|p^{\pi_c}(C_k,\pi_c^{-1}(z))|}}_{\text{Part I}} \\
&+ \underbrace{\sum_{\substack{\pi_c \in \Pi_{N,\mathscr{C}}: \\ \pi_c(i) \neq |p^{\pi_c}(N,C_k)|+1}} \frac{1}{|\Pi_{N,\mathscr{C}}|} \sum_{r=|p^{\pi_c}(N,C_k)|+1}^{\pi_c(i)-1} \frac{M^{\pi_c}_{\pi_c^{-1}(r)}(N) - M^{\pi_c}_{\pi_c^{-1}(r)}(C_k)}{|p^{\pi_c}(N,C_k)|+|C_k|-r}}_{\text{Part II(1)}} \\
&+ \underbrace{\sum_{\substack{\pi_c \in \Pi_{N,\mathscr{C}}: \\ \pi_c(i) = |p^{\pi_c}(N,C_k)|+|C_k|}} \frac{1}{|\Pi_{N,\mathscr{C}}|} (M_i^{\pi_c}(N) - M_i^{\pi_c}(C_k) - v(p^{\pi_c}(N,C_k)))}_{\text{Part II(2)}}
\end{aligned}$$

Next, we will show the Part I is consistent with player i's payoff which results from the internal bargaining according to the subgame $(C_k, v_{|C_k})$, while Part II, i.e., the sum of Part $II(1)$ and Part $II(2)$, coincides with the surplus that player i can obtain due to the union's collective bargaining.

Let us focus on Part I first. For every $S \subseteq C_k$ such that $j \in S$, it is worth noting that there are $m! \prod_{p \neq k} |C_p|!(s-1)!(|C_k|-s)!$ consistent permutations for which player j is a successor of the players in $S \setminus j$ and the players in $C_k \setminus S$ are the successors of player j. Hence, it means that for each consistent permutation $\pi_c \in \Pi_{N,\mathscr{C}}$ such that $p^{\pi_c}(C_k, j) = S$ the player j's marginal contribution with respect to the union C_k is given by $v(S) - v(S \setminus j)$.

$$
\begin{aligned}
&\text{Part I}\\
&= \frac{1}{|\Pi_{N,\mathscr{C}}|} \sum_{\pi_c \in \Pi_{N,\mathscr{C}}} \frac{v(p^{\pi_c}(C_k, i)) - v(p^{\pi_c}(C_k, i) \setminus i)}{|p^{\pi_c}(C_k, i)|} + \sum_{\substack{\pi_c \in \Pi_{N,\mathscr{C}}: \\ \pi_c(i) \neq |p^{\pi_c}(N, C_k)| + |C_k|}} \Bigg(\frac{1}{|\Pi_{N,\mathscr{C}}|} \\
&\sum_{z=\pi_c(i)+1}^{|p^{\pi_c}(N,C_k)|+|C_k|} \frac{v(p^{\pi_c}(C_k, \pi_c^{-1}(z))) - v(p^{\pi_c}(C_k, \pi_c^{-1}(z)) \setminus \pi_c^{-1}(z))}{|p^{\pi_c}(C_k, \pi_c^{-1}(z))|} \Bigg)\\
&= \frac{(s-1)!(|C_k|-s)!}{|C_k|!} \left(\sum_{S \subseteq C_k : i \in S} \frac{v(S) - v(S \setminus i)}{s} + \sum_{j \in C_k \setminus i} \sum_{\substack{S \subseteq C_k: \\ \{i,j\} \subseteq S}} \frac{v(S) - v(S \setminus j)}{s} \right)\\
&= \frac{(s-1)!(|C_k|-s)!}{|C_k|!} \left(\sum_{S \subseteq C_k : i \in S} \frac{v(S) - v(S \setminus i)}{s} + \sum_{\substack{S \subseteq C_k: \\ i \in S, s \geq 2}} \sum_{j \in S \setminus i} \frac{v(S) - v(S \setminus j)}{s} \right)\\
&= \sum_{S \subseteq C_k : i \in S} \frac{(s-1)!(|C_k|-s)!}{|C_k|!} \sum_{j \in S} \frac{v(S) - v(S \setminus j)}{s}\\
&= Sol_i(C_k, v_{|C_k})
\end{aligned}
$$

Then, for Part II, we look at Part $II(1)$ and Part $II(2)$ separately.

Part II(1)

$$= \sum_{\substack{\pi_c \in \Pi_{N,\mathscr{C}}: \\ \pi_c(i) \neq |p^{\pi_c}(N,C_k)|+1}} \frac{1}{|\Pi_{N,\mathscr{C}}|} \sum_{r=|p^{\pi_c}(N,C_k)|+1}^{\pi_c(i)-1} \left(\frac{v(p^{\pi_c}(N,\pi_c^{-1}(r))) - v(p^{\pi_c}(N,\pi_c^{-1}(r)) \setminus \pi_c^{-1}(r))}{|p^{\pi_c}(N,C_k)| + |C_k| - r} \right.$$

$$\left. - \frac{v(p^{\pi_c}(C_k,\pi_c^{-1}(r))) - v(p^{\pi_c}(C_k,\pi_c^{-1}(r)) \setminus \pi_c^{-1}(r))}{|p^{\pi_c}(N,C_k)| + |C_k| - r} \right)$$

$$= \sum_{Q \subseteq M \setminus k} \sum_{j \in C_k \setminus i} \sum_{\substack{S \subseteq C_k: \\ i \notin S, j \in S}} \frac{q!(m-q-1)!}{m!} \cdot \frac{(s-1)!(|C_k|-s)!}{|C_k|!} \cdot$$

$$\left(\frac{v(\cup_{h \in Q} C_h \cup S) - v(\cup_{h \in Q} C_h \cup S \setminus j)}{|C_k| - s} - \frac{(v(S) - v(S \setminus j))}{|C_k| - s} \right)$$

$$= \sum_{Q \subseteq M \setminus k} \sum_{\substack{S \subseteq C_k: \\ i \notin S}} \frac{q!(m-q-1)!}{m!} \cdot \frac{(s-1)!(|C_k|-s-1)!}{|C_k|!} \cdot s \cdot (v(\cup_{h \in Q} C_h \cup S) - v(S))$$

$$- \sum_{Q \subseteq M \setminus k} \sum_{\substack{T \subseteq C_k : i \notin T, \\ t < |C_k|-1}} \sum_{\substack{j \in C_k \setminus i: \\ j \notin T}} \frac{q!(m-q-1)!}{m!} \cdot \frac{t!(|C_k|-t-2)!}{|C_k|!} \cdot (v(\cup_{h \in Q} C_h \cup T) - v(T))$$

$$= \sum_{Q \subseteq M \setminus k} \sum_{\substack{S \subseteq C_k: \\ i \notin S}} \frac{q!(m-q-1)!}{m!} \cdot \frac{s!(|C_k|-s-1)!}{|C_k|!} \cdot (v(\cup_{h \in Q} C_h \cup S) - v(S))$$

$$- \sum_{Q \subseteq M \setminus k} \sum_{\substack{T \subseteq C_k: \\ i \notin T,\, t < |C_k|-1}} \frac{q!(m-q-1)!}{m!} \cdot \frac{t!(|C_k|-t-1)!}{|C_k|!} \cdot (v(\cup_{h \in Q} C_h \cup T) - v(T))$$

$$= \sum_{Q \subseteq M \setminus k} \frac{q!(m-q-1)!}{m!} \cdot \frac{1}{|C_k|} \cdot (v(\cup_{h \in Q} C_h \cup C_k \setminus i) - v(C_k \setminus i)).$$

The second equality comes from the fact that, for any coalition $\cup_{h \in Q} C_h \cup S$ where $Q \subseteq M \setminus k$, and $j \in S \subseteq C_k$, there are $q!(m-q-1)! \prod_{p \neq k} |C_p|!(s-1)!(|C_k|-s)!$ permutations for which the predecessors of the union C_k consist of the players in $\cup_{h \in Q} C_h$, and player $j \in S$ is both a successor of the players in $S \setminus j$ and a predecessor of the players in $C_k \setminus S$. Hence, for each permutation such that $p^{\pi_c}(N, C_k) = \cup_{h \in Q} C_h$, $p^{\pi_c}(C_k, j) = S$ and $\pi_c(j) = |\cup_{h \in Q} C_h| + s$, the marginal contributions of player $j \in S \subseteq C_k$ with respect to N and C_k are given by $v(\cup_{h \in Q} C_h \cup S) - (\cup_{h \in Q} C_h \cup S \setminus j)$ and $v(S) - v(S \setminus j)$ respectively. For Part $II(2)$, observing that there are $q!(m-q-1)! \prod_{p \neq k} |C_p|!(|C_k|-1)!$ permutations where player $i \in C_k$ is the last entrant among his union members and players in

$\cup_{h\in Q} C_h$ are the predecessors of the union C_k, we have

$$\begin{aligned}
&\text{Part II(2)}\\
=&\sum_{\substack{\pi_c\in\Pi_{N,\mathscr{C}}:\\ \pi_c(i)=|p^{\pi_c}(N,C_k)|+|C_k|}} \frac{1}{|\Pi_{N,\mathscr{C}}|}\Big((v(p^{\pi_c}(N,i))-v(p^{\pi_c}(N,i)\setminus i))\\
&\qquad -(v(p^{\pi_c}(C_k,i))-v(p^{\pi_c}(C_k,i)\setminus i))-v(p^{\pi_c}(N,C_k))\Big)\\
=&\sum_{Q\subseteq M\setminus k}\frac{q!(m-q-1)!}{m!}\cdot\frac{1}{|C_k|}\cdot\Big(v(\cup_{h\in Q}C_h\cup C_k)-v(\cup_{h\in Q}C_h\cup C_k\setminus i)\\
&\qquad -(v(C_k)-v(C_k\setminus i))-v(\cup_{h\in Q}C_h)\Big)
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\text{Part II} =&\text{Part II(1)}+\text{Part II(2)}\\
=&\sum_{Q\subseteq M\setminus k}\frac{q!(m-q-1)!}{m!}\cdot\frac{1}{|C_k|}\cdot\Big(v(\cup_{h\in Q}C_h\cup C_k)-v(\cup_{h\in Q}C_h)-v(C_k)\Big)\\
=&\frac{1}{|C_k|}\left\{\sum_{Q\subseteq M\setminus k}\frac{q!(m-q-1)!}{m!}\cdot\Big(v(\cup_{h\in Q}C_h\cup C_k)-v(\cup_{h\in Q}C_h)\Big)-v(C_k)\right\}\\
=&\frac{1}{|C_k|}\left\{\left(\sum_{Q\subseteq M\setminus k}\frac{q!(m-q-1)!}{m!}\cdot(v^{\mathscr{C}}(Q\cup k)-v^{\mathscr{C}}(Q))\right)-v(C_k)\right\}\\
=&\frac{Sh_k(M,v^{\mathscr{C}})-v(C_k)}{|C_k|}
\end{aligned}$$

Putting all this together, it is immediate that

$$\psi_i(N,v,\mathscr{C})=Sol_i(C_k,v_{|C_k})+\frac{Sh_k(M,v^{\mathscr{C}})-v(C_k)}{|C_k|}=TSS_i(N,v,\mathscr{C}),$$

which completes the proof. □

4.4 Axiomatizations

As we can see, the two-step Shapley-solidarity value and the two-step Shapley value exactly differ in which principle is agreed to be used in bargaining on each union's worth. Hence, we next propose three axiomatizations to further indicate the precise similarities and differences between these two values. One of the axiomatizations is based on a variation of the null player axiom, and the other two are related to balanced contributions.

4.4.1 Coalitional A-null player axiom

The two-step Shapley value was firstly characterized in [72] by efficiency, additivity, coalitional symmetry and other two axioms introduced by Kamijo, called internal equity and coalitional null player axiom.

Unlike the intracoalitional symmetry axiom requiring two players within the same union to be symmetric in the original game (N, v), Kamijo [72] suggested an axiom called internal equity which only requires the symmetric relation of the two players to be true in the subgame defined on the union they belong to. The coalitional value ψ is said to satisfy

- *internal equity* (**IE**) if for all $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, $k \in M$ and two symmetric players $\{i, j\} \subseteq C_k$ in $(C_k, v_{|C_k})$, $\psi_i(N, v, \mathscr{C}) = \psi_j(N, v, \mathscr{C})$.

It states that two players who are regarded to be symmetric in the internal situation should be treated equally and thus receive equal payoff. As for the coalitional null player axiom, it requires that a null player in (N, v) gets zero payoff if the union he belongs to is a dummy player in quotient game. The coalitional value ψ is said to satisfy

- *the coalitional null player axiom* (**CNP**) if for all $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, a dummy player $k \in M$ in $(M, v^{\mathscr{C}})$, and a null player $i \in C_k$ in (N, v), $\psi_i(N, v, \mathscr{C}) = 0$.

In the statement of the coalitional null player axiom, note that it is still possible that a null player receives nonzero payoff. Hence, it is not necessarily the case that a zero payoff is given to all null players as shown by the null player axiom [94]. Actually, identifying which kind of players are supposed to get zero payoff or how to deal with the payoff of a null player is one of the key issues in axiomatizations with additivity. This pops up in quite a number of papers which apply variants of null player axiom to characterize values or coalitional values. For example, the δ-reducing player proposed by van den Brink and Funaki [109], the p-null player proposed by Béal et al. [20] for a class of solidarity values, two types of null players proposed by Borkotokey et al. [28] for a class of k-lateral Shapley values, the partial A-null player introduced by Hu and Li [67] for the Shapley-solidarity value, to name just a few.

In line with these works, we introduce the coalitional A-null player axiom for cooperative games with coalition structures. Let us recall that a player is called an A-null player if his average marginal contribution to any coalition containing him is zero. The coalitional value ψ is said to satisfy

- *the coalitional A-null player axiom* (**CANP**) if for all $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$, a dummy player $k \in M$ in $(M, v^{\mathscr{C}})$, and an A-null player $i \in C_k$ in $(C_k, v_{|C_k})$, $\psi_i(N,v,\mathscr{C}) = 0$.

It states that if a player is an A-null player in the subgame with a player set consisting of his union members, and the union he belongs to is a dummy player in the quotient game, then this player should obtain zero payoff.

With the aid of this axiom, we obtain an axiomatization of the two-step Shapley-solidarity value. Before we give the formal axiomatization, some definitions and lemmas are needed. For $\mathscr{C} \in \mathscr{C}^N$, we firstly define a family of TU-games $\{(N, \tilde{u}_T)\}_{T \in 2^N \setminus \emptyset}$ with respect to $\mathscr{C}$ as follows.

Given $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$, for any $T \subseteq C_k$, $k \in M$,

$$\tilde{u}_T(S) = \begin{cases} \binom{s}{t}^{-1} \cdot \binom{|C_k|}{t}, & T \subseteq S \subseteq C_k; \\ 1, & T \subseteq S \nsubseteq C_k; \\ 0, & T \nsubseteq S, \end{cases}$$

and if $T \nsubseteq C_k$, $\forall k \in M$,

$$\tilde{u}_T(S) = \begin{cases} 1, & T \subseteq S; \\ 0, & T \nsubseteq S. \end{cases}$$

Note that $(N, \tilde{u}_T)$ is an ordinary unanimity game when $T \nsubseteq C_k$. Next, we show that the family of $\{(N, \tilde{u}_T)\}_{T \in 2^N \setminus \emptyset}$ forms a basis of $\mathscr{G}^N$.

Lemma 4.2. *For each $\mathscr{C} \in \mathscr{C}^N$, the family of TU-games $\{(N, \tilde{u}_T)\}_{T \in 2^N \setminus \emptyset}$ is a basis of the linear space $\mathscr{G}^N$.*

Proof. It is well-known that $\mathscr{G}^N$ is a $(2^n - 1)$-dimensional linear space. Similar to the spirit of the proof of Lemma 2.2 in [92], we just show that TU-games $\{(N, \tilde{u}_T)\}_{T \in 2^N \setminus \emptyset}$ consist of a set of $2^n - 1$ independent vectors in $\mathscr{G}^N$. To that end, let $S_1, S_2, \cdots, S_{2^n-1}$ be a fixed sequence containing all non-empty set of N such that $n = |S_1| \geq |S_2| \geq \cdots \geq |S_{2^n-1}|$. Moreover, define a $(2^n - 1) \times (2^n - 1)$ matrix $A = [a_{i,j}]$ whose entries are given by

$$a_{i,j} = \tilde{u}_{S_i}(S_j),\ i, j = 1, 2, \ldots, 2^n - 1.$$

Notice that A is a triangular matrix and its diagonal entries equal $\binom{|C_k|}{t}$ if $T \subsetneq C_k$ $(k \in M)$ and 1 otherwise. Hence, we see that $\det(A) = \prod_{k=1}^{m} \prod_{\emptyset \neq T \subsetneq C_k} \binom{|C_k|}{t} \neq 0$. It follows that vectors $\{\tilde{u}_T\}_{T \in 2^N \setminus \emptyset}$ are independent, and thus, $\{(N, \tilde{u}_T)\}_{T \in 2^N \setminus \emptyset}$ forms a basis of $\mathscr{G}^N$. This holds for all coalition structures $\mathscr{C}$. □

Then, we have the following, main theorem.

Theorem 4.3. *A coalitional value ψ on $\mathscr{C}\mathscr{G}$ satisfies efficiency, additivity, coalitional symmetry, internal equity, and the coalitional A-null player axiom if and only if $\psi(N, v, \mathscr{C})$ is the two-step Shapley-solidarity value.*

Proof. Existence. Firstly, we show that the two-step Shapley-solidarity value satisfies the above five axioms. Efficiency and additivity are trivial due to the definition of the two-step Shapley-solidarity value. Coalitional symmetry and internal equity can be easily verified since both Shapley value and solidarity value satisfy symmetry. It also turns out to be true that the two-step Shapley-solidarity value satisfies coalitional A-null player axiom, because the solidarity value satisfies the A-null player axiom and the Shapley value assigns a dummy player his stand-alone worth.

Uniqueness. Let ψ be a coalitional value over $\mathscr{C}\mathscr{G}$ which satisfies the five axioms. Lemma 4.2 immediately implies that, given $\mathscr{C} \in \mathscr{C}^N$, for each $(N, v) \in$

$\mathcal{G}^N$, there exists $\{\lambda_T \mid \lambda_T \in \mathbb{R}, T \in 2^N \setminus \emptyset\}$ such that $v = \sum_{T\in 2^N\setminus\emptyset} \lambda_T \tilde{u}_T$. According to additivity, it is now sufficient to prove that, for each TU-game $(N, \lambda_T\tilde{u}_T, \mathscr{C})$, $\psi(N, \lambda_T\tilde{u}_T, \mathscr{C})$ is uniquely determined by efficiency, coalitional symmetry, internal equity and the coalitional A-null player axiom.

We recall that $D_T = \{h \in M \mid C_h \cap T \neq \emptyset\}$. Note that the corresponding quotient game $(M, (\lambda_T\tilde{u}_T)^{\mathscr{C}})$ for $(N, \lambda_T\tilde{u}_T, \mathscr{C})$ is equivalent to the unanimity game $(M, \lambda_T u_{D_T})$ because, for each $T \in 2^N \setminus \emptyset$, there is

$$(\lambda_T\tilde{u}_T)^{\mathscr{C}}(Q) = \lambda_T\tilde{u}_T(\cup_{k\in Q}C_k) = \begin{cases} \lambda_T, & T \subseteq \cup_{k\in Q}C_k; \\ 0, & \text{otherwise}, \end{cases}$$

for all $Q \subseteq M$. Hence, each $k \notin D_T$ is a null player in $(M, (\lambda_T\tilde{u}_T)^{\mathscr{C}})$. Moreover, for each $k \notin D_T$, the subgame $(C_k, (\lambda_T\tilde{u}_T)_{|C_k})$ is a null game, namely $(\lambda_T\tilde{u}_T)_{|C_k}(S) = 0$ for all $S \subseteq C_k$. By the coalitional A-null player axiom, we have $\psi_i(N, \lambda_T\tilde{u}_T, \mathscr{C}) = 0$ for each $i \in C_k$ $(k \notin D_T)$. For $k \in D_T$, there is $\sum_{i\in C_k} \psi_i(N, \lambda_T\tilde{u}_T, \mathscr{C}) = \frac{\lambda_T}{|D_T|}$, which derives from efficiency and coalitional symmetry.

Now, let us focus on the internal distribution of the payoff that one union obtains from their collective bargaining. For each $T \in 2^N \setminus \emptyset$, there is the corresponding D_T, and we consider the following two cases.

(i) $|D_T| = 1$. Let $D_T = \{k\}$, notice that each player $i \in C_k \setminus T$ is an A-null player in $(C_k, (\lambda_T\tilde{u}_T)_{|C_k})$ since, for each coalition $S \subseteq C_k$ satisfying $T \subseteq S$ and $i \in S$,

$$\begin{aligned}(\lambda_T\tilde{u}_T)_{|C_k}(S) =& \lambda_T \binom{s}{t}^{-1} \cdot \binom{|C_k|}{t} \\ =& \lambda_T \frac{t!(s-t)!}{s!} \cdot \frac{|C_k|!}{t!(|C_k|-t)!} \\ =& \lambda_T \frac{1}{s} \cdot (s-t) \cdot \frac{t!(s-t-1)!}{(s-1)!} \frac{|C_k|!}{t!(|C_k|-t)!} \\ =& \frac{1}{s}\sum_{j\in S}(\lambda_T\tilde{u}_T)_{|C_k}(S \setminus j).\end{aligned}$$

Besides, k is a dummy player in the quotient game $(M, (\lambda_T\tilde{u}_T)^{\mathscr{C}})$. By the coalitional A-null player axiom, we have $\psi_i(N, \lambda_T \tilde{u}_T, \mathscr{C}) = 0$ for each $i \in C_k \setminus T$. Furthermore, the symmetry of any two players $i, j \in T$ in subgame $(C_k, (\lambda_T \tilde{u}_T)_{|C_k})$ immediately implies $\psi_i(N, \lambda_T \tilde{u}_T, \mathscr{C}) = \frac{\lambda_T}{t}$.

(ii) $|D_T| \geq 2$. For each C_k $(k \in D_T)$ and $\{i, j\} \subseteq C_k$, we have $(\lambda_T \tilde{u}_T)_{|C_k}(S \cup i) = (\lambda_T \tilde{u}_T)_{|C_k}(S \cup j)$ for each $S \subseteq C_k \setminus \{i, j\}$. Thus, by internal equity, there is $\psi_i(N, \lambda_T \tilde{u}_T, \mathscr{C}) = \frac{\lambda_T}{|D_T| \cdot |C_k|}$ for each $i \in C_k$ $(k \in D_T)$.

Hence, it is clear that $\psi(N, \lambda_T \tilde{u}_T, \mathscr{C})$ is unique, which completes the proof. □

4.4.2 Quasi-balanced contributions for the grand coalition

The quasi-balanced contributions axiom, a variation of the balanced contributions axiom proposed by Myerson [90], was introduced by Xu et al. [120], which gives rise to an axiomatization of the solidarity value by combining with efficiency. In this subsection, we provide two other axiomatizations for two-step Shapley-solidarity value based on this axiom's formulation in a coalition structure setting.

To begin with, we firstly pay attention to two axiomatizations for the two-step Shapley value which are provided by Calvo and Gutiérrez [32]. They prove that the two-step Shapley value can be characterized with the **CBC** axiom, in which two other axioms are also involved, called population solidarity within unions and coherence. The coalitional value ψ is said to satisfy

- *population solidarity within unions* (**PSU**) if for all $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, $C_h, C_r \in \mathscr{C}$ with $r \neq h$, and $\{i, j, k\} \subseteq N$ such that $\{i, j\} \subseteq C_h$ and $k \in C_r$,

$$\psi_i(N, v, \mathscr{C}) - \psi_i(N \backslash k, v_{|N \backslash k}, \mathscr{C}_{|N \backslash k}) = \psi_j(N, v, \mathscr{C}) - \psi_j(N \backslash k, v_{|N \backslash k}, \mathscr{C}_{|N \backslash k}).$$

- coherence (**C**) if for all $(N, v) \in \mathscr{G}$, $\psi(N, v, \mathscr{C}_N) = \psi(N, v, \mathscr{C}_n)$.

Population solidarity within unions states that players in the same union follow the solidarity principle in such a way that all members in the union experience the same gains or losses when the game changes due to addition or deletion of players outside the union. Coherence means that it is indistinguishable between games in which all players belong to one union and when all of them act as singletons. The following theorem is due to Calvo and Gutiérrez [32].

Theorem 4.4. *[32] The two-step Shapley value is the unique value that satisfies efficiency, coalitional balanced contributions, population solidarity within unions and coherence.*

Besides, they also introduce the axiom of null coalition which requires null coalitions should get nothing. Formally, the coalitional value ψ is said to satisfy

- *null coalition axiom* (**NC**) if for all $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$ and a null coalition $C_k \in \mathscr{C}$, $\sum_{i \in C_k} \psi_i(N, v, \mathscr{C}) = 0$.

Then the following theorem holds.

Theorem 4.5. *[32] A coalitional value ψ satisfies efficiency, additivity, coalitional symmetry, null coalitional axiom, population solidarity within unions and coherence if and only if $\psi(N, v, \mathscr{C}) = TSh(N, v, \mathscr{C})$.*

Both Theorems 4.4 and 4.5 invoke the coherence axiom which is violated by the two-step Shapley-solidarity value. Next, we will show if we replace the coherence in the above two theorems with a coalitional version of the quasi-balanced contributions for TU-games with coalition structures, called quasi-balanced contributions for the grand coalition, we can get corresponding axiomatizations of the two-step Shapley-solidarity value. First, we formulate the mentioned axiom. The coalitional value ψ is said to satisfy

- *quasi-balanced contributions for the grand coalition* (**QCGC**) if for all $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$ with $|\mathscr{C}| = 1$, and $i, j \in C_k \in \mathscr{C}$,

$$\begin{aligned}&\psi_i(N, v, \mathscr{C}) - \psi_i(N \setminus j, v_{|N \setminus j}, \mathscr{C}_{|N \setminus j}) + \frac{1}{n} v(N \setminus j)\\ =\ &\psi_j(N, v, \mathscr{C}) - \psi_j(N \setminus i, v_{|N \setminus i}, \mathscr{C}_{|N \setminus i}) + \frac{1}{n} v(N \setminus i).\end{aligned}$$

Note that this axiom has exactly the same requirement as the condition for the solidarity value. As we know, the solidarity value can be characterized by quasi-balanced contributions and efficiency. Hence, quasi-balanced contributions for the grand coalition is the corresponding feature for TU-games with coalition structures in which the coalition structure is just one union.

Theorem 4.6. *A coalitional value ψ satisfies efficiency, additivity, coalitional symmetry, null coalitional axiom, population solidarity within unions and quasi-balanced contributions for the grand coalition if and only if*

$$\psi(N, v, \mathscr{C}) = TSS(N, v, \mathscr{C}).$$

Proof. The proof follows the same spirit as the proof of Theorem 4 in [32] (Theorem 4.5 above). For clarity, we here restate it in order to highlight the difference.

Existence. It is straightforward to verify that the two-step Shapley-solidarity value satisfies efficiency, additivity, coalitional symmetry, null coalition axiom and the population solidarity within unions. As for the quasi-balanced contributions for the grand coalition, if $\mathscr{C} = \{C_1\} = \mathscr{C}_N$, then $M = \{1\}$ and $Sh_1(M, v^{\mathscr{C}}) = v(N)$, and there is $TSS_i(N, v, \mathscr{C}) = Sol_i(N, v)$ for each $i \in N$. Hence, the two-step Shapley-solidarity value satisfies the quasi-balanced contributions for the grand coalition because the solidarity value satisfies the quasi-balanced contributions.

Uniqueness. Let ψ be a coalitional value satisfying the above six axioms. Given $(N, v, \mathscr{C}) \in \mathscr{C}\mathscr{G}$, define value ϕ on $\mathscr{G}^M$ by, for each $k \in M$, $\phi_k(M, v^{\mathscr{C}})$ $= \sum_{i \in C_k} \psi_i(N, v, \mathscr{C})$.

It turns out that the value ϕ is well-defined by efficiency, additivity, coalitional symmetry and null coalition axiom, and there is $\phi_k(M, v^{\mathscr{C}}) = Sh_k(M, v^{\mathscr{C}})$. Thus, when $\mathscr{C} = \mathscr{C}_n$, we have

$$\psi_i(N, v, \mathscr{C}_n) = \phi_i(M, v^{\mathscr{C}_n}) = Sh_i(N, v) = TSS_i(N, v, \mathscr{C}_n).$$

On the other hand, when $\mathscr{C} = \mathscr{C}_N$, because ψ satisfies efficiency and quasi-balanced contributions for the grand coalition, then by Theorem 4.2 in [120], we can obtain

$$\psi_i(N, v, \mathscr{C}_N) = Sol_i(N, v) = TSS_i(N, v, \mathscr{C}_N).$$

Now we focus on the cases when the coalition structure is not trivial. Assume that $|\mathscr{C}| \geq 2$, for each $\{h,r\} \subseteq M$, each $i \in C_h$ and $k \in C_r$, according to population solidarity within unions, there exists $\gamma_h \in \mathbb{R}$ such that $\psi_i(N,v,\mathscr{C}) - \psi_i(N\backslash k, v_{|N\backslash k}, \mathscr{C}_{|N\backslash k}) = \gamma_h$, and hence, for each $i \in C_h$,

$$\psi_i(N,v,\mathscr{C}) = \psi_i(N\backslash k, v_{|N\backslash k}, \mathscr{C}_{|N\backslash k}) + \frac{1}{|C_h|}[Sh_h(M,v^{\mathscr{C}}) - Sh_h(M,v^{\mathscr{C}_{|N\backslash k}})].$$

Using the population solidarity within unions repeatedly until only C_h is in the game, we obtain

$$\begin{aligned}\psi_i(N,v,\mathscr{C}) =& \psi_i(C_h, v_{|C_h}, \mathscr{C}_{|C_h}) + \frac{1}{|C_h|}[Sh_h(M,v^{\mathscr{C}}) - Sh_h(\{h\}, v^{\mathscr{C}_{|C_h}})] \\ =& Sol_i(C_h, v_{|C_h}) + \frac{1}{|C_h|}[Sh_h(M,v^{\mathscr{C}}) - v(C_h)] \\ =& TSS_i(N,v,\mathscr{C})\end{aligned}$$

for each $i \in C_h \in \mathscr{C}$. Hence, there is $\psi(N,v,\mathscr{C}) = TSS(N,v,\mathscr{C})$ for each $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$, which completes the proof. □

Theorem 4.7. *A coalitional value ψ satisfies efficiency, coalitional balanced contributions, population solidarity within unions and quasi-balanced contributions for the grand coalition if and only if $\psi(N,v,\mathscr{C}) = TSS(N,v,\mathscr{C})$.*

Proof. Existence. By Theorem 4.6, it is left to show the two-step Shapley-solidarity value satisfies the coalitional balanced contributions axiom. By definition, for all $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$ and $C_h, C_r \in \mathscr{C}$ with $r \neq h$, $\sum_{i\in C_h} TSS_i(N,v,\mathscr{C}) = Sh_h(M,v^{\mathscr{C}})$ and $\sum_{i\in C_r} TSS_i(N,v,\mathscr{C}) = Sh_r(M,v^{\mathscr{C}})$. Hence, the coalitional balanced contributions of the two-step Shapley-solidarity value immediately follows from the balanced contributions of the Shapley value [90].

Uniqueness. Let ψ be a coalitional value satisfying the above four axioms. We show $\psi(N,v,\mathscr{C}) = TSS(N,v,\mathscr{C})$ for all $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$ by induction on $|\mathscr{C}|$.

Let $|\mathscr{C}| = 1$. This means that the coalition structure is trivial and $\mathscr{C} = \mathscr{C}_N$. Given $(N,v,\mathscr{C}_N) \in \mathscr{C}\mathscr{G}$, quasi-balanced contributions for the grand coalition together with efficiency implies $\psi_i(N,v,\mathscr{C}_N) = Sol_i(N,v)$ for all $i \in N$. Hence, we have $\psi(N,v,\mathscr{C}) = TSS(N,v,\mathscr{C})$ for all $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$ with $|\mathscr{C}| = 1$.

Now, assume $\psi(N,v,\mathscr{C}) = TSS(N,v,\mathscr{C})$ holds for all TU-games with coalition structures $(N,v,\mathscr{C}) \in \mathscr{C}\mathscr{G}$ when $|\mathscr{C}| \leq m-1$, we prove $\psi(N,v,\mathscr{C}) = TSS(N,v,\mathscr{C})$ can also be established for $(N,v,\mathscr{C})$ with $|\mathscr{C}| = m$.

Let $(N,v,\mathscr{C})$ be a TU-game with a coalition structure where $|\mathscr{C}| = m$. Since both ψ and TSS satisfy **CBC**, we have

$$\begin{aligned}&\sum_{i\in C_h}\psi_i(N,v,\mathscr{C}) - \sum_{i\in C_r}\psi_i(N,v,\mathscr{C})\\ =&\sum_{i\in C_h}\psi_i(N\backslash C_r, v_{|N\backslash C_r}, \mathscr{C}_{|N\backslash C_r}) - \sum_{i\in C_r}\psi_i(N\backslash C_h, v_{|N\backslash C_h}, \mathscr{C}_{|N\backslash C_h}),\end{aligned}$$

and

$$\begin{aligned}&\sum_{i\in C_h}TSS_i(N,v,\mathscr{C}) - \sum_{i\in C_r}TSS_i(N,v,\mathscr{C})\\ =&\sum_{i\in C_h}TSS_i(N\backslash C_r, v_{|N\backslash C_r}, \mathscr{C}_{|N\backslash C_r}) - \sum_{i\in C_r}TSS_i(N\backslash C_h, v_{|N\backslash C_h}, \mathscr{C}_{|N\backslash C_h}).\end{aligned}$$

Moreover, according to the induction hypothesis, we have

$$\begin{aligned}&\sum_{i\in C_h}\psi_i(N\backslash C_r, v_{|N\backslash C_r}, \mathscr{C}_{|N\backslash C_r}) - \sum_{i\in C_r}\psi_i(N\backslash C_h, v_{|N\backslash C_h}, \mathscr{C}_{|N\backslash C_h})\\ =&\sum_{i\in C_h}TSS_i(N\backslash C_r, v_{|N\backslash C_r}, \mathscr{C}_{|N\backslash C_r}) - \sum_{i\in C_r}TSS_i(N\backslash C_h, v_{|N\backslash C_h}, \mathscr{C}_{|N\backslash C_h}).\end{aligned}$$

The above three equations yield

$$\sum_{i\in C_h}\psi_i(N,v,\mathscr{C}) - \sum_{i\in C_h}TSS_i(N,v,\mathscr{C}) = \sum_{i\in C_r}\psi_i(N,v,\mathscr{C}) - \sum_{i\in C_r}TSS_i(N,v,\mathscr{C}),$$

for all $C_h, C_r \in \mathscr{C}$. Then, fixing h in the left part in the above equation and summing over $r \in M$ of the right, we have

$$|M|\left(\sum_{i\in C_h}\psi_i(N,v,\mathscr{C}) - \sum_{i\in C_h}TSS_i(N,v,\mathscr{C})\right)$$

$$=\sum_{r\in M}\sum_{i\in C_r}\psi_i(N,v,\mathscr{C})-\sum_{r\in M}\sum_{i\in C_r}TSS_i(N,v,\mathscr{C}).$$

Combining with efficiency, we get

$$\sum_{i\in C_h}\psi_i(N,v,\mathscr{C})=\sum_{i\in C_h}TSS_i(N,v,\mathscr{C}), \tag{4.4}$$

for all $h\in M$.

Then, it remains to show $\psi_i(N,v,\mathscr{C})=TSS_i(N,v,\mathscr{C})$ for all $i\in C_h\in\mathscr{C}$. This can be obtained by induction on $|C_h|$. Given $C_h\in\mathscr{C}$ with $|C_h|=1$, Eq. (4.4) yields $\psi_i(N,v,\mathscr{C})=TSS_i(N,v,\mathscr{C})$ for $\{i\}=C_h$. We now assume $|C_h|\geq 2$. For each $C_h, C_r\in\mathscr{C}$, and $\{i,j\}\subseteq C_h$, by repeatedly using **PSU** on ψ and TSS until the players within union C_r are ruled out, we have

$$\begin{aligned}&\psi_i(N,v,\mathscr{C})-\psi_j(N,v,\mathscr{C})\\=&\psi_i(N\backslash C_r,v_{|N\backslash C_r},\mathscr{C}_{|N\backslash C_r})-\psi_j(N\setminus C_r,v_{|N\backslash C_r},\mathscr{C}_{|N\backslash C_r}),\end{aligned}$$

and

$$\begin{aligned}&TSS_i(N,v,\mathscr{C})-TSS_j(N,v,\mathscr{C})\\=&\,TSS_i(N\setminus C_r,v_{|N\backslash C_r},\mathscr{C}_{|N\backslash C_r})-TSS_j(N\setminus C_r,v_{|N\backslash C_r},\mathscr{C}_{|N\backslash C_r}).\end{aligned}$$

Again, by induction hypothesis, we have

$$\begin{aligned}\psi_i(N\setminus C_r,v_{|N\backslash C_r},\mathscr{C}_{|N\backslash C_r})&=TSS_i(N\setminus C_r,v_{|N\backslash C_r},\mathscr{C}_{|N\backslash C_r}),\\ \psi_j(N\setminus C_r,v_{|N\backslash C_r},\mathscr{C}_{|N\backslash C_r})&=TSS_j(N\setminus C_r,v_{|N\backslash C_r},\mathscr{C}_{|N\backslash C_r}).\end{aligned}$$

Hence, there is

$$\psi_i(N,v,\mathscr{C})-TSS_i(N,v,\mathscr{C})=\psi_j(N,v,\mathscr{C})-TSS_j(N,v,\mathscr{C}).$$

Then, fixing i and summing over $j\in C_h$, we obtain

$$|C_h|(\psi_i(N,v,\mathscr{C})-TSS_i(N,v,\mathscr{C}))=\sum_{j\in C_h}(\psi_j(N,v,\mathscr{C})-TSS_j(N,v,\mathscr{C})).$$

By Eq. (4.4), we conclude that $\psi_i(N, v, \mathscr{C}) = TSS_i(N, v, \mathscr{C})$ for all $i \in C_h \in \mathscr{C}$, which completes the proof.

□

4.5 Conclusions

The two-step Shapley-solidarity value is in our opinion a conceptually simple value for cooperative games with coalition structures that captures the solidarity concept within unions. The given axiomatizations exactly pinpoint this, and show similarities and also the subtle difference when compared to the two-step Shapley value as defined by Kamijo [72]. The Example 4.1 also highlights the difference to the two closest relatives, but of course, other examples can be constructed to show opposite effects, too. It is an interesting question for further research to find subclasses of games to turn the “empirical” observations of Example 4.1 into a firm theorem. In this context, observe that for *anonymous* games where $v(S) = |S|$, and a specific class of *simple* games, namely when $v(S) = 1$ for all $|S| \geq 2$ and $v(S) = 0$ otherwise, all three values that we defined in Table 4.1 are identical. Moreover, for *additive* games where $v(S) = \sum_{i \in S} v(i)$, the two-step Shapley value is given by the stand-alone worth of the players, so $TSh_i(N, v, \mathscr{C}) = v(i)$, and the Shapley-solidarity value and two-step Shapley-solidarity value are identical, but different from the former. Finally, note that also another two-step coalitional value can be defined by using the solidarity value for both, the quotient game and the induced internal game, implementing solidarity also among the coalitions. It turns out that this value can be characterized along the same lines as the two-step Shapley-solidarity value.

Chapter 5

Efficient Extensions of the Myerson Value Based on Endogenous Claims from Players

The combination of a TU-game and an undirected graph was introduced by Myerson [89] who assumed that a coalition is feasible if and only if its members are connected in the graph. This assumption suggests that graph game values should be component efficient. One representative graph game value is the Myerson value [89]. However, communication links sometimes do not function as a generator to enable cooperation but as a promotor to improve players' bargaining power. In this case, graph game values are expected to be efficient and efficient extensions of the Myerson value have drawn a lot of attention. In this chapter, we extend the Myerson value to be efficient through so-called bankruptcy rules.

5.1 Introduction

As discussed earlier, solutions for TU-games suggest possible allocation schemes to deal with the problem to distribute revenue among all players. Different solutions generally have different fairness standards, but most of them have one desirable property, namely efficiency. It states that the players distribute among themselves exactly what they earn when they all cooperate together. One of the most well-known efficient solutions for TU-games is the Shapley value [102], which offers each player his expected marginal contribution with all possible orders of the players happening with the same probability. This performance-based value has been widely generalized to cooperative games with restrictions on coalition formation, see [9], [94], [55], [24] etc. for references.

Myerson [89] generalized the Shapley value to cooperative games with communication restrictions. A communication restriction is described by an undirected graph, and it is assumed that only coalitions that are connected in the communication graph are feasible, and a non-connected coalition can just achieve the worth that equals the sum of the worths generated by its connected components. The Myerson value [89] is defined as the Shapley value of the induced graph restricted game and it is characterized by component efficiency and fairness.

Component efficiency states that the worth of each component is distributed among its members. Although component efficiency coincides with efficiency for connected graph games, they are not the same when TU-games are restricted by unconnected communication graphs. As a consequence, the Myerson value generally does not satisfy efficiency, and a surplus can be derived from the difference between the worth of the grand coalition of the graph restricted game and that of the underlying game.

Indeed, there may exist situations where the worth of the grand coalition is available even though a cooperative game is restricted by a communication graph. It implies that the productive unit is the grand coalition instead of the components of the communication graph. In such cases, it is natural to require a solution to be efficient. Hence, efficient extensions of the Myerson value have been developed. An efficient extension of the Myerson value refers to a graph game value which is efficient and coincides with the Myerson value on connected graph games.

We briefly discuss some of the efficient extensions of the Myerson value that have appeared in the literature. Casajus [35] provided the first efficient extension of the Myerson value which coincides with the Owen value [94] for completely connected components. Van den Brink et al. [114] extended the Myerson value to be efficient with an egalitarian division of the surplus, which leads to the efficient egalitarian Myerson (EEMy) value. Its characterization involves three axioms, including efficiency, fairness and fair distribution of the surplus. The efficient two-step egalitarian surplus Myerson (ESMy) value firstly appeared in [35] and was later axiomatized in [67]. Shan et al. [106] unified the two efficient extensions (EEMy value and ESMy value) with a measure function α determined by the graph, in which the surplus is distributed in proportion to this measure function. Recently, Li and Shan [79] established the efficient quotient Myerson value which equally distributes the surplus of each connected component among all players within that component.

Another line of research on the efficient extension of the Myerson value was initiated by Béal et al. [15]. It is shown that the efficient egalitarian Myerson value is the unique efficient extension of the Myerson value which admits fairness and coincides with the Myerson value for connected games. Béal et al. [16] then extended this result by proving that all graph game values equipped with efficiency and fairness on connected graph games admit a unique efficient extension on the class of all graph games. Moreover, this approach has been used by Béal et al. [17] to analyze several other graph game values.

Observing the existing literature on efficient extensions of the Myerson value, two distribution principles, i.e., the egalitarian and proportional principles, are dominant when dividing the surplus. As a matter of fact, the Myerson value puts emphasis on the marginal effect due to its dependency on the Shapley value. Hence, we believe that incorporating marginalism into the process of surplus division is consistent with the distribution as originally proposed by Myerson.

In this chapter, we put this idea into practice by setting the players' Shapley payoffs of the underlying game as their claims to the surplus. This value quantifies players' contribution in a cooperative situation without any restriction, so it is an embodiment of their potential performance. It is reasonable for players to make it serve as a benchmark and claim for surplus. Since these claims are actually determined by their own performance instead of being exogenously imposed, we refer them to be endogenous. Obviously, the surplus to be divided is not sufficient to cover all endogenous claims. Therefore, we firstly introduce a graph-induced bankruptcy problem that consists of players' endogenous claims and an endowment that equals the surplus. Then, classical bankruptcy rules as proposed by Aumann and Maschler [10], namely the constrained equal awards (CEA) rule and the constrained equal losses (CEL) rule, are invoked to distribute the surplus. We then obtain the efficient constrained equal awards Myerson value and the efficient constrained equal losses Myerson value. The two efficient extensions of the Myerson value are axiomatically justified.

The remainder of this chapter is organized as follows. Section 5.2 recaps some basic definitions about the standard bankruptcy problems and defines the graph-induced bankruptcy problem. Sections 5.3 and 5.4 give two efficient extensions of the Myerson value, based on distributing the surplus through the constrained equal awards rule and the constrained equal losses rule respectively. Conclusions are given in Section 5.5.

5.2 Graph-induced bankruptcy problem

Let $\mathcal{N} = \{1,2,\cdots\}$ be the (infinite) set of potential creditors (players). Given a finite set of the creditors $N \subsetneq \mathcal{N}$, a bankruptcy problem for N is a pair (c,E), where $c = (c_i)_{i\in N} \in \mathbb{R}^N_+$ is the claims vector and $E \in \mathbb{R}_+$ is the endowment which has to be divided among its creditors N, satisfying $\sum_{i\in N} c_i \geq E$. Let $\mathscr{B}^N$ be the class of all bankruptcy problems for N.

A rule is a function ψ that associates every bankruptcy problem for $N \subsetneq \mathcal{N}$ with an awards vector $(\psi_i(c,E))_{i\in N} \in \mathbb{R}^N$ such that $0 \leq \psi_i(c,E) \leq c_i$ for all $i \in N$ and $\sum_{i\in N} \psi_i(c,E) = E$. Then, we recall two classical rules in bankruptcy problems which are central to our efficient extensions. The first is the constrained equal awards rule [10] which makes awards as equal as possible subject to no one receiving more than his claim.

Definition 5.1. [10] Given $N \subsetneq \mathcal{N}$ and $(c,E) \in \mathscr{B}^N$, for every $i \in N$, the constrained equal awards (CEA) rule is given by

$$CEA_i(c,E) = min\{c_i, \lambda_A\},$$

where $\lambda_A \in \mathbb{R}_+$ satisfies $\sum_{i \in N} CEA_i(c,E) = E$.

The constrained equal losses rule [10] imposes equal losses but subject to no creditor ending up with a negative award.

Definition 5.2. [10] Given $N \subsetneq \mathcal{N}$ and $(c,E) \in \mathscr{B}^N$, for every $i \in N$, the constrained equal losses (CEL) rule is given by

$$CEL_i(c,E) = max\{c_i - \lambda_L, 0\},$$

where $\lambda_L \in \mathbb{R}_+$ satisfies $\sum_{i \in N} CEL_i(c,E) = E$.

Facing with the unequal claims, the CEA rule gives priority to the players with smaller claims, instead the CEL rule gives more protection to players with larger claims. Different rules have their own criterion and applicability under specific circumstances. As suggested by Herrero and Villar [65], in the situation of bankrupt savings bank, a rule should give priority to households rather than firms. Even though the firms usually hold larger claims, the claims of households are larger shares of their wealth. However, when the claims represent needs like medical treatment in a given population or the expenditure of a public health system, a rule may firstly consider the larger claims. Herrero and Villar [65] translated these value judgements into distributional results by the following two axioms, called sustainability and independence of residual claims.

Firstly, a claim c_i ($i \in N$) is called sustainable and residual if $\sum_{j \in N} min\{c_i, c_j\} \leq E$ and $\sum_{j \in N} max\{c_j - c_i, 0\} \geq E$ respectively. Then, axioms that are related to the two kinds of claims are given as follows.

The bankruptcy rule ψ is said to satisfy

- *sustainability* [65] if for all $N \subsetneq \mathcal{N}$, $(c,E) \in \mathscr{B}^N$ and $i \in N$ with c_i being sustainable, $\psi_i(c,E) = c_i$.
- *independence of residual claims* [65] if for all $N \subsetneq \mathcal{N}$, $(c,E) \in \mathscr{B}^N$ and $i \in N$ with c_i being residual, $\psi_i(c,E) = 0$.

Sustainability implies that the sustainable claims deserve to be honored fully, and independence of residual claims deems it appropriate to dismiss residual claims. These two axioms pave the way for axiomatizing CEA rule and CEL rule, respectively.

We implement efficient extensions of the Myerson value with the help of CEA and CEL rules. The efficient extension of the Myerson value actually aims to explore a suitable way to conduct the division of the surplus. We embed a marginalism-based principle in the process, namely dividing the surplus based on the Shapley value of the original game. Note that the TU-games we refer to in this chapter are non-negative and superadditive. Next, we introduce a bankruptcy problem with respect to a graph game as follows.

Definition 5.3. Graph-induced bankruptcy problem. For any $(N, v, \Gamma) \in \mathcal{GL}$, the graph-induced bankruptcy problem is a pair $(c^{N,v,\Gamma}, E^{N,v,\Gamma})$, where each element of claims vector $c_i^{N,v,\Gamma} = Sh_i(N, v)$ for all $i \in N$ and $E^{N,v,\Gamma} = v(N) - v^{\Gamma}(N)$.

The Shapley value of the original game and the surplus are treated as players' claims and endowment respectively. Due to the superadditivity and non-negativity of the TU-game (N, v), it turns out that $c_i^{N,v,\Gamma} \in \mathbb{R}_+$ for each $i \in N$, and $E^{N,v,\Gamma} \in \mathbb{R}_+$. By efficiency of the Shapley value, it is easy to check that $\sum_{i \in N} c_i^{N,v,\Gamma} \geq E^{N,v,\Gamma}$. Thus, $(c^{N,v,\Gamma}, E^{N,v,\Gamma})$ is a bankruptcy problem.

We call player i sustainable if $c_i^{N,v,\Gamma}$ is sustainable in $(c^{N,v,\Gamma}, E^{N,v,\Gamma})$. Denote the set of all sustainable players by $S(N, v, \Gamma)$. Likewise, player i is called residual if $c_i^{N,v,\Gamma}$ is residual in $(c^{N,v,\Gamma}, E^{N,v,\Gamma})$, and the set of all residual players is denoted as $R(N, v, \Gamma)$. By the definitions of CEA rule and CEL rule, it is straightforward to obtain the following two lemmas.

Lemma 5.1. *Given $(N, v, \Gamma) \in \mathcal{GL}$ and $i \in N$, if $i \in S(N, v, \Gamma)$, we have $c_i^{N,v,\Gamma} \leq \lambda_A$, and if $i \notin S(N, v, \Gamma)$, we have $c_i^{N,v,\Gamma} > \lambda_A$.*

Proof. For any $i \in S(N, v, \Gamma)$, there is $\sum_{j \in N} min\{c_i^{N,v,\Gamma}, c_j^{N,v,\Gamma}\} \leq E^{N,v,\Gamma}$, but $\sum_{j \in N} CEA_j(c^{N,v,\Gamma}, E^{N,v,\Gamma}) = \sum_{j \in N} min\{\lambda_A, c_j^{N,v,\Gamma}\} = E^{N,v,\Gamma}$. Therefore, we have $c_i^{N,v,\Gamma} \leq \lambda_A$, and $\lambda_A < c_i^{N,v,\Gamma}$ holds for any $i \in N \setminus S(N, v, \Gamma)$. □

Lemma 5.2. *Given* $(N, v, \Gamma) \in \mathscr{GL}$ *and* $i \in N$, *if* $i \in R(N, v, \Gamma)$, *we have* $c_i^{N,v,\Gamma} \leq \lambda_L$, *and if* $i \notin R(N, v, \Gamma)$, *we have* $c_i^{N,v,\Gamma} > \lambda_L$.

Proof. The proof is similar to that of Lemma 5.1, we omit details here. □

Next, we propose two efficient extensions of the Myerson value by applying two different bankruptcy rules to the graph-induced bankruptcy problem.

5.3 Efficient extension with constrained equal awards in surplus

The first efficient extension of the Myerson value provides players with awards. We use the **CEA** rule to carry out a constrained egalitarian division of the graph-induced endowment.

Definition 5.4. For any $(N, v, \Gamma) \in \mathscr{GL}$, and the corresponding graph-induced bankruptcy problem $(c^{N,v,\Gamma}, E^{N,v,\Gamma})$, the efficient constrained equal awards Myerson value is given by

$$E\mu_i^{CEA}(N, v, \Gamma) = \mu_i(N, v, \Gamma) + CEA_i(c^{N,v,\Gamma}, E^{N,v,\Gamma}), \text{ for all } i \in N.$$

Note that for a graph game (N, v, Γ), if the sustainable player set $S(N, v, \Gamma) = \emptyset$, then the efficient constrained equal awards Myerson value is equivalent to the efficient egalitarian Myerson value [114]. We proceed the characterization of the efficient constrained equal awards Myerson value by introducing the following two axioms. Let us firstly recall that $C(i)$ is the component in N/Γ containing player i. The graph game value f is said to satisfy

- *sustainability in surplus* (**SS**) if for all $(N, v, \Gamma) \in \mathscr{GL}$ and $i \in S(N, v, \Gamma)$, $f_i(N, v, \Gamma) - f_i(C(i), v_{|C(i)}, \Gamma_{C(i)}) = f_i(N, v, \Gamma^N)$.

Since players' claims depend on the Shapley value of the underlying game, a player's claim indicates his expected marginal contribution to the surplus to some extent. If a player is sustainable, it means that this player makes a relatively small contribution which enables the surplus to cover all truncated claims. In reality, some players may be unable to generate substantial contribution as a consequence of some objective reasons such as physical limitations, even though they dedicate themselves to the cooperation. **SS** is inclined to honor these players for the protection of their interest. However, instead of directly honoring them what they claim, it is more convincing to honor them after an evaluation made in a cooperative situation same as the one where their claims are measured. Hence, **SS** assigns a sustainable player a surplus share of his payoff obtained in the cooperation without restrictions.

Then, we introduce the next axiom with the following example.

Example 5.1. Consider a graph game (N, v, Γ) with the player set $N = \{1, 2, 3, 4, 5\}$, $\Gamma = \{\{1, 2\}, \{3, 4\}, \{4, 5\}\}$, as is shown in Figure 5.1, and the characteristic function v is given by $v(\{1\}) = v(\{3\}) = v(\{4\}) = v(\{5\}) = v(\{1, 4\}) = v(\{4, 5\}) = 0$, $v(\{2\})$=1; $v(N) = 10$; $v(S) = v(N) - \sum_{i \in N \setminus S} d_i$, where $d_1 = 1$, $d_2 = 5$, $d_3 = 4$, $d_4 = 0.5$, $d_5 = 3.5$, otherwise.

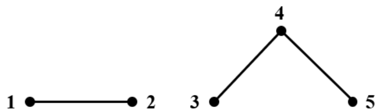

Figure 5.1: Communication graph for the player set N

By the definition of the given graph game, the endowment of the graph-induced bankruptcy problem is $v(N) - v^{\Gamma}(N) = v(N) - (v(\{1, 2\}) + v(\{3, 4, 5\})) = 4$. The claims, the Myerson value and the efficient constrained equal awards Myerson value are shown in the following table.[1]

Table 5.1 Claims and the (efficient constrained equal awards) Myerson value

Players	1	2	3	4	5
Claims	0.9250	3.6333	2.9667	0.2167	2.2583
μ	0.5	1.5	$1.416\dot{6}$	$1.416\dot{6}$	$1.166\dot{6}$
$E\mu^{CEA}$	1.4250	$2.4527\dot{6}$	$2.3694\dot{3}$	$1.633\dot{6}$	$2.1194\dot{3}$

[1] $1.416\dot{6}$ is a short form of $1.4166666\cdots$, and so forth.

It is observed that the players with relatively larger claims (players 2, 3 and 5) get the same level of surplus $E\mu_i^{CEA}(N,v,\Gamma) - \mu_i(N,v,\Gamma) = 0.9527\dot{6}$, but their claims actually are distinct from one another. This situation can be illustrated by the following axiom. The graph game value f is said to satisfy

- *weak fair distribution of surplus* **(WFDS)** if for all $(N,v,\Gamma) \in \mathscr{GL}$ and $i,j \in N \setminus S(N,v,\Gamma)$,

$$f_i(N,v,\Gamma) - f_i(C(i), v_{|C(i)}, \Gamma_{C(i)}) = f_j(N,v,\Gamma) - f_j(C(j), v_{|C(j)}, \Gamma_{C(j)}).$$

This axiom requires that a pair of non-sustainable players should get the same surplus for participating in the cooperation. Note that each player in $N \setminus \{1,4\}$ in Example 5.1 is a non-sustainable player. Because the endowment of the surplus 4 is still insufficient, even though the larger claims are truncated. In this sense, no matter how much one player claims, it will make no difference. Hence, all these players get the same surplus. The idea of dividing surplus among related players equally can also be seen from fair distribution of surplus within component (**FDSI**) in [67], which restricts a pair of players to the same component instead of the set of non-sustainable players in **WFDS**.

Theorem 5.3. *Given $(N,v,\Gamma) \in \mathscr{GL}$, a graph game value $f(N,v,\Gamma)$ satisfies efficiency (**E**), coherence with the Myerson value for connected graphs (**CMC**), sustainability in surplus (**SS**) and weak fair distribution of surplus (**WFDS**) if and only if $f(N,v,\Gamma) = E\mu^{CEA}(N,v,\Gamma)$.*

Proof. Existence. For any $(N,v,\Gamma) \in \mathscr{GL}$, we firstly show that $E\mu^{CEA}(N,v,\Gamma)$ is a graph game value satisfying the above four axioms. It is straightforward to verify the efficiency of $E\mu^{CEA}(N,v,\Gamma)$ by its definition. For any $(N,v,\Gamma) \in \mathscr{GL}_C$, there is $v(N) = v^{\Gamma}(N)$, which implies that $E\mu^{CEA}(N,v,\Gamma)$ is equivalent to $\mu(N,v,\Gamma)$. Hence, **CMC** holds for $E\mu^{CEA}(N,v,\Gamma)$.

For the axiom of sustainability in surplus, let $i \in S(N,v,\Gamma)$. By Lemma 5.1, we have $min\{c_i^{N,v,\Gamma}, \lambda_A\} = c_i^{N,v,\Gamma}$. Then,

$$\begin{aligned} &E\mu_i^{CEA}(N,v,\Gamma) - E\mu_i^{CEA}(C(i), v_{|C(i)}, \Gamma_{C(i)}) \\ =&\mu_i(N,v,\Gamma) + CEA_i(c^{N,v,\Gamma}, E^{N,v,\Gamma}) - \mu_i(C(i), v_{|C(i)}, \Gamma_{C(i)}) \end{aligned}$$

$$
\begin{aligned}
=&CEA_i(c^{N,v,\Gamma}, E^{N,v,\Gamma}) \\
=&c_i^{N,v,\Gamma} \\
=&E\mu_i^{CEA}(N, v, \Gamma^N).
\end{aligned}
$$

It shows $E\mu^{CEA}(N, v, \Gamma)$ satisfies **SS**, and the second equality in the above equation derives from the **CD** of the Myerson value.

Let $i, j \in N \setminus S(N, v, \Gamma)$. For any $k \in N \setminus S(N, v, \Gamma)$, it is similar to check that $c_k^{N,v,\Gamma} > \lambda_A$. Combining with the fact that the **CMC** and **CD** are satisfied by the efficient constrained equal awards Myerson value and the Myerson value respectively, we have

$$
\begin{aligned}
&E\mu_i^{CEA}(N, v, \Gamma) - E\mu_i^{CEA}(C(i), v_{|C(i)}, \Gamma_{C(i)}) \\
=&\mu_i(N, v, \Gamma) + CEA_i(c^{N,v,\Gamma}, E^{N,v,\Gamma}) - \mu_i(C(i), v_{|C(i)}, \Gamma_{C(i)}) \\
=&\lambda_A + \mu_j(C(j), v_{|C(j)}, \Gamma_{C(j)}) - \mu_j(C(j), v_{|C(j)}, \Gamma_{C(j)}) \\
=&\mu_j(N, v, \Gamma) + CEA_j(c^{N,v,\Gamma}, E^{N,v,\Gamma}) - \mu_j(C(j), v_{|C(j)}, \Gamma_{C(j)}) \\
=&E\mu_j^{CEA}(N, v, \Gamma) - E\mu_j^{CEA}(C(j), v_{|C(j)}, \Gamma_{C(j)}).
\end{aligned}
$$

Hence, **WFDS** is valid for $E\mu^{CEA}(N, v, \Gamma)$.

Uniqueness. Assume that a graph game value f satisfies **E**, **CMC**, **SS** and **WFDS** as well. It is sufficient to show that $f_i(N, v, \Gamma) = E\mu_i^{CEA}(N, v, \Gamma)$ for all $i \in N$.

Let $i \in C(i) \in N/\Gamma$, since both f and $E\mu^{CEA}$ satisfy **CMC**, we have

$$
f_i(C(i), v_{|C(i)}, \Gamma_{C(i)}) = E\mu_i^{CEA}(C(i), v_{|C(i)}, \Gamma_{C(i)}) = \mu_i(N, v, \Gamma). \tag{5.1}
$$

Then, we consider the following cases.

- If $i \in S(N, v, \Gamma)$, by **SS** and Eq. (5.1), we have

$$
\begin{aligned}
f_i(N, v, \Gamma) &= f_i(C(i), v_{|C(i)}, \Gamma_{C(i)}) + f_i(N, v, \Gamma^N) \\
&= \mu_i(N, v, \Gamma) + \mu_i(N, v, \Gamma^N).
\end{aligned}
$$

  It immediately follows that $f_i(N, v, \Gamma)$ is unique for all $i \in S(N, v, \Gamma)$.

- If $i \in N \setminus S(N, v, \Gamma)$, by Eq. (5.1), we have $f_i(C(i), v_{|C(i)}, \Gamma_{C(i)}) = \mu_i(N, v, \Gamma)$.

Picking $i, j \in N \setminus S(N,v,\Gamma)$, combining with **WFDS** and Lemma 5.1, we obtain

$$f_i(N,v,\Gamma) - E\mu_i^{CEA}(N,v,\Gamma) = f_j(N,v,\Gamma) - E\mu_j^{CEA}(N,v,\Gamma).$$

Then, fix i and let j run over $N \setminus S(N,v,\Gamma)$, together with **E**, there is

$$\begin{aligned} & |N \setminus S(N,v,\Gamma)|(f_i(N,v,\Gamma) - E\mu_i^{CEA}(N,v,\Gamma)) \\ = & \sum_{j \in N \setminus S(N,v,\Gamma)} (f_j(N,v,\Gamma) - E\mu_j^{CEA}(N,v,\Gamma)) \\ = & 0, \end{aligned}$$

which implies $f_i(N,v,\Gamma) = E\mu_i^{CEA}(N,v,\Gamma)$ for all $i \in N \setminus S(N,v,\Gamma)$.

This completes the proof. □

The axioms of Theorem 5.3 are logically independent.

- For any $(N,v,\Gamma) \in \mathcal{GL}$ and $i \in N$, a graph game value f^1 is given by

$$f_i^1(N,v,\Gamma) = \begin{cases} \mu_i(N,v,\Gamma) + c_i^{N,v,\Gamma} & i \in S(N,v,\Gamma) \\ \mu_i(N,v,\Gamma) + \frac{E^{N,v,\Gamma}}{|N \setminus S(N,v,\Gamma)|} & \text{otherwise.} \end{cases}$$

  It is straightforward to check that f^1 satisfies **CMC**, **SS** and **WFDS** but violates **E**, since $\sum_{i \in N} f_i^1(N,v,\Gamma) = v(N) + \sum_{i \in S(N,v,\Gamma)} c_i^{N,v,\Gamma}$.

- For any $(N,v,\Gamma) \in \mathcal{GL}$ and $i \in C(i) \subseteq N$, a graph game value f^2 is given by

$$f_i^2(N,v,\Gamma) = Sh_i(C(i), v_{|C(i)}) + CEA_i(c^{N,v,\Gamma}, E^{N,v,\Gamma}).$$

  Obviously, f^2 satisfies **E**, **SS** and **WFDS** but violates **CMC**, because $f_i^2(N,v,\Gamma) = Sh_i(N,v) \neq \mu_i(N,v,\Gamma)$ for any $(N,v,\Gamma) \in \mathcal{GL}_C$.

- For any $(N,v,\Gamma) \in \mathcal{GL}$ and $i \in N$, a graph game value f^3 is given by

$$f_i^3(N,v,\Gamma) = \begin{cases} \mu_i(N,v,\Gamma) + \frac{E^{N,v,\Gamma} - |N \setminus S(N,v,\Gamma)|\lambda_A}{|S(N,v,\Gamma)|} & i \in S(N,v,\Gamma) \\ \mu_i(N,v,\Gamma) + \lambda_A & \text{otherwise,} \end{cases}$$

where $\lambda_A \in \mathbb{R}_+$ satisfies $\sum_{i\in N} CEA_i(c^{N,v,\Gamma}, E^{N,v,\Gamma}) = E^{N,v,\Gamma}$. Since for any $i \in S(N,v,\Gamma)$,

$$f_i^3(N,v,\Gamma) - f_i^3(C(i), v_{|C(i)}, \Gamma_{C(i)}) = \frac{E^{N,v,\Gamma} - |N \setminus S(N,v,\Gamma)|\lambda_A}{|S(N,v,\Gamma)|}$$
$$\neq \mu_i(N,v,\Gamma^N) = f_i^3(N,v,\Gamma^N),$$

it is clear that f^3 violates **SS**. Moreover, **E**, **CMC** and **WFDS** for f^3 can be easily verified.

- For any $(N,v,\Gamma) \in \mathscr{GL}$ and $i \in N$, a graph game value f^4 is given by

$$f_i^4(N,v,\Gamma) = \begin{cases} \mu_i(N,v,\Gamma) + CEA_i(c^{N,v,\Gamma}, E^{N,v,\Gamma}) & i \in S(N,v,\Gamma) \\ \mu_i(N,v,\Gamma) + \delta_i & \text{otherwise,} \end{cases}$$

where $\delta_i = \frac{c_i^{N,v,\Gamma}(E^{N,v,\Gamma} - \sum_{j\in S(N,v,\Gamma)} CEA_i(c^{N,v,\Gamma}, E^{N,v,\Gamma}))}{\sum_{j\in N\setminus S(N,v,\Gamma)} c_j^{N,v,\Gamma}}$. For any $i, j \in N\setminus S(N,v,\Gamma)$, we have $\frac{f_i^4(N,v,\Gamma) - f_i^4(C(i), v_{|C(i)}, \Gamma_{C(i)})}{f_j^4(N,v,\Gamma) - f_j^4(C(j), v_{|C(j)}, \Gamma_{C(j)})} = \frac{c_i^{N,v,\Gamma}}{c_j^{N,v,\Gamma}}$. Thus, it is straightforward to prove that f^4 satisfies **E**, **CMC**, and **SS** but violates **WFDS**.

5.4 Efficient extension with constrained equal losses in surplus

In contrast to the constrained equal division of the graph-induced endowment, we implement a constrained egalitarian division of the graph-induced deficit[2] in this section. That is to say, constrained equal losses in claims are experienced by players in this efficient extension, which follows the spirit of the **CEL** rule in bankruptcy problems.

Definition 5.5. For any $(N,v,\Gamma) \in \mathscr{GL}$, and the corresponding graph-induced

[2]The amount is the difference between the sum of all the claims and the graph restricted endowment, thus it is $v^\Gamma(N)$.

bankruptcy problem $(c^{N,v,\Gamma}, E^{N,v,\Gamma})$, the efficient constrained equal losses Myerson value is given by

$$E\mu_i^{CEL}(N,v,\Gamma) = \mu_i(N,v,\Gamma) + CEL_i(c^{N,v,\Gamma}, E^{N,v,\Gamma}), \text{ for all } i \in N.$$

Following the step of independence of residual claims axiom used to axiomatize the CEL rule in [65], we consider the following adapted version in the graph-restricted setting to characterize this efficient graph game value. The graph game value f is said to satisfy

- *residual in surplus* (**RS**) if for all $(N,v,\Gamma) \in \mathscr{GL}$ and $i \in R(N,v,\Gamma)$, $f_i(N,v,\Gamma) = f_i(C(i), v_{|C(i)}, \Gamma_{C(i)})$.

Residual in surplus means that, if player i's claim is too small to make the surplus cover all players' remaining claims obtained by deducting player i's claim from the original ones[3], player i is dismissed when dividing the surplus. For any pair of non-residual players, the following axiom requires that the difference of their surplus should be in accordance with the gap of their claims. The graph game value f is said to satisfy

- *surplus marginality within non-residual players* (**SMNR**) if for all $(N,v,\Gamma) \in \mathscr{GL}$, and $i,j \in N \setminus R(N,v,\Gamma)$, $(f_i(N,v,\Gamma) - f_i(C(i), v_{|C(i)}, \Gamma_{C(i)})) - (f_j(N,v,\Gamma) - f_j(C(j), v_{|C(j)}, \Gamma_{C(j)})) = c_i^{N,v,\Gamma} - c_j^{N,v,\Gamma}$.

Theorem 5.4. *Given $(N,v,\Gamma) \in \mathscr{GL}$, a graph game value $f(N,v,\Gamma)$ satisfies efficiency (**E**), coherence with the Myerson value for connected graphs (**CMC**), residual in surplus (**RS**) and surplus marginality within non-residual players (**SMNR**) if and only if $f(N,v,\Gamma) = E\mu^{CEL}(N,v,\Gamma)$.*

Proof. Existence. Firstly, we show that $E\mu^{CEL}(N,v,\Gamma)$ satisfies **E**, **CMC**, **RS** and **SMNR**. It is easy to check that **E** and **CMC** hold for the efficient constrained equal losses Myerson value. The details are similar to those of $E\mu^{CEA}(N,v,\Gamma)$, thereby we omit them.

As for **RS**, pick $i \in R(N,v,\Gamma)$, by Lemma 5.2, we have $max\{c_i - \lambda_L, 0\} = 0$. Together with the **CD** of the Myerson value and the **CMC** of $E\mu^{CEL}(N,v,\Gamma)$, it is straightforward to obtain $E\mu_i^{CEL}(N,v,\Gamma) = E\mu_i^{CEL}(C(i), v_{|C(i)}, \Gamma_{C(i)})$.

[3]The remaining claims is zero for those who are with smaller claims than $c_i^{N,v,\Gamma}$.

Then, for any $k \in N \backslash R(N,v,\Gamma)$, by Lemma 5.2, we have $CEL_k(c^{N,v,\Gamma}, E^{N,v,\Gamma}) = c_k^{N,v,\Gamma} - \lambda_L$. Therefore, we obtain

$$\begin{aligned} & E\mu_k^{CEL}(N,v,\Gamma) - E\mu_k^{CEL}(C(k), v_{|C(k)}, \Gamma_{C(k)}) \\ =& \mu_k(N,v,\Gamma) + CEL_k(c^{N,v,\Gamma}, E^{N,v,\Gamma}) - \mu_k(C(k), v_{|C(k)}, \Gamma_{C(k)}) \\ =& c_k^{N,v,\Gamma} - \lambda_L. \end{aligned}$$

Setting $k = i$ and $k = j$ in the above equation, then we can validate **SMNR** for $E\mu^{CEL}(N,v,\Gamma)$ by subtracting the two equations.

Uniqueness. Let f be an efficient graph game value satisfying **CMC**, **RS** and **SMNR** for any $(N,v,\Gamma) \in \mathcal{GL}$.

For any $i \in R(N,v,\Gamma) \subseteq N$, since f satisfies **RS** and **CMC**, we have

$$\begin{aligned} f_i(N,v,\Gamma) &= f_i(C(i), v_{|C(i)}, \Gamma_{C(i)}) \\ &= \mu_i(C(i), v_{|C(i)}, \Gamma_{C(i)}). \end{aligned}$$

It immediately implies the uniqueness of $f_i(N,v,\Gamma)$ for all $i \in R(N,v,\Gamma)$.

For any $k \in N \setminus R(N,v,\Gamma)$, according to **CMC**, we get

$$f_k(C(k), v_{|C(k)}, \Gamma_{C(k)}) = E\mu_k^{CEL}(C(k), v_{|C(k)}, \Gamma_{C(k)}). \tag{5.2}$$

Then, for $i,j \in N \setminus R(N,v,\Gamma)$, using **SMNR** for f and $E\mu^{CEL}$ and combining with Eq. (5.2), we obtain

$$f_i(N,v,\Gamma) - E\mu_i^{CEL}(N,v,\Gamma) = f_j(N,v,\Gamma) - E\mu_j^{CEL}(N,v,\Gamma),$$

which implies that there exists a constant ξ such that $f_i(N,v,\Gamma) - E\mu_i^{CEL}(N,v,\Gamma) = \xi$ for all $i \in N \setminus R(N,v,\Gamma)$.

Hence, summing over $i \in N \setminus R(N,v,\Gamma)$, we have

$$\sum_{i \in N \setminus R(N,v,\Gamma)} f_i(N,v,\Gamma) - \sum_{i \in N \setminus R(N,v,\Gamma)} E\mu_i^{CEL}(N,v,\Gamma) = |N \setminus R(N,v,\Gamma)|\xi.$$

Since both f and $E\mu^{CEL}$ satisfy **E** and $f_i(N,v,\Gamma) = E\mu_i^{CEL}(N,v,\Gamma)$ for all $i \in R(N,v,\Gamma)$, the left-hand side of the above equation is equal to zero. This means $\xi = 0$, which equivalently implies $f_i(N,v,\Gamma) = E\mu_i^{CEL}(N,v,\Gamma)$ for all $i \in N \setminus R(N,v,\Gamma)$. This completes the proof. □

The axioms of Theorem 5.4 are logically independent.

- For any $(N,v,\Gamma) \in \mathcal{GL}$ and $i \in N$, a graph game value f^5 is given by

$$f_i^5(N,v,\Gamma) = \begin{cases} \mu_i(N,v,\Gamma) & i \in R(N,v,\Gamma) \\ \mu_i(N,v,\Gamma) + CEL_i(c^{N,v,\Gamma}, E^{N,v,\Gamma}) - \eta_1 & \text{otherwise,} \end{cases}$$

  where $\eta_1 = \frac{min_{i \in N\setminus R(N,v,\Gamma)} CEL_i(c^{N,v,\Gamma}, E^{N,v,\Gamma})}{2}$. It is easy to verify that f^5 satisfies **CMC**, **RS** and **SMNR** but violates **E** because $\sum_{i\in N} f_i^5(N,v,\Gamma) = v(N) - |N \setminus R(N,v,\Gamma)|\eta_1 < v(N)$.

- For any $(N,v,\Gamma) \in \mathcal{GL}$ and $i \in C(i) \subseteq N$, a graph game value f^6 is given by

$$f_i^6(N,v,\Gamma) = \frac{v(C(i))}{|C(i)|} + CEL_i(c^{N,v,\Gamma}, E^{N,v,\Gamma}).$$

  f^6 satisfies **E**, **RS** and **SMNR** but violates **CMC**, because $f_i^6(N,v,\Gamma) = \frac{v(N)}{n} \neq \mu_i(N,v,\Gamma)$ for any $(N,v,\Gamma) \in \mathcal{GL}_C$.

- For any $(N,v,\Gamma) \in \mathcal{GL}$ and $i \in N$, a graph game value f^7 is given by

$$f_i^7(N,v,\Gamma) = \begin{cases} \mu_i(N,v,\Gamma) + \frac{E^{N,v,\Gamma}}{n} & i \in R(N,v,\Gamma) \\ \mu_i(N,v,\Gamma) + CEL_i(c^{N,v,\Gamma}, E^{N,v,\Gamma}) - \eta_2 & \text{otherwise,} \end{cases}$$

  where $\eta_2 = \frac{E^{N,v,\Gamma}|R(N,v,\Gamma)|}{n|N\setminus R(N,v,\Gamma)|}$. Since for any $i \in R(N,v,\Gamma)$, $f_i^7(N,v,\Gamma) \neq \mu_i(N,v,\Gamma) = f_i^7(C(i), v_{|C(i)}, \Gamma_{C(i)})$, f^7 violates **RS**. However, it is not difficult to check that f^7 satisfies the other three axioms, namely **E**, **CMC** and **SMNR**.

- For any $(N,v,\Gamma) \in \mathcal{GL}$ and $i \in N$, a graph game value f^8 is given by

$$f_i^8(N,v,\Gamma) = \begin{cases} \mu_i(N,v,\Gamma) & i \in R(N,v,\Gamma) \\ \mu_i(N,v,\Gamma) + \frac{c_i^{N,v,\Gamma} E^{N,v,\Gamma}}{\sum_{j\in N\setminus R(N,v,\Gamma)} c_j^{N,v,\Gamma}} & \text{otherwise.} \end{cases}$$

For any $i, j \in N \backslash R(N, v, \Gamma)$, there is $(f_i^8(N, v, \Gamma) - f_i^8(C(i), v_{|C(i)}, \Gamma_{C(i)})) - (f_j^8(N, v, \Gamma) - f_j^8(C(j), v_{|C(j)}, \Gamma_{C(j)})) = (c_i^{N,v,\Gamma} - c_j^{N,v,\Gamma}) \frac{E^{N,v,\Gamma}}{\sum_{j \in N \backslash R(N,v,\Gamma)} c_j^{N,v,\Gamma}}$. Hence, f^8 violates **SMNR**. **E**, **CMC**, and **RS** can be verified to be valid for f^8.

5.5 Conclusions

In our efficient extensions, we use the CEA rule and CEL rule to implement the division of the surplus, by which two efficient extensions of the Myerson value are achieved. Following the spirit of the CEA rule and CEL rule, the two efficient extensions of the Myerson value end up with giving priority to specific groups of players when dividing the surplus. The efficient constrained equal awards Myerson value turns out to be more favourable for players who make a smaller expected marginal contribution in the underlying game, whereas the efficient constrained equal losses Myerson value is preferred by larger contributors. Note that there is an alternative bankruptcy rule which can be employed to avoid such preference among players, i.e., the proportional rule [122]. This rule makes awards proportional to players' claims. If we apply it to the graph-induced bankruptcy problem, we could get another efficient extension of the Myerson value in which all players experience a same percentage of awards or losses in their own claims, and thereby they are treated symmetrically. Moreover, it turns out that this efficient extension of Myerson value belongs to the class of efficient β Myerson value [79].

Chapter 6

Algorithmic Solutions for Cost Sharing Beyond the Core

In contrast to the previous chapters, this chapter is on cost sharing rather than value distribution. This is due to the fact that the problems studied in this chapter are more natural in a cost sharing context. In the previous chapters, we considered solutions for TU-games in which cooperation among players is restricted by coalition structures and communication structures. The proposed values were studied under the assumption that the grand coalition is formed. As a matter of fact, this assumption is standard for the field of cooperative of game theory, hence also for most other solutions for TU-games, both with or without cooperation restrictions.

In this chapter, we return to this assumption and move on to the topic of the stability of the grand coalition. Basically, it seems reasonable to assume that the grand coalition will form as long as the stability of the grand coalition is "guaranteed", meaning that no player, or coalition of players can be better off by deviating from the grand coalition and acting on their own. That being said, one can wonder what the extremal solutions are that still satisfy this type of stability. Effectively, our approach is to drop the efficiency constraint from the core, and study an optimization problem to maximize the total shareable costs while maintaining coalitional rationality. We refer to the set of all allocations satisfying coalitional rationality as the almost core. As we focus on cost rather than value TU-games, note that the worth of a coalition represents the cost incurred by players of this coalition if they cooperate among themselves.

6.1 Introduction

There are many situations where all players are required to cooperate so that the total cost incurred by completing a joint task can be reduced. When TU-games are employed to model such cooperative situations, they are usually referred to as cost TU-games. Given a cost TU-game (N,c), a payoff vector $x \in \mathbb{R}^N$ is also called an allocation for (N,c), where each component x_i is the cost share allocated to each player $i \in N$. An efficient allocation x, i.e., $x(N) = c(N)$, is also said to be budget balanced. It is called stable if it satisfies coalitional rationality, i.e., $x(S) \leq c(S)$ for all $S \subsetneq N$.

The *core* [56] of game (N,c), arguably one of the most important concepts in cooperative game theory, consists of all budget balanced allocations satisfying coalitional rationality. The core of a cost TU-game is given by

$$C(N,c) = \{x \in \mathbb{R}^N \mid x(N) = c(N),\ x(S) \leq c(S), \forall S \subsetneq N\}.$$

The core of a TU-game is non-empty iff the game is balanced[1] [29, 105]. In fact, being balanced is just a dual characterization of the non-emptiness of the polyhedron $C(N,c)$.

Core allocations are required to be budget balanced, i.e., the total cost allocated to all players is exactly the worth of the grand coalition. However, there may be a gap between the total cost incurred by all players and the shareable costs for them. In some cases, $c(N)$ is too high to be shared by all players since for any efficient allocation there may exist coalitions finding it more appealing to choose to deviate and thereby blocking the grand coalition. In other words, the core may be empty. Moreover, the total shareable costs for all players may be higher than $c(N)$. This is possible when in a given allocation no subset of

[1] A cost TU-game is balanced if $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_S c(S) \geq c(N)$ holds for every balanced collection of weights λ.

players is able to be better off by leaving, so no coalition can pose a threat to the stability of the grand coalition. When dropping the equality constraint of being budget balanced, it allows to vary the total cost that is divided over the set of players, resulting a problem that always has a feasible solution. In particular, this captures the idea that, depending on the underlying game, one may have to, or want to, divide either less or more than $c(N)$.

Hence, we put emphasis on allocations which are stable but need not be budget balanced. For convenience, we refer to the set of all stable allocations as the almost core. Formally, for a cost TU-game $(N,c) \in \mathcal{G}$, the *almost core* is given by

$$AC(N,c) = \{x \in \mathbb{R}^N : x(S) \leq c(S), \forall S \subsetneqq N\}.$$

Obviously, $C(N,c) \subseteq AC(N,c)$. The major motivation for this definition is to systematically study the algorithmic complexity of cooperative games without having to obey to budget balance, so optimization over the polyhedron $AC(N,c)$. Let us motivate the relevance of this problem.

On the one hand, if the total costs $c(N)$ of the grand coalition cannot be distributed over the set of players while maintaining coalitional stability, i.e., the game is unbalanced, it is a natural question to ask what fraction of the total cost $c(N)$ can be maximally distributed while maintaining coalitional stability. For this case maximizing $x(N)$ over the almost core is in fact equivalent to some of the earlier proposed core relaxations; see Section 6.2.

On the other hand, also if the core is non-empty one may be interested in maximizing the total cost that can be distributed over the set of players. It answers the question by how much one could maximally tax the total cost $c(N)$ of the grand coalition, without any subset of players $S \subsetneqq N$ wanting to deviate.

That said, the object of interest of this chapter is the following optimization problem defined by a linear program over the almost core. We call it the almost core optimization problem. For a cost TU-game, *the almost core optimization problem* is given by

$$\max\{x(N) : x \in AC(N,c)\}. \tag{6.1}$$

The objective value of the linear program (6.1) indicates the largest value that can be shared among the players while retaining stability in the sense that no subset of players $S \subsetneq N$ would prefer to deviate, as no subset can realize smaller costs on its own. We call an optimal solution value for this linear program the *almost core optimum*, and any maximizer an *optimal almost core allocation*. The set of all optimal almost core allocations is called the *optimal almost core*, and denoted by $AC^*(N,c)$. Sometimes we also consider the restricted problem where we require that $x \geq 0$, which means that players must not receive subsidies.

Clearly, the core of a game is non-empty if and only if the almost core optimum is larger than or equal to $c(N)$. In particular, when equality holds, the set of optimal almost core allocations equals the core, which is then just a facet of $AC(N,c)$. We study Problem (6.1) mainly for games with non-empty cores, while for games with empty core we give a fairly complete overview of its relation to earlier work in Section 6.2.

The contribution and structure of this chapter are as follows. We briefly review the related core relaxations for unbalanced games in Section 6.2, and discuss how they relate to the almost core optimization problem. In comparison to most of these papers, an interesting aspect of our approach is to specifically address balanced games. Section 6.3 then relates linear optimization over the almost core to the core, and we also derive some of the algorithmic consequences. Section 6.4 addresses the almost core optimization problem (6.1) for minimum cost spanning tree (mcst) games. These are known to have a non-empty core, and finding a core element is easy, while linear optimization over the core is hard. Moreover, mcst games show that the computational complexity results which hold for games with superadditive or submodular cost functions, are no longer true for subadditive case. Indeed, as we will argue, computing an optimal almost core allocation is NP-hard for mcst games. Under the additional assumption of nonnegativity, we show how to derive a 2-approximation algorithm. Section 6.5 gives some final conclusions.

6.2 Equivalent and related relaxations of the core

In this section we review several well-known and related concepts that were introduced in order to deal with games having an empty core and discuss their relationship to the almost core (optimum).

For TU-games with empty cores, several core relaxations have been proposed in order to restore the stability. Typically, two approaches are suggested: i) injecting an external subsidy to the grand coalition; ii) imposing taxes on players or coalitions so that their bargaining power is weakened and thereby enforcing them to stay in the grand coalition. Let us proceed with recalling some of them.

The first relaxation of the core, introduced by Shapley and Shubik [103], is the *strong ε-core*, defined as

$$C_s^{\varepsilon}(N,c) = \{x \in \mathbb{R}^N : \ x(N) = c(N),\, x(S) \leq c(S) + \varepsilon, \forall S \subsetneq N\}.$$

Given an allocation in the strong ε-core, no coalition will improve its payoff by leaving the grand coalition if a fixed tax of ε is imposed on each coalition except for the grand coalition. We denote the smallest $\varepsilon \geq 0$ for which this set is non-empty by ε_s^*. The corresponding set $C_s^{\varepsilon_s^*}(N,c)$ is called the *least core* [84], which is also described as the intersection of all nonempty strong ε-cores.

Shapley and Shubik [103] also introduced the *weak ε-core* as

$$C_w^{\varepsilon}(N,c) := \{x \in \mathbb{R}^N : \ x(N) = c(N),\, x(S) \leq c(S) + |S|\varepsilon, \forall S \subsetneq N\}.$$

For a weak ε-core allocation, all individuals pay the same tax of ε, and thereby a coalition is taxed with rate ε proportionally to its size. We denote the smallest $\varepsilon \geq 0$ for which this set is non-empty by ε_w^*. Note that by definition, for any $\varepsilon \geq 0$, $C_s^{\varepsilon}(N,v) \subseteq C_w^{\varepsilon}(N,v)$, and hence $\varepsilon_w^* \leq \varepsilon_s^*$.

Instead of using an additive relaxation of the constraints, Faigle and Kern [49] defined the multiplicative ε-core as

$$C_m^{\varepsilon}(N,c) := \{x \in \mathbb{R}^N : x(N) \geq c(N),\ x(S) \leq (1+\varepsilon)c(S), \forall S \subsetneq N\}.$$

Here, the tax imposed on coalition S is proportional to its worth $c(S)$. We denote the smallest $\varepsilon \geq 0$ for which this set is non-empty by ε_m^*.

A different viewpoint is called *approximate core* or *γ-core* [70] for some $\gamma \in [0,1]$, it is defined as

$$C_a^{\gamma}(N,c) := \{x \in \mathbb{R}^N : \gamma \cdot c(N) \leq x(N),\ x(S) \leq c(S),\ \forall S \subsetneq N\}.$$

Denote the largest $\gamma \leq 1$ for which this set is non-empty by γ_a^*. Here, the budget balance constraint of the core is relaxed to be γ-budget balanced, i.e., $\gamma c(N) \leq x(N) \leq c(N)$. By generalizing the Bondareva-Shapley theorem [29, 104] in a straightforward way, it is shown in [70] that the γ-core ($\gamma \leq 1$) for a cost sharing game $(N,c) \in \mathscr{G}$ is nonempty if and only if there is $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_S c(S) \geq \gamma c(N)$ for every $\lambda \in \mathscr{B}(N)$.

The gap between the almost core optimum and the total cost of the grand coalition $c(N)$ was also called the *cost of stability* for an unbalanced cooperative game by Bachrach et al. [12]. For (unbalanced) cost TU-games it is defined by Meir et al. [85] as

$$CoS(N,c) := c(N) - \max\{x(N) : x(S) \leq c(S)\ \forall S \subsetneq N\}\,.$$

An alternative viewpoint was independently introduced in a paper by Bejan and Gómez [21] who considered the so called *extended core*. They studied individual taxation schemes. In their model, taxes for individuals are distinguished and a coalition may be taxed differently depending on the payoff that its members receive. In order to define the extend core for cost TU-games, they denote the minimum subsidy as

$$\begin{aligned}\delta_{ec}^*(N,c) = \min\{t(N) : \exists (x,t) \in \mathbb{R}^N \times \mathbb{R}_{\geq 0}^N,\ x(N) = c(N),\\ (x-t)(S) \leq c(S)\ \forall S \subsetneq N\}\end{aligned} \tag{6.2}$$

The *extended core* is now the set of all budget balanced allocations for which the minimum above is attained (for suitable $t \in \mathbb{R}_{\geq 0}^n$).

Yet another comparable concept to stabilize an unbalanced game was considered by Zick et al. [123]. In contrast to the individual taxation scheme proposed by Bejan and Gómez [21], they suggested that the tax on a coalition $S \subseteq N$ is a lump sum tax t_S, instead of the sum of individual taxes over the members of the coalition. Denote the minimum total tax by

$$\delta_{ct}^*(N,c) = \min\{\sum_{S \subseteq N} t_S : \exists (x,t) \in \mathbb{R}^N \times \mathbb{R}_{\geq 0}^{2^N},\ x(N) = c(N),$$

$$x(S) - t_S \leq c(S) \ \forall S \subseteq N\} \tag{6.3}$$

This is an exponential blowup of the solution space, which however gives more flexibility.

For unbalanced games, computing the almost core optimum is by definition equivalent to computing the cost of stability $CoS(N,c)$. The following theorem further summarizes how the different core relaxations are related; the last equality is from [86, Section 4].

Theorem 6.1. *For any TU-game (N,c) with empty core, the optimization problems for the weak ε-core, the multiplicative ε-core, the cost of stability and the extended core are equivalent. In particular, the values satisfy*

$$\delta^*_{ec}(N,c) = (1-\gamma^*_a)\cdot c(N) = \frac{\varepsilon^*_m}{1+\varepsilon^*_m}\cdot c(N) = CoS(N,c) = \varepsilon^*_w \cdot n.$$

Proof. First, we establish $CoS(N,c) = \delta^*_{ec}(N,c)$. We substitute $x - t$ by x' in Eq. (6.2) and obtain

$$\delta^*_{ec}(N,c) = \min\{t(N) : \exists (x',t) \in \mathbb{R}^N \times \mathbb{R}^N_{\geq 0},\ x'(N) + t(N) = c(N), \\ x'(S) \leq c(S)\ \forall S \subsetneq N\}\,.$$

Now it is easy to see that the actual entries of t do not matter (except for nonnegativity), but only the value $t(N)$ is important. This yields $CoS(N,c) = \delta^*_{ec}(N,c)$.

Second, we show $CoS(N,c) = (1-\gamma^*_a)\cdot c(N)$. To this end, observe

$$\gamma^*_a = \max\{\gamma \in \mathbb{R} : \exists x \in \mathbb{R}^N,\ x(S) \leq c(S)\ \forall S \subseteq N,\ x(N) = \gamma c(N)\}\,.$$

Clearly, the maximum is attained by $x^\star \in \mathbb{R}^N$ with $x^\star(N)$ maximum. Moreover, the value of γ^*_a is then equal to $x^\star(N)/c(N)$. This shows $CoS(N,c)/c(N) = 1-\gamma^*_a$.

Third, we show $1-\gamma^*_a = \varepsilon^*_m/(1+\varepsilon^*_m)$. Observe that the map $\pi : \mathbb{R}^N \to \mathbb{R}^N$ defined by $\pi(x) = (1+\varepsilon)x$ induces a bijection between allocations $x \in \mathbb{R}^N$ with $x(S) \leq c(S)$ for all $S \subsetneq N$ and allocations $\pi(x)$ with $\pi(x)(S) \leq (1+\varepsilon)c(S)$ for all $S \subsetneq N$. Moreover, $\pi(x)(N) = (1+\varepsilon)x(N)$. Hence, $C^\varepsilon_m(N,c)$ is (non-)empty if and only if $C_a^{\gamma}(N,c)$ is (non-)empty, where $\gamma = 1/(1+\varepsilon)$ holds. This implies $\gamma_a^* = 1/(1+\varepsilon_m^*)$.

We finally show $CoS(N,c) = \varepsilon_w^* \cdot n$. To this end, in

$$\varepsilon_w^* = \min\{\varepsilon \geq 0 : \exists x \in \mathbb{R}^N,\ x(S) \leq c(S) + \varepsilon \cdot |S|\ \forall S \subsetneq N,\ x(N) = c(N)\}$$

we substitute x by $x' + (\varepsilon, \varepsilon, \ldots, \varepsilon)$ which yields

$$\varepsilon_w^* = \min\{\varepsilon \geq 0 : \exists x' \in \mathbb{R}^N,\ x'(S) \leq c(S)\ \forall S \subsetneq N,\ x'(N) + \varepsilon \cdot n = c(N)\}\,.$$

Clearly, the minimum ε_w^* is attained if and only if $\varepsilon \cdot n = CoS(N,c)$ holds. □

Moreover, it was shown in [86, Section 4] that $\varepsilon_w^* \geq \frac{1}{n-1}\varepsilon_s^*$. Further relations between the cost of stability $CoS(N,c)$ and other core relaxations for specific classes of games appear in [13, 86].

Let us next briefly discuss some of the related work. To the best of our knowledge, most of the previous work in this direction was about determining bounds on the (relative) cost of stability[2] for several classes of TU-games, including standard (cost) TU-games with simple axiomatic condition like superadditive, subadditive and anonymous games etc. [12, 85, 86], combinatorial optimization games like weighted voting games [12], facility games [85], threshold network flow games [99], and games with restrictions like coalitional size, partition and communication graph [30, 86, 87]. Algorithmic aspects of the cost of stability for weighted voting games and the cost of stability for TU-games with coalition structures were also provided in [12]. Aziz et al. [11] conducted a computational analysis of the cost of stability and the least core for several subclass of monotone TU-games and threshold simple game versions. Chalkiadakis et al. [38] analysed several complexity issues for core-related problems over compact games under the restriction of various interaction graphs such as lines, cycles, nearly-acyclic graphs, trees and complete graphs.

Approximations of ε_m^* for the multiplicative $(1+\varepsilon)$-core and corresponding allocations have also been obtained for the symmetric traveling salesman game by Faigle et al. [51], and for the asymmetric case by Bläser et al. [27]. There are

[2] The ratio between the almost core optimum and the cost of the grand coalition is referred to as the relative cost of stability, see [86].

also papers that attack the problem from a computational point of view. Under the name “optimal cost share problem”, Caprara and Letchford [34] computationally obtained γ-core solutions for a generalization of certain combinatorial games, named integer minimization games. Under the name “optimal cost allocation problem”, also Liu et al. [80] gave computational results using Lagrangian relaxation, which also works for nonlinear objective functions.

The problem to compute allocations in the least core has been considered also in the literature. For cooperative games with submodular cost functions, it can be computed in polynomial time [40], while for supermodular cost functions it is NP-hard to compute, and even hard to approximate [100]. Finally, Faigle et al. [52] showed NP-hardness to compute a cost allocation in the so-called f-least core for minimum cost spanning tree games, which is a tightening of the core constraints to $x(S) \leq c(S) - \varepsilon f(S)$ for certain non-negative functions f. As we will argue later, their result also implies hardness of computing optimal almost core allocations for minimum cost spanning tree games.

6.3 Computational complexity considerations

In this section we investigate the computational complexity of optimization problems related to the (nonnegative) core and almost core. To capture results for the general and the nonnegative case, we consider linear optimization over the polyhedra

$$AC(N,c) \quad \text{and} \quad P(N,c) := \{x \in \mathbb{R}^N : x(S) \leq c(S)\ \forall S \subseteq N\}.$$

as well as optimization over $P(N,c) \cap \mathbb{R}^N_{\geq 0}$ and $AC(N,c) \cap \mathbb{R}^N_{\geq 0}$ for families of games (N,c). Note that if the core is non-empty then it is the set of optimal solutions when we maximize $\mathbb{1} \cdot x$ over $P(N,c)$. Also note that whenever the core of a game (N,c) is empty, this means that the constraint $x(N) < c(N)$, and hence also the constraint $x(N) \leq c(N)$ are implied by the set of constraints $x(S) \leq c(S)$, $S \subsetneq N$, which in turn implies $P(N,c) = AC(N,c)$. For games with a non-empty core, we get the following correspondence between the optimization problems for the two polyhedra.

Theorem 6.2. *For a family of games (N,c), linear optimization problems over $AC(N,c)$ can be solved in polynomial time if and only if linear optimization problems over $P(N,c)$ can be solved in polynomial time.*

Proof. In order to prove the result we make use of the equivalence of optimization and separation [59, 74, 96]. This means, we only need to show that we can solve the separation problem for $P(N,c)$ if and only if we can solve the separation problem for $AC(N,c)$. Since $P(N,c) = \{x \in AC(N,c) : x(N) \leq c(N)\}$ holds, separation over $P(N,c)$ reduces to separation over $AC(N,c)$ plus an explicit check of a single inequality.

It remains to show how to solve the separation problem for $AC(N,c)$ when we can solve the separation problem for $P(N,c)$. For given $\hat{x} \in \mathbb{R}^n$, we construct n points $\hat{x}^k \in \mathbb{R}^N$ $(k = 1,2,\ldots,n)$ which are copies of $\hat{x}$ except for $\hat{x}^k_k := \min(\hat{x}_k, c(N) - \sum_{i \in N\setminus\{k\}} \hat{x}_i)$. Note that by construction $\hat{x}^k \leq \hat{x}$ and $\hat{x}^k(N) \leq c(N)$ hold for all $k \in N$. We then query a separation oracle of $P(N,c)$ with each $\hat{x}^k$.

Suppose such a query yields $\hat{x}^k(S) > c(S)$ for some $S \subseteq N$. Due to $\hat{x}^k(N) \leq c(N)$ we have $S \neq N$. Moreover, $\hat{x} \geq \hat{x}^k$ implies $\hat{x}(S) > c(S)$, and we can return the same violated inequality.

Otherwise, we have $\hat{x}^k \in P(N,c)$ for all $k \in N$. We claim that then $\hat{x} \in AC(N,c)$. To prove this we assume that, for the sake of contradiction, $\hat{x}(S) > c(S)$ holds for some $S \subsetneq N$. Let $k \in N \setminus S$. Since $\hat{x}^k_i = \hat{x}_i$ holds for all $i \in S$, we have $\hat{x}^k(S) = \hat{x}(S) > c(S)$. This contradicts the fact that $\hat{x}^k \in P(N,c)$. □

It turns out that almost the same result is true when we also require that there are no subsidies, that is $x \geq 0$. For linking the non-negative core to the non-negative almost core, we require an assumption on the characteristic function.

$$c(N \setminus \{k\}) \leq c(N) \qquad \forall k \in N. \tag{6.4}$$

This condition holds, for instance, for monotone functions c, and implies that the core is contained in $\mathbb{R}^N_{\geq 0}$ (see Lemma 2 and Theorem 1 in [43]).

Theorem 6.3. *For a family of games (N,c) satisfying* (6.4)*, linear optimization problems over $AC(N,c) \cap \mathbb{R}^N_{\geq 0}$ can be solved in polynomial time if and only if* *linear optimization problems over* $P(N,c) \cap \mathbb{R}^N_{\geq 0}$ *can be solved in polynomial time.*

The proof is an extension of that of Theorem 6.2, additionally making use of condition (6.4) to guarantee nonnegativity. We obtain an immediate consequence from these two theorems.

Corollary 6.1. *For a family of games* (N,c) *for which* $c(\cdot)$ *is submodular (and* (6.4) *holds) one can find a (non-negative) optimal almost core allocation in polynomial time.*

Proof. For submodular $c(\cdot)$ one can optimize any linear objective function over $P(N,c)$ using the Greedy algorithm [47]. The result follows from Theorem 6.2 and Theorem 6.3. □

These results only make statements about optimizing arbitrary objective vectors over these polyhedra. In particular we cannot draw conclusions about hardness of the computation of an almost core allocation. However, it is easy to see that this problem cannot be easier than deciding non-emptiness of the core.

Theorem 6.4. *Consider a family of games* (N,c) *for which deciding nonemptiness of the core is (co)NP-hard. Then finding an optimal almost core allocation is also (co)NP-hard.*

Proof. By the premise of the theorem there exists a Karp reduction from some NP-hard problem $\mathscr{P}$ to the non-emptiness decision problem for our family of games. The reduction turns (in polynomial time) an instance $\mathscr{I}$ of $\mathscr{P}$ into a game (N,c) such that $\mathscr{I}$ is a YES-instance (resp. NO-instance) if and only if (N,c) has a non-empty core. The same reduction works for the almost core since $\mathscr{I}$ is a YES-instance (resp. NO-instance) if and only if the almost core optimum is at least $c(N)$. □

It is well known that there exist games for which it is NP-hard to decide non-emptiness of the core, e.g., the weighted graph game [41]. Hence, and maybe not surprisingly, we cannot hope for a polynomial-time algorithm that computes an optimal almost core allocation for arbitrary games.

In contrast, the maximization of $x(N)$ becomes trivial for games (N,c) with superadditive characteristic function $c(\cdot)$, as the set of constraints $x(\{i\}) \leq c(\{i\})$, $i = 1,\dots,n$, already imply all other constraints $x(S) \leq c(S)$, $S \subsetneqq N$, and one can simply define $x_i := c(\{i\})$ for all $i \in N$. In particular, the constraint $x(N) \leq c(N)$ is implied and $P(N,c) = AC(N,c)$. Moreover, for superadditive games with non-empty cores, since the non-emptiness of the core implies $x(N) \geq c(N)$ for all $x \in AC^*(N,c)$, and $x(N) \leq c(N)$ is derived from $AC^*(N,c) \subseteq AC(N,c)$, we conclude that $x(N) = c(N)$ for all $x \in AC^*(N,c)$. This implies $AC^*(N,c) = C(N,c)$ for superadditive balanced games.

Actually, the equivalence of the two polyhedra, $P(N,c)$ and $AC(N,c)$, holds for all classes of games where a polynomial number of constraints can be shown to be sufficient to define the complete core. As an example of such a game, we mention *matching games* in undirected graphs [75], where the core is completely defined by the polynomially many core constraints induced by all edges of the graph, as these can be shown to imply all other core constraints.

Proposition 6.1. *Whenever $P(N,c)$ is described by a polynomial number of inequalities, finding an optimal (almost) core allocation can be done in polynomial time by linear programming.*

Note that Proposition 6.1 also includes supermodular cost functions. It is therefore interesting to note that for supermodular cost games, it is NP-hard to approximate the least core value ε_s^* better than a factor $17/16$ [100].

It also turns out that Condition (6.4) implies that the value of an almost core allocation cannot exceed that of a core allocation by much.

Proposition 6.2. *Let (N,c) be a game that satisfies* (6.4). *Then every $x \in AC(N,c)$ satisfies $x(N) \leq \left(1 + \frac{1}{n-1}\right)c(N)$.*

Proof. Let $x \in AC(N,c)$. We obtain

$$(n-1) \cdot x(N) = \sum_{k \in N} x(N \setminus \{k\}) \leq \sum_{k \in N} c(N \setminus \{k\}) \leq \sum_{k \in N} c(N) = n \cdot c(N),$$

where the first inequality follows from feasibility of x and the second follows from (6.4). □

Condition (6.4) implies non-negativity for all core allocations and all optimal almost core allocations. However, this does not mean that a non-negativity requirement implies that the almost core optimum is close to $c(N)$. In the next section, we will see that this gap can even be arbitrarily large (see Proposition 6.3).

6.4 Minimum cost spanning tree games and approximation

In this section we address a well known special class of games known as minimum cost spanning tree (mcst) games [26, 39, 58], where the cost of a set of players is determined by the cost of a minimum cost spanning tree for these players. In particular, mcst games are known to have a non-empty core. The optimization problem that we address asks for the maximal amount that can be charged to the players while no subsets of all players would prefer their outside option. What makes the almost core optimization problem for mcst games interesting is the fact that finding an element in the core is computationally easy, while linear optimization over the core is hard. This led us to believe that an algorithm to compute an almost core optimum might be in reach. Moreover, mcst games are a class of games with subadditive but not submodular cost functions. Given our earlier results that settled the cases of arbitrary submodular (and superadditive) cost functions, it is worthwhile noting that comparable results cannot be expected for the subadditive case, as we will show.

Following a run of Prim's algorithm and by the fact that Bird's rule is a core element [58], it is clear that computing *some* core allocation can be done efficiently. However, linear optimization over the core of mcst games is co-NP hard [50]. We are interested in a linear optimization problem with the specific objective function $\mathbb{1} \cdot x$ but for the case that the budget balance constraint is absent. So we seek solutions to the almost core optimization problem

$$\max x(N) \; s.t. \; x \in AC(N,c), \tag{6.5}$$

when $c(\cdot)$ is the characteristic function defined by costs of minimum cost spanning trees. The interpretation of the lacking constraint $x(N) = c(N)$ is that the grand coalition cannot establish the solution with cost $c(N)$ on its own.

6.4.1 Computational complexity

As a first result, and not surprising, linear optimization over the almost core is NP-hard under Turing reductions for mcst games.

Corollary 6.2. *For minimum cost spanning tree games (N,c), a polynomial time algorithm for linear optimization over $AC(N,c)$ would yield $P = NP$.*

Proof. The result follows from Theorem 6.2 and the fact that the membership problem for the core of (N,c) is a coNP-hard problem for mcst games [50]. □

What is more interesting is that optimizing $\mathbb{1} \cdot x$ over the almost core is hard for mcst games.

Theorem 6.5. *Computing an optimal almost core allocation in Problem* (6.5) *for minimum cost spanning tree games is NP-hard.*

Proof. Let $\varepsilon^\star$ be the largest ε for which the linear inequality system

$$x(S) \leq (1-\varepsilon)c(S) \ \forall S \subsetneq N, \quad x(N) = c(N) \tag{6.6}$$

has a solution. In [52] it is shown that finding a feasible solution x for (6.6) with respect to $\varepsilon^\star$ is NP-hard. Note that in the reduction leading to this hardness result, $c(N) > 0$. Then, given an optimum almost core allocation x^{AC}, $x^{\mathrm{AC}}(N) \geq c(N) > 0$, and we can obtain $\varepsilon^\star := 1 - c(N)/x^{\mathrm{AC}}(N)$. It is now easy to see that the vector $x' := (1-\varepsilon^\star)x^{\mathrm{AC}}$ is a feasible solution for (6.6). To see that the so-defined $\varepsilon^\star$ is indeed maximal, observe that scaling any feasible vector in (6.6) by $1/(1-\varepsilon^\star)$ yields an almost core allocation. Hence, computation of an almost core optimum for mcst games yields a solution for an NP-hard problem. □

Next, we note that in general, the almost core optimum may be arbitrarily larger than $c(N)$ for mcst games. This is interesting in view of Proposition 6.2, which shows that under Condition (6.4), any core allocation yields a good approximation for an optimal almost core allocation, as they differ by a factor at most $n/(n-1)$. A fortiori, the same holds for the monotonized mcst games $(N, \bar{c})$. For general mcst games (N, c), and without Condition (6.4), this gap can be large.

Proposition 6.3. *The almost core optimum for Problem (6.5) can be arbitrarily larger than* $c(N)$*, even when we require that* $x \geq 0$.

Proof. Consider the instance depicted in Figure 6.1, for some value $k > 0$. Then $c(N) = 0$, while $x = (0, 0, k)$ is an optimal non-negative almost core allocation with value k.

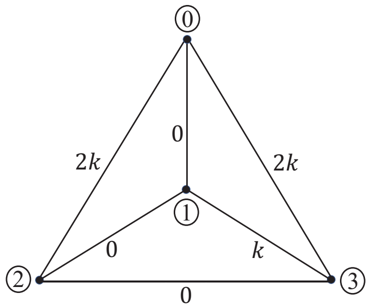

Figure 6.1: Mcst game with large relative gap between almost core optimum and $c(N)$

□

In the following we consider Problem (6.5) but with the added constraint that $x \geq 0$.

$$\max x(N)\ s.t.\ x \in AC(N, c), \text{and } x \geq 0\,. \tag{6.7}$$

The presence of the constraint $x \geq 0$ means that agents must not be subsidized. First, we show that even this restricted problem remains NP-hard for mcst games.

Theorem 6.6. *Computing an optimal non-negative almost core allocation in (6.7) for minimum cost spanning tree games is NP-hard.*

Proof. The claim follows by showing that Problem (6.5) can be reduced in polynomial time to Problem (6.7). The reduction works as follows. Given an instance of (6.5) with cost function c which is induced by cost matrix $W = (w_{ij})_{i,j\in N_0}$, we define a new cost matrix $W' = (w'_{ij})_{i,j\in N_0}$ where $w'_{ij} = w_{ij} + M$, $i, j \in N_0$, for some large enough constant M. This induces cost function $c' : 2^N \to \mathbb{R}_{\geq 0}$. Now consider an optimal solution x' to Problem (6.7) for cost function c', and define $x := x' - (M, \ldots, M)$. Now we have $x(S) = x'(S) - |S| \cdot M \leq c'(S) - |S| \cdot M = c(S)$ for all $S \subsetneq N$, so x is feasible for Problem (6.5). We claim that x is also optimal for Problem (6.5). This is true since for *any* solution $\tilde{x}$ that is optimal for (6.5), we have $y' := \tilde{x} + (M, \ldots, M) \geq 0$ for large enough M, and $y'(S) = \tilde{x}(S) + |S| \cdot M \leq c(S) + |S| \cdot M = c'(S)$, so y' is feasible for (6.7) with cost function c'. Hence, $\tilde{x}(N) > x(N)$ yields the contradiction $y'(N) > x'(N)$. Finally, observe in Problem (6.5) we maximize $x(N)$, hence for any optimal solution $\tilde{x}$ there exists $M > 0$ so that $\tilde{x}_i \geq -M$ for all $i \in N$, e.g. one can easily see that $M := \sum_{j\in N} c(\{j\})$ suffices. This is true because for an optimal solution $\tilde{x}$, for all $i \in N$ there exists some $S \ni i$ so that constraint $x(S) \leq c(S)$ is tight, and $\tilde{x}_i \leq c(\{i\})$ for all $i \in S$. □

Remark 6.1. The above reduction of computing arbitrary allocations to computing non-negative allocations generalizes to general cost sharing games (N, c), by defining $c'(S) := c(S) + |S| \cdot M$ for all subsets $S \subsetneq N$.

6.4.2 2-Approximation algorithm

We next propose the following polynomial time algorithm to compute an approximately optimal almost core allocation for Problem (6.7). For notational convenience, let us define for all $i = 1, \ldots, n$,

$$N_{-i} := N \setminus \{i\}.$$

Algorithm 1: Approximation algorithm for the almost core maximization Problem (6.7) for mcst games

Input: A minimum cost spanning tree problem (N_0, W)

Output: Almost core allocation x.

1 Initialize $I_0 := \{0\}$ and $T := \emptyset$.
2 **for** $k = 1, 2, \ldots, n$ **do**
3   Let $i \in I_{k-1}$, $j \in N \setminus I_{k-1}$ with minimum w_{ij} (among those i, j).
4   Let $I_k := I_{k-1} \cup \{j\}$ and augment the tree $T := T \cup \{\{i, j\}\}$.
5   Assign player j the cost share $x_j := w_{ij}$.
6 **end**
7 Let $\ell \in I_n \setminus I_{n-1}$ be the last player assigned.
8 Update player ℓ's cost share $x_\ell := \min_{k \in N_{-\ell}} \{c(N_{-k}) - x(N \setminus \{k, \ell\})\}$.

The backbone of Algorithm 1 is effectively Prim's algorithm to compute a minimum cost spanning tree [98]. The additional Line 5 yields the core allocation by Granot and Huberman [58], which we extend by adding Lines 7 and 8.

Let us first collect some basic properties of Algorithm 1. Henceforth, we assume w.l.o.g. that the players get assigned their cost shares in the order $1, \ldots, n$ (so that $\ell = n$ in Lines 7 and 8). We denote by x^{ALG} a solution computed by Algorithm 1.

Lemma 6.7. *We have that $x^{ALG}(I_k) = c(I_k)$ for all $k = 1, \ldots, n-1$, and for all $S \subseteq \{1, \ldots, n-1\}$ we have $x^{ALG}(S) \leq c(S)$.*

Proof. The first claim follows directly because Algorithm 1 equals Prim's algorithm to compute a minimum cost spanning tree on the vertex set $\{0, 1, \ldots, n-1\}$, and $x^{\text{ALG}}(I_k)$ equals the cost of the minimum spanning tree on vertex set $\{0, 1, \ldots, k\}$, Hence by Prim's algorithm [98], $x^{\text{ALG}}(I_k) = c(I_k)$. The second claim follows by [58, Thm. 3], since the cost allocation for players $\{1, \ldots, n-1\}$ is the same as in [58]. □

Lemma 6.8. *Suppose $x^{ALG}(S) > c(S)$ for some set S with $n \in S \subsetneq N$. Then there is a superset $N_{-k} \supseteq S$ $(k \in N_{-n})$ such that $x^{ALG}(N_{-k}) > c(N_{-k})$.*

Proof. Recall the players got assigned their cost shares in order $1, \ldots, n$. Define $k := max\{i \mid i \notin S\}$ to be the largest index of a player not in S. Let $i_1, \ldots, i_m$

be the set of players so that $N_{-k} = N \setminus \{k\} = S \cup \{i_1, \ldots, i_m\}$ and w.l.o.g. $i_1 < \cdots < i_m$. We show that $x^{\text{ALG}}(S) > c(S)$ implies $x^{\text{ALG}}(S \cup \{i_1\}) > c(S \cup \{i_1\})$. Then repeating the same argument, we inductively arrive at the conclusion that $x^{\text{ALG}}(N_{-k}) > c(N_{-k})$. So observe that

$$x^{\text{ALG}}(S \cup \{i_1\}) = x^{\text{ALG}}(S) + x_{i_1} > c(S) + x_{i_1},$$

and $c(S)$ is the cost of a minimum cost spanning tree for S, call it MCST(S). Moreover, as $i_1 \neq n$, x_{i_1} is the cost of the edge, call it e, that the algorithm used to connect player i_1. We claim that MCST$(S) \cup \{e\}$ is a tree spanning vertices $S \cup \{0, i_1\}$, hence $c(S) + x_{i_1}$ is the cost of some tree spanning $S \cup \{0, i_1\}$. Then, as required we get

$$x^{\text{ALG}}(S \cup \{i_1\}) > c(S) + x_{i_1} \geq c(S \cup \{i_1\}),$$

because $c(S \cup \{i_1\})$ is the cost of a *minimum cost* tree spanning $S \cup \{0, i_1\}$. If MCST$(S) \cup \{e\}$ was not a spanning tree for vertices $S \cup \{0, i_1\}$, then edge e would connect i_1 to some vertex outside $S \cup 0$, but this contradicts the choice of i_1 as the vertex outside $S \cup 0$ with smallest index. □

Lemma 6.9. *We have $x^{ALG} \geq 0$.*

Proof. Recall that in minimum cost spanning tree games [39, 58], the weight of edges are non-negative. Since Algorithm 1 computes the allocation for players in Line 5 by the edge weight of the first edge on the unique path to 0, there is $x_k^{\text{ALG}} \geq 0$ for all $k = 1, 2, \cdots, n-1$. So we only need to argue about x_n^{ALG}. To that end, note that an equivalent definition of x_n^{ALG} in Line 8 of the algorithm is

$$\text{max. } x_n \text{ s.t. } x_n \leq c(N_{-k}) - x^{\text{ALG}}(N \setminus \{k, n\}) \text{ for all } k = 1, \ldots, n-1. \quad (6.8)$$

We claim that $\tilde{x}_n := c(N) - c(N_{-n}) \geq 0$ is a feasible solution to this maximization problem, hence the actual value of x_n^{ALG} after the update in Line 8 can only be larger, and therefore in particular it is non-negative. First, note that indeed, $\tilde{x}_n \geq 0$, as this is the cost of the last edge that Prim's algorithm uses to connect the final vertex n to the minimum cost spanning tree. That $\tilde{x}_n$ is feasible in (6.8) follows from the fact that $\tilde{x}_n$ is the cost share that is assigned to player

n in the core allocation of [58]. Indeed, letting $\tilde{x}$ be equal to x^{ALG} except for $\tilde{x}_n = c(N) - c(N_{-n})$, we have that $\tilde{x}$ is precisely the cost allocation as proposed in [58]. By the fact that this yields a core allocation, we have that $\tilde{x}(S) \leq c(S)$ for all $S \subseteq N$, so in particular for all $k = 1, \ldots, n-1$,

$$\tilde{x}_n + x^{\text{ALG}}(N \setminus \{k, n\}) = \tilde{x}(N_{-k}) \leq c(N_{-k}),$$

and hence the claim follows. □

Theorem 6.10. *Algorithm 1 is a 2-approximation for the almost core maximization Problem* (6.7) *for minimum cost spanning tree games, and this bound is tight.*

Proof. Denote by x^{ALG} a solution by Algorithm 1. We first argue that Algorithm 1 yields a feasible solution. For $S \not\ni n$, this follows from Lemma 6.7. For $S \ni n$, assume $x(S) > c(S)$. Then Lemma 6.8 yields that there exists some $N_{-k} \ni n$ with $x^{\text{ALG}}(N_{-k}) > c(N_{-k})$. However by definition of x_n in Line 8 of the algorithm, we have for all $k = 1, \ldots, n-1$

$$x_n^{\text{ALG}} \leq c(N_{-k}) - x^{\text{ALG}}(N \setminus \{k, n\}),$$

which yields a contradiction to $x^{\text{ALG}}(N_{-k}) > c(N_{-k})$.

To show that the performance guarantee is indeed 2, let x^{OPT} be some optimal solution to the almost core maximization problem. Let $k^\star \in N_{-n}$ be the index for which the minimum in Line 8 is attained. Observe that x_n^{ALG} is updated such that $x^{\text{ALG}}(N_{-k^\star}) = c(N_{-k^\star})$ holds. Then by non-negativity of x^{OPT} and because of Lemma 6.9,

$$x_n^{\text{OPT}} \leq x^{\text{OPT}}(N_{-k^\star}) \leq c(N_{-k^\star}) = x^{\text{ALG}}(N_{-k^\star}) \leq x^{\text{ALG}}(N).$$

Moreover, by definition of x^{ALG}, we have $x^{\text{ALG}}(N_{-n}) = c(N_{-n})$, and by Lemma 6.9,

$$x^{\text{OPT}}(N_{-n}) \leq c(N_{-n}) = x^{\text{ALG}}(N_{-n}) \leq x^{\text{ALG}}(N).$$

Hence we get $x^{\text{OPT}}(N) = x_n^{\text{OPT}} + x^{\text{OPT}}(N_{-n}) \leq 2x^{\text{ALG}}(N)$. To see that the performance bound 2 is tight for Algorithm 1, consider the instance in Figure 6.2.

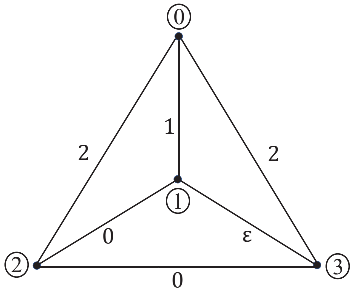

Figure 6.2: Mcst game showing that the analysis of Algorithm 1 cannot be improved.

Here, Algorithm 1 computes the solution $x^{\mathrm{ALG}} = (1, 0, \varepsilon)$ with value $1 + \varepsilon$, as the order in which players get assigned their cost shares is $1, 2, 3$, and in Line 8 of the algorithm we get $x_3^{\mathrm{ALG}} = c(\{1, 3\}) - x_1 = (1 + \varepsilon) - 1 = \varepsilon$. An optimal almost core allocation would be $x^{\mathrm{OPT}} = (0, 1, 1)$ with value 2. □

6.5 Conclusions

For minimum cost spanning tree games, we give a 2-approximation algorithm for the almost core optimization problem under the additional assumption that the subsidies are not allowed. As a matter of fact, the requirement that $x \geq 0$ is important in Theorem 6.10. Allowing that players receive subsidies, so allowing $x_i < 0$ for some players i, Algorithm 1 does not provide any approximation guarantee. To see that, consider the instance given in Figure 6.3. Observe that Algorithm 1 yields a cost allocation $x^{\mathrm{ALG}} = (0, 0, 0)$, while $x = (-k, k, k)$ is an optimal almost core allocation. Here, it is necessary to give player 1 a subsidy of k in the optimal almost core allocation. Note that the corresponding monotonized mcst game is trivial as $c(S) = 0$ for all coalitions S, and the almost core optimum is 0 with an optimal almost core allocation $(0, 0, 0)$. Hence, it would be interesting to extend the result in Theorem 6.10 to the general, unconstrained case. Moreover, one could define an even more general problem to drop not only the efficiency constraint $x(N) = c(N)$ but have an arbitrary set system $\mathscr{F} \subsetneq 2^N$ that describes all those subsets of players that are able to cooperate and hence

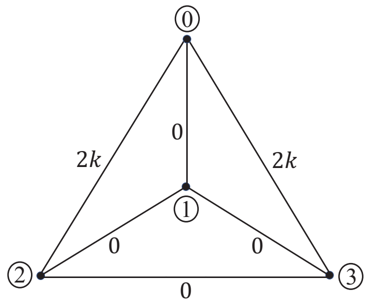

Figure 6.3: Mcst game where subsidies are necessary for the almost core optimization problem

have access to an outside option, while all other subsets of players do not have that option.

Summary

The thesis focuses on cooperative games with transferable utility and incorporates two topics: Solutions for TU-games restricted by cooperation structures and the stability of the grand coalition. While Chapters 3-5 contribute to the former topic with coming up with new solutions and justifying their reasonableness by different approaches, Chapter 6 deals with the latter topic by studying an optimization problem over the set of all stable allocations which we refer to as the almost core.

Chapter 3 is devoted to a new solution for cooperative games with coalition structures, called the α-egalitarian Owen value. This coalitional value extends the α-egalitarian Shapley value [71] to a coalition structure setting. Three approaches are used to characterize this coalitional value. Firstly, we provide two axiomatizations by introducing the α-indemnificatory null player axiom, and the (intra) coalitional quasi-balanced contributions axiom. Secondly, we define an α-guarantee potential function to show that the payoff vector $SP^*(N, v, \mathscr{C})$ equals the α-egalitarian Owen value. Finally, the coalitional value is implemented by a punishment-reward bidding mechanism. As to future work, it would be interesting to axiomatize the proposed coalitional value based on other axiomatic systems without additivity, such as axioms related to monotonicity [121] and associated consistency [61].

In Chapter 4, we continue to work with TU-games restricted by coalition structures and propose a coalitional value called the two-step Shapley-solidarity value. In comparison to the two-step Shapley value proposed by Kamijo [72], the difference lies in the idea to distribute the worth of a union among its members based on the solidarity value [92], which implements the solidarity principle

for the unions. We provide a procedural interpretation of the two-step Shapley-solidarity value. Moreover, we introduce a new axiom called the coalitional A-null player axiom to axiomatize the value based on additivity. Two other axiomatizations on the basis of quasi-balanced contributions for the grand coalition are also provided to further highlight the precise similarities and differences between our two-step coalitional value and the two-step Shapley value. Except for the given axiomatizations, a non-cooperative implementation of the two-step Shapley-solidarity value based on a bidding mechanism could be considered for future work. In addition, one may study the two coalitional values proposed in the thesis for fuzzy cooperative games with coalition structures.

In Chapter 5, we focus on cooperative games with communication structures and provide efficient extensions of the Myerson value [89]. The idea lies in introducing the Shapley payoffs of the underlying game as players' claims to derive a graph-induced bankruptcy problem. Then, two efficient extensions of the Myerson value are achieved through bankruptcy rules, including the CEA rule and the CEL rule [10]. In line with the spirit of the axioms satisfied by the CEA rule, we introduce two axioms called the sustainability in surplus axiom and the weak fair distribution surplus axiom to axiomatize the efficient constrained equal awards Myerson value. Similarly, we provide an axiomatization for the efficient constrained equal losses Myerson value by defining the so-called residual in surplus axiom and surplus marginality with non-residual players axiom. It remains interesting to consider other bankruptcy rules in a future work, the Talmud rule [10] and the random arrival rule [93], for example. Moreover, this approach could also be applied to deal with efficient extensions of the Aumann-Drèze value (AD-value) [9] for cooperative games with coalition structures.

Chapter 6 proceeds with studying the stability of the grand coalition for cost TU-games by addressing an optimization problem to maximize the total shareable costs over what we called the almost core. We analyse the computational complexity of this optimization problem, in relation to the computational complexity of related problems for the core. For games with an empty core, it turns out that it is equivalent to optimization problems for several core relaxations that have been proposed earlier. For games with a non-empty core, we specifically consider the minimum cost spanning tree games as an example of subadditive games with non-empty cores for which it is computationally easy to find a core element. We show that maximizing the total shareable costs over the (non-negative) almost core is NP-hard for mcst games, and we provide a 2-approximation algorithm for this almost core optimization problem under the additional assumption that no subsidy is allowed. It would be interesting to gain more insights into the computational complexity of the almost core problem, possible generalizations, and also for other classes of games.

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Acknowledgements

Until this moment, I feel so strongly that my doctoral career is coming to an end. It is my great privilege to be able to spend my PhD life both at Northwestern Polytechnical University and University of Twente with the support of China Scholarship Council. When I look back on this colorful and impressive journey, I would like to express my sincere appreciation to those who have helped, encouraged and supported me to complete this thesis.

Above all, I would like to show my huge thanks to my supervisors: Prof. Dr. Genjiu Xu and Prof. Dr. Marc Uetz.

It was Genjiu Xu that brought me into the field of game theory. There is a saying goes, everything is hard in the beginning. With his patient guidance and kind help, I got through the novice period and gradually learned the way of doing research. He would not hesitate to point out what I do not do well, and to give me appropriate suggestions to help me make progress. He always tries his best to provide us with a good research environment and supports me to conduct academic exchanges. Without his support and encouragement, I could not get the precious opportunity to study in the Netherlands.

I spend my last two PhD years as a joint PhD candidate at University of Twente. To be honest, during the Covid-19 pandemic, it is harder than I expected to study abroad. However, it is fortunate for me that I got a lot of kind help from my supervisor Marc Uetz. Those weekly meetings and discussions are still vivid in my mind. He is so patient to listen to my thoughts and give me feedback. He always gives me encouragement when I am self-doubting. He also provides many detailed suggestions to help to improve our papers. I am grateful to him for sharing with me his way of thinking and writing skills. Honestly, it is really enjoyable and relaxing to work with him. I here would like to show my heartfelt thanks to Marc Uetz who took me to experience the fun of scientific research.

Then, I want to express my sincere thanks to Alexander Skopalik and Walter Matthias. They are generous to share with me inspirations and offer helpful comments to our projects, which helps me go a step further in my research. I also thank for their great patience and kindness. Besides, I do appreciate Marjo Mulder, the secretary of our department, who helped me a lot to organize all kinds of things before I came to the Netherlands and also gave me timely help when I asked for her help during my stay here.

My deep thanks next go to all my other colleagues in the team of Game Theory at Northwestern Polytechnical University and the group of Discrete Mathematics and Mathematical Programming at University of Twente. They gave me a lot of help both in life and work. I would also like to thank all my friends who showed their concern and brought warmth to my life.

Last but not least, I am deeply indebted to my family members. I will never be able to find the words to thank my parents for their unconditional love and support so that I become who I am today. I owe my special appreciation to my husband, Jikun Xie. He always encourages me when I am depressed and gives me the courage to preserve when I want to give up. Thanks for your understanding and company all the way!

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October 2022, Enschede

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